Mathematics 3
for the Swedish 'Gymnasium'

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Chapter 1

Algebra

1.1 Polynomials

Expressions of the type $5x+4$, $3x^2-8x+1$, $25t^3-8t^2-2t+12$ are called *polynomials* (in one variable). To be a polynomial, the variable in the expression can only be raised up to a *natural number* (0, 1, 2, 3, 4, ...).
The highest power of the variable that occurs is called the *degree of the polynomial*.
Polynomial expressions appear often in mathematics and when studying functions, which is the main purpose of the C-course, polynomial functions are a nice (but nevertheless very important) class of functions, easy to understand and easy to work with.
We need to practise the basic skills of algebra to be able to handle polynomial functions in the next chapters.

Remember these rules:

The quadratic rules
(1) $(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$
(2) $(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2$

The conjugate rule
(3) $(a+b)(a-b)=(a-b)(a+b)=a^2-b^2$

Example 1
Which of the following expressions are *not* polynomials?

A $6x^7-3x^5+x-\frac{1}{2}$ B $x^2-\sqrt{x}+1$

C $5x^4-2x^3-x+\frac{1}{x}-5$ D $t^3-1.3t^2+0.2t+\pi$

Solution:
B, because $\sqrt{x}=x^{\frac{1}{2}}$, and ½ is not a natural number.
C, because $\frac{1}{x}=x^{-1}$, and –1 is not a natural number.

Example 2
Expand the expression $(x+1)^2-2(x-1)^2$

Solution:
$(x+1)^2-2(x-1)^2=x^2+2x+1-2(x^2-2x+1)=x^2+2x+1-2x^2+4x-2=$
$=-x^2+6x-1$

Example 3
Simplify $(2x+1)(2x-1)+2(1-2x^2)-1$

Solution:
$(2x+1)(2x-1)+2(1-2x^2)-1=4x^2-1+2-4x^2-1=0$

Example 4
Factorise completely $2x^3-8x$

Solution:
$2x^3-8x=2x(x^2-4)=2x(x+2)(x-2)$

Example 5
If $f(x)=x^2-2x+1$, calculate
a) $f(-2)$
b) $f(x+1)-f(x)$

Solution:
a) $f(-2)=(-2)^2-2(-2)+1=4+4+1=9$
b) $f(x)=x^2-2x+1$
$f(x+1)=(x+1)^2-2(x+1)+1=x^2+2x+1-2x-2+1=x^2$
$f(x+1)-f(x)=x^2-(x^2-2x+1)=2x-1$

Exercises

A

1101 Expand and simplify the following expressions
a) $3(x+4)^2$ b) $-2(3y-1)^2$
c) $(10+4x)^2$ d) $(2y^2+x^3)^2$

1102 Simplify
a) $(x+4)(x-4)$ b) $(x-7)(7+x)$
c) $4(2x-1)(2x+1)$ d) $3(3x-3)(3x+3)-2(x+3)(x-3)$

1103 Simplify
a) $(x-3)^2+(x+2)^2$ b) $3(2x-4)^2-4(3x+2)^2$
c) $2(x+3)^2-(x-4)(x+4)$ d) $4(3t+1)(3t-1)-(4t+1)(9t-4)$

1104 Factorise
a) $xy+y^2$ b) $16-x^2$
c) $2x^2-18$ d) $50-2x^2$

1105 Factorise completely

a) $x^2 - 10x + 25$ b) $2x^2 + 16x + 32$

c) $12a^2 + 12ab + 3b^2$ d) $12 - 48x + 48x^2$

1106 If $f(x) = x^2 - 5x + 2$, find

a) $f(3)$ b) $f(-2)$

1107 If $f(x) = -x^2 - x - 3$, find

a) $f(5)$ b) $f(-3)$

c) $f(\frac{1}{2})$ d) $f(-\frac{1}{3})$

1108 If $f(x) = 3x - x^2$, and $g(x) = x^3 - 2x + 1$, find

a) $f(2) - g(2)$ b) $f(1) - g(-1)$

c) $f(-0.5) + g(0.5)$ d) $g(x) - f(x) - 1$

B

1109 Factorise completely

a) $p^4 - 1$ b) $80t^5 - 5t$

c) $x^2 + 7x + 10$ d) $12x + 52x^2 - 40x^3$

1110 Simplify

a) $(1.5x - 0.2)(2.6 - 0.8x)$ b) $(a+b)(a-b)(a+b)$

c) $(p-1)(p+1)(p^4+1)(p^2+1)$ d) $(x-5)(2x-3)(5+x)(3+2x)$

1111 Expand

a) $(a+b)^3$ b) $(a-b)^3$

c) $(1+x)^3$ d) $(2x-3)^3$

1112 If $f(x) = x^2 - x - 1$, find

a) $f(x-1)$ b) $f(2x)$

c) $f(-x)$ d) $f(x+h) - f(x)$

Polynomial equations

You have already solved some types of equations and throughout this text you will meet old types as well as new ones. Solving equations is one of the keystones of mathematics so it's a good idea to spend some extra time working on these.

We will meet many equations involving polynomials. *The degree* of a polynomial equation is the same as the degree of the polynomial, which is the highest power of the variable that occurs.

The solutions to an equation are sometimes called the *roots* of the equation.

Equations of 1st degree (Linear equations)

Example 1

Solve the following equation:

a) $3(2x-1) - 4(x+3) = 0$ (expand)

$6x - 3 - 4x - 12 = 0$ (simplify and rearrange)

$2x = 15$ (divide by 2)

$x = 7.5$

b) $-0.3(3t - 1.2) - 0.2(7 - 5t) = 1.5(t - 0.4)$ (expand)

$-0.9t + 0.36 - 1.4 + t = 1.5t - 0.6$ (simplify and rearrange)

$-0.44 = 1.4t$ or $1.4t = -0.44$ (divide by 1.4)

$t \approx -0.31$

Equations of 2nd degree (Quadratic equations)

Remember the three types of quadratic equations, an example of each type is illustrated below.

Example 2

Solve the following equations:

a) *(Type 1)*

$125 - 5x^2 = 0$ (add $5x^2$ to both sides and divide by 5)

$x^2 = 25$ (take square roots of both sides)

$x = \pm 5$ or $\begin{cases} x_1 = 5 \\ x_2 = -5 \end{cases}$

b) *(Type 2)*

$\frac{36x^2 + 6x}{7} = 0$ (factorise)

$\frac{6x(6x+1)}{7} = 0$ (Think!)

$\begin{cases} x_1 = 0 \\ x_2 = -\frac{1}{6} \end{cases}$

c) *(Type 3)*
$x^2 - 5x + 6 = 0$ (use the formula with $a = 1, b = -5, c = 6$)

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$

Equations of higher degree

Solving polynomial equations of higher degree than 2 is complicated and the algebraic methods for 3rd and 4th degree equations demand much more algebra theory. The general 5th degree equation cannot even be solved at all using algebraic methods.

We can, however, solve some higher degree equations if we are aware of a basic mathematical fact:

If $a \cdot b = 0$ *then either* $a = 0$ *or* $b = 0$ *(or both)*

Note: This is only true for zero! If $ab = 1$ then neither a nor b have to be 1.

Example 3
Solve the following equations:

a) $4x^3 - 12x^2 - 7x = 0$ (factorise)
$x(4x^2 - 12x - 7) = 0$ (either $x = 0$ or $4x^2 - 12x - 7 = 0$)

Solving $4x^2 - 12x - 7 = 0$ by formula gives $x = 3.5$ or $x = -0.5$.

The complete solution is:

$$\begin{cases} x_1 = 0 \\ x_2 = 3.5 \\ x_3 = -0.5 \end{cases}$$

b) $(x^2 - 4)(2x + 5) = 0$ (either $x^2 - 4 = 0$ or $2x + 5 = 0$)

$x^2 - 4 = 0$ gives $x = 2$ or $x = -2$.
$2x + 5 = 0$ gives $x = -2.5$

The complete solution is:

$$\begin{cases} x_1 = -2 \\ x_2 = 2 \\ x_3 = -2.5 \end{cases}$$

c) $t^4 - 1 = 0$ (factorise)
$(t^2 + 1)(t^2 - 1) = 0$ (factorise completely!)
$(t^2 + 1)(t + 1)(t - 1) = 0$ (either $t^2 + 1 = 0$ or $t + 1 = 0$ <u>or</u> $t - 1 = 0$)

$t + 1 = 0$ gives $t = -1$
$t - 1 = 0$ gives $t = 1$
$t^2 + 1 = 0$ is impossible! (has no real roots!)

The complete solution is:

$$\begin{cases} t_1 = -1 \\ t_2 = 1 \end{cases}$$

Exercises

Solve the following equations.

A

1113 a) $13(3x - 5) + 3(23 - 11x) = 0$ b) $4x - (2x - 1) \cdot 2 - 3(1 - x) = 0$
c) $2(3x - 2) - 5(1 - x) = 2x - 3(3 - 3x)$ d) $3x - 2(3 - 7x) = 5(2x - 3) + 1$

1114 a) $2x - 0.9(x - 1) = 3.54$ b) $0.2(2a - 1) = 3 - 0.25(3a - 1)$
c) $0.5(1 + x) = 0.2(2x - 1)$ d) $3.2x - (x - 2.7) = 3.1(4.5 - 0.4x)$

1115 a) $29 + x(x - 4) = (x + 3)^2$ b) $x^2 + 4x - 5 = 0$

1116 a) $x^2 - 2x - 15 = 0$ b) $3x^2 - 3x - 18 = 0$

1117 a) $1 + x^2 = 2x$ b) $x + 4 = x(x + 4)$

1118 a) $(y - 2)^2 - 1 = -(y - 1)^2$ b) $x(x + 3) = 0$

1119 a) $4x(1 + x) = 0$ b) $517x(9 - 27x) = 0$

1120 a) $(x + 4)(x + 2) = 0$ b) $x^3 - 2x^2 = 0$

1121 a) $4x^2 - 12x^3 = 0$ b) $8x^2 = 27x^5$

1.2 Rational expressions

We define *rational numbers* as quotients of two integers, like $\frac{1}{3}$, $\frac{3}{1}$ and $\frac{119}{100}$.

In a similar way we define *rational expressions* as quotients of two polynomials.
A rational number is not defined if the denominator is zero (we can never divide by zero), and for the same reason a rational expression is not defined if the polynomial in the denominator is equal to zero.
If this happens for certain variable values we have to avoid those values for the variable when evaluating the expression.

Example 1

$\frac{x^2-9}{x+3}$, $\frac{3t^5+2t^2-t+9}{t^3+2t^2-t-2}$ and $\frac{p}{p^2+1}$ are all rational expressions.

$\frac{x^2-9}{x+3}$ is not defined if $x=-3$.

$\frac{3t^5+2t^2-t+9}{t^3+2t^2-t-2}$ is not defined if $t=-1$, 1 or -2

$\frac{p}{p^2+1}$ is defined for all values of p

In the study of functions we often meet rational expressions so we need to practise the associated algebra; how to simplify, add, subtract, multiply and divide rational expressions.
We compare the calculations with those for rational numbers (fractions).

Simplifying

We often want to write a fraction as simply as possible, we write $\frac{1}{2}$ rather than $\frac{3}{6}$, for example. In the same way we prefer rational expressions written in their simplest form.
We say that we *cancel* as many common factors as possible in the numerator and denominator. To do that we must firstly factorise both the numerator and denominator.

Example 2

a) $\frac{60}{36}=\frac{3\cdot4\cdot5}{3\cdot4\cdot3}=\frac{5}{3}$

b) $\frac{6x}{4x^2-2x}=\frac{2\cdot3\cdot x}{2\cdot x\cdot(2x-1)}=\frac{3}{2x-1}$

Addition and subtraction

When we add or subtract fractions we need to have a common denominator (not necessarily the lowest one). A common denominator can always be obtained by multiplying the denominators by each other.

Example 3

a) $\frac{2}{3}+\frac{1}{5}=\frac{2\cdot5}{3\cdot5}+\frac{1\cdot3}{5\cdot3}=\frac{10}{15}+\frac{3}{15}=\frac{13}{15}$

b) $\frac{2}{x}+\frac{1}{1+x}=\frac{2\cdot(1+x)}{x\cdot(1+x)}+\frac{1\cdot x}{(1+x)\cdot x}=\frac{2+2x}{x(1+x)}+\frac{x}{x(1+x)}=\frac{2+3x}{x(1+x)}$

Multiplication and division

When multiplying fractions we simply multiply numerator by numerator and denominator by denominator.
Division could be called *"multiplication by the inverse"*, which maybe makes better sense if we consider this:

Dividing by 2	is the same as	multiplying by $\frac{1}{2}$.
Dividing by $\frac{1}{2}$	is the same as	multiplying by 2 .
Dividing by $\frac{3}{2}$	is the same as	multiplying by $\frac{2}{3}$.
Dividing by $\frac{1}{100}$	is the same as	multiplying by 100 .
Dividing by x	is the same as	multiplying by $\frac{1}{x}$.
Dividing by $\frac{1}{x}$	is the same as	multiplying by x .
	and so on ...	

Example 4

a) $\frac{3}{5}\cdot\frac{2}{7}=\frac{3\cdot2}{5\cdot7}=\frac{6}{35}$

b) $2\cdot\frac{5}{13}=\frac{2}{1}\cdot\frac{5}{13}=\frac{2\cdot5}{1\cdot13}=\frac{10}{13}$

c) $\frac{2}{x}\cdot\frac{x+1}{x^2}=\frac{2\cdot(x+1)}{x\cdot x^2}=\frac{2x+2}{x^3}$

d) $(x-3)\cdot\frac{4x^2}{(2+3x)}=\frac{(x-3)}{1}\cdot\frac{4x^2}{(2+3x)}=\frac{(x-3)\cdot4x^2}{1\cdot(2+3x)}=\frac{4x^3-12x^2}{(2+3x)}$

Example 5

a) $\frac{1}{2}\Big/3=\frac{1}{2}\Big/\frac{3}{1}=\frac{1}{2}\cdot\frac{1}{3}=\frac{1\cdot1}{2\cdot3}=\frac{1}{6}$

b) $\frac{2}{5}\Big/\frac{3}{4}=\frac{2}{5}\cdot\frac{4}{3}=\frac{2\cdot4}{5\cdot3}=\frac{8}{15}$

c) $\frac{1}{2+x}\Big/(x+1)=\frac{1}{2+x}\Big/\frac{x+1}{1}=\frac{1}{2+x}\cdot\frac{1}{x+1}=\frac{1\cdot1}{(2+x)\cdot(x+1)}=\frac{1}{(x+1)(x+2)}$

d) $\frac{2x^2}{x-1}\Big/\frac{8x}{x-1}=\frac{2x^2}{x-1}\cdot\frac{x-1}{8x}=\frac{2x^2\cdot(x-1)}{(x-1)\cdot8x}=\frac{x}{4}$

Exercises

A

1201 Find the x-values for which the following expressions are not defined

a) $\dfrac{1}{x-1}$ b) $\dfrac{x}{(x+1)^2}$

c) $\dfrac{3+x}{x(x-5)}$ d) $\dfrac{1-x}{(x-2)(x^2+1)}$

1202 Write as simple as possible

a) $\dfrac{8}{12}$ b) $\dfrac{48}{84}$

c) $\dfrac{21}{77}$ d) $\dfrac{18}{72}$

1203 Write as a fraction with denominator 24

a) $\dfrac{1}{4}$ b) $\dfrac{3}{8}$

c) $\dfrac{5}{6}$ d) $\dfrac{5}{-4}$

1204 Write as a fraction with denominator $6x$

a) $\dfrac{1}{x}$ b) $\dfrac{2}{3}$

c) $\dfrac{5}{3x}$ d) $\dfrac{x}{2}$

1205 Factorise the numerator and simplify

a) $\dfrac{3h+h^2}{h}$ b) $\dfrac{5hx+h^2}{h}$

c) $\dfrac{h^2-8h}{h}$ d) $\dfrac{4xh-3h+h^2}{h}$

1206 Simplify

a) $\dfrac{(3+h)^2-9}{h}$ b) $\dfrac{2(3+h)^2-2\cdot 3^2}{h}$

1207 Calculate

a) $\dfrac{3}{5}-\dfrac{1}{10}$ b) $\dfrac{5}{7}+\dfrac{4}{5}$

1208 Simplify

a) $\dfrac{1}{x}+\dfrac{3}{x}-\dfrac{2}{x}$ b) $\dfrac{3}{4x}+\dfrac{7}{4x}$

1209 Simplify

a) $\dfrac{1}{x}-\dfrac{1}{3x}$ b) $\dfrac{1}{3a}+\dfrac{1}{4a}-\dfrac{1}{6a}$

1210 Evaluate and simplify the following

a) $\dfrac{3}{5}\cdot\dfrac{7}{9}$ b) $\dfrac{21}{54}\cdot\dfrac{90}{112}$

c) $\dfrac{x}{4}\cdot\dfrac{2}{5}$ d) $\dfrac{3}{8x}\cdot\dfrac{20x}{21}$

1211 a) $2x\cdot\dfrac{x}{6}$ b) $\dfrac{3x}{4}\cdot\dfrac{6x}{5}$

c) $\dfrac{3}{15}\cdot\dfrac{x-3}{12}$ d) $\dfrac{1+x}{x}\cdot\dfrac{3x}{2}$

1212 a) $\dfrac{2}{3}\Big/\dfrac{10}{21}$ b) $\dfrac{18}{35}\Big/\dfrac{30}{49}$

c) $5\Big/\dfrac{5}{9}$ d) $\dfrac{2}{9}\Big/6$

1213 a) $\dfrac{1}{x}\Big/\dfrac{2}{3x}$ b) $\dfrac{x}{3}\Big/\dfrac{2x^2}{1+x}$

c) $\dfrac{3x}{2}\Big/x$ d) $2x^2\Big/\dfrac{3x}{5}$

B

1214 Find the x-values for which the following expressions are not defined

a) $\dfrac{(x+2)^7}{(x-3)(x+7)^3}$ b) $\dfrac{x(x-4)}{(x-4)}$

c) $1+\dfrac{x-1}{2+3x^2}$ d) $\dfrac{1}{x^2+4x+4}$

1215 Simplify the following

a) $\dfrac{x+2}{5x}\cdot\dfrac{10x^2}{x+2}$ b) $\dfrac{t+1}{4t}\cdot\dfrac{2t}{3t+3}$

c) $\dfrac{x+5}{6x}\Big/\dfrac{x+5}{2x^3}$ d) $\dfrac{x(2-x)}{3}\Big/\dfrac{5x}{6}$

1216 Simplify

a) $\dfrac{3x}{4}+\dfrac{x-2}{6}$ b) $\dfrac{3x}{4}-\dfrac{x-2}{6}$

1217 Simplify

a) $\dfrac{x-2}{2x}+\dfrac{x+3}{3x}$ b) $\dfrac{x+1}{3x}-\dfrac{x-2}{5x}$

Equations

Equations with rational expressions are solved using standard methods. The main idea is to eliminate the fractions or rational expressions and then solve as a straightforward polynomial equation.

Warning: Always check your solutions in the original equation. So called *false roots* may appear as they give a zero value in the denominator.

Example 6

Solve the following equations:

a) $\frac{x}{3}+\frac{x}{2}=\frac{1}{5}$ (if we multiply both sides by 3, 2 and 5 (= 30), all fractions will disappear)

$30\cdot\left(\frac{x}{3}+\frac{x}{2}\right)=30\cdot\frac{1}{5} \quad \Rightarrow$

$\frac{30\cdot x}{3}+\frac{30\cdot x}{2}=\frac{30\cdot 1}{5} \quad \Rightarrow \quad 10x+15x=6 \quad \Rightarrow$

$25x=6$

$x=\frac{6}{25}$

b) $\frac{3}{x}-5=\frac{2}{3}$ (add 5 to both sides)

$\frac{3}{x}=\frac{2}{3}+5=\frac{17}{3}$ (multiply by x and by 3)

$9=17x$ (divide by 17)

$x=\frac{9}{17}$

(Check: The solution works in the original equation.)

c) $\frac{1}{x}+\frac{1}{2x}=\frac{1}{3}$ (multiply both sides by a common denominator, $6x$)

$6x\left(\frac{1}{x}+\frac{1}{2x}\right)=6x\cdot\frac{1}{3} \quad \Rightarrow \quad \frac{6x}{x}+\frac{6x}{2x}=\frac{6x}{3} \quad \Rightarrow$

$6+3=2x$

$x=\frac{9}{2}$

(Check: The solution works in the original equation.)

d) $\frac{2}{x}=\frac{1}{x^2+x}$ (multiply by x and (x^2+x))

$2(x^2+x)=x$

$2x^2+x=0$ (factorise)

$x(2x+1)=0$ (either $x=0$ or $2x+1=0$)

This gives that $x=0$ or $x=-\frac{1}{2}$.

However, checking the solutions in the original equation shows that $x=0$

doesn't work (a false root).

The (only) solution is:

$x=-\frac{1}{2}$

Exercises

Solve the equations

A

1218 a) $\frac{x}{3}-5=\frac{x}{4}$ b) $\frac{7x}{5}-\frac{3x}{2}=\frac{9}{10}$

1219 a) $\frac{5}{6}+\frac{p}{3}=\frac{p}{4}+\frac{17}{12}-\frac{p}{2}$ b) $\frac{13y}{4}-2y-\frac{44}{5}=-\frac{19}{5}$

1220 a) $\frac{x+3}{2}=1+\frac{x+4}{5}$ b) $\frac{2t}{3}-\frac{t}{4}=t$

1221 a) $\frac{1}{3x}=\frac{2}{9}$ b) $\frac{4a+3}{a}=\frac{9}{2}$

1222 a) $x+\frac{x}{2}-\frac{x}{3}=\frac{7}{3}$ b) $\frac{x}{2}-\frac{x}{3}=\frac{1}{5}$

1223 a) $4x-\frac{(x+4)}{5}=41$ b) $\frac{2x-1}{5}=3-\frac{3x-1}{4}$

B

1224 a) $\frac{1}{x-x^2}=\frac{2}{x}$ b) $\frac{2}{x-3}=\frac{6}{x^2-3x}$

1225 a) $\frac{1}{x}+\frac{1}{2x}=\frac{1}{2}$ b) $\frac{x-2}{2x}+\frac{x+3}{3x}=\frac{x}{6}$

1226 a) $\frac{3}{x-1}=\frac{2}{x+2}$ b) $\frac{3}{2x}-4=3-\frac{9}{x}$

1227 a) $1-\frac{1}{x}=1+\frac{1}{x}$ b) $1+\frac{1}{x}=1-\frac{1}{x^2}$

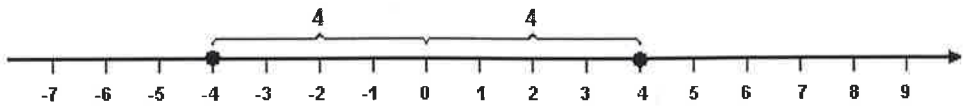
1.3 Absolute value

The *absolute value* or *modulus* of a number is its size, ignoring its sign.
For example, the absolute value of -4 is 4 , and the absolute value of 4 is also 4.

We use the symbol $|\;|$ to denote the absolute value of a number a as $|a|$, so

$$|4| = 4 \quad , \text{ and } \quad |-4| = 4$$

The absolute value of a number can be understood as the distance from zero on the number line, since a distance cannot be negative.

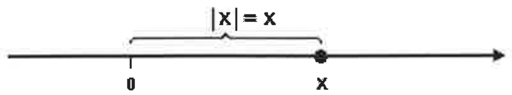


To define the absolute value of an unknown number x we have to consider x being either positive or negative:

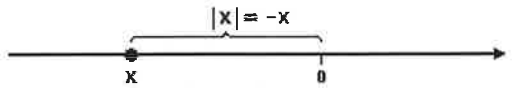
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

By this definition, $|x|$ will always be positive and may be interpreted as the distance from x to 0 on the number line.

If $x \geq 0$



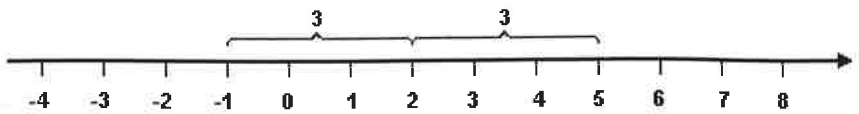
If $x < 0$



The absolute value $|7-3| = 4$ indicates that the difference between, or distance between, 7 and 3 is 4. Similarly we have that $|-1-3| = 4$, which can be interpreted in the same way since the difference, or distance, between -1 and 3 is also 4.
Note that $|7-3| = |3-7| = 4$. The distance from 7 to 3 s of course the same as from 3 to 7.
From this we can conclude that $|a-b|$ will always denote the distance from a to b, no matter if $a > b$ or vice versa.

Equations

It is clear that the equation $|x| = 3$ has two solutions, $x = 3$ and $x = -3$, since $|3| = 3$ and $|-3| = 3$. If the equation is $|x-2| = 3$, then $(x-2)$ must be either 3 or -3 so x can be 5 or -1.
If we think of absolute value as distance, we may interpret $|x-2| = 3$ as "the distance from x to 2 must be 3" and the solutions are 5 and -1, whose distances to 2 both are 3.



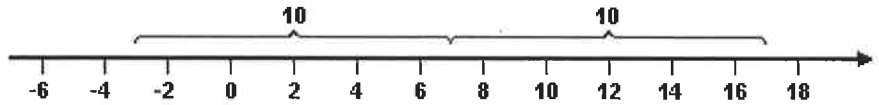
To find x we can either use the definition of absolute value, or use the distance interpretation.

Example 1
Solve the following equations by distance interpretation

a) $|x-7| = 10$ b) $|x+1| = 4$

Solution:

a) The distance from x to 7 shall be 10, which means that x must be 10 more or less than 7.



$x = 17$ or $x = -3$

b) The equation can be written $|x-(-1)| = 4$, which indicates that the distance from x to -1 is 4.

$x = 3$ or $x = -5$

Example 2
Solve the following equations by using the definition of absolute value

a) $|x-3| = 5$ b) $|2x-1| = 4$

Solution:

a) If $(x-3) \geq 0$ (that is, $x \geq 3$), $|x-3| = x-3$ and the equation becomes
$x-3 = 5 \rightarrow x = 8$
If $(x-3) < 0$ (that is, $x < 3$), $|x-3| = -(x-3)$ and the equation becomes
$-(x-3) = 5 \rightarrow -x+3 = 5 \rightarrow x = -2$

Alt.
$|x-3| = 5 \rightarrow x-3 = \pm 5 \rightarrow x = 3 \pm 5 \rightarrow x = 8$ *or* $x = -2$

b) If $2x-1 \geq 0$ (that is, $x \geq \frac{1}{2}$), $|2x-1| = 2x-1$ and the equation becomes
$2x-1 = 4 \rightarrow x = 2{,}5$
If $2x-1 < 0$ (that is, $x < \frac{1}{2}$), $|2x-1| = -(2x-1)$ and the equation becomes
$-(2x-1) = 4 \rightarrow -2x+1 = 4 \rightarrow x = -1{,}5$

Alt.
$|2x-1| = 4 \rightarrow 2x-1 = \pm 4 \rightarrow 2x = 1 \pm 4 \rightarrow x = 2{,}5$ *or* $x = -1{,}5$

From what is said above, we can conclude that the following properties hold for absolute values of any number:

$$|x| \geq 0$$
$$|-x| = |x|$$
$$|a-b| = |b-a|$$
$$\text{If } |x| = a \text{ then } x = \pm a \qquad (a \geq 0)$$

This also means that equations of the type $|2x-3| = |x-1|$ can be rewritten as $2x-3 = \pm(x-1)$ which in turn can be divided into the two equations

$2x-3 = x-1$ and
$2x-3 = -(x-1)$, and so on.

Equations of the type $|x+1| = 7-2x$ can be solved in a similar way but must be a little more carefully examined.

Example 3
Solve the equation $|x+1| = 7-2x$

Solution:
$|x+1| = \begin{cases} x+1 & \text{if } x \geq 1 \\ -(x+1) & \text{if } x < 1 \end{cases}$, which leads to the following two equations:

(1) $x+1 = 7-2x \quad \rightarrow \quad 3x = 6 \quad \rightarrow \quad x = 2$
The solution conform to the initial condition, $x \geq 1$, so it is valid.

(2) $-(x+1) = 7-2x \quad \rightarrow \quad -x-1 = 7-2x \quad \rightarrow \quad x = 8$
In this case, however, the initial condition was $x < 1$, why this is *not* a valid solution.

The (only) solution to the original equation is $x = 2$.

Inequalities
Absolute value can be of practical use in inequalities. If we in a short and simple way want to denote all numbers between –2 and 2, we may use an absolute value sign and write

$$|x| \leq 2$$

which means the same as

$$-2 \leq x \leq 2$$

Similarly

$$|x| > 2 \text{ means that } x > 2 \text{ or } x < -2$$

Like equations we are sometimes interested in finding the values of *x* that satisfy a certain inequality. If absolute values are involved, we usually start by rewriting the inequality without absolute value signs

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Example 4
Find *x* if

a) $|x-2| \leq 4$ b) $|3-2x| \leq 1$

Solution:

a) The inequality may be written
$-4 \leq x-2 \leq 4$
Adding 2 to each part gives the solution
$-2 \leq x \leq 6$
That is, *x* could be any value between –2 and 6.

b) The inequality may be written
$|2x-3| \leq 1$ (since $|a-b| = |b-a|$)
Which means that
$-1 \leq 2x-3 \leq 1$
Adding 3 to each part gives
$2 \leq 2x \leq 4$
Finally, dividing each part by 2 gives the solution
$1 \leq x \leq 2$

Exercises

A

1301 Find the value of
a) $|2-9|$ b) $|-1-4|$
c) $|(-3)\cdot(-5)|$ d) $2\cdot|1-3|-3\cdot|5-2|$

1302 If $a = -2$, $b = 3$ and $c = -4$ find the value of
a) $|a|$ b) $|ab|$
c) $|a-b|$ d) $|a|+|b|$
e) $|a+b|$ f) $|bc|$
g) $|b|\cdot|c|$ h) $|c|^2$

1303 Use a number line to illustrate, and express in words what we mean by
a) $|x| = 2$ b) $|x-1| = 3$
c) $|x+1| = 3$ d) $|x-7| = 1$

1304 Use a modulus (absolute value) sign to write
a) The distance from *x* to 3 b) The distance from *x* to –7
c) The distance from *x* to 0 is 5 d) The distance from *x* to 3 is 4

1305 Solve for *x*
a) $|x| = 5$ b) $|x| = -5$
c) $|x-1| = 2$ d) $|3-x| = 4$

1306 Solve the following equations

a) $|x+2| = 2x+1$ b) $|x-6| = 2x$

c) $|2x+3| = x$ d) $|1-2x| = x+1$

B **1307** Try to figure out if the following expressions are (always) equal:

a) $|a+b|$ and $|a|+|b|$ b) $|ab|$ and $|a|\cdot|b|$

c) $|a-b|$ and $|a|-|b|$ d) $\left|\frac{a}{b}\right|$ and $\frac{|a|}{|b|}$

1308 Use a number line to illustrate, and express in words what we mean by

a) $|x| \le 2$ b) $|x-4| \le 1$

c) $|x+2| \le 2$ d) $|x-1| = |x-3|$

1309 Use a modulus sign to express

a) $-3 \le x \le 3$ b) $-4 < x < 4$

c) $x \le -6$ *or* $x \ge 0$ d) $-\frac{3}{2} < x < \frac{9}{2}$

1310 Solve the following equations

a) $|3x-2| = 1$ b) $|2-5x| = 12$

c) $|x+1| = |2-x|$ d) $|2x+5| = |1-x|$

1311 Write without a modulus sign

a) $|x| < 4$ b) $|x+4| \ge 2$

c) $|2x+1| < 3$ d) $|3-7x| \le 4$

1.4 Geometric sequences and sums

A sequence of numbers can often follow some kind of pattern, or rule. In the sequence 1, 4, 7, 10, 13 we can immediately discover the pattern, or rule, to be "*add 3*" to get the next number in the sequence. If the rule is to add (or subtract) a certain number the sequence is called an *arithmetic sequence*.
In the sequence 3, 12, 48, 192, 768, 3072 we see that the rule is "*multiply by 4*" to get the following number. A sequence where the rule is to multiply like this is called a *geometric sequence*. The number we multiply by to get the next number in the sequence is usually called *the common ratio* of the geometric sequence, because it can always be calculated by taking one number of the sequence and dividing it by the previous number.

Example 1

3, 6, 12, 24	is a geometric sequence with quotient 2
10, 20, 40, 60, 120, 240, 480	is another geometric sequence with quotient 2
4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$	is a geometric sequence with quotient $\frac{1}{2}$
1, 5, 25, 125, 620	is not a geometric sequence

Geometric sequences appear in certain problems and we are often interested in finding the sum of such a sequence. We can, for example, find the sum

$$2 + 8 + 32 + 128 + 512 + 2048$$

rather quickly, but wouldn't it be nice to have some kind of formula to help out the calculation?
You will get one here that is very useful, especially when the sequences are long.

Example 2
Since "the rule" in a geometric sequence is to multiply by the same number over and over again, we may write the sum above like this:

$$2+8+32+128+512+2048 = 2+2\cdot4+2\cdot4^2+2\cdot4^3+2\cdot4^4+2\cdot4^5$$

We call this sum s and write

$$s = 2+2\cdot4+2\cdot4^2+2\cdot4^3+2\cdot4^4+2\cdot4^5 \quad (1)$$

Now, multiply this sum by 4 (the quotient of the sequence) and get

$$4\cdot s = 4\cdot(2+2\cdot4+2\cdot4^2+2\cdot4^3+2\cdot4^4+2\cdot4^5) \quad \text{so}$$

$$4s = 2\cdot4+2\cdot4^2+2\cdot4^3+2\cdot4^4+2\cdot4^5+2\cdot4^6 \quad (2)$$

If we take *(2) - (1)* we will get

$$4s - s = 2\cdot4+2\cdot4^2+2\cdot4^3+2\cdot4^4+2\cdot4^5+2\cdot4^6 - \\ -(2+2\cdot4+2\cdot4^2+2\cdot4^3+2\cdot4^4+2\cdot4^5)$$

$$(4-1)s = 2\cdot4^6 - 2$$

$$s = \frac{2(4^6-1)}{4-1} = 2730$$

Let's try to generalise this method to a formula.

Compare with the example above, but let

the first term in the sequence be	a	instead of	2
the quotient of the sequence be	k	instead of	4
the number of terms in the sequence be	n	instead of	6

then the sum, s_n , is

$$s_n = \frac{a(k^n - 1)}{k - 1}$$ (as long as k is not equal to 1)

Example 3
Suppose you deposit 5000 kr at the beginning of each year into a bank account that gives you an interest rate of 6% per year. How much money do you have on this account at the beginning of the 6th year?

Solution:
The instalments will grow by 6% each year. The values of the instalments at the beginning of the 6th year will be as follows:

Instalment	*Value at 6th year*	
1st year	$5000 \cdot 1.06^5$	(5 years interest)
2nd year	$5000 \cdot 1.06^4$	(4 years interest)
3rd year	$5000 \cdot 1.06^3$	(3 years interest)
4th year	$5000 \cdot 1.06^2$	(2 years interest)
5th year	$5000 \cdot 1.06$	(1 year interest)
6th year	5000	

The total value is

$$5000 + 5000 \cdot 1.06 + 5000 \cdot 1.06^2 + 5000 \cdot 1.06^3 + 5000 \cdot 1.06^4 + 5000 \cdot 1.06^5$$

which is a geometric sum with first number 5000, quotient 1.06 and 6 terms.
According to the formula above, the sum is

$$5000 + 5000 \cdot 1.06 + 5000 \cdot 1.06^2 + 5000 \cdot 1.06^3 + 5000 \cdot 1.06^4 + 5000 \cdot 1.06^5 =$$
$$\frac{5000(1.06^6 - 1)}{1.06 - 1} \approx \frac{2092.6}{0.06} \approx 34877 \text{ (kr)}$$

Example 4
Kip and Diana are going to take out a loan of 300 000 *kr* to rebuild their cave. The money is to be paid back over 10 years, one instalment at the end of each year. This type of loan is a so-called *annuity loan*, which means that the loan is paid back in equally large instalments (*annuities*) each year, including an interest of 8% per year.
How large will the instalments be?

Solution:
The idea is that the total value of the instalments will be the same as the loan after ten years, including interest. The value of the loan after ten years, with 8% interest each year is

$$300000 \cdot 1.08^{10} \approx 647677 \ (kr)$$

We can call the yearly instalment a (*kr*). The value of the instalments at the end of the 10th year will be:

Instalment	*Value after the 10th year (kr)*	
1st year	$a \cdot 1.08^9$	(9 years interest)
2nd year	$a \cdot 1.08^8$	(8 years interest)
3rd year	$a \cdot 1.08^7$	(7 years interest)
4th year	$a \cdot 1.08^6$	(6 years interest)
...	...	...
10th year	a	(no interest)

The total sum of the instalments is

$$\frac{a(1.08^{10} - 1)}{1.08 - 1} \approx \frac{a \cdot 1.1589}{0.08} \approx 14.49a$$

This must be equal to the value of the loan after 10 years, which gives the equation

$$14.49a = 647677$$
$$a = \frac{647677}{14.49} \approx 44709 \ (kr)$$

The annuities will be 44709 *kr* at the end of each year.

Exercises

A

1401 Which of the following sequences are geometric?
A 2, 8, 32, 128, 512 B -1, 1, -1, 1, -1, 1, -1, 1, -1
C 1.5, 4.5, 13.5, 27, 40.5 D $\frac{1}{2}, \frac{3}{4}, \frac{9}{8}, \frac{27}{16}$

1402 Find the common ratio of the following geometric sequences:
a) $5, 5\cdot 10, 5\cdot 10^2, 5\cdot 10^3, 5\cdot 10^4, 5\cdot 10^5$ b) 81, 27, 9, 3, 1
c) $x, 2x, 4x, 8x, 16x, 32x$ d) -3, 12, -48, 192

1403 Calculate the following sums:
a) $1.12+1.12^2+....+1.12^{10}$
b) $1+0.5+0.5^2+0.5^3+....+0.5^7$
c) $2+1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^8}$
d) $1-2+4-8+....+1024$

1404 How many terms of the sequence 1, 3, 9, 27, … must be included to make the sum larger than 100000?

1405 Mr Pig wants to decorate one wall of his house with a pyramid of bricks. How many rows of bricks can be laid, if he wants each row to contain half as many bricks as the row below, and he has 1279 bricks to work with?

B

1406 The magician Merlin has invented a new trick. The first time it took him 30 seconds to perform, but he has discovered that after each practise it took him 2 % less time to do it. How much practise will Merlin have to do to enable him to perform his trick in 5 seconds?

1407 Joakim is investing 1000 *kr* at 6% p.a. at the beginning of each year, starting in 1994. How much money will there be by the end of 2002?

1408 The first term of a geometric sum is 2 and common ratio 1.3. How many terms must be included to give a sum greater than 500?

1409 Solve the equation $x+x\cdot 1.1+x\cdot 1.1^2+....+x\cdot 1.1^5=8538$

1410 A ball is dropped from a height of 3 *m*. It bounces up to 1/3 of its height, and continues doing so until it stops, after 10 bounces. Find the total distance the ball has moved.

1411 A loan of 50000 *kr* was taken out on the 1st of April 1999 and is to be paid back in 10 equal annuities, the first annuity 1 April 2000 and the last 1 April 2009. Find the amount of the annuity, if the annual interest rate is 8 %.

1412 On the Maths Teachers Society's anniversary last year, everybody took 1/25 of what was left of the jubilee cake. How much was left of the cake after 20 people had taken their piece?

Chapter exercises 1

A

1 Evaluate
a) $|-9|$ b) $|2\cdot 3-8|$

2 a) $-3\cdot|1-2|-3\cdot|4-1|$ b) $|-2-3|\cdot|7-3|-\frac{|5-9|}{|12-8|}$

3 Use a modulus sign to write
a) "The distance from x to 0 is 7" b) "The distance from x to 3 is 7"
c) Which x satisfies a) ? d) Which x satisfies b) ?

4 Which of the following expressions can be simplified to a^2+b?
A $a\cdot a+b\cdot b-b$ B $2\cdot a+b$
C $a^2+0.5b^2$ D $2\cdot 0.5b+2a\cdot a$
E $0.5a\cdot 0.5a+b$ F $0.5b+0.5b+0.5a\cdot 2a$

5 a) Simplify the expression $3ab-2a^2b+5ab+a^2b-8ab$.
b) Find the value of the expression when $a=1.5$ and $b=0.2$.
c) Find the exact value of the expression when $a=\frac{2}{3}$ and $b=\frac{3}{4}$.

6 Simplify
a) $\frac{3s^2t}{6st}$ b) $\frac{3xyz}{12xy^2z}$
c) $\frac{3\pi}{\pi^2\cdot r}$ d) $\frac{2a^2\cdot 3b^2}{3ab\cdot 2ab^2}$

7 Expand and simplify
a) $x(x-y)-(x-y)-(x-y)^2-2x\cdot 2y$
b) $(a^2-1)(a^2+1)$

8 Solve the equations
a) $0.1(5x-0.2)=0.3(x-0.2)$ b) $\frac{x}{7}-\frac{2}{3}=1$

9 Solve the equations
a) $28x^2=63$ b) $(2t+3)^2=9+t(t+2)$
c) $x^2-10x+9=0$ d) $\frac{2x^2}{3}-4x=-\frac{10}{3}$

10 $f(x)=ax^2+bx$. Find a and b if
a) $f(1)=0$ and $f(2)=1$ b) $f(-1)=2$ and $f(-2)=6.5$

11 In a right-angled triangle the hypotenuse is 18 cm longer than one of the other sides, which is 82 cm longer than the third side. Find the lengths of the sides of the triangle.

B

12 Expand and simplify

a) $\left(\frac{1}{3}+a\right)(1-a)$

b) $\frac{p}{3}(p-2)-\left(\frac{p}{3}-1\right)^2$

13 Simplify

a) $\frac{5}{1-x}+\frac{3}{3-3x}$

b) $\frac{-3a^2}{a^2-4b^2}+\frac{3a}{a-2b}$

14 Solve the equations

a) $\frac{2}{3-x}+\frac{3}{9-3x}=\frac{1}{3}$

b) $\frac{6}{5r-2}-\frac{1}{4}=\frac{3}{4-10r}$

15 Solve the equations

a) $\frac{y+4}{2}=\frac{4.5}{y+4}$

b) $\frac{5}{x^2+9-6x}-\frac{4}{3-x}=1$

16 The function f is defined by $f(x)=\frac{2x+3}{x-2}$.

a) Find $f(z)$ if $z=f(3)$.

b) Simplify $f(f(x))$ as much as possible.

17 Use a modulus sign to express

a) $-10\le x\le -2$

b) $x\le -1$ *or* $x\ge 13$

18 Solve the following equations

a) $|2x-5|=1$

b) $|2x+3|=|5-x|$

19 The area of a rectangle is 687 m^2 and the perimeter is 135 m. One of the sides of the rectangle is x m.

a) Write down an equation in x from this information.

b) Find the length of the sides of the rectangle.

Chapter 2

Functions