B

12 Expand and simplify

a) $\left(\frac{1}{3}+a\right)(1-a)$

b) $\frac{p}{3}(p-2)-\left(\frac{p}{3}-1\right)^2$

13 Simplify

a) $\frac{5}{1-x}+\frac{3}{3-3x}$

b) $\frac{-3a^2}{a^2-4b^2}+\frac{3a}{a-2b}$

14 Solve the equations

a) $\frac{2}{3-x}+\frac{3}{9-3x}=\frac{1}{3}$

b) $\frac{6}{5r-2}-\frac{1}{4}=\frac{3}{4-10r}$

15 Solve the equations

a) $\frac{y+4}{2}=\frac{4.5}{y+4}$

b) $\frac{5}{x^2+9-6x}-\frac{4}{3-x}=1$

16 The function *f* is defined by $f(x)=\frac{2x+3}{x-2}$.

a) Find $f(z)$ if $z=f(3)$.

b) Simplify $f(f(x))$ as much as possible.

17 Use a modulus sign to express

a) $-10 \le x \le -2$

b) $x \le -1$ *or* $x \ge 13$

18 Solve the following equations

a) $|2x-5|=1$

b) $|2x+3|=|5-x|$

19 The area of a rectangle is 687 m^2 and the perimeter is 135 m. One of the sides of the rectangle is x m.

a) Write down an equation in x from this information.

b) Find the length of the sides of the rectangle.

Chapter 2

Functions

2.1 Functions and graphs

Remember that a function is a rule, where each input value of x gives one output value for y. We say that the value of y depends on the value of x, or "y is a function of x".
Depending on the rule of the function we put them in different groups and give them different names. Functions of the same type have similar properties and the purpose here is to try to characterise different functions and to understand why each type behaves differently.

As each x-value of a function gives one y-value, we may let this pair of numbers (x, y) represent a point in a coordinate system. Working through a sufficient number of x-values let us see what *the graph of the function* looks like. How many points we need depends on how complicated the function is.

Important properties of a function

The graph of a function enables us to see how the function "behaves". By looking at the graph we can find out some of the properties of the function and answer questions like:

- *What is the domain and range of the function?*
- *When is the function positive/negative/zero?*
- *Is the function increasing or decreasing?*
- *Does the function have any maximum or minimum points?*

X-values where the value of the function is 0 are called the *zeroes* of the function. Maximum and minimum points are often called *turning points* with a common name.

To answer questions like these more exactly, however, we need a little more math, which will be introduced in the next chapter.
Here we will take a quick look at some of the *elementary functions* of mathematics, and by looking at their graphs try to answer the above questions.

Drawing graphs of functions

It is often worthwhile to spend a minute thinking before starting to draw a graph of a function. You need some points to get a good picture of the function, but if the idea is to get a sketch of the graph you shouldn't calculate more points than necessary but of course enough points to include the interesting parts of the curve. You need to make a table, choose some x-values and calculate the corresponding y-values. How many x-values you need depends on how complicated the function is. A linear function can be drawn using only 2 points, while a 4th degree polynomial function requires at least 8-10 points to get a good picture.
If the domain is not restricted, you may choose any x-values, and in general it is hard to say which to choose. You need to use your experience and mathematical knowledge to choose the most sensible ones. Your aim must be to include all interesting parts of the curve (like zeroes and turning points).

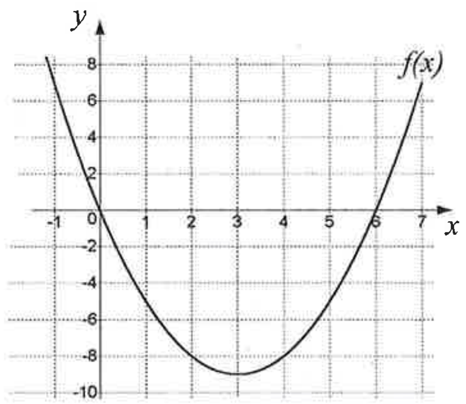
A good method to follow is:

1) Look at the function and see if you know anything about it (are there turning points, zeros, ...?)
2) If you can, try to find the x-values of the zeroes and turning points.
3) Make a table, using the x-values you have found, and maybe a few more in the same neighbourhoods.
4) Draw a coordinate system, choose a suitable scale on each axis (by looking at the values in your table) and plot your points.
5) Do you have enough points to be sure of the shape of the curve? If not, take some more x-values where you seem to need more points and work out the y-values.

Exercises

A

2101 Look at the graph below and find
a) $f(1)$
b) $f(-1)$
c) the turning point of $f(x)$
d) the zeros of $f(x)$



2102 Draw a possible graph of a function that satisfies the following: $f(0)=1$, $f(1)=4$ and $f(4)=0$.

2103 What is the domain of the following functions?
a) $f(x)=x^2$
b) $f(x)=\sqrt{x}$
c) $f(x)=\dfrac{x}{1-x}$
d) $f(x)=\dfrac{1}{\sqrt{1-x}}$

Polynomial functions

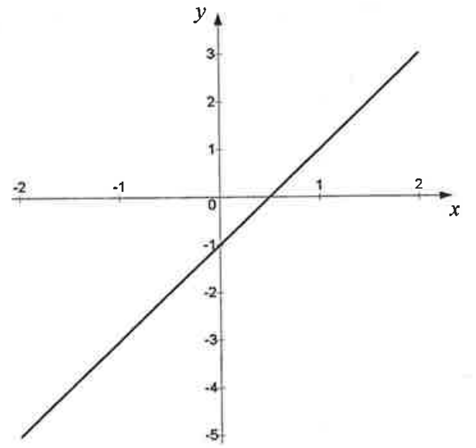
Functions of type $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$,
where a_0 , a_1 , ..., a_n are coefficients and *n* is a positive integer, are called polynomial functions. The number *n* is said to be the *degree* of the polynomial function.

Linear functions *(polynomial functions of degree 1)*

$y = 2x - 1$

Table: Graph:

x	y
-2	-5
-1	-3
0	-1
1	1
2	3



Domain (= allowed *x*-values):
Since all *x*-values can be used, the domain of the function is *all real numbers*, which sometimes is written $-\infty < x < \infty$ or simply $x \in \mathrm{R}$, where R stands for all real numbers (read: "*x* belongs to the set of real numbers").

Range (= possible *y*-values):
It is a pretty good guess that *y* can be any number, so we write
$-\infty < y < \infty$ or $y \in \mathrm{R}$.

When is the function positive/negative/zero?
The value of the function (*y*) is negative when *x* is less than 0.5, zero when *x* is 0.5 and positive when *x* is greater than 0.5. In mathematical symbols that is:
Negative when $x < 0.5$
Zero when $x = 0.5$
Positive when $x > 0.5$

Is the function increasing or decreasing?
The function is increasing (= pointing up) for all *x*-values.

Does the function have any maximum or minimum points?
No, there isn't any point where the value of the function is smaller/greater than any other point in the neighbourhood.

Exercises

A

2104 Draw the graph of
a) $f(x) = 2x - 7$ b) $f(x) = 25x + 250$

2105 Draw the graph of $y = -3x + 12$ and answer the following questions:
a) Is the graph increasing or decreasing ?
b) Where does the graph cut the *y*-axis ?
c) Where does the graph cut the *x*-axis ?

2106 Below are some equations of straight lines. Which are increasing and which decreasing?
A $y = -x + 37$ B $3y - 5 = 2x$
C $x + y + 1 = 0$ D $\frac{3y+1}{4} = x$

2107 Find in each case the gradient of the line between each of the following pairs of co-ordinate points. Plot each pair onto a co-ordinate grid.
a) (-1, 3) and (2, 0) b) (3, 46) and (5, 152)
c) (2, 5) and (2.5, 7.25) d) (3, 0.2) and (3.1, 1.4)

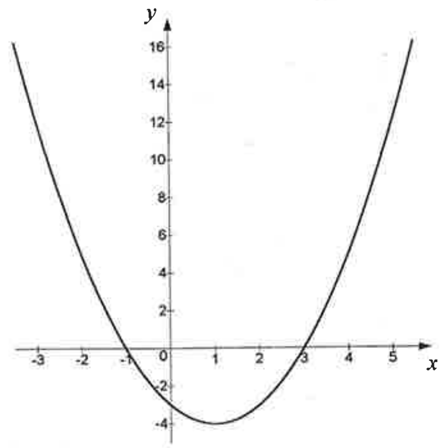
Quadratic functions *(polynomial functions of degree 2)*

$y = x^2 - 2x - 3$

Table:

x	y
-2	5
-1	0
0	-3
1	-4
2	-3
3	0

Graph:



Domain:

$-\infty < x < \infty$

Range:

It looks like y cannot be less than -4 but can have any value larger than this, so

$-4 < y < \infty$

When is the function positive/negative/zero?

The function is negative when x is between -1 and 3, zero when x is -1 and 3, positive when x is smaller than -1 and when x is greater than 3.

We write this:

Negative: $-1 < x < 3$

Zero: $x = -1$ or $x = 3$

Positive: $x < -1$ or $x > 3$

(Note that this cannot be written $3 < x < -1$. Why not?)

Is the function increasing or decreasing?

The function is increasing (= pointing up) when x is greater than 1 and decreasing when x is smaller than 1.

In maths language:

Increasing: $x \geq 1$

Decreasing: $x \leq 1$

Does the function have any maximum or minimum points?

Yes, a minimum point when x is 1 (y is -4).

Minimum point: (1, -4)

Exercises

A

2108 Draw the graph of

a) $f(x) = 2x^2 - 7$ b) $f(x) = 0.5 - 0.1x^2$

2109 Draw graphs of the following functions and find where the functions are increasing/decreasing.

a) $g(x) = x^2 - 3x - 10$ b) $p(x) = 4 - 3x - x^2$

2110 Draw the graph of $f(x) = x - x^2 + 1$ and find from the graph

a) the zeroes of $f(x)$ (the x-values where $f(x) = 0$)

b) the turning point of $f(x)$.

B

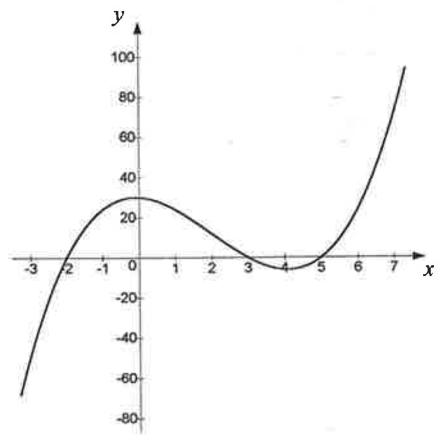
2111 Draw the graph of the curve $y = x^2 - 2x - 8$. Draw a line between the points on the curve where $x = -1$ and $x = 5$. Find the equation of that line.

Higher degree polynomial functions

$y = x^3 - 6x^2 - x + 30$

Table: Graph:

x	y
-3	-48
-2	0
0	30
3	0
4	-6
5	0
6	24



Domain:

$-\infty < x < \infty$

Range:

$-\infty < y < \infty$

When is the function positive/negative/zero?

Negative: $x < -2$ or $3 < x < 5$

Zero: $x = -2$ or $x = 3$ or $x = 5$

Positive: $-2 < x < 3$ or $x > 5$

Is the function increasing or decreasing?

As far as we can see from the graph this seems to be true:

Increasing: $x \leq 0$ or $x \geq 4$

Decreasing: $0 \leq x \leq 4$

Does the function have any maximum or minimum points?

It looks like a maximum point when x is 0 and a minimum point when x is 4.

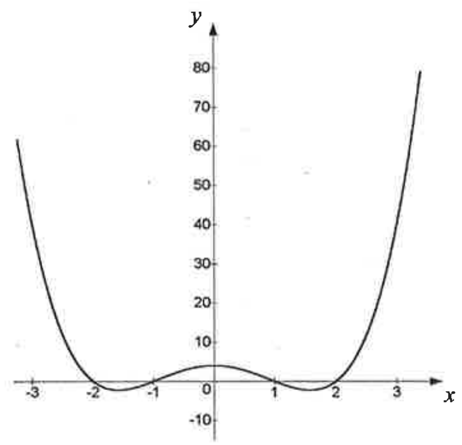
Maximum point: (0, 30)

Minimum point: (4, -6)

$y = x^4 - 5x^2 + 4$

Table: Graph:

x	y
-3	40
-2	0
-1.5	-2.2
-1	0
0	4
1	0
1.5	-2.2
2	0
3	40



Domain:

$-\infty < x < \infty$

Range:

y cannot be less than approximately -2.2 , so

$-2.2 < y < \infty$

When is the function positive/negative/zero?

Negative: $-2 < x < -1$ or $1 < x < 2$

Zero: $x = -2$ or $x = -1$ or $x = 1$ or $x = 2$

Positive: $x < -2$ or $-1 < x < 1$ or $x > 2$

Is the function increasing or decreasing?

It looks like

Increasing: $-1.5 \leq x \leq 0$ or $x \geq 1.5$

Decreasing: $x \leq -1.5$ or $0 \leq x \leq 1.5$

Does the function have any maximum or minimum points?

There are two minimum points; when $x \approx -1.5$ and when $x \approx 1.5$.

There is also a maximum point when $x = 0$.

Minimum points: (-1.5, -2.2) and (1.5, -2.2)

Maximum point: (0, 4)

Exercises

A

2112 Draw the graphs of

a) $f(x) = x^3 - 6$ b) $f(x) = x^4 - 5x^3$

2113 Draw the graph of $f(x) = 2x^3 - 3x^2$ and find from the graph

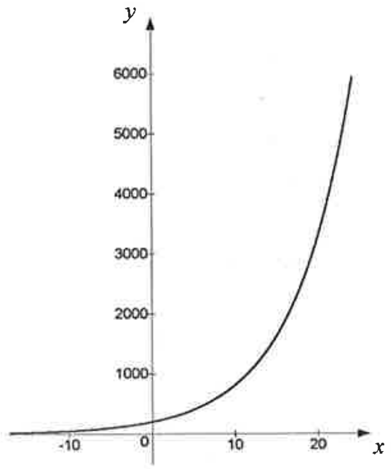
a) the turning points of $f(x)$ b) the zeroes of $f(x)$

Exponential functions
In exponential functions the variable is in the exponent, as in the function $y = 5^x$.
More generally, an exponential function may be written in the form $y = C \cdot a^{k \cdot x}$, where C, a and k are constants.
Exponential functions appear frequently in many applications.

$y = 200 \cdot 1.15^x$

Table: Graph:

x	y
-10	49
0	200
10	809
15	1627
20	3273



Domain:
$-\infty < x < \infty$

Range:
y cannot be negative or zero, so
$0 < y < \infty$

When is the function positive/negative/zero?
Negative: never
Zero: never
Positive: at all x-values

Is the function increasing or decreasing?
Increasing at all x-values.

Does the function have any maximum or minimum points?
No.

Exercises

A **2114** Draw the graphs of
a) $f(x) = 1.3^x$ b) $f(x) = 10 \cdot 2.5^{-x}$

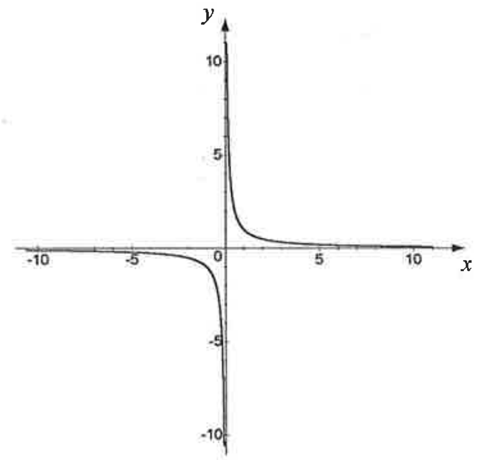
2115 Decide if the following functions are increasing or decreasing
A $f(x) = 0.99^x$ B $f(x) = 25000 \cdot 1.02^x$
C $f(x) = 0.75 \cdot 3^{2x}$ D $f(x) = 10^{-2x}$

Some other functions
There are many other types of functions. In the C-course we will take a brief look at some of them, especially functions of the type $y = x^a$, where a can take any value.

$y = \frac{1}{x}$

Table: Graph:

x	y
-5	-0.2
-2	-0.5
-1	-1
-0.5	-2
0	not def.
0.5	2
1	1
2	0.5
5	0.2



(Note: Since we cannot divide by zero we write *not def.* (not defined) for y when x is 0.)

Domain:
x can be any value but 0. We may write
$-\infty < x < 0$ and $0 < x < \infty$, or simply $x \neq 0$.

Range:
y can never be zero, so
$y \neq 0$

When is the function positive/negative/zero?
Negative: $x < 0$
Zero: never
Positive: $x > 0$

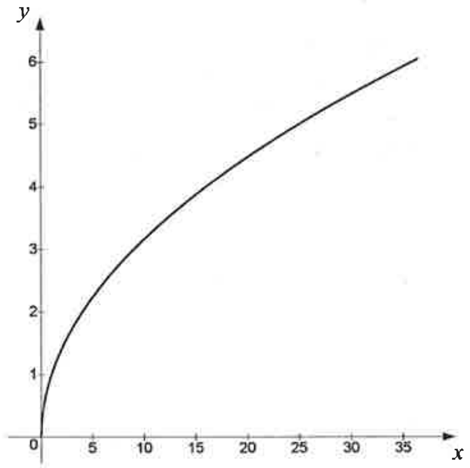
Is the function increasing or decreasing?
Increasing: never
Decreasing: at all x-values in its domain (that is: all x-values except 0).

Does the function have any maximum or minimum points?
No.

$y=\sqrt{x}$

Table: Graph:

x	y
0	0
1	1
2	1.4
4	2
10	3.2
25	5
36	6



Domain:
x cannot be negative,
$x \geq 0$

Range:
y cannot be negative,
$y \geq 0$

When is the function positive/negative/zero?
Negative: never
Zero: $x=0$
Positive: $x>0$

Is the function increasing or decreasing?
Increasing: $x \geq 0$
Decreasing: never

Does the function have any maximum or minimum points?
No.

Exercises

A **2116** Draw the graphs of the following functions

a) $\sqrt{x+2}$ b) $\dfrac{x^{1.5}}{2}$

B **2117** Draw the graph of $f(x)=\dfrac{3-x}{x-1}$ and answer the following questions:

a) Where is $f(x)$ positive ? b) Where is $f(x)$ negative ?
c) Where is $f(x)$ increasing ? d) Where is $f(x)$ decreasing ?

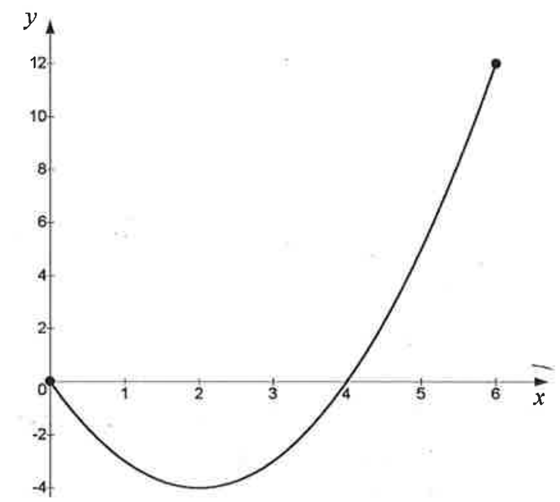
Restricted domains
Sometimes, often in practical problems (and for obvious reasons), we are only interested in the value of the function at some x-values. In such cases we say that we *restrict the domain* to those x-values, and this will of course also affect the range of the function.
When we draw the graph of a function in a restricted domain, we usually mark the endpoints with a point to show that the curve actually stops there. If we don't draw a point, we mean that the curve continues outside our figure.
Let us look at such an example, where we also answer the same questions as above.

$y=x^2-4x$, $0 \leq x \leq 6$ (x can be between 0 and 6)

(This means that although any x-value could be used in the function $y=x^2-4x$, we are now only interested in x-values from 0 to 6.)

Table: Graph:

x	y
0	0
1	-3
2	-4
3	-3
4	0
5	5
6	12



Domain:
$0 \leq x \leq 6$

Range:
Between the x-values 0 and 6, y cannot be less than -4 or greater than 12, so we may write
$-4 \leq y \leq 12$

When is the function positive/negative/zero?
Negative: $0<x<4$
Zero: $x=0$ or $x=4$
Positive: $4<x \leq 6$

Is the function increasing or decreasing?
Increasing: $2 \leq x \leq 6$
Decreasing: $0 \leq x \leq 2$

Does the function have any maximum or minimum points?
Minimum point: (2, -4)

(Actually, as we shall see later, the endpoints here are normally regarded as maximum points)

Exercises

A **2118** Draw the graph of the following functions between $x=-1$ and $x=2$

a) $f(x)=4x-x^3$ b) $g(x)=2x^2-\dfrac{3x^3}{2}$

B **2119** Draw the graph of the following functions between $x=-1$ and $x=2$ and try to figure out whether the function has any turning points or zeroes in this interval.

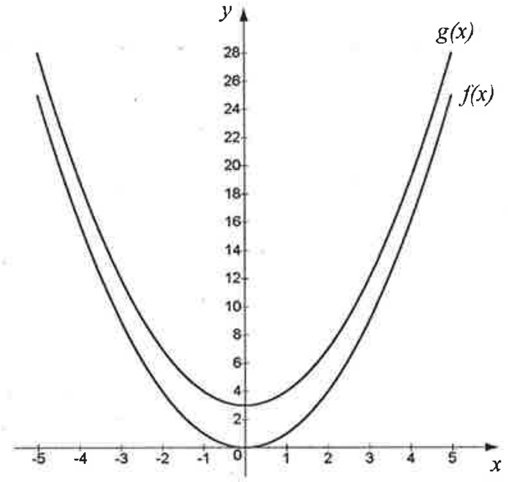
a) $f(x)=3x^4+4x^3-12x^2+12$ b) $g(x)=x^3-2^x$

Adding a constant to a function
How will the graph of a function change if we add a constant number to the function? To see this, we start by comparing the two functions $f(x)=x^2$ and $g(x)=x^2+3$.

Table:

x	$f(x)$	$g(x)$
-4	16	19
-2	4	7
0	0	3
2	4	7
4	16	19

Graph:



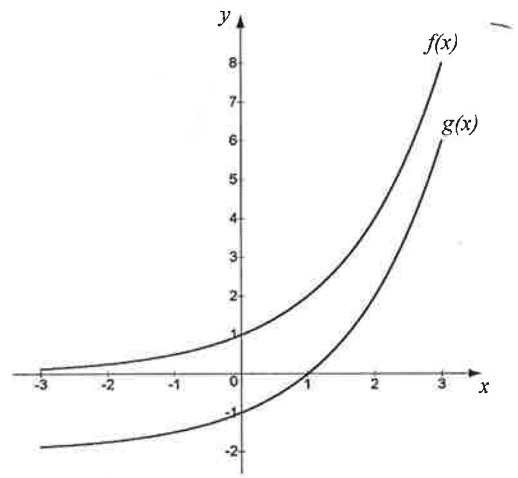
As we can see, the shape of the two graphs is the same though $g(x)$ is lying 3 units above $f(x)$ at each x-value.
Try to figure out where the graph of $h(x)=x^2-4$ would be using the same coordinate system.

If we compare the graphs of the two exponential functions $f(x)=2^x$ and $g(x)=2^x-2$ the result is similar, as you can see below.

Table:

x	$f(x)$	$g(x)$
-3	0.1	-1.9
-1	0.5	-1.5
0	1	-1
1	2	0
3	8	6

Graph:



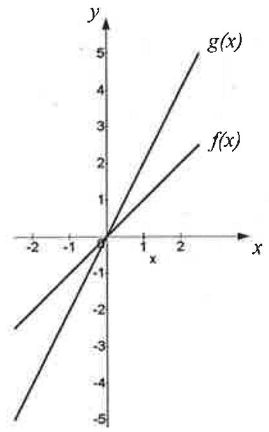
When we add a constant to a function this means that this constant will be added to each value of the function, and the graph will simply be moved up (or down, if the constant is negative) this many steps in the coordinate system. The shape of the curve will be the same.

Multiplying a function by a constant

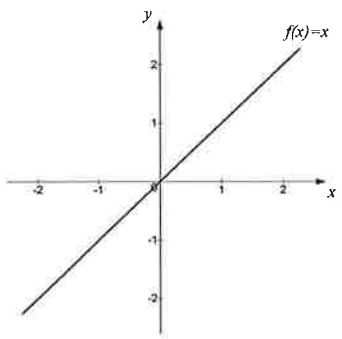
Multiplying a function $f(x)$ by a constant will affect the graph in a slightly different way. Let us compare the functions $f(x)=x$ and $g(x)=2x$.

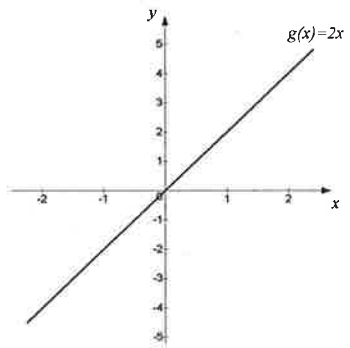
Table: Graph:

x	f(x)	g(x)
-2	-2	-4
-1	-1	-2
0	0	0
1	1	2
2	2	4



We see that $g(x)$ has a steeper slope than $f(x)$.
If we multiply the scale of the y-axis by 2, and draw $g(x)$ in this new coordinate system instead, the graphs look identical.



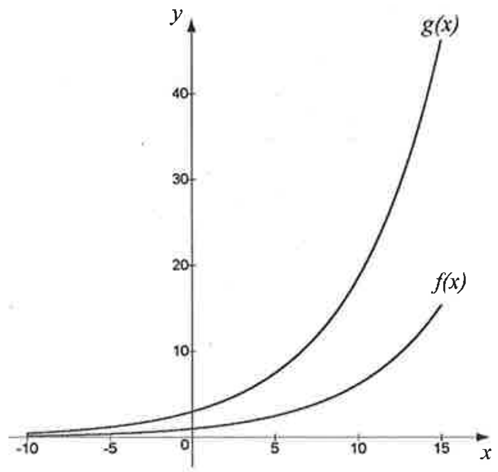


We could now make a guess that if we want to multiply a function by a constant, we can get the new graph by copying the graph of the original function and multiplying the y-scale by that constant.

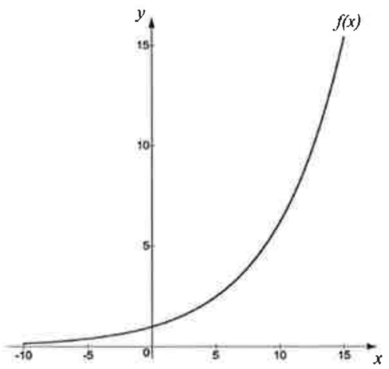
Let us check our guess with another function. We compare $f(x)=1.2^x$ with $g(x)=3\cdot 1.2^x$.

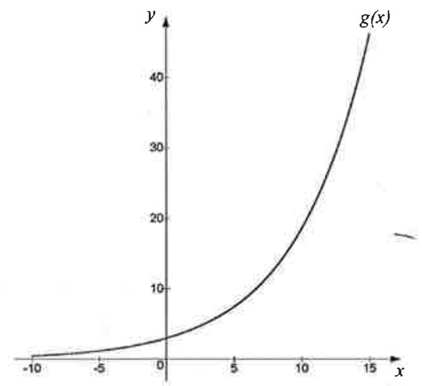
Table: Graph:

x	f(x)	g(x)
-5	0.4	1.2
0	1	3
5	2.5	7.5
10	6.2	18.6
15	15.4	46.2



If we multiply the y-scale by 3, and redraw $g(x)=3\cdot 1.2^x$ on this coordinate system we can conclude that our guess seems to be correct.





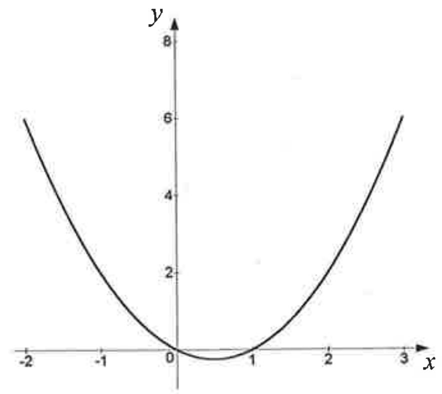
Exercises

A

2120 The figure shows the graph of $f(x)=x^2-x$. Try, without any calculations, to sketch

a) $f(x)=x^2-x-2$

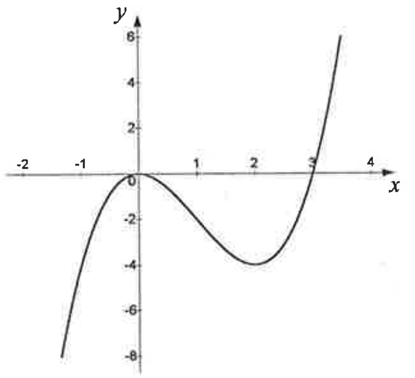
b) $f(x)=3(x^2-x)$



2121 The figure shows the curve $y=x^3-3x^2$. Try, without any calculations, to draw a sketch of

a) $y=x^3-3x^2+3$

b) $y=2x^3-6x^2$



B

2122 Draw the graph of $y=27x^2-3x^3-72x+60$ between $x=1$ and $x=5$. Try, without any calculations, to draw a sketch of

a) $y=x^3-9x^2+24x-20$ b) $y=9x^2-x^3-24x$

2.2 Continuous and discrete functions

Most functions we meet have continuous domains which means that the variable can take any value on each part of the number line. For example, the function $f(x)=x^2$ is continuous everywhere, since the variable x can take any value.

A *discrete* function is only defined for distinct and separate values of the variable, which means that the domain of a discrete function consists of a set of unconnected values. Often we meet discrete functions where the domain is the set of natural numbers (0, 1, 2, 3, ...).

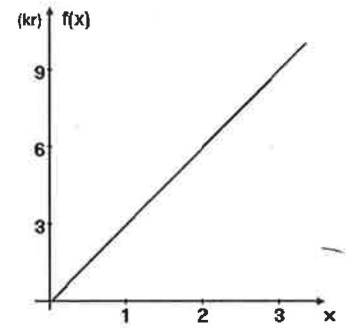
If for example $f(x)$ is the cost for x persons to visit a concert, then $f(x)$ is a discrete functions, since x can only be natural numbers.
The graph of a discrete function is therefore a set of points which are not connected, while the graph of a continuous function is a smooth curve.

If nothing else is stated, normally the functions we meet are continuous.

Example 1

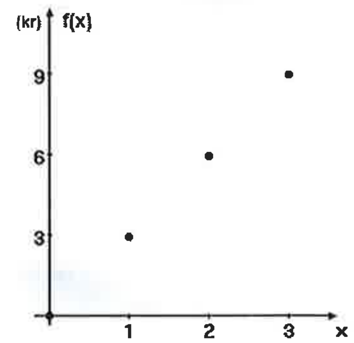
If potatoes are sold for 3 kr/kg, then the total price is a continuous function of the weight;
$f(x)=3x$, where $f(x)$ is the total cost for x kg and x can take any value ≥ 0.

The graph of this function is a continuous line where any point along the line has a meaning for the function.



If apples are sold for 3 kr each, then the total price is a discrete function of the number of apples.
$f(x)=3x$, where $f(x)$ is the total cost for x apples, but x can only be a natural number.

The graph is a set of points which are not connected, since only these points have a meaning for the function.



We can summarize the differences between these two concepts as follows:

	Continuous function	*Discrete function*
Domain	Any value within an interval.	Only distinct, separate values, unconnected.
Applications	Length depending on age Temperature depending on time Weight depending on size	People in a shop depending on hours Income from number of sold units Number of biscuits depending on size of package
Range (y-values)	Can take any value. In applications often measured	Often counted, which gives natural numbers
Graph	A smooth line or curve which can be drawn without lifting the pencil	A series of unconnected points

Functions may be continuous over the whole real line, or on some interval. A function is said to have a discontinuity if its graph has a jump or hole in it. Thus, functions can be continuous everywhere or piecewise. However, discrete functions are nowhere continuous.

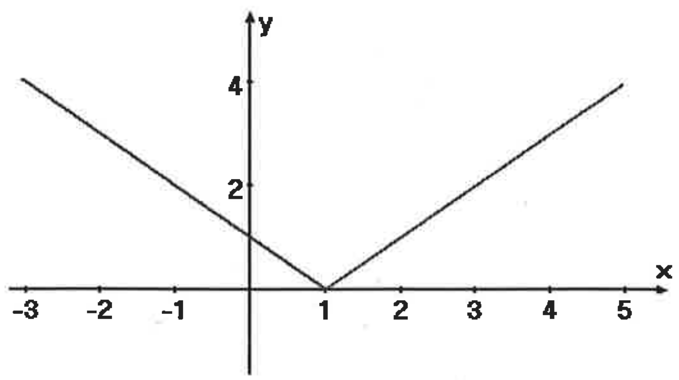
Here follows some different types of functions.

Example 2

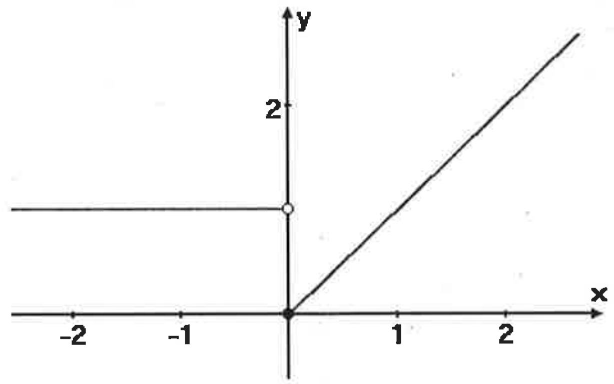
a) The absolute value function $f(x)=|x-1|$ can be written

$$f(x)=\begin{cases} x-1 & \text{if} \quad x\geq 1 \\ -(x-1) & \text{if} \quad x<1 \end{cases}$$

The graph of $f(x)$ consists of two parts, $y=x-1$ for $x\geq 1$ and $y=-x+1$ for $x<1$. Although the graph changes at $x = 1$, it is connected at this point so it is continuous over the whole real line.

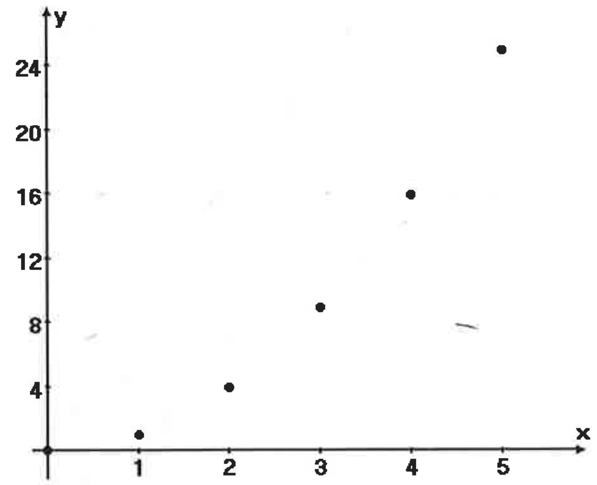


b) The function $f(x)=\begin{cases} 1 & \text{if} \quad x<0 \\ x & \text{if} \quad x\geq 0 \end{cases}$ is piecewise defined and consists of two parts. For $x < 0$ by the line $y = 1$ and for $x\geq 0$ by the line $y = x$. The graph has a gap at $x = 0$ where it jumps from 1 to 0. Although the function is defined everywhere it is not continuous at $x = 0$.



Example 3

a) The *quadratic numbers* 0, 1, 4, 9, 16, 25, ... can be regarded as a discrete function $f(x)=x^2$, with the domain $\mathbb{N}$ (the natural numbers).

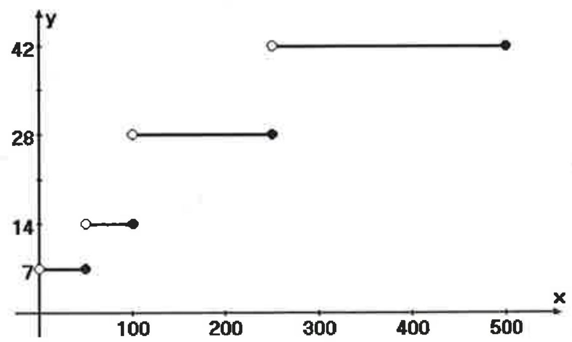


b) The cost for sending letters in Sweden is a piecewise defined function, where the postage is a function of the weight, as follows:

Weight (*g*)	50	100	250	500
Postage (*kr*)	7	14	28	42

Or, in a mathematical way:

$$f(x)\begin{cases} 7 & , \quad 0<x\leq 50 \\ 14 & , \quad 50<x\leq 100 \\ 28 & , \quad 100<x\leq 250 \\ 42 & , \quad 250<x\leq 500 \end{cases}$$



The graph is defined for all $x\leq 500$ but has several discontinuities.

Exercises

A

2201 Decide whether the following functions are discrete or continuous
a) The speed of a skydiver, t seconds after takeoff.
b) The number of cars at a parking lot, x hours after midnight.
c) The sum of the first n positive integers.
d) The height of a tree as a function of its age.

2202 Decide if the following functions are continuous over the whole real line or not.

a) $f(x)=\dfrac{1}{x}$ b) $f(x)=\dfrac{x^3-2x^2+x-1}{x-1}$

c) $f(x)=|x|$ d) $f(x)=\begin{cases} 2 & \text{if } x<1 \\ x+1 & \text{if } x\geq 1\end{cases}$

2203 If $f(x)$ and $g(x)$ are continuous everywhere. What can you say about the function
a) $y=f(x)+g(x)$ b) $y=f(x)-g(x)$
c) $y=f(x)\cdot g(x)$ d) $y=\dfrac{f(x)}{g(x)}$

B

2204 Decide if the following functions are continuous over the whole real line or not. Sketch the graph of each function.

a) $f(x)=\dfrac{1}{1-x^2}$ b) $f(x)=\dfrac{1}{1+x^2}$

c) $f(x)=\begin{cases} 2x+1 & \text{if } x\leq 1 \\ 4-x & \text{if } x>1\end{cases}$ d) $f(x)=\dfrac{x}{\sqrt{x^2}}$

2205 Are the following functions continuous, discrete, or neither?

a) $f(x)=\dfrac{4}{2^x}$, $D_f=\mathbb{N}$ b) $f(n)=\dfrac{1}{2^n}$, $n=\{0,1,2,3,4\}$

c) $f(x)=\begin{cases} x^2-2x+1 & , x<2 \\ 5-2x & , x\geq 2\end{cases}$ d) $f(x)=\begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational}\end{cases}$

2.3 Polynomial functions

We shall now take a look at polynomial functions once again, and see how they may appear in practical problems. If you don't feel familiar enough with the basics of polynomial functions you can always go back to the revision chapter to refresh your memory.

Linear functions

A linear function can always be written in the form
$y=mx+c$
where m is the slope (or gradient) of the line and c is the y-intercept. This is often called *the equation of a straight line.*

Example 1
Find the slope and y-intercept of the straight line $4x+2y-3=0$.

Solution:
If we rewrite the equation as
$y=-2x+3$
we can immediately find the slope, –2, and y-intercept, 3.

Example 2
Find the equation of the straight line passing through the points (2, 25) and (5, 61).

Solution:

The slope is $m=\dfrac{\textit{change in } y}{\textit{change in } x}=\dfrac{61-25}{5-2}=\dfrac{36}{3}=12$

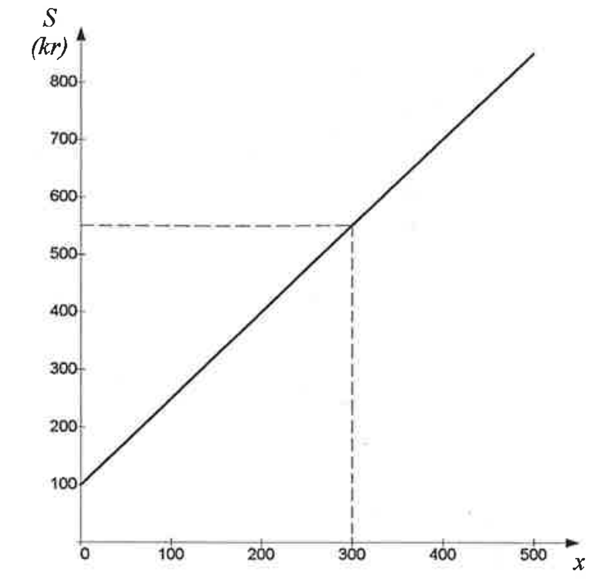
The equation is $y=mx+c$, but $m=12$, so $y=12x+c$.
The point (2, 25) is on the line, so $25=12\cdot 2+c$, which gives $c=1$.
The equation is $y=12x+1$.

Example 3
Bob is selling newspapers. His salary each week consists of a basic fee of 100 kr and 1.50 kr for each newspaper sold.
a) Describe his salary, S, as a function of the number of newspapers sold, x. What is the domain and range for this linear function?
b) Draw a graph of the function in a coordinate system and use the graph to find out how many newspapers he has to sell to earn 550 kr one week.

Solution:
a) $S(x)=100+1.5x$ (a straight line with slope 1.5 and y-intercept 100)
Theoretically he could sell any number of newspapers (not a negative number, though), so we write
Domain: $0\leq x<\infty$
The range here is how much money he could earn, which could be anything from 100 kr and up.
Range: $100\leq S<\infty$

b) Table: Graph:

x	y
0	100
100	250
200	400
500	850



From the graph we find that a salary of 550 kr corresponds to 300 newspapers sold.

Example 4
Holy Underwear, Ltd, a clothing firm, has *fixed costs* of \$2000 per year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. When producing their only article, a plastic bra, it costs \$4 per unit in addition to the fixed costs. That is, the *variable cost* for producing x of these units is $4x$ dollars.

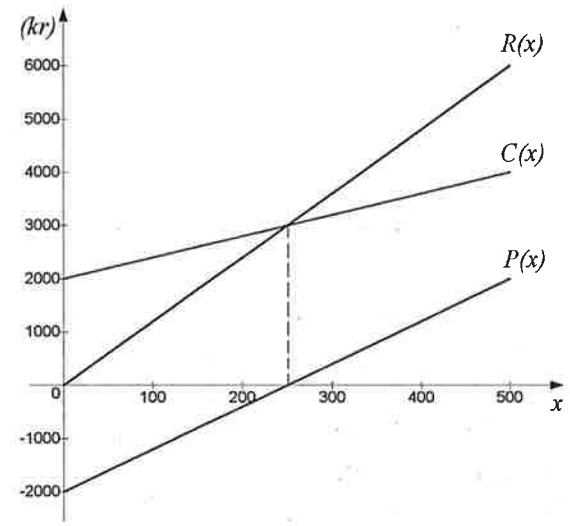
a) Describe the total cost per year, C, as a function of the number of bras produced in a year.
b) What is the total cost of producing 100 units? 500 units?
How much more does it cost to produce 500 units than 100 units?
c) Find the total cost per bra when producing 100 units and 500 units respectively.

Solution:

a) The *total cost* $C(x)$ of producing x bras in a year will be
$C(x) = (\textit{variable costs}) + (\textit{fixed costs}) = 4x + 2000$
b) The total cost of producing 100 units is
$C(100) = 4 \cdot 100 + 2000 = 2400$
The total cost of producing 500 units is
$C(500) = 4 \cdot 500 + 2000 = 4000$
The extra cost of producing 500 units rather than 100 units is given by
$C(500) - C(100) = 4000 - 2400 = 1600$
c) The total production cost per bra when producing 100 units is
$$\frac{C(100)}{100} = \frac{2400}{100} = 24$$
When producing 500 units this cost will be
$$\frac{C(500)}{500} = \frac{4000}{500} = 8$$

Example 5
Referring to Example 4, Holy Underwear, Ltd., determines that they can sell x bras at a price of \$12 per bra. That is, the total *revenue* $R(x)$ is given by the function
$$R(x) = 12x$$
a) Graph $R(x)$ and $C(x)$ in the same coordinate system.
b) The total *profit* $P(x)$ is the difference between the revenue and the cost. Determine $P(x)$ and draw its graph in the same coordinate system as above.
c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). Find the *break-even value* of x.

Solution:



The graphs of $C(x)$ and $R(x)$ cut each other at the break even point. The corresponding x-value is 250 (units). To the left of that point, where $C(x)$ is above $R(x)$, a loss will occur. To the right, where $R(x)$ is above $C(x)$, a gain will occur.
This can also be viewed from the graph of $P(x)$, given by
$P(x) =$ (Total revenue) - (Total cost) $=$
$= R(x) - C(x) = 12x - (4x + 2000) = 8x - 2000$
$P(x)$ is negative means "negative profit", or loss. $P(x)$ is positive means "positive profit", or gain.
At the break even point, $P(x) = 0$, so
$$8x - 2000 = 0$$
$$8x = 2000$$
$$x = \frac{2000}{8} = 250$$

Exercises

A

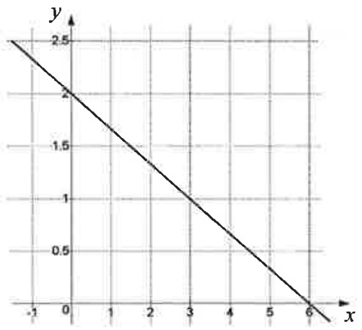
2301 Find m and c for the functions

a) $y = 3x - 7$ b) $y = 5 - 2x$

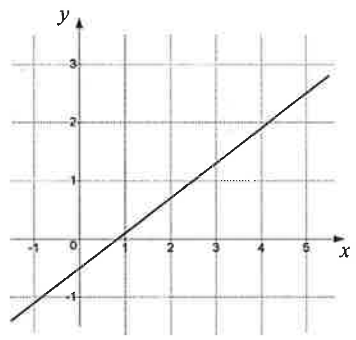
c) $y = \dfrac{5-2x}{7}$ d) $y = -\dfrac{x-3}{4}$

2302 Find the equations of the lines below

a)



b)



2303 Find the gradient of the line through the points (-1, 3) and (3, 9).

2304 A line passes through the point (2, -3). Find the equation of the line, if

a) $m = 4$ b) $m = -2$

2305 Find where the following lines cut the coordinate axes:

a) $2x - 9y - 13 = 0$ b) $\dfrac{2y}{3} = \dfrac{x}{2} - 5$

2306 The charge for a TV repairman is $y = 150 + 200x$ (kr), where x is the number of working hours.

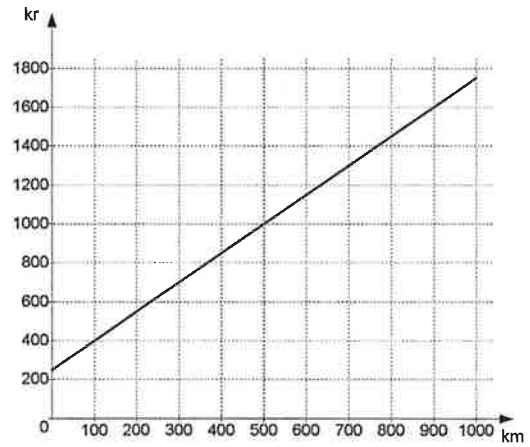
a) What is the charge per hour?

b) How many hours of work was done if the charge was 650 kr.

2307 The graph shows the cost of renting a car.

a) Find the fixed cost and the cost per km.

b) Express the cost as a function of the distance travelled.



2308 To rent a car at a certain company will cost 430 kr if you drive 100 km, and 538 kr if the distance is 160 km. If the company charges a fixed cost and a cost per km, find

a) the price per km b) the total cost as a function of the distance travelled.

B

2309 Bob has figured out that he can sell his mother's marmalade. The cost (in *Kr*) of producing x *dl* marmalade is $C(x) = 500 + 2.5x$. He figures that the marmalade can be sold at a price of 6 *Kr/dl*.

a) Write down a formula for the profit from the production and sale of x *dl* of marmalade.

b) How many *dl* must be sold in order to make a profit?

c) If he believes he can sell at least 25 *l* of marmalade, what is the minimum price he can charge to be sure of making a profit?

2310 A test tube containing 40 ml of a certain liquid weighs 97 g. A similar test tube with 55 ml of the same liquid weighs 116.5 g. Find

a) the weight of 1 ml of the liquid b) the total weight as a function of the amount of liquid contained.

Quadratic functions
Quadratic functions can always be written in the form
$f(x) = ax^2 + bx + c$
where *a*, *b* and *c* are constants.
The graph of a quadratic function is called a *parabola.*
It always has one turning point, which is a minimum point if $a > 0$, or a maximum point if $a < 0$.
The graph is symmetric around the vertical line $x = -b/2a$. The turning point can therefore be found along this line.

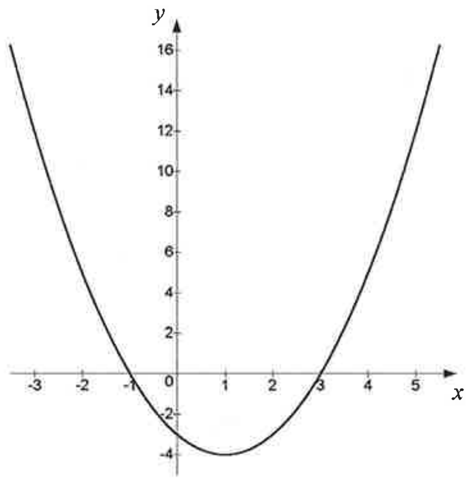
Example 6
Find the zeros and the turning point of the function $f(x) = x^2 - 2x - 3$.
Draw the graph of the function.

Solution:
A zero of a function is an *x*-value where $f(x) = 0$, so we have to solve the equation $x^2 - 2x - 3 = 0$.
Using the quadratic equation formula gives $x_1 = -1$ and $x_2 = 3$.
Since the graph is symmetric, we can easily find the turning point by taking the *x*-value halfway between the two zeros, which is $x = 1$. This is the *x*-coordinate of the turning point. The *y*-coordinate is $f(1) = -4$, so the turning point is (1, -4).
It is a minimum point, since the coefficient for x^2 (*a*) is positive.
We now know 3 important points of the graph, so by calculating a few more sensible points we can get a good sketch of the graph:

Table:

x	*y*
-3	12
-1	0
0	-3
1	-4
2	-3
3	0
5	12

Graph:

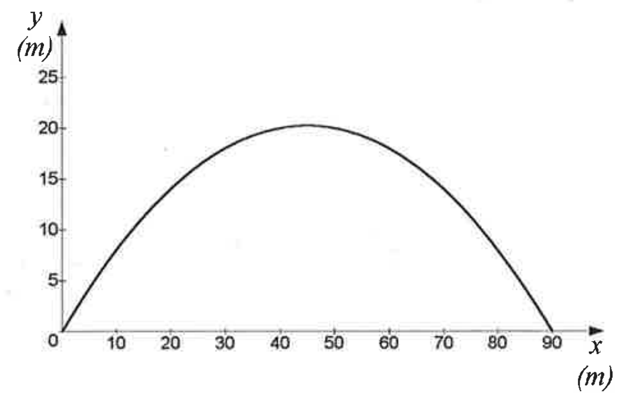


Example 7
Seppo is throwing the javelin. His coach (who has studied mathematics) discovered when watching Seppo on video, that the best throw for the javelin is to follow the graph of the quadratic function
$h(x) = -0.01x^2 + 0.9x$
where $h(x)$ is the height of the javelin above the ground, and *x* is the length of the throw (all in metres).
Draw the graph of the function $h(x)$ and calculate the length and height of Seppo´s throw.

Solution:
To simplify the drawing we start by finding the zeros of $h(x)$, that is when $h(x) = 0$. This leads to the quadratic equation $-0.01x^2 + 0.9x = 0$ which can be solved (without using the formula) by factorising:
$-0.01x^2 + 0.9x = 0$
$x(0.9 - 0.01x) = 0$ (either $x = 0$ or $0.9 - 0.01x = 0$)
$0.9 - 0.01x = 0$
$x = \frac{0.9}{0.01} = 90$ $\Rightarrow$ $x_1 = 0$ and $x_2 = 90$.
The zeros of $h(x)$ tell us where the height of the javelin is zero (= when the javelin is at the ground). $x_1 = 0$ is of course when the throw starts, and $x_2 = 90$ is when the javelin has landed. The length of the throw must be 90 m.

We now turn to finding the maximum point of the curve (which is the maximum height of the javelin). Since the curve is symmetric, the maximum height must occur at $x = 45$.
$h(45) = 20.25 \approx 20$
The maximum height is approximately 20 m.

We can now get a rough picture of the function:



Exercises

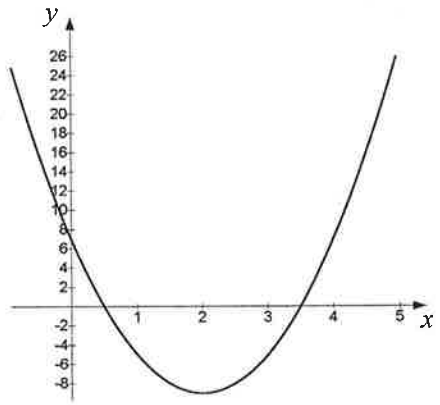
A

2311 The figure shows the graph of $f(x)$. Find from the figure

a) $f(0)$

b) the zeroes of the function

c) the minimum point.



2312 Which of the following quadratic functions has a *maximum* point?

A $y = x^2 + 4x + 3$ B $y = x^2 - 4x + 3$

C $y = -x^2 + 4x + 3$ D $y = x^2 - 4x - 3$

2313 Which of the following quadratic functions has a *minimum* point?

A $y = x^2 + 2x + 7$ B $y = -x^2 - x - 11$

C $y = 2x - x^2$ D $y = 15 - 3x^2$

2314 The cost (in kr) of producing x units in a company is given by

$$K(x) = 25000 + 80x - 0.1x^2 \text{ .}$$

Find, and express with words

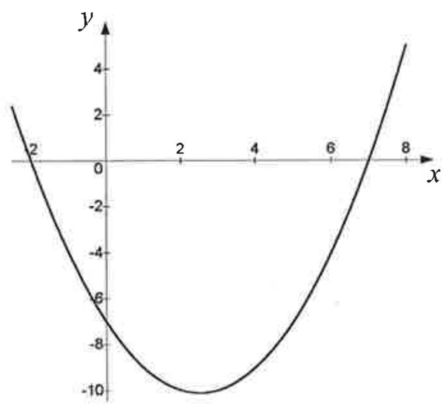
a) $K(200)$ b) $K(150) - K(100)$

c) $\dfrac{K(200) - K(100)}{100}$ d) $\dfrac{K(250) - K(200)}{50}$

2315 The figure shows the graph of $f(x)$. Solve the following problems graphically.

a) Find $f(0)$

b) Solve the equation $f(x) = 0$



B

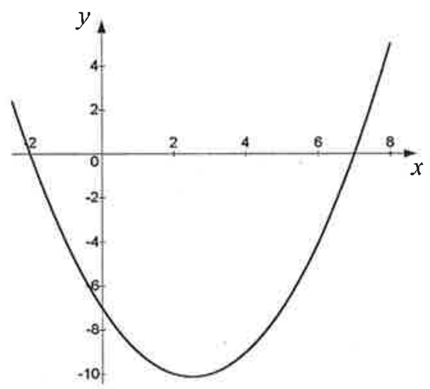
2316 Find the minimum point of the following functions.

a) $y = x^2 - 4x - 12$ b) $y = 2x^2 - 16x + 30$

2317 A quadratic curve cuts the y axis at $y = 1$. The curve passes also the points (2, 1) and (4, -7). Find the function of the graph and draw the graph.

2318 For which values of d has the quadratic function $f(x) = x^2 - (2d + 2)x + 1$ only one zero?

2319 The graph of the quadratic function $f(x)$ is shown in the figure. Find $f(x)$.



2320 Find a quadratic function that fits the data points (1, 5), (2, 9) and (3, 4).

Polynomial functions of higher degree

Any polynomial function may be written in the form

$$f(x) = a_n x^n + \ldots.. + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer, and coefficients $a_n \ldots.. a_2, a_1, a_0$ are constants.

In general, graphing higher degree polynomial functions is very time-consuming and difficult. The curves may or may not be symmetric, and may have several turning points or none. Chapter 3 will enable us to sketch these graphs in an easier way.

However, let us look at a few simple examples.

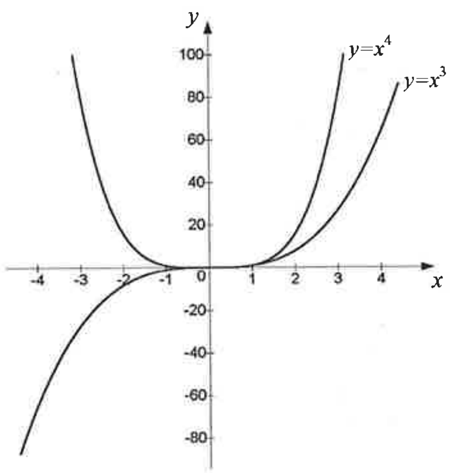
Example 8

Draw the graphs of $y = x^3$ and $y = x^4$ on the same coordinate system.
Can you figure out something about polynomial functions from this graph?

Solution:

Table: Graph:

x	x^3	x^4
-3	-27	81
-2	-8	16
-1	-1	1
0	0	0
1	1	1
2	8	16
3	27	81



A (true) guess is that a polynomial function of *odd degree* is negative to the left and positive to the right (or vice versa), while a polynomial function of *even degree* is either positive on both sides or negative on both sides.

Example 9

Draw a rough sketch of the graph of $f(x) = x^3 - 3x^2 - x + 3$, given that

$$x^3 - 3x^2 - x + 3 = (x+1)(x-1)(x-3) \ .$$

Solution:

Since we are told that $f(x)$ could be written

$$f(x) = (x+1)(x-1)(x-3)$$

we can easily find the zeroes of $f(x)$. The equation $f(x) = 0$ becomes

$$(x+1)(x-1)(x-3) = 0$$

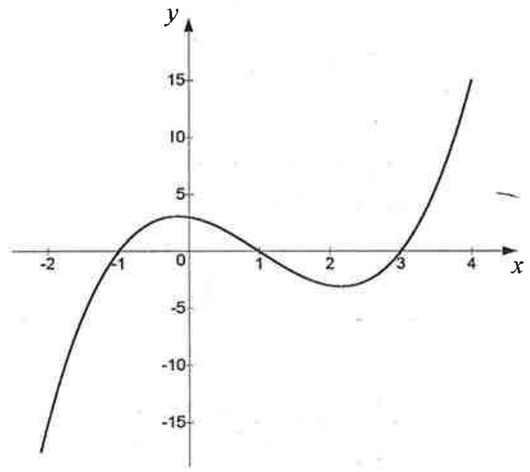
and the solutions are evident:

$$\begin{cases} x_1 & = & -1 \\ x_2 & = & 1 \\ x_3 & = & 3 \end{cases}$$

We know now where the function is zero, so any turning points must appear at x-values between these zeroes (not necessarily halfway between.Why?). If we work out some points between the zeroes (and some outside) we get a pretty good graph.

Table: Graph:

x	y
-2	-15
-1	0
0	3
1	0
2	-3
3	0
4	15



Exercises

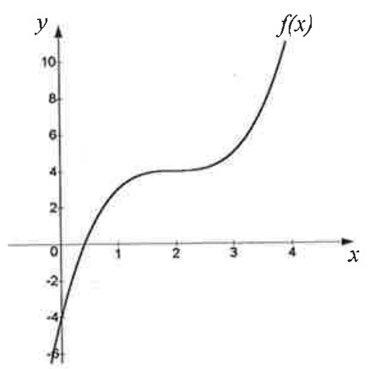
A

2321 Find the zeroes of the following functions.
a) $f(x) = x^3 + 1$ b) $f(x) = (x-1)(x-5)$

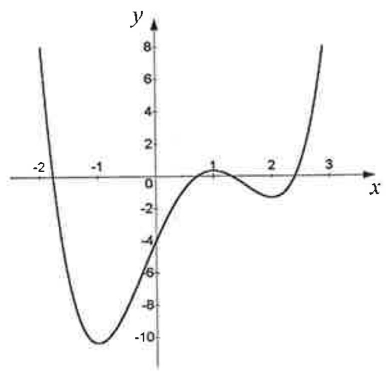
2322 Draw a rough sketch of $f(x)$, given that the zeroes of f are -2, 0 and 3. What is the lowest possible degree of $f(x)$?

2323 What is the lowest possible degree of the polynomial functions shown below ?

a)



b)



B

2324 Sketch the graph of the following functions
a) $f(x) = (x-1)(x+2)(x-2)$ b) $f(x) = (x^2-1)(x^2-4)$
c) $f(x) = x^3 - 3x^2 + \frac{5x}{4}$ d) $f(x) = (x^2+2x+1)(x^2-2x+1)$

2325 The number of units of a certain product that can be sold (*S*), is depending on the price (*p Kr*), following the function $S(p) = 800 - p^3$, $0 \le p \le 9.28$.
a) Find the number of units sold when the price is 6.50 *Kr*.
b) Find the price per unit when 720 units are sold.
c) Express the revenue, *R*, as a function of the price, *p*. Draw the graph of *R(p)* and find from the graph the price that gives the maximum revenue.

2326 Find all cubic functions that fit the data points (0, 1), (1, 0) and (-1, 0).

2.4 Exponential functions

An exponential function is a function of the form

$f(x) = a^x$

where a is a positive number ($a > 0$) and the variable x can take any value.
The big difference compared to a polynomial or power function is that the variable here is in the exponent.
Some constants are often involved, like in the function

$f(x) = C \cdot a^{kx}$

where C and k are constants.

Exponential functions appear frequently in many applications and perhaps form the most important class of functions in mathematics.

A former winner of the Nobel Prize in physics has said:
"Man can never cope with the future problems of the earth until he has fully understood the exponential function"

Example 1
Draw and compare the graphs of the functions

$f(x) = 2^x$

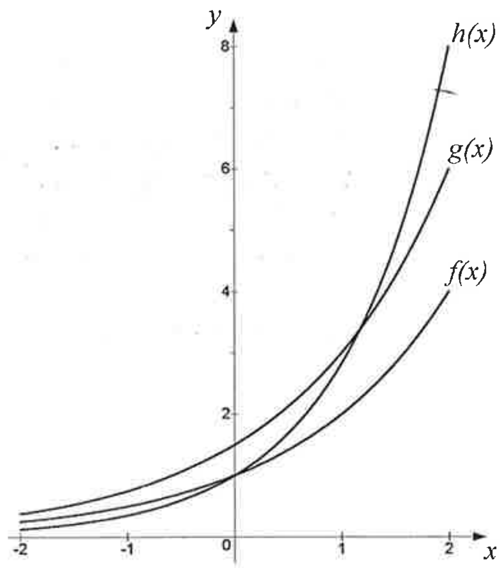
$g(x) = 1.5 \cdot 2^x$

$h(x) = 2^{1.5x}$

Solution:
Table:

x	f(x)	g(x)	h(x)
-2	0.25	0.38	0.13
-1	0.5	0.75	0.35
0	1	1.5	1
1	2	3	2.8
2	4	6	8

Graph:

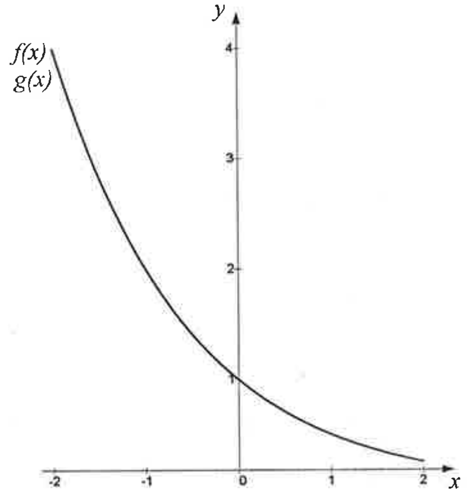


Notice here how the appearance of the constant 1.5 affects the curve in different ways.

Example 2
Draw the graphs of the functions
$f(x) = 2^{-x}$ and $g(x) = 0.5^x$
and explain the result.
Compare the two curves with the graph of $f(x) = 2^x$ above.

Solution:
Table: Graph:

x	$f(x)$	$g(x)$
-2	4	4
-1	2	2
0	1	1
1	0.5	0.5
2	0.25	0.25



It is the same curve!
The explanation is

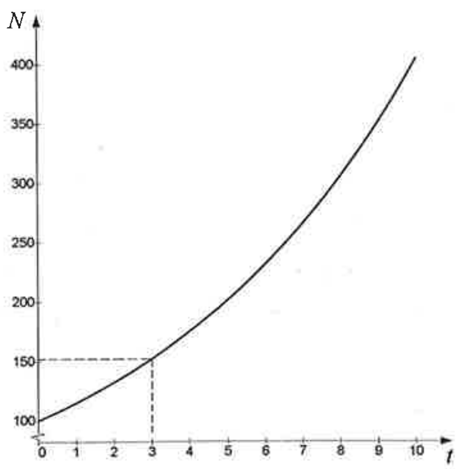
$$0.5^x = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}$$

Compared to the curve 2^x the curve 2^{-x} is mirrored through the *y*-axis.

Example 3
In a laboratory the number of a certain bacterium is growing by 15% each hour. At the start of an experiment the number of bacteria was 100.
a) Write down a formula describing the number of bacteria (*N*) as a function of time (*t*).
b) Draw the graph of the function between 0 and 10 hours and find from the graph the number of bacteria after 3 hours.
c) Calculate how many hours it will take to get 1000 bacteria.

Solution:
a) $N(t) = 100 \cdot 1.15^t$
b) Table: Graph:

t	$N(t)$
0	100
2	132
4	175
6	231
8	306
10	405



The number of bacteria after 3 hours is approximately 150.

c) $N(t) = 1000$ gives the equation

$$100 \cdot 1.15^t = 1000$$
$$1.15^t = 10$$
$$\lg 1.15^t = \lg 10 = 1$$
$$t \cdot \lg 1.15 = 1$$
$$t = \frac{1}{\lg 1.15} \approx 16.48 \approx 16.5$$

After 16.5 hours the number of bacteria will exceed 1000.

Example 4
Radioactive materials, such as radium, decay without external influence owing to the emission of radiation. Measurements show that the quantity of radium decays to half its original value in 1622 years. This time is called the *half-life* of radium. If the number of radium atoms in a material is 200 at some point of time, the number of atoms, *N*, after *t* years could be described by the function

$$N(t) = 200 \cdot 2^{kt}$$

where *k* is a constant.
Knowing that half of the radium remains after 1622 years, we can find the constant *k* from the equation $N(1622) = 100$, or

$$200 \cdot 2^{1622k} = 100$$
$$2^{1622k} = \frac{1}{2} = 2^{-1}$$
$$1622k = -1$$
$$k = -\frac{1}{1622} \approx -0.000617$$

The decay function of radium then becomes

$$N(t) = 200 \cdot 2^{-0.000617t}$$

or

$$N(t) = 200 \cdot 2^{-\frac{t}{1622}}$$

From the examples above we can summarise the following facts about the exponential function $f(x) = C \cdot a^{kx}$:

- *C* is the "starting value" ($f(0) = C \cdot a^0 = C$.)
- If $a > 1$ and *k* is positive, the function is increasing.
- If either $a < 1$ or *k* is negative, the function is decreasing.

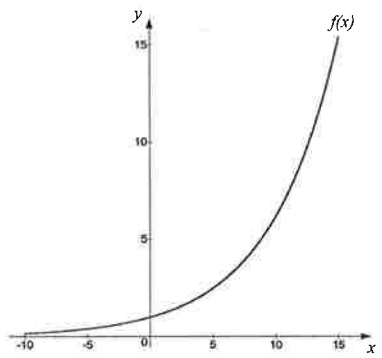
Exercises

A

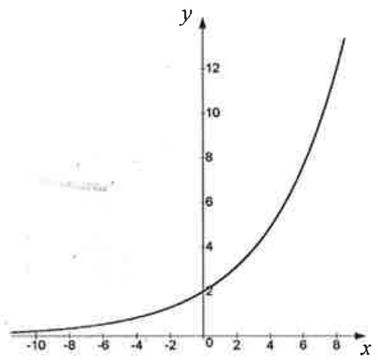
2401 Draw the graph of the following functions.
a) $f(x) = 100 \cdot 1.05^x$ b) $g(x) = 250 \cdot 0.9^x$

2402 Match up the following functions with their graphs.
a) $f(x) = 1.2^x$ b) $f(x) = 1.2^{-x}$
c) $f(x) = 2 \cdot 0.8^x$ d) $f(x) = 2 \cdot 0.8^{-x}$

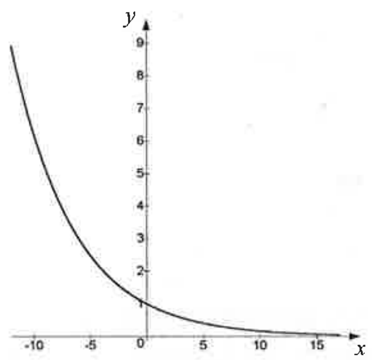
A



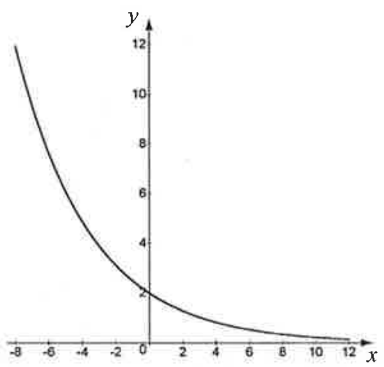
B



C



D



2403 Which of the following functions are *decreasing*?
A $y = 210 \cdot 0{,}75^x$ B $y = 0{,}95 \cdot 1{,}8^x$
C $y = 0{,}01 \cdot 18^x$ D $y = 3{,}5 \cdot 0{,}9^{4x}$

2404 Which of the following functions are *increasing*?
A $f(x) = 1{,}9 \cdot 0{,}91^x$ B $f(x) = 0{,}91 \cdot 1{,}9^x$
C $f(x) = 5 \cdot 10^{-3} \cdot 3^x$ D $f(x) = 75000 \cdot 0{,}12^x$

2405 Find the value of $7500 \cdot 1{,}09^x$ when $x = 4{,}5$. Answer with three significant figures.

2406 Find $f(-4)$ when $f(x) = 19{,}1 \cdot 10^{-0{,}5x}$.

2407 In Summer Night City the population, which is 32500 today, decreases each year by 4.0 %. How many will there be after 10 years if this continues?

2408 The value of a yacht is expected to fall according to the formula $V(t) = 250000 \cdot 10^{-0{,}09t}$, where V is the value in Kr after t years. Find the value of the boat after 8 years.

2409 Sir Henry bought an old black horse six years ago, costing 15000 Kr.
The value of the horse has increased by 27 % per year during this period.
Find the value today.

2410 Bono is putting £5000 into a savings account with an annual interest of 6.5 %. How much money will he have after 6 years?

2411 A car was bought for 50000 Kr and sold two years later for 40000 Kr. By how many percent did the value decrease each year?

2412 The graph of an exponential function passes through the points (0, 150) and (1, 225). Find the function.

2413 The number of flies in a barn was during a period increasing by 4% each day. If there were 100 flies from the beginning, write down a function describing the number of flies a after t days.

B

2414 The number of flowers of an apple tree could one sunny week be described with the formula $N(x) = 5 \cdot 2^{0.7x}$, where N is the number of flowers after x days.
Find the daily percentage increase.

2415 One of Jimi Hendrix' guitars, which cost £150 in 1968, was recently (1998) sold at an auction for £250000. Find the annual percentage increase!

2416 The graph of an exponential function passes through the points (1, 26) and (4, 208). Find the function.

2417 The graph of an exponential function passes through the points (2, 192) and (4, 12288). Does it also pass through the point (3, 1535) ?

2.5 Some other functions

In this chapter we take a brief look at some special functions which you can happen to meet further on in this course.
Rational functions are functions which are quotients of polynomial functions. A rational function can be very complicated, like, for example,

$$f(x)=\frac{x^3-2x^2+2x-1}{3x^2-7x+5} \quad \text{(too complicated for this course)}.$$

Here we stick to more simple rational functions, like

$$f(x)=\frac{1}{x} \quad \text{and similar.}$$

Power functions are functions of the type

$$f(x)=x^a$$

where a may be any number (not necessarily an integer) and x is positive.
A typical power function is

$$f(x)=x^{\frac{1}{2}}=\sqrt{x}$$

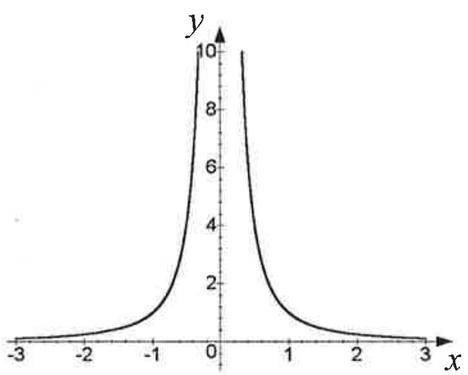
Example 1

Find the zeroes and turning points of $f(x)=\dfrac{1}{x^2}$.

Solution:
We notice first that the domain consists of all x except 0. By choosing some x-values, positive and negative, and calculating the function values we get the graph below.

Table: Graph:

x	$f(x)$
-2	0.25
-1	1
-0.5	4
0	ej def.
0.5	4
1	1
2	0.25



It is obvious that $f(x)$ doesn't have any zeroes or turning points

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Example 2
An owner of a fruit tree plantation has found out that each fruit tree needs a certain free area (without vegetation) to give an acceptable amount of fruit. This area seems to be a function of the diameter of the trunk of each tree, following

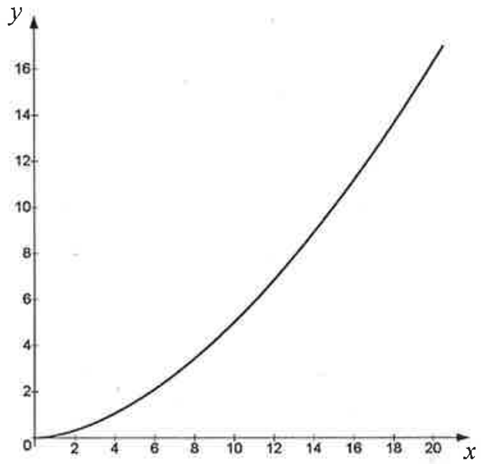
$$A(x)=0.1\cdot x^{1.7} \quad , \quad 0\le x\le 20$$

where $A(x)$ is the area needed (in m^2) and x is the diameter (in cm).

a) Draw the graph of this function.
b) Calculate the maximum diameter of a fruit tree, according to this model, on an area of 12 m^2.

Solution:
a) Table: Graph:

x	$A(x)$
0	0
2	0.3
4	1.1
6	2.1
10	5.0
14	8.9
20	16.3



b) If $A(x)=12$ we have the equation

$$0.1\cdot x^{1.7}=12$$
$$x^{1.7}=120$$
$$x=120^{\frac{1}{1.7}}\approx 16.7\approx 17$$

The maximum diameter is 17 cm.

Exercises

A

2501 Find the value of $5.3 \cdot x^{-7}$ when $x = 1.92$. Give your answer correct to three significant figures.

2502 Find the value of $2 + \frac{3}{x^{1.3}}$ when $x = 5$. Give your answer correct to three significant figures.

2503 Solve the following equations. Answer with three significant figures.
a) $x^3 = 14$ b) $x^7 = 550$
c) $200 \cdot x^{11} = 2500$ d) $3.2 \cdot a^5 + 7 = 65$

2504 Draw the graphs of the following functions in the given domain and find graphically the zeroes of the functions.
a) $f(x) = \sqrt{x^2} - 2$, $x \geq 0$ b) $g(x) = x^{1.5} - 5$, $x \geq 0$

2505 The number of needles in a certain haystack can be approximated by the function $N(x) = x^{0.3}$, where x is the volume of the haystack, in m^3.
a) Draw the graph of the function for $0 \leq x \leq 400$.
b) Find from the graph the volume of the smallest haystack containing 5 needles.

B

2506 Draw the graphs of the following functions in the given domain and find graphically the zeroes of the functions.
a) $f(x) = \sqrt{10x^3 - 28x^2 + 19.6x}$, $0 \leq x \leq 5$
b) $g(x) = x - x^{1.8} + 2$, $0 \leq x \leq 4$

2507 Solve the following equations. Answer with three significant figures.
a) $x^{-3.1} = 3.1$ b) $\frac{7}{x^{7.5} + 2} = 3$
c) $3.2 \cdot x^{1.,5} = 160 \cdot x^{-2}$ d) $7.6 \cdot a^{0.03} + 2 = 55.2$

2508 The territorial area of an animal is defined to be its defended region, or exclusive region. It has been found that the territorial area of a lion can be approximated by the power function $T = 100 \cdot W^{1.31}$, where T is the area (in m^2) and W is the weight of the animal (in *kg*).
a) Find the territorial area of a 100 kg male lion called Leon.
b) Find the weight of the Lion King, who needs a territorial area of 140000 m^2.

2.6 Linear programming (linear optimization)

Linear inequalities

We have earlier studied linear equations in two variables, $Ax + By = C$ or $y = kx + m$, where x and y are the unknown variables to be found. Such an equation has infinitely many solutions. Each solution is a pair of numbers (x, y) that fits into the equation. We can think of these solutions as points in a coordinate system which all lies along a straight line. This is just another way of saying that $Ax + By = C$ (or $y = kx + m$) is the equation of a straight line.

If we in such an equation replace the equal sign by an inequality symbol ($<$, $>$, $\leq$ or $\geq$) we obtain a *linear inequality* in two variables, x and y.

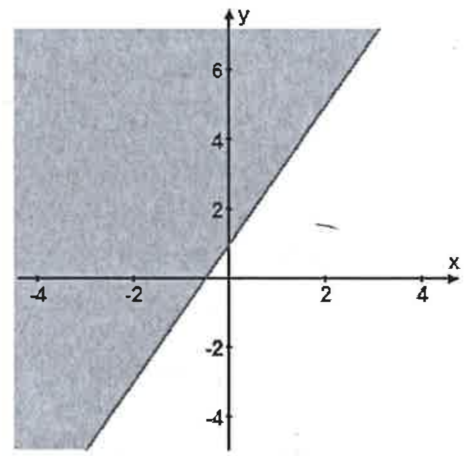
The solutions to the linear inequality $y - 2x \geq 1$ are all points (x, y) that satisfy the inequality. How do we find these points? The following example shows how to work it out.

Example 1

In a coordinate system, graph the inequality $y - 2x \geq 1$ (that is, mark all points that satisfy the inequality).

Solution:

$y - 2x \geq 1$ can also be written $y \geq 2x + 1$.
First we draw the line $y = 2x + 1$.
On this line we have all points where y is equal to $2x + 1$ (or, where $y - 2x$ is equal to 1).
Above this line we have all points where y is greater than $2x + 1$ (or, where $y - 2x$ is greater than 1). At points below this line, y is less than $2x + 1$.
Therefore, the graph of the inequality (= the solution to the inequality) is the set of points on and above the line $y = 2x + 1$.



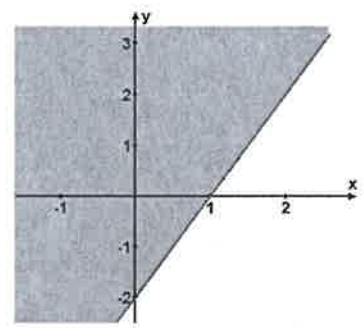
Systems of linear inequalities

A system of linear inequalities is a collection of two or more linear inequalities. To graph a system of linear inequalities we must find all points that obey all the inequalities involved.

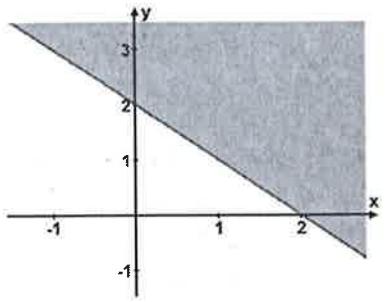
To graph the system $\begin{cases} 2x - y \leq 2 \\ x + y \geq 2 \end{cases}$

we first locate the points satisfying each inequality.

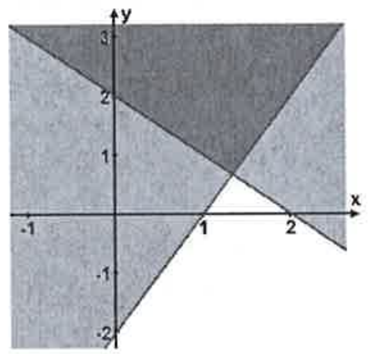
The graph of $2x - y \leq 2$, or $y \geq 2x - 2$ consists of all points on and above the line $y = 2x - 2$.



The graph of $x+y\geq 2$, or $y\geq -x+2$ consists of all points on and above the line $y=-x+2$.

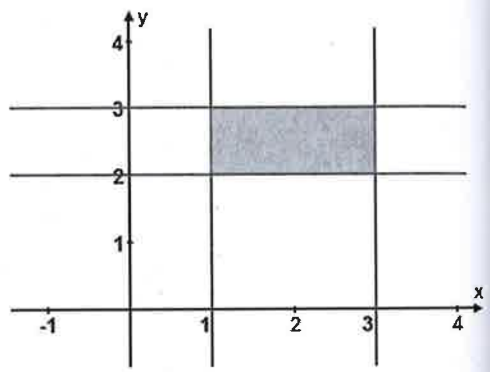


The solution to the original system of inequalities is then the region common to the two separate inequalities (heavily shaded in the figure).



Example 2

Graph the system $\begin{cases} 1\leq x\leq 3 \\ 2\leq y\leq 3 \end{cases}$



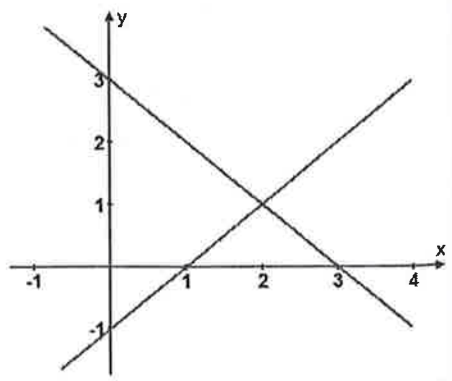
Solution:
Since x must be between 1 and 3, and y between 2 and 3, the region must be the set of points bounded by the lines $x = 1$, $x = 3$, $y = 2$ and $y = 3$, including parts of the lines.

Example 3

Graph the following system of inequalities: $\begin{cases} x+y\leq 3 \\ y-x\geq -1 \\ x>0 \\ y>0 \end{cases}$

Solution:
We begin by graphing the four lines associated with each inequality:

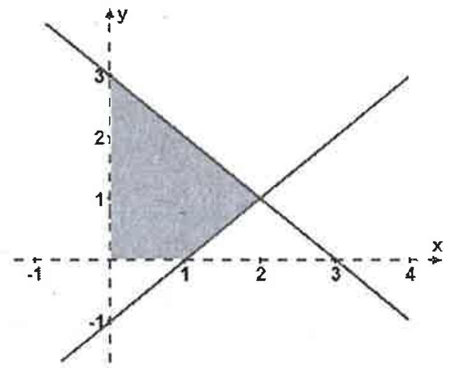
$$\begin{cases} y=-x+3 \\ y=x-1 \\ x=0 \\ y=0 \end{cases}$$



Now, the graph of the system of inequalities is the region where

$y\leq -x+3$ (on and below the line $y=-x+3$)
$y\geq x-1$ (on and above the line $y=x-1$)
$x>0$ (to the right of the y axis)
$y>0$ (above the x axis)
(Note that the axes are *not* included, since both x and y must be > 0)

This area, the *intersection* of the four inequalities above, contains all points that satisfy the system of inequalities.



Example 4
Johnson's Nuts have 75 kg of cashews and 120 g of peanuts. These are to be mixed in 1 kg packages as follows:
A low-grade mixture that contains 2 hg of cashews and 8 hg of peanuts, and a high-grade mixture that contains 5 hg of each kind.

a) Using x to denote the number of packages of low-grade mixture, and using y to denote the number of packages of high-grade mixture, write down a system of inequalities that describes the possible number of each kind of package.

b) Graph the system of inequalities.

Solution:
a) Total number of cashews must be less than or equal to 75 kg (= 750 hg).
Total number of peanuts must be less than or equal to 120 kg (= 1200 hg).
Using the variables x and y and the fact that x and y can't be negative, we get

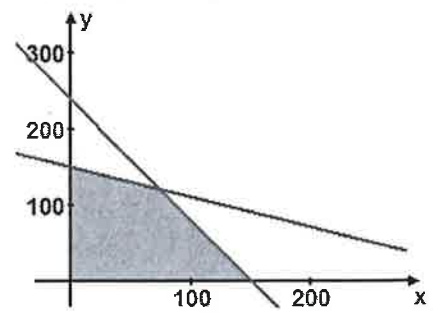
$$\begin{cases} 2x+5y\leq 750 \\ 8x+5y\leq 1200 \\ x\geq 0 \\ y\geq 0 \end{cases}$$

b) The system of inequalities is equivalent to

$$\begin{cases} y\leq -0,4x+150 \\ y\leq -1,6x+240 \\ x\geq 0 \\ y\geq 0 \end{cases}$$

The graph of this system of inequalities is the region bounded by the four lines

$$\begin{cases} y=-0,4x+150 \\ y=-1,6x+240 \\ x=0 \\ y=0 \end{cases}$$



Exercises

A

2601 Graph the following inequalities

a) $x \geq 1$ b) $y \geq 2$

c) $x \geq -2$ d) $-1 \leq y \leq 3$

2602 Graph the following systems of inequalities

a) $\begin{cases} x+y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{cases}$ b) $\begin{cases} x+y \geq 3 \\ x \leq 2 \\ y \leq 3 \end{cases}$

c) $\begin{cases} 3x+y \leq 12 \\ 2x+3y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$ d) $\begin{cases} x+2y \leq 10 \\ 3x+2y \geq 8 \\ x \leq 3 \\ y \geq 0 \end{cases}$

B

2603 Graph the following systems of inequalities

a) $\begin{cases} 2x-y \leq 5 \\ 2y-x \leq 2 \\ x \geq 1 \\ y \geq 1 \end{cases}$ b) $\begin{cases} x+y \geq 2 \\ 2x+3y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$

c) $\begin{cases} 3x+y \leq 12 \\ 2x+3y \leq 12 \\ x+y \geq 2 \\ x \geq 0 \end{cases}$ d) $\begin{cases} x-y \leq 2 \\ x+5y \leq 20 \\ 2x+y \geq 4 \end{cases}$

2604 Mike's Famous Toy Trucks manufactures two kinds of toy trucks – a standard model and a deluxe model. In the manufacturing process, each standard model requires 4 hours of grinding and 3 hours of finishing, and each deluxe model needs 2 hours of grinding and 4 hours of finishing. The company has 2 grinders and 3 finishers, each of whom work 40 hours per week.
Let x be the number of standard models, and y the number of deluxe models made. Set up and graph a system of inequalities that describes the possible numbers of each truck that can be manufactured.

2605 A factory manufactures two products, each requiring the use of three machines. The first machine can be used at most 70 h, the second machine at most 40 h and the third machine at most 90 h.
Product A requires 2 hours on Machine 1, 1 hour on Machine 2 and 1 hour on Machine 3. Product B requires 1 hour each on Machines 1 and 2 and 3 hours on Machine 3.
Set up and graph a system of inequalities that describes the given data.

Linear programming
Linear programming, or linear optimization, is about finding the maximum or minimum value of a function within a region bounded by a system of linear inequalities.
If we, for example, want to find the maximum value of the function $z = x + y$ (called the *objective function*), but we have the restrictions (or *constraints*) that

$$\begin{cases} y \leq -2x+4 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Then we have a linear programming problem. It can be restated as:

Maximize the function

$z = x + y$ (*objective function*)

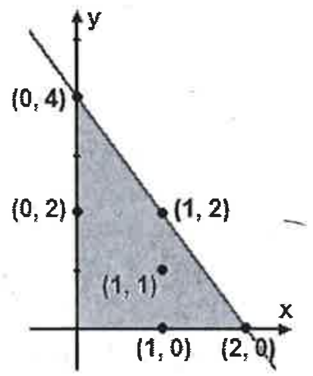
Subject to the conditions that

$\begin{cases} y \leq -2x+4 \\ x \geq 0 \\ y \geq 0 \end{cases}$ (*constraints*)

The task is to find the point (x, y) that gives the maximum value of $z = x + y$ within the region defined by the given system of inequalities.

The graph of this system of inequalities is shown in the figure (some points within this region are marked).

At what point inside this region can we find the maximum value of $z = x + y$?



If we check the value of $x + y$ at some points we find the largest value, $0 + 4 = 4$, at one of the corner points of the region. This is not a coincidence. The following fact, which is not to be proved here, holds for any linear programming problem:

If a linear programming problem has a solution, it is located at a corner point (vertex) of the region bounded by the constraints of the problem.

Example 5
In *Ex. 4* we determined the possible alternatives for Johnson's Nuts to produce a number of low-grade mixture packages (x) and a number of high-grade mixture packages (y). Suppose that in addition to this information, we also know that the profit on each low-grade package is 2 kr and is 3 kr on each high-grade package. The question of importance to the manager is "How many packages of each type of mixture should be prepared to maximize the profit?"

If P symbolizes the profit, x and y the number of packages of each type, then this is a typical linear programming problem, where we want to find the maximum value of an expression bounded by a system of linear inequalities:

Maximize

$P = 2x + 3y$ (*objective function*)

subject to the following restrictions:

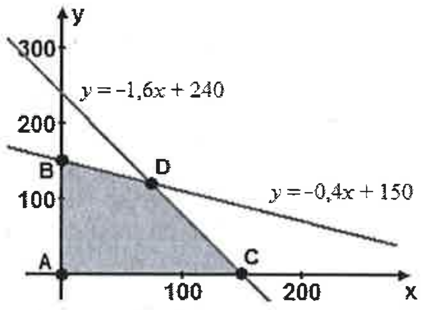
$$\begin{cases} 2x+5y \le 750 \\ 8x+5y \le 1200 \\ x \ge 0 \\ y \ge 0 \end{cases} \quad (\textit{constraints})$$

The solution is to be found within the graph of the system of inequalities given, and the theory tells us that the maximum is located at one of the vertices (corner points).

We have A = (0, 0), B = (0, 150) and C = (150, 0). To find the coordinates of point D, we have to solve the system of equations

$$\begin{cases} y = -1,6x + 240 \\ y = -0,4x + 150 \end{cases} \rightarrow \begin{cases} x = 75 \\ y = 120 \end{cases}$$

The value of the profit $P = 2x + 3y$ at each point is



A: $P = 2\cdot 0 + 3\cdot 0 = 0$

B: $P = 2\cdot 0 + 3\cdot 150 = 450$

C: $P = 2\cdot 150 + 3\cdot 0 = 300$

D: $P = 2\cdot 75 + 3\cdot 120 = 510$

A maximum profit is obtained at point D, that is, if 75 packages of low-grade mixture and 120 packages of high-grade mixture are made.

The maximum profit is 510 kr.

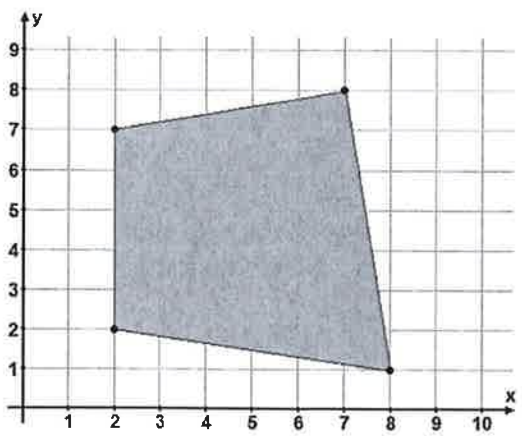
Exercises

A

2606 Maximize the quantity $z = 5x + 7y$ subjective to the given constraints

a) $\begin{cases} x+y \le 2 \\ x \ge 0 \\ y \ge 0 \end{cases}$ b) $\begin{cases} 2x+3y \le 6 \\ x \ge 0 \\ y \ge 0 \end{cases}$

2607 A linear programming problem is bounded by some constraints which gives a region of possible solutions, shown in the figure.
Find the maximum value if the objective function is



a) $z = 2x + 3y$

b) $z = 3y - 2x$

c) $z = 3x + y$

d) $z = 3x - y$

2608 Suppose that, in addition to the information given in problem 2605, the profit is 200 kr per unit for product A and 300 kr per unit for product B, decide how many units of each product should be manufactured to maximize profit.

B

2609 Minimize the function $z = 5x + 7y$ subject to

$$\begin{cases} x+y \ge 2 \\ 2x+3y \le 12 \\ 3x+y \le 12 \\ y \ge 0 \\ x \ge 0 \end{cases}$$

2610 Gwendolyn is knitting Christmas stars and Santa Clauses to be sold at the Christmas market. Christmas stars are sold for 65 kr and Santa Clauses for 30 kr each. She usually sells everything and she has already received orders for 100 Santa Clauses.
Each Santa Claus takes 3 m of red yarn and 2 m of green yarn, while each Christmas star takes 3 m of red and 5 m of green. She has 1200 m of red and 950 m of green yarn.
How many of each should she make to maximize the profit?

2611 A diet is to contain at least 400 units of vitamins, 500 units of minerals and 1400 calories. Two foods are available: Food 1, which costs 0,50 kr per unit, and Food 2, which costs 0,30 kr per unit. A unit of Food 1 contains 2 units of vitamins, 1 unit of minerals and 4 calories. A unit of Food 2 contains 1 unit of vitamins, 2 units of minerals and 4 calories.
Find the minimum cost for a diet that consists of a mixture of these two foods and also meet the minimum nutrition requirements.

2.7 Limits

Limits at a point

If we study the two functions $f(x)=x^2$ and $g(x)=\dfrac{x^2-4}{x-2}$ close to $x=2$ we get the following table:

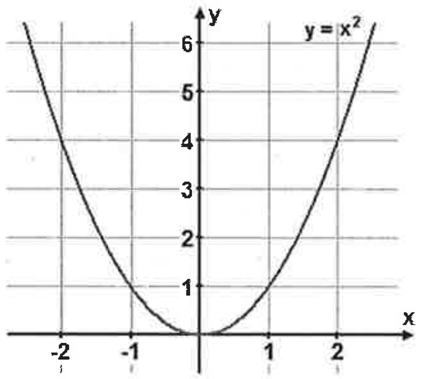
x	1,9	1,99	1,999	1,9999	2,0001	2,001	2,01	2,1
$f(x)$	3,61	3,9601	3,99600	3,99960	4,00040	4,00400	4,0401	4,41
$g(x)$	3,9	3,99	3,999	3,9999	4,0001	4,001	4,01	4,1

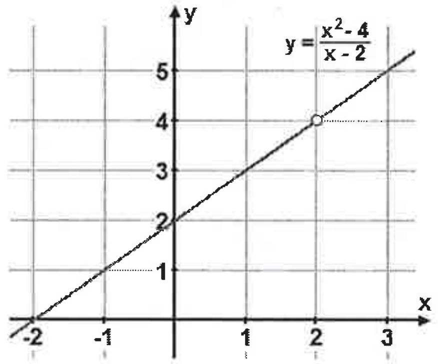
We notice that both functions approaches 4 as x approaches 2 from below, and both functions also approaches 4 as x approaches 2 from above.

We say that as x approaches 2 from either direction, both $f(x)$ and $g(x)$ approaches a *limit* of 4, and we write

$$\lim_{x\to 2} x^2 = 4 \quad \text{and} \quad \lim_{x\to 2}\frac{x^2-4}{x-2}=4$$

The difference here between $f(x)$ and $g(x)$ however, is that while $f(x)$ is defined at $x=2$, $g(x)$ is not. Still, the *limit* of $g(x)$ at $x=2$ exists.
The graphs of the two functions are:





Observe that, $\dfrac{x^2-4}{x-2}=\dfrac{(x+2)(x-2)}{x-2}$ which if $x\neq 2$ can be simplified to $(x+2)$, so we have

$$g(x)=\begin{cases} x+2 & \text{if } x\neq 2 \\ \text{not defined} & \text{if } x=2 \end{cases}$$

The function $g(x)$ has a gap (hole) at $x=2$. In maths language we say that $g(x)$ is *discontinuous* at $x=2$. Still, $g(x)$ has a *limit* at $x=2$, because it is obvious that the value of $g(x)$ "points at" 4 when x is getting close to 2.

Example 1
What happens to the following expressions as x is getting close to 1?

a) x^2+1 b) $x(x-1)$

c) $\dfrac{x-3}{2x}$ d) $\dfrac{x^2-1}{x-1}$

Solution:

a) x^2+1 is defined at $x=1$, so as x approaches 1, x^2+1 will approach $1^2+1=2$.

b) $x(x-1) \quad \to \quad 1\cdot(1-1)=1\cdot 0=0$

c) $\dfrac{x-3}{2x} \quad \to \quad \dfrac{1-3}{2\cdot 1}=\dfrac{-2}{2}=-1$

d) $\dfrac{x^2-1}{x-1}$ is undefined at $x=1$ (substituting x for 1 gives $\frac{0}{0}$),

but close to 1 we have $\dfrac{x^2-1}{x-1}=\dfrac{(x+1)(x-1)}{x-1}=x+1$

From this we see that, as x approaches 1, $\dfrac{x^2-1}{x-1} \quad \to \quad 1+1=2$.

Example 2
Find the following limits:

a) $\displaystyle\lim_{x\to 0}\frac{x^2-3x}{2x}$ b) $\displaystyle\lim_{x\to 1}\frac{1}{x-1}$

Solution:

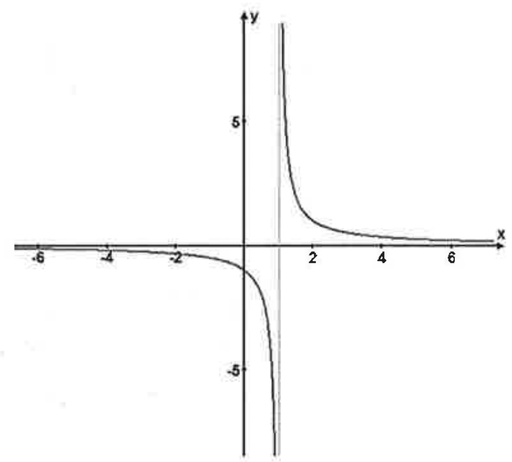
a) $\dfrac{x^2-3x}{2x}=\dfrac{x(x-3)}{2x}=\dfrac{x-3}{2}$, so $\displaystyle\lim_{x\to 0}\frac{x^2-3x}{2x}=\lim_{x\to 0}\frac{x-3}{2}=\frac{0-3}{2}=-\frac{3}{2}$

b) As x approaches1 from the left (values less than 1), the denominator will be smaller and smaller, but negative. Therefore $\dfrac{1}{x-1} \quad \to \quad -\infty$ ("minus infinity", that is, very large and negative).
As x approaches 1 from the right (values greater than 1), the denominator will be smaller and smaller, but positive,

so $\dfrac{1}{x-1} \quad \to \quad \infty$.

Since there is no common value for $\dfrac{1}{x-1}$ as $x\to 1$, we say that the limit *does not exist*.

If we look at the graph of $y=\dfrac{1}{x-1}$ (see fig.), this is obvious.



Limits at infinity
We can use the idea of limits to study the behaviour of functions for very large negative or positive values of x.
We write $x \to \infty$ to mean as x gets as large as we like, and positive,
and $x \to -\infty$ to mean as x gets as large as we like, and negative.
Notice that as $x \to \infty$, $1 < x < x^2 < x^3 < \ldots$ and so on.

This means that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$, $\lim\limits_{x\to\infty} \frac{x}{x^2} = 0$, ... and so on.

Example 3
Find the following limits

a) $\lim\limits_{x\to\infty} \frac{x^3}{x^5}$ b) $\lim\limits_{x\to\infty} \frac{x}{x^2 - x}$

c) $\lim\limits_{x\to\infty} \frac{x+1}{x-2}$ d) $\lim\limits_{x\to\infty} \frac{3x^2 + x - 1}{2x^3 - 4x + 7}$

Solution:

a) $\lim\limits_{x\to\infty} \frac{x^3}{x^5} = \lim\limits_{x\to\infty} \frac{1}{x^2} = 0$

b) $\lim\limits_{x\to\infty} \frac{x}{x^2 - x} = \lim\limits_{x\to\infty} \frac{x}{x(x-1)} = \lim\limits_{x\to\infty} \frac{1}{x-1} = 0$

c) When x becomes large, the numerator and denominator will be more and more equal, so

$\lim\limits_{x\to\infty} \frac{x+1}{x-2} \to \lim\limits_{x\to\infty} \frac{x}{x} = 1$

d) When x becomes large, $3x^2$ will dominate the numerator and $2x^3$ will dominate the denominator. Since $2x^3$ will grow faster than $3x^2$ when $x \to \infty$, the quotient $\frac{3x^2 + x - 1}{2x^3 - 4x + 7} \to 0$

We may write

$$\lim_{x\to\infty} \frac{3x^2 + x - 1}{2x^3 - 4x + 7} \approx \lim_{x\to\infty} \frac{3x^2}{2x^3} = \lim_{x\to\infty} \frac{3}{2x} = 0$$

Alternatively, divide both numerator and denominator by x^2:

$$\frac{3x^2 + x - 1}{2x^3 - 4x + 7} = \frac{3 + \frac{1}{x} - \frac{1}{x^2}}{2x - \frac{4}{x} + \frac{7}{x^2}}$$

As $x \to \infty$ it is obvious that this will tend to $\frac{3}{\infty}$, which approaches 0.

Exercises

A

2701 Evaluate the following limits

a) $\lim\limits_{x\to 2}(6 - 3x)$ b) $\lim\limits_{x\to -1}(4x^3 - 2x^2 - 3x)$

c) $\lim\limits_{x\to 0} \frac{x}{2}$ d) $\lim\limits_{x\to\infty} \frac{1}{5x}$

e) $\lim\limits_{x\to\infty} \frac{x+10}{x^2+4}$ f) $\lim\limits_{x\to\infty} \frac{x}{-x}$

2702 Find the following limits, if they exist

a) $\lim\limits_{x\to 0} \frac{x^2 - 4x}{x}$ b) $\lim\limits_{x\to\infty} \frac{x}{3x}$

c) $\lim\limits_{x\to\infty} \frac{x^2}{100x}$ d) $\lim\limits_{x\to 1} \frac{x - x^2}{1 - x}$

B

2703 Evaluate the following limts

a) $\lim\limits_{x\to 3} \frac{x^2 - 9}{(x-3)(x-1)}$ b) $\lim\limits_{x\to -4} \frac{12x + 3x^2}{2x + 8}$

c) $\lim\limits_{x\to 1} \frac{x^4 - 1}{3x^2 - 3}$ d) $\lim\limits_{x\to -2} \frac{x^2 + 4x + 4}{x + 2}$

2704 Find the following limits, if they exist

a) $\lim\limits_{x\to 1} \frac{x^2 + 1}{x - 1}$ b) $\lim\limits_{x\to\infty} \frac{1000x^2 + 500x + 875}{0{,}1x^3}$

c) $\lim\limits_{x\to 1} \frac{1 - x^2}{x - 1}$ d) $\lim\limits_{x\to 3}\left(\frac{x^2 - 9}{x - 3} + \frac{4x - 12}{x^2 - 9}\right)$

Chapter exercises 2

A

1 28 kg of apples cost £7. Write down a formula for the cost, C, of x kg of apples.

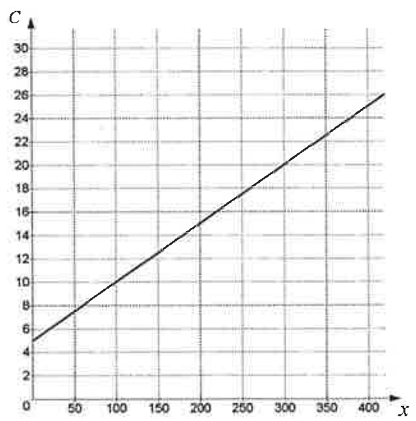
2 The figure shows the relation between the number of units of electricity used and the cost of a certain electricity bill.

a) Find the cost when 360 units are used.

b) Find the number of units used when the cost is £17.

c) The bill is made up of a standing charge plus a cost per unit. What is the standing charge and what is the cost per unit?

d) Write down the cost, C, as a function of the number of units used, x.



3 How can you find out, without drawing a graph, if two lines are parallel?

4 Which of the following lines are parallel?

A $y=-2x+1$ B $y-2x=3$

C $3x+\frac{3y}{2}+5=0$ D $5y+10x=-9$

5 A straight line between the points $(2, 1)$ and $(5, a)$ has gradient 6. Find a.

6 Find from the polynomial $x^3+\frac{2x^2}{3}-3x+0.5$

a) the degree of the polynomial   b) the coefficient for x

c) the constant term   d) the coefficient for x^2

7 What are the domains of the following functions ?

a) $f(x)=\frac{3x^2}{2x-1}$ b) $f(x)=\frac{x^2+4}{x^2+8x-9}$

8 If $f(x)=\frac{x^2-4x+3}{2x^2+3}$, find

a) $f(2)$ b) $f(-2)$

9 Decide whether the following functions are continuous everywhere or not:

a) $y=|x-2|$ b) $g(x)=\begin{cases} x-x^2+3 & \text{if } x\le 1 \\ 5-2x & \text{if } x>1 \end{cases}$

10 Match the following functions with their graph:

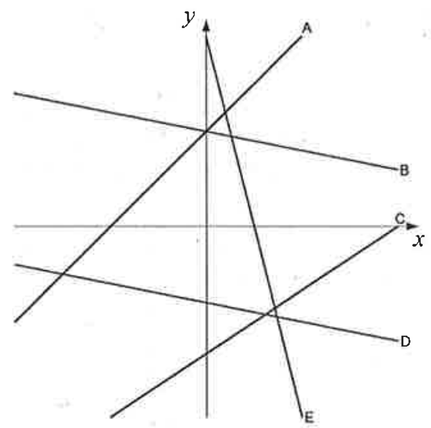
a) $y=x+1$

b) $y=2-4x$

c) $y=\frac{2x-4}{3}$

d) $y=-0.2x+1$

e) $y=-\frac{4+x}{5}$

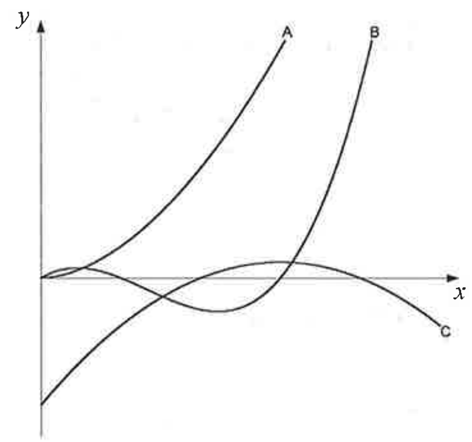


11 Match the following types of functions with a graph:

a) quadratic function

b) polynomial function of degree 3

c) exponential function



12 Find the equation of the line parallel to the line $y=-3x+5$, containing the point

a) $(-1,-1)$ b) $(\frac{1}{2}, \frac{1}{4})$

13 Solve graphically the following equations:

a) $-x^2+10x-24=0$ b) $x^2-6x+9=0$

B

14 For each of the functions below, find $f(-1), f(\frac{1}{2}), f(2a), f(x+h)$.

a) $f(x)=3x-1$ b) $f(x)=x^2-x$

c) $f(x)=\frac{x^3}{2}$ d)   d) $f(x)=(x+1)^2$

15 The graph of $f(x)=x^3+x^2-12x$ cuts the x-axis at three points. Find the x-coordinates of the three points.

16 The straight line $3x+2y-6=0$ forms together with the coordinate axes a triangle. Draw the line and find the area and perimeter of the triangle.

17 A record company has fixed costs of \$10000 for producing a record master. Thereafter, the variable costs are \$0.50 per record for duplicating the record master. Revenue from each record is expected to be \$1.30.

a) Formulate a function $C(x)$ for the total cost of producing x records.
b) Formulate a function $R(x)$ for the total revenue from the sale of x records.
c) Formulate a function $P(x)$ for the total profit from the production and sale of x records.
d) How many records must the company sell in order to break even?

18 Find the slope of the line between the points $(x, 4x)$ and $(x+h, 4(x+h))$.

19 A quadratic function has the form $f(x) = ax^2 + bx + c$, where a, b and c are constants. What can be said about a, b and c, if

a) $f(x)$ has the only zero $x = 0$?
b) $f(x)$ has a maximum point at the origin ?
c) $f(x)$ cuts the y-axis at $(0, 3)$

20 Find the point(s) of interception between the line $y = -3x + 3.25$ and the curve $y = -x^2 - 2x + 3$.

21 a) What is the domain of $f(x) = \sqrt{1-x^2}$?
b) Draw the graph of $f(x)$ in its domain. What is the shape of the graph?
c) Describe the graph of $g(x) = \sqrt{a^2 - x^2}$, where a is a positive constant.

22 Evaluate the following limits:

a) $\lim\limits_{x \to 4} \dfrac{6x^2 - 96}{3x^2 - 12x}$ b) $\lim\limits_{x \to \infty} \dfrac{\frac{x^2}{3} - 2x + 10}{x(1-x^2)}$

Chapter 3

The Derivative