17 A record company has fixed costs of \$10000 for producing a record master. Thereafter, the variable costs are \$0.50 per record for duplicating the record master. Revenue from each record is expected to be \$1.30.

a) Formulate a function $C(x)$ for the total cost of producing x records.
b) Formulate a function $R(x)$ for the total revenue from the sale of x records.
c) Formulate a function $P(x)$ for the total profit from the production and sale of x records.
d) How many records must the company sell in order to break even?

18 Find the slope of the line between the points $(x, 4x)$ and $(x+h, 4(x+h))$.

19 A quadratic function has the form $f(x)=ax^2+bx+c$, where a, b and c are constants. What can be said about a, b and c, if

a) $f(x)$ has the only zero $x=0$?
b) $f(x)$ has a maximum point at the origin ?
c) $f(x)$ cuts the y-axis at $(0, 3)$

20 Find the point(s) of interception between the line $y=-3x+3.25$ and the curve $y=-x^2-2x+3$.

21 a) What is the domain of $f(x)=\sqrt{1-x^2}$?
b) Draw the graph of $f(x)$ in its domain. What is the shape of the graph?
c) Describe the graph of $g(x)=\sqrt{a^2-x^2}$, where a is a positive constant.

22 Evaluate the following limits:

a) $\lim\limits_{x\to 4}\dfrac{6x^2-96}{3x^2-12x}$ b) $\lim\limits_{x\to\infty}\dfrac{\frac{x^2}{3}-2x+10}{x(1-x^2)}$

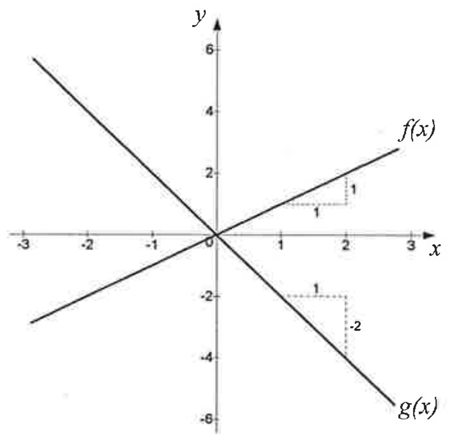
Chapter 3

The Derivative

3.1 Rates of change

In our studies of functions we have seen that functions may increase or decrease, which is shown by its graph going up or down. We also know that the steeper the curve is, the more the function is increasing (or decreasing). But what if we want to know how much, or how fast, a function changes?
The purpose of this chapter is to find a way to measure how fast a function changes. We call that measure the *rate of change* of the function. We shall also see what it really means in practical problems.
Since a function may both increase and decrease, the rate of change of a function may vary.

Example 1
The linear functions $f(x)=x$ and $g(x)=-2x$ are drawn using the same coordinate system as shown below.



A reasonable way of describing the rate of change of the two functions is to find out how much the function value changes on each step along the x-axis. This is also known as the slope (or gradient) of the straight line. We can therefore conclude that
"The rate of change of a linear function is the same as the slope of the line"

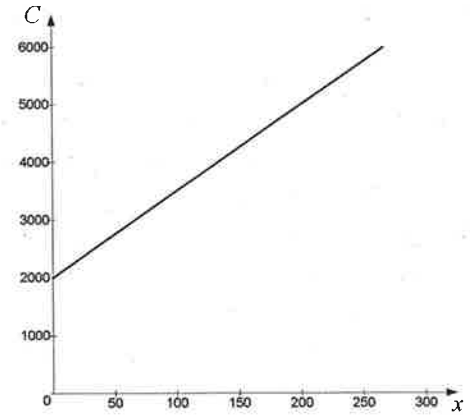
We can calculate the rate of change of a linear function in the same way as we calculate the slope of a straight line: $\dfrac{\text{change in } y}{\text{change in } x}$.

In our example the rates of change of $f(x)$ and $g(x)$ are 1 and -2, respectively.

Example 2
The cost C, in kr, for water in a household during a year can be described by the linear function

$$C(x)=15x+2000$$

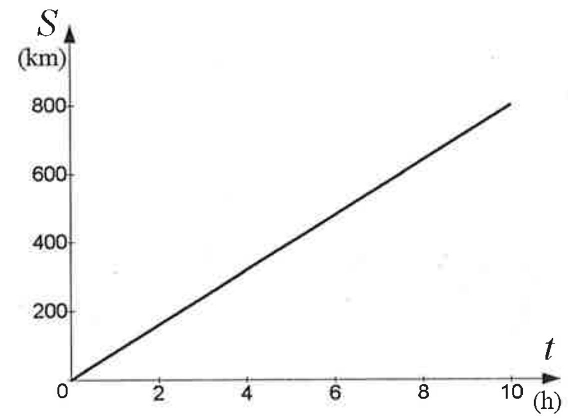
where x is the amount of water used, in m^3.
The rate of change of this function is 15, which here has the meaning "cost per m^3 of water".



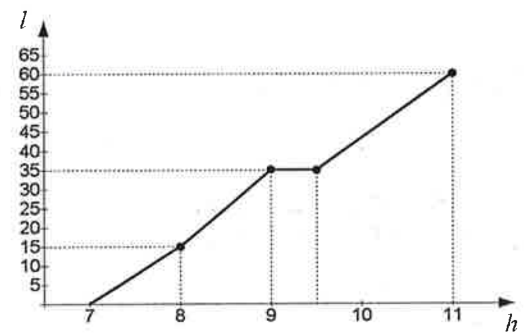
Example 3
If we have a function involving time, it feels natural to talk about the *speed* of change, because the rate of change tells how *fast* the function is changing.
If a car travels with a speed of 80 km/h, the distance, S km, travelled by the car in t hours can be written as a function

$$S(t)=80t\ .$$

The rate of change is of course how many km the car travels in 1 hour, which obviously is the same as the speed of the car, 80 km/h, so the rate of change tells us how fast the car is moving.



Example 4
Cherry has a summer job picking strawberries. The graph below shows how much she picked before lunch one day. We can say that the amount of berries (in l) picked is a function of time (in h).



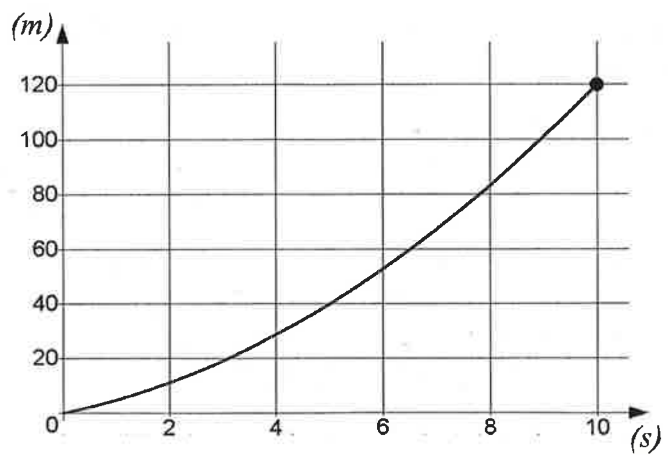
a) Describe the rate of change of this function, and what it stands for in this example.
b) When was Cherry picking fastest?
c) Find the *average rate of change* during this period.

Solution:

a) The rate of change is the same as the slope of the graph, which tells how many litres of strawberries she picks per hour. Thus, the rate of change in this case is a measure of her speed of picking.
We can see that the speed varies during the day, and is even zero between 9.00 and 9.30 (probably a break).
b) Between 8.00 and 9.00. Because the graph has the steepest slope during that period. The rate of change is here
$$\frac{\textit{change in } y}{\textit{change in } x} = 20\ l/h.$$
c) In 4 hours she picked 60 l. The *average* rate of change is $\frac{60}{4} = 15\ l/h$.
This means that her speed of picking changed during the day, but her average speed of picking during the 4 hours was 15 l/h.

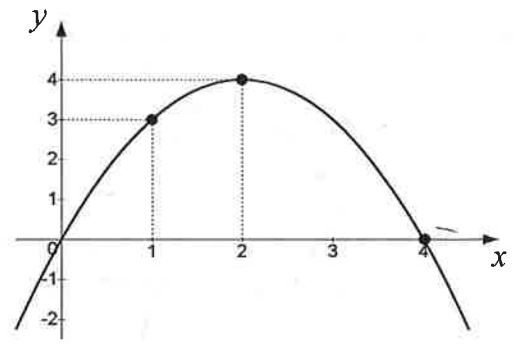
So far, we have looked at linear functions, where the rate of change may be read from the slope of the line. But what if the graph of the function is a smooth curve, where the rate of change changes all the time?
We start attacking the problem by once again looking at the *average rate of change.*

Example 5
A car is accelerating from rest. The graph shows the distance travelled, in m, after t seconds. It is the graph of the distance as a function of time.
We can see from the graph that the rate of change (= the speed of the car) is increasing all the time.



We cannot find the speed of the car at any moment, but we can calculate the *average speed* of the car during a certain period of time. This is the same as the *average rate of change* of the function.
If we want to find the average speed during the whole period, we simply divide the distance travelled by the time passed, which is
$$\frac{120}{10} = 12\ m/s$$
The average speed during the first 5 seconds is
$$\frac{\textit{change in } y}{\textit{change in } x} = \frac{40}{5} = 8\ m/s$$
The average speed during the last 5 seconds is
$$\frac{\textit{change in } y}{\textit{change in } x} = \frac{80}{5} = 16\ m/s$$
The average speed during the last second (from $t = 9$ to $t = 10$) is
$$\frac{\textit{change in } y}{\textit{change in } x} = \frac{20}{1} = 20\ m/s$$

Example 6
The figure shows the graph of the function $f(x) = 4x - x^2$. Find the average rate of change of the function
a) between $x = 1$ to $x = 2$.
b) between $x = 2$ to $x = 4$.



Solution:

a) From $x = 1$ to $x = 2$ the value of the function has changed from $f(1) = 3$ to $f(2) = 4$. The average rate of change is
$$\frac{\textit{change in } y}{\textit{change in } x} = \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 3}{1} = 1$$
This can be interpreted as "the slope of a line between the points $(1, 3)$ and $(2, 4)$ is 1".
b) From $x = 2$ to $x = 4$ the value of the function has changed from $f(2) = 4$ to $f(4) = 0$. The average rate of change is
$$\frac{\textit{change in } y}{\textit{change in } x} = \frac{f(4) - f(2)}{4 - 2} = \frac{0 - 4}{2} = -2$$
The rate of change is negative, and the reason is of course that the function is decreasing from $x = 2$ to $x = 4$ (remember that a negative gradient means that a straight line is pointing down).
Thus, the slope of the line between the points $(2, 4)$ and $(4, 0)$ is -2.

Exercises

A

3101 The table shows the population of Littlewood between 1986 and 1995. Calculate the average annual increase
a) from 1986 to 1990
b) from 1987 to 1992
c) during the whole period.

Year:	1986	1987	1990	1992	1995
Population:	16200	18100	18300	17900	18000

3102 The table shows the growth of bacteria in a laboratory.

Time (h):	0	2	4	6	8
Number:	1200	1550	2000	2750	4200

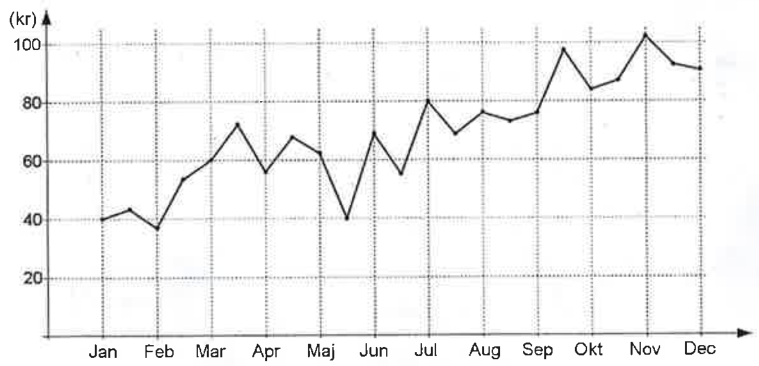
Calculate the speed of growth
a) from 0 to 4 hours
b) from 6 to 8 hours.

3103 Mercedes is driving her Fiat from Stockholm to Göteborg. The table shows the total distance travelled at certain points in time.

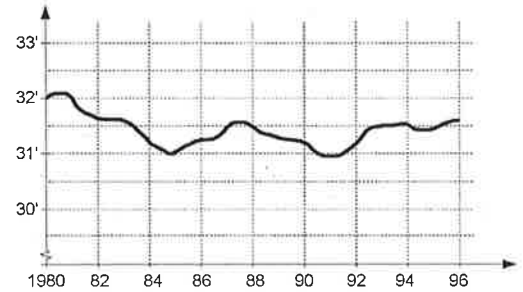
Time from start (h):	1	2	3	4	5
Distance (km):	103	198	307	402	480

Find the average speed (in km/h)
a) during the first 3 hours
b) during the whole journey

3104 The graph below shows the value of one share of Wheelfolio's Cars during a year. Find the rate of change of this graph, and what it stands for
a) from January to March
b) from July to December

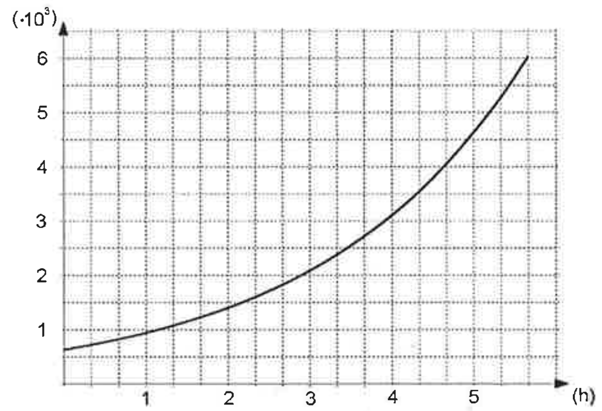


3105 The change in population in a town from 1980 to 1996 is shown in the figure. Find the average rate of change of the population between
a) 1980 and 1985
b) 1986 and 1993



3106 The figure shows how the number of insects in a biological experiment increases. Find the average speed of growth during the following periods

a) from 1h to 3 h
b) from 3 h to 3.5 h
c) during the whole period
d) from 5 h to 5.5 h
e) *at* 4 h



B

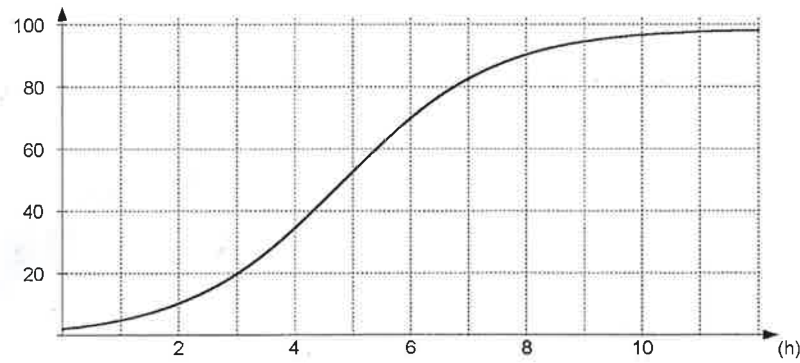
3107 The table shows the distance a ball has rolled down a furrow.

Time (s):	0	1	2	3	4
Distance (m):	0	0.21	0.81	1.84	3.24

a) Find the average speed of the ball from 0 to 2 seconds.
b) Estimate the speed of the ball *at* 3 seconds.

3108 One day the number of flies in Old Macdonald's kitchen increased as described in the figure. Find the rate of growth

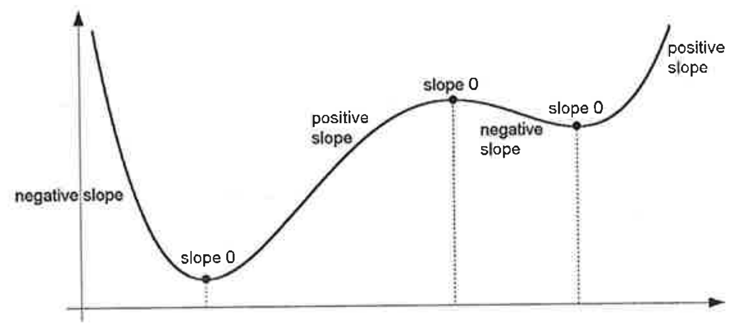
a) from 1 h to 3 h
b) from 3 h to 6 h
c) from 9 h to 10 h
d) at 5 h



3.2 The slope of a curve.

We are now getting close to the crux of the problem:
How to calculate the slope at each point of a curve.
Before we proceed with calculations we will summarise some facts and introduce a few new terms.

We know that a straight line pointing upwards has a positive slope, a line pointing down has negative slope, while a horizontal line has zero slope.
If a curve of a function goes up and down, it would be natural to describe the slope of the curve as in the figure below.



An important fact which we will use later is that at each turning point the slope must be zero. The remaining problem is to find out *how much*, or better, *at what rate* the curve is sloping at other points of the curve.

The difference quotient
The average rate of change of a function was calculated by taking the difference between two points of the curve and dividing, exactly as we would to find the slope of a straight line. The average rate of change (or slope) between the points (x_1, y_1) and (x_2, y_2) is

$$\frac{\Delta y}{\Delta x} = \frac{\textit{difference in } y}{\textit{difference in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

(the Greek letter delta, Δ, is often used in maths to symbolise differences)
This quotient is often called *the difference quotient* and can also be written by using $f(x)$ instead of y

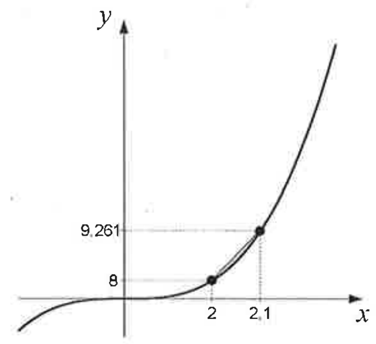
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example 1
For $f(x) = x^3$, find the difference quotient from $x = 2$ to $x = 2.1$.

Solution:
$f(2) = 8$
$f(2.1) = 9.261$

The difference quotient is

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.261 - 8}{0.1} = 12.61$$



This means that the slope between the points (2, 8) and (2.1, 9.261) is 12.61 .

Exercises

A

3201 For the function $f(x) = 2x^2 - 1$, find the difference quotient

a) from $x = 0$ to $x = 1$ b) from $x = -1$ to $x = -\frac{1}{2}$

3202 Calculate the difference quotient $\frac{f(5.5) - f(4.0)}{1.5}$ if $f(x) = x^2 - 3x$.
What does this difference quotient mean?

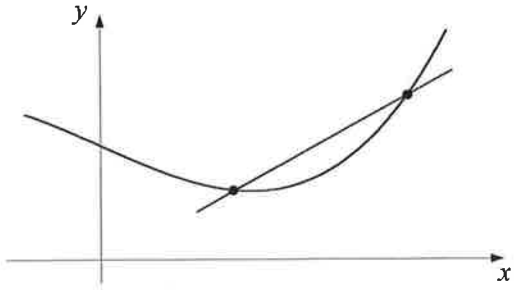
3203 Find the difference quotient $\frac{f(2.01) - f(2.00)}{0.01}$ if $f(x) = 3x^2 - 2x^3 - 1$.
Answer correct to two significant figures.

3204 Find the difference quotient for the function $f(x) = -x^3 - 3x^2 - 3$ when x changes from 3.0 to 5.0 .

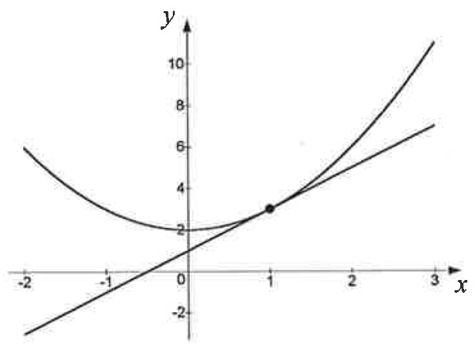
Secant lines and tangent lines

Since we know how to calculate the slope of a straight line, we use this information when trying to find the slope of a curve.

When we calculate the average rate of change of a curve we also calculate the slope of the line between two points of the curve. Such a line is called a *secant line* (or just secant).



Now, if we want to find the slope of a curve in *one point* of the curve, it would be convenient to draw a line that touches the curve *only in that point*, and then try to find the slope of that line. Such a line, that touches a curve in only one point, is called a *tangent line* (or just tangent).



When we speak of the slope (or gradient) of a curve at one point of the curve, we mean the slope of the tangent to the curve at that point.

Example 2

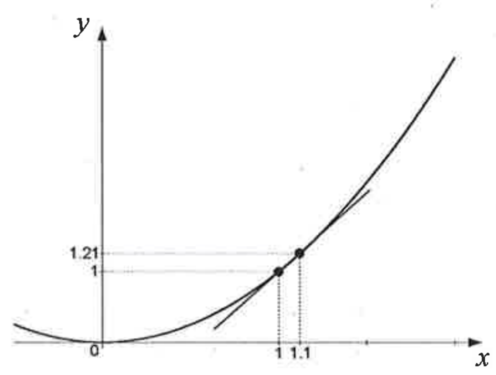
Draw the graph of the function $f(x)=x^2$. Try to find the slope of the curve *at the point* $(1, 1)$ by

a) drawing a secant line and using the difference quotient.

b) drawing a tangent line at $(1, 1)$ and estimating the slope of that line.

Solution:

a)

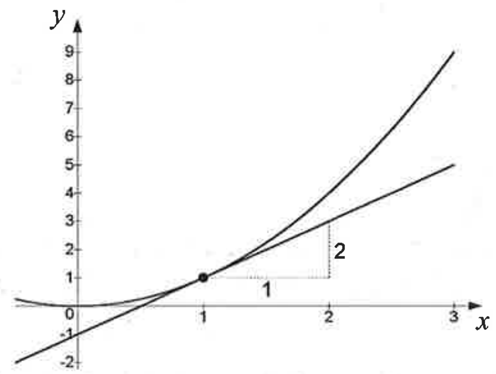


The secant here is drawn between the points $(1, 1)$ and $(1.1, 1.21)$.
The slope of the secant is

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{1.21-1}{1.1-1}=\frac{0.21}{0.1}=2.1$$

(How can we get a better result?)

b)



The slope of the tangent is $m=\frac{\Delta y}{\Delta x}=\frac{4}{2}=2$.

From a) and b) it seems reasonable to believe that the slope of the curve at the point $(1, 1)$ is 2 .

Exercises

A

3205 Sketch the following curves and estimate the slope of the curve at $x = 1$ by drawing secant lines through two suitable points of the curve and using the difference quotient.

a) $y = x^2 - x - 1$ b) $y = 2x - \frac{x^2}{2} + 1$

3206 A math book is dropped from a tower. The distance fallen, *s* metres, can be found using the formula $s(t) = 4{,}9t^2$, where *t* is the time passed, in seconds, since the book was dropped.
Find, and interpret, the difference quotient
a) from $t = 4.0\ s$ to $t = 5.0\ s$ b) from $t = 4.0\ s$ to $t = 4.5\ s$
c) from $t = 4.0\ s$ to $t = 4.1\ s$ d) from $t = 4.0\ s$ to $t = 4.01\ s$

3207 Draw the graph of $y = x^2 - 6x$. Find the slope of the curve at the point of the curve where $x = 2$ by drawing a tangent line and calculating the slope of the tangent.

B

3208 Sketch the following curves and estimate the slope of the curve at $x = 0$ by using the difference quotient on some suitable secant line.
a) $y = x^3 + 6x^2 + 8x$ b) $y = \sqrt{2x+1}$

3209 The following curves all passes through the point (0, 1). Estimate their slope at that point.
a) $y = x^2 - x + 1$ b) $y = 2x - x^2 + 1$
c) $y = 2^x$ d) $y = 3^{-x}$

3210 What is the slope of the curve $y = x^2 - x$ at the point where $x = a$?

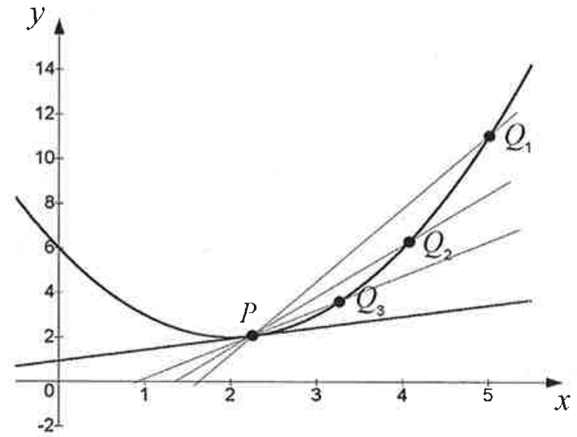
3211 The number of bacteria in a laboratory experiment can be estimated by the formula $N(t) = 91t^2 + 25t + 1900$, where *t* is the number of hours from start.
Calculate the difference quotient and explain what it means
a) from $t = 0\,\text{h}$ to $t = 5.0\,\text{h}$ b) from $t = 5.0\,\text{h}$ to $t = 5.1\,\text{h}$

3.3 The derivative

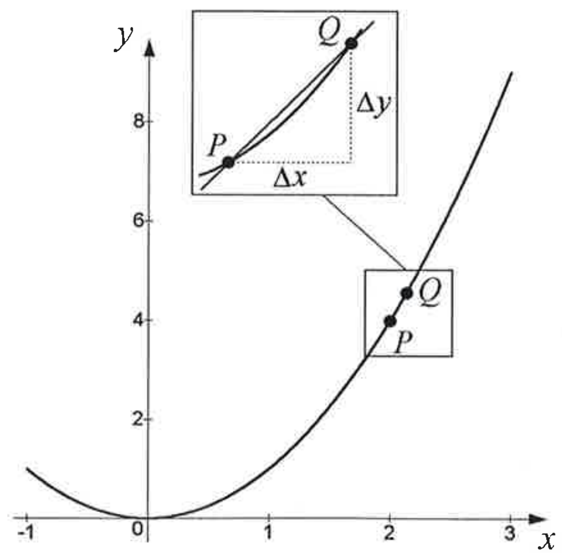
We saw in the previous chapter that we could find a good approximation for the slope of a curve at a point by drawing a tangent line and estimating the slope of that line, or by calculating the slope of a secant line close to the tangent. Our aim in this chapter is to find a method to calculate the *exact* value of the slope at each point of the curve.
Let us look at the problem graphically once again.

The figure shows the graph of a function $f(x)$. We want to find the slope of the curve at the point P , or more precisely: the slope of the tangent at the point *P*.
If we draw a secant line between the point *P* and some other point *Q* on the curve, it is pretty clear that the closer *Q* is to *P*, the better the result.

In other words, *as the point Q approaches the point P, by moving from Q_1 to Q_2 to Q_3 and so on, the slope of the secant approaches the slope of the tangent.*



Example 1
We look at the graph of $f(x) = x^2$ and want to find the slope of the tangent at the point $P = (2, 4)$. Let us see what happens if we put a point *Q* closer and closer to *P* and calculate the slope of each secant line (see fig.). The slope is calculated by the difference quotient $\left(\frac{\Delta y}{\Delta x}\right)$.



We start at a point fairly close to P.
Let Q_1 have the x-value 2.1 .
The y-value of Q_1 is then $f(2.1) = 4.41$.
The slope of the secant line from $P = (2, 4)$ to $Q_1 = (2.1, 4.41)$ is

$$\frac{4.41-4}{2.1-2} = \frac{0.41}{0.1} = 4.1$$

To choose Q_2 even closer to P, we let Q_2 have the x-value 2.01, which gives the y-value $f(2.01) = 4.0401$.
The slope of the secant from P to Q_2 is

$$\frac{4.0401-4}{2.01-2} = \frac{0.0401}{0.01} = 4.01 \ .$$

Putting Q_3 at $x = 2.001$ and Q_4 at $x = 2.0001$ we can summarise as follows:

Secant	*Slope*
P to Q_1	$\frac{4.41-4}{2.1-2} = \frac{0.41}{0.1} = 4.1$
P to Q_2	$\frac{4.0401-4}{2.01-2} = \frac{0.0401}{0.01} = 4.01$
P to Q_3	$\frac{4.004001-4}{2.001-2} = \frac{0.004001}{0.001} = 4.001$
P to Q_4	$\frac{4.00040001-4}{2.0001-2} = \frac{0.00040001}{0.0001} = 4.0001$

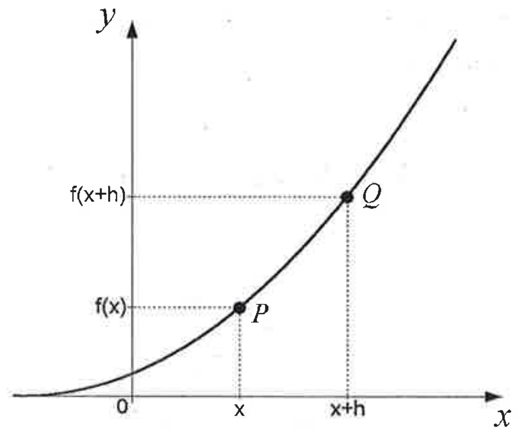
We discover that the slope of the secant is getting closer and closer to the value 4. Although not proven yet, I am sure you are convinced that the slope of the tangent at the point (2, 4) is 4.

What we did here was to choose x-values closer and closer to $x = 2$, starting at $x = 2.1$. We can of course move even closer than $x = 2.0001$, but let us save some time here.
We start at $x = 2 + h$, where h is "a little bit more", and then imagine what will happen if h gets smaller and smaller.
The y-value at $x = 2 + h$ is $f(2+h) = (2+h)^2$.
The slope of the secant line (= the difference quotient) from $P =$ (2, 4) to $Q = (2+h, (2+h)^2)$ is

$$\frac{(2+h)^2-4}{(2+h)-2} = \frac{4+4h+h^2-4}{2+h-2} = \frac{4h+h^2}{h} = \frac{h(4+h)}{h} = 4+h$$

It is now obvious that as the point Q is moved closer to P, the value of h will approach 0 and the value of the difference quotient will approach 4.

The definition of the derivative
If we look back at the problem of how to find the slope of a curve at some point, the time has come to make a general formula for the slope of *any* curve at *any* point. To do that, we must not use any specific curve or any specific point, so let us find the slope of any curve $y = f(x)$ at the point $P = (x, y)$ of the curve.
If P has the x-coordinate x, the y-coordinate is $f(x)$ (since P is on the curve).
Let Q have the x-coordinate $(x+h)$, and consequently the y-coordinate $f(x+h)$.



The slope of the secant from P to Q is then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

This difference quotient will tell us the slope of the curve at P, if we let h approach zero. We can write

The slope of the (tangent line to the) curve $f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

The slope of the curve of a function is called *the derivative of the function f*, and is written $f'(x)$. The formal definition is

Definition of the derivative
For a function $f(x)$, its *derivative* at x is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \ .$$

If $f'(x)$ exists, we say that $f(x)$ is *differentiable* at x.

There are several different symbols used for the derivative:

Function	*Derivative*
$f(x)$	$f'(x)$
y	y'
y	Dy
y	$\frac{dy}{dx}$

The meaning of the derivative, $f'(x)$

The derivative of a function tells us the slope of its graph, which is the same as the rate of change of the function, at each point of the graph. We can say that the *value* of $f'(x)$ tells us how fast $f(x)$ changes at each moment, not just on the average.

The *sign* of $f'(x)$ tells us if the curve is pointing up or down:

If $f'(x) > 0$	(positive slope)	$f(x)$ is increasing.
If $f'(x) < 0$	(negative slope)	$f(x)$ is decreasing.
If $f'(x) = 0$	(zero slope)	$f(x)$ is horizontal.

Example 3

$f(2) = 3$ means that the value of *the function* when $x = 2$ is 3.

$f'(2) = 3$ means that the value of *the derivative* when $x = 2$ is 3, which is the same as *the slope of the curve* when $x = 2$ is 3.

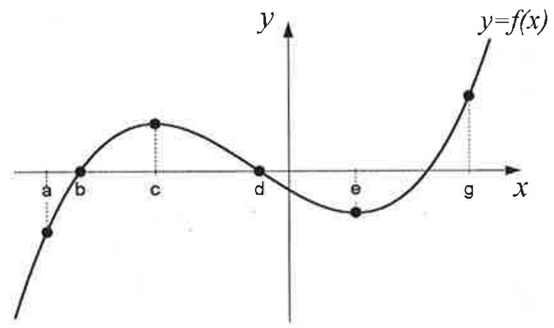
Example 4

If $f(x)$ has a turning point when $x = a$, we know that $f'(a) = 0$, because the tangent at a turning point is horizontal (slope zero).

Example 5

From the graph below we can find that

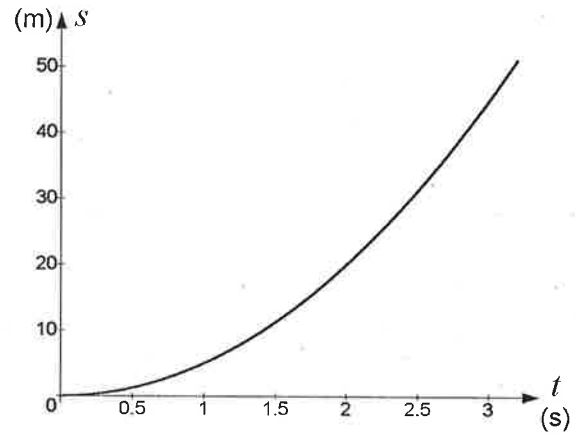
$f'(a) > 0$
$f(b) = 0$
$f'(c) = 0$
$f(d) = 0$
$f'(e) = 0$
$f(e) < 0$
$f'(g) > 0$



Note the difference between $f(x)$ and $f'(x)$.

Example 6

When dropping a ball from a balcony, the distance fallen s metres after t seconds can be described by the function $s(t) = 5t^2$. The graph is shown in the figure.



Here, $s(t)$ means the distance the ball has fallen after t seconds, while $s'(t)$ means the speed of the ball after t seconds, since the rate of change here is how much the distance changes each second (m/s).

Example 7

The temperature of water in a thermos can be described by a function $T(t)$, where T is the temperature after t minutes. Write the following in mathematical terms:

a) The temperature after 10 minutes is 80°.
b) After 2 minutes the temperature is decreasing by 2° per minute.

Solution:

a) $T(10) = 80$
b) $T'(2) = -2$ (the function is decreasing, so the derivative is negative)

Exercises

A

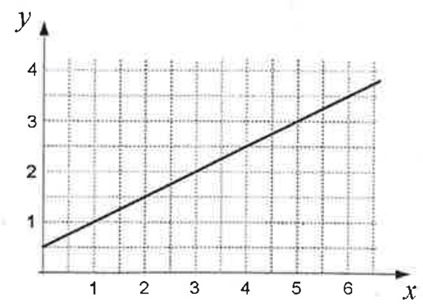
3301 What happens with the following expressions when x becomes infinitely small? (Check with your calculator!)

a) $x^2 - 3x + 5$ b) $\frac{3x}{5}$

c) $\frac{5}{x}$ d) $\frac{x^2 + 3x}{x}$

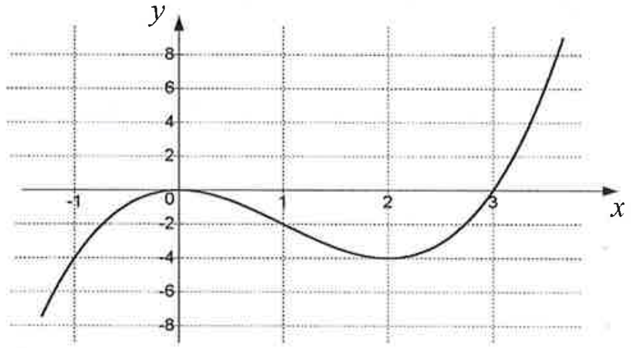
3302 The figure shows the graph of $y = f(x)$. Find from the figure

a) $f(3)$
b) $f'(3)$



3303 The figure shows the graph of $y = f(x)$. Find from the figure

a) $f(1)$ b) $f'(1)$



3304 The temperature in an oven is $T(t)$ °C, t minutes after it has been turned on. Explain in words the following

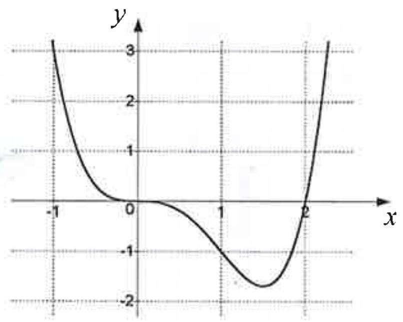
a) $T(0) = 22$ b) $T(15) = 145$
c) $T'(15) = 6$ d) $T'(80) = -5$

3305 One day the temperature at the centre court of Wimbledon was $W(x)$, where x is the number of hours after 8.00. Write the following in mathematical symbols

a) At 13.00 the temperature was 30 °C.
b) At 11.00 the temperature rose by 3 °C per hour.
c) At 19.00 the temperature fell by 2.5 °C per hour.
d) At 15.00 the temperature didn't change.

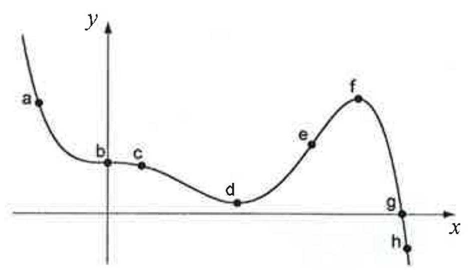
3306 The figure below shows the graph of $y = f(x)$. Solve the following problems by looking at the figure.

a) Find $f(1)$
b) Find $f'(1)$
c) Solve the equation $f(x) = 0$
d) Solve the equation $f'(x) = 0$



3307 The figure shows the graph of $f(x)$. At which points is it true that

a) $f(x) > 0$
b) $f'(x) < 0$
c) $f'(x) = 0$
d) $f'(x) > 0$



3308 The famous rap artist Cool Candy is strictly mathematical. On his latest hit "Cool Curves" he follows the function $W(t)$, where W is the number of words "rapped" after t seconds of the song. Explain in words

a) $W(30) = 62$ b) $W'(60) = 3$

3309 If $s(t)$ is the distance travelled by a caterpillar after t seconds, what is $s'(t)$?

3310 If $V(t)$ is the amount of water (in l) poured into a bathtub after t minutes, what does $V'(2)$ mean?

B

3311 Find the following limits

a) $\lim_{x\to 0} 2^x$ b) $\lim_{x\to 0} \sqrt{(x-2)^2}$

c) $\lim_{x\to 2} \dfrac{x^2-4}{3x-6}$ d) $\lim_{h\to 0} \dfrac{h^3+3h^2x+7hx}{h}$

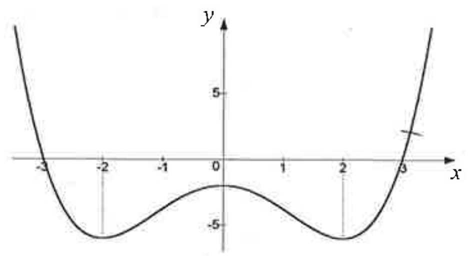
3312 $f(x) = 7 - x - 3x^2$. Estimate $f'(1)$ by calculating the difference quotient

a) $\dfrac{f(1.1)-f(1.0)}{0.1}$

b) $\dfrac{f(1+h)-f(1)}{h}$ and let h approach zero.

3313 Solve the following problems from the graph of $f(x)$ below

a) Find $f'(0)$
b) Solve the equation $f'(x) = 0$
c) Solve the inequality $f(x) > 0$
d) Solve the inequality $f'(x) > 0$



3314 In an experiment $V(T)$ describes the volume (in cm^3) of a metal piece as a function of the temperature (in °C).

a) What is the meaning of $V'(80)$? b) What is the unit?
c) What does $V'(50) = 0.2$ mean?

3.4 Differentiation of polynomial functions.

We now turn to finding rules of differentiation, easy ways of finding the derivative of a function. We start with polynomial functions.

Example 1

Find $f'(x)$, if $f(x)=x^2$.

Solution:

According to the definition of the derivative, $f'(x)$ is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \ .$$

If $f(x)=x^2$ we get

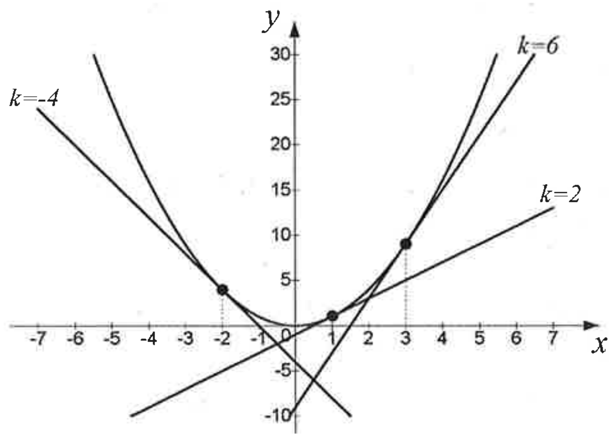
$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=$$

$$=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}(2x+h)=2x$$

Here we have found out that if the function is $f(x)=x^2$, then its derivative is $f'(x)=2x$.

This means that the slope of the curve $y=x^2$ at any point of the curve is 2 times the x-value of the point.

This seems to agree well with the shape of the curve (see figure):



Example 2

Find the derivative of $f(x)=x$.

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)-x}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

So, if $f(x)=x$, then $f'(x)=1$.

This means that the slope of $f(x)$ is 1 at every point of the graph (independent of the x-value). Since we already know that the graph of $f(x)=x$ is a straight line with slope 1 this is perfectly correct.

Example 3

Differentiate $f(x)=2$.

Solution:

The graph of this function is a horizontal line at $y=2$ with slope zero everywhere. Knowing this, we can immediately tell that $f'(x)=0$.

Checking with the definition of the derivative:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{2-2}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$$

The examples have shown that

If	$f(x)=2$	then	$f'(x)=0$
If	$f(x)=x$	then	$f'(x)=1$
If	$f(x)=x^2$	then	$f'(x)=2x$

Working in the same way with $f(x)=x^3$ and $f(x)=x^4$ gives

If	$f(x)=x^3$	then	$f'(x)=3x^2$
If	$f(x)=x^4$	then	$f'(x)=4x^3$

We start to see a pattern here. We have reason to believe that, for example, the function $f(x)=x^{100}$ has the derivative $f'(x)=100x^{99}$. It can also be shown that this is true.

In general, the rule is:

If	$f(x)=x^n$ (n is a positive integer)
then	$f'(x)=nx^{n-1}$

Example 4
Find the derivative of $f(x)=7x^2$.

Solution:
The definition of the derivative gives

$$f'(x)=\lim_{h\to 0}\frac{7(x+h)^2-7x^2}{h}=\lim_{h\to 0}\frac{7x^2+14xh+7h^2-7x^2}{h}=$$

$$=\lim_{h\to 0}\frac{14xh+7h^2}{h}=\lim_{h\to 0}(14x+7h)=14x$$

Thus, the derivative of $7x^2$ is $7\cdot 2x=14x$. In fact

If $f(x)=k\cdot x^n$ (where n is a positive integer and k is a constant)
then $f'(x)=k\cdot nx^{n-1}$

Example 5
What happens if we have a polynomial function with several terms?
Let us differentiate $f(x)=x^2-2x+3$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^2-2(x+h)+3]-(x^2-2x+3)}{h}=$$

$$=\lim_{h\to 0}\frac{(x^2+2xh+h^2)-(2x+2h)+3-(x^2-2x+3)}{h}=$$

$$=\lim_{h\to 0}\frac{2xh+h^2-2h}{h}=\lim_{h\to 0}(2x-2+h)=2x-2\ .$$

The derivative of x^2-2x+3 is $2x-2$.
In general we say that a polynomial function may be differentiated "term by term".

The second derivative
The second derivative is what you get if you differentiate twice. We denote the second derivative by $f''(x)$ (or y'', D^2 etc.). The second derivative has also a meaning, as we shall see later.

Example 6
Find $f''(x)$ if

a) $f(x)=x^2-3x+1$

b) $f(x)=3x^3+3x^2-5x+7$

Solution:

a) $f'(x)=2x-3$
$f''(x)=2$

b) $f'(x)=9x^2+6x-5$
$f''(x)=18x+6$

Example 7
Find $D^2(x^3-2x+19)$.

Solution:
$D(x^3-2x+19)=3x^2-2$
$D(3x^2-2)=6x$

so, $D^2(x^3-2x+19)=6x$.

Example 8
Find $f''(-2)$ if

a) $f(x)=x^3-1$

b) $f(x)=\frac{3x^2}{2}-4x+11$

Solution:

a) $f'(x)=3x^2$
$f''(x)=6x$
$f''(-2)=6\cdot(-2)=-12$

b) $f'(x)=\frac{6x}{2}-4=3x-4$
$f''(x)=3$
$f''(-2)=3$

Exercises

A

3401 Find $f'(x)$ if
a) $f(x)=x^3$
b) $f(x)=x^2-x^4$
c) $f(x)=3x^3-5x^2+7x-1$
d) $f(x)=0.5x^2+0.8x^5-1758$

3402 Find $f'(x)$ if
a) $f(x)=\frac{x-3x^2}{5}$
b) $f(x)=\frac{2x^5-4x^3}{7}$
c) $f(x)=\frac{2x}{3}$
d) $f(x)=\frac{5x-3}{4}$

3403 Find $f'(2)$ if
a) $f(x)=5x^2$
b) $f(x)=4x$
c) $f(x)=-3x^3$
d) $f(x)=3$

3404 Find $f'(-2)$ if
a) $f(x)=4x^3-7x$
b) $f(x)=-x^4-x^3$
c) $f(x)=\frac{3x^2-2x+2}{4}$
d) $f(x)=\frac{-3x^2}{4}-\frac{x^3}{2}$

3405 Find x, so that $f'(x)=0$, if
a) $f(x)=3x^2-12x$
b) $f(x)=20x-2x^2+3$
c) $f(x)=x^3-12x$
d) $f(x)=2x^3-21x^2+36x-1$

3406 Find $f''(x)$, given that

a) $f(x)=2x-12$

b) $f(x)=3x^3-2x^2+7x-2$

c) $f(x)=25x^5-12x^2$

d) $f(x)=\frac{x^4}{3}-\frac{x^3}{4}$

3407 Calculate

a) $D(3x^5-x^6)$

b) $D^2(3x^5-x^6)$

B **3408** Differentiate the function

a) $f(x)=(3x+1)^2$

b) $y=\frac{1}{2}(2x-1)^2$

c) $g(x)=x^3(2x^2-x+1)$

d) $y=\left(\frac{2-x}{5}\right)^2$

3409 Solve the equation $f'(x)=0$ when

a) $f(x)=\frac{x^2-2x-1}{7}$

b) $f(x)=-\frac{3x^3}{4}$

c) $f(x)=3.5x^2-\frac{x^3}{3}+5x-2$

d) $f(x)=x^4-2x^3-4$

3410 Find y'', if

a) $y=\frac{2}{3}(x^2-5x-9)$

b) $y=3(x-x^2)^2$

c) $y=x-3a$, where a is a constant.

d) $y=ax^3+bx^2$, where a and b are constants.

3411 Calculate $s''(t)$ when

a) $s(t)=2t^3-t-8$

b) $s(t)=v_0t+\frac{at^2}{2}$, where v_0 and a are constants.

3.5 Differentiation of $1/x, 1/x^2, \sqrt{x}$

How do we differentiate functions like $f(x)=\frac{1}{x}$, $f(x)=\frac{1}{x^2}$, $f(x)=\sqrt{x}$?

As powers they can be written $f(x)=x^{-1}$, $f(x)=x^{-2}$, $f(x)=x^{\frac{1}{2}}$.
We saw on page 118 that $f(x)=x^n$ has the derivative $f'(x)=nx^{n-1}$ if n is a *positive integer.*
Can we use this rule even if n is *not* a positive integer?

We investigate the function $f(x)=x^{-1}$.

If the rule can be used, the derivative would be $f'(x)=-1\cdot x^{-2}=-\frac{1}{x^2}$.

The difference quotient $\frac{f(x+h)-f(x)}{h}$ becomes

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\frac{\frac{x-(x+h)}{x(x+h)}}{h}=\frac{-h}{hx(x+h)}=\frac{-1}{x(x+h)}=\frac{-1}{x^2+xh}.$$

As h approaches 0, this quotient becomes $-\frac{1}{x^2}$.

One can show that the rule from page 118 is good for any value of n, so we may write

If $f(x)=x^n$ (n is any real number)
then $f'(x)=nx^{n-1}$

Example 1

Find the derivative of $f(x)=\frac{3}{x^2}$.

Solution:

The function can be written $f(x)=3\cdot\frac{1}{x^2}=3x^{-2}$.

The differentiation rule gives $f'(x)=-2\cdot 3x^{-3}=-\frac{6}{x^2}$

Example 2

Differentiate $f(x)=\sqrt{x}$.

Solution:

We write the function as a power, $f(x)=x^{\frac{1}{2}}$, and use the differentiation rule:

$$f'(x)=\frac{1}{2}\cdot x^{-\frac{1}{2}}=\frac{1}{2}\cdot\frac{1}{x^{1/2}}=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Example 3

Find $f'(x)$, if $f(x)=\dfrac{2}{5x}$.

Solution:

$$f(x)=\frac{2}{5x}=\frac{2}{5}\cdot\frac{1}{x}=\frac{2}{5}\cdot x^{-1}\ .$$

Then

$$f'(x)=-1\cdot\frac{2}{5}\cdot x^{-2}=-\frac{2}{5}\cdot\frac{1}{x^2}=-\frac{2}{5x^2}$$

Exercises

A

3501 Differentiate

a) $f(x)=x^{-3}$

b) $f(x)=3x^{-4}$

c) $f(x)=-x^{-23}$

d) $f(x)=x^2-x^{-2}$

3502 Find $f'(x)$ if

a) $f(x)=-\dfrac{2}{x}$

b) $f(x)=\dfrac{1}{x^5}$

c) $f(x)=x-\dfrac{3}{x^2}$

d) $f(x)=\dfrac{1}{2x}$

3503 Calculate $f'(3)$, if

a) $f(x)=2x^{-2}$

b) $f(x)=\dfrac{2}{x^2}-1$

3504 Calculate $f'(-0.5)$, if

a) $f(x)=x+\dfrac{2}{x}$

b) $f(x)=5-\dfrac{1}{x}-\dfrac{3}{x^2}$

B

3505 Differentiate

a) $\dfrac{1-3x}{3x}$

b) $\dfrac{x^2-5x+2}{4x}$

c) $x\sqrt{x}$

d) $\dfrac{2x-3\sqrt{x}-1}{\sqrt{x}}$

3506 Find the first and second derivative of the following functions

a) $y=x^2-3-\dfrac{5}{x^2}$

b) $p(t)=\dfrac{t-1}{t}$

c) $V(r)=\sqrt{r}-\dfrac{r^{-2}}{2}$

d) $N(t)=0.5t^2-\dfrac{2}{5t}+2t\sqrt{t}$

3507 If $f(x)=\dfrac{1}{x}$, find and interpret

a) $\lim\limits_{x\to 0} f'(x)$

b) $\lim\limits_{x\to\infty} f'(x)$

3.6 Differentiation of exponential functions

An exponential function has the form $f(x)=a^x$. What is the rule for differentiation such a function? We shall see that on our way to that rule we discover a number, which is very important. We call that number *e* ($\approx$ 2.71828....)
and like π it is an irrational number.

The number *e*

If we try to differentiate $f(x)=a^x$ by using the definition of the derivative, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}a^x\left(\frac{a^h-1}{h}\right).$$

Taking the function $f(x)=2^x$ we have

$$f'(x)=\lim_{h\to 0}2^x\left(\frac{2^h-1}{h}\right)=2^x\cdot\lim_{h\to 0}\left(\frac{2^h-1}{h}\right).$$

So, the derivative of 2^x is $2^x\cdot k$, where $k=\lim\limits_{h\to 0}\left(\dfrac{2^h-1}{h}\right)$.

In the same way we can see that the derivative of 3^x is $3^x\cdot k$,
where $k=\lim\limits_{h\to 0}\left(\dfrac{3^h-1}{h}\right)$.

But what is $\lim\limits_{h\to 0}\left(\dfrac{2^h-1}{h}\right)$ and $\lim\limits_{h\to 0}\left(\dfrac{3^h-1}{h}\right)$? Let us investigate what happens when *h* becomes very small (approaches zero).

h	$\left(\frac{2^h-1}{h}\right)$	$\left(\frac{3^h-1}{h}\right)$
0.1	0.7177	1.1612
0.010.6956	1.1047	
0.001	0.6934	1.0992
0.0001	0.6932	1.0987
0.00001	0.6931	1.0986
0.000001	0.6931	1.0986

The conclusion is, that if

$f(x)=2^x$ then $f'(x)=2^x\cdot 0.6931$

$f(x)=3^x$ then $f'(x)=3^x\cdot 1.0986$

From this we might guess that there is a number between 2 and 3 (we call it *e*) such that if

$f(x)=e^x$ then $f'(x)=e^x\cdot 1$

which means that $f'(x)$ is the same as $f(x)$!

Trying with different values between 2 and 3 shows that $e\approx 2.72$.

To get a better approximation of e we do the following:

We know that $\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1$, which means that, if h is small, then

$\frac{e^h-1}{h}\approx 1$ or $e^h\approx 1+h$.

Solving for e gives

$e\approx(1+h)^{\frac{1}{h}}$

Putting in small values of h gives

h	e
0.1	2.5937
0.01	2.7048
0.001	2.7169
0.0001	2.7181
0.00001	2.7183
0.000001	2.7183

Continuing in the same way shows that $e\approx 2.718281828459045....$.
e is an irrational number (like $\pi, \sqrt{2}$, for example) which means that its decimals continue without any order or pattern, and therefore cannot be written as a rational number.

We have now established the following important fact:

$$D(e^x)=e^x$$

We also need to differentiate function expressions like $3e^x$, e^{2x} or $5.4e^{0.03x}$. It can easily be shown (using the same method as above) that if C and k are constants, then

$$D(Ce^{kx})=k\cdot Ce^{kx}$$

Example 1
Differentiate

a) $y=2\cdot e^{3x}$ b) $y=3e^{-\frac{x}{4}}$
c) $y=e^{2+x}$

Solution:

a) $y'=3\cdot 2\cdot e^{3x}=6e^{3x}$

b) $y=3e^{-\frac{1}{4}x}$

$y'=\left(-\frac{1}{4}\right)\cdot 3e^{-\frac{x}{4}}=-\frac{3}{4}e^{-\frac{x}{4}}$

c) $y=e^2\cdot e^x$

$y'=e^2\cdot e^x=e^{2+x}$

Example 2
Find $f'(2)$ if

a) $f(x)=10-e^{0.2x}$ b) $f(x)=x+3.5e^{-2x}$

Solution:

a) $f'(x)=-0.2e^{0.2x}$
$f'(2)=-0.2e^{0.4}\approx -0.298$

b) $f'(x)=1+(-2)\cdot 3.5e^{-2x}=1-7e^{-2x}$
$f'(2)=1-7e^{-4}\approx 0.872$

Natural logarithms
The logarithms we have used so far are associated with the base 10, as the following examples show:

$\lg 100=2$ because $10^2=100$
$\lg 0.001=-3$ because $10^{-3}=0.001$
$\lg 50\approx 1.699$ because $10^{1.699}\approx 50$
$10^{\lg 4}=4$
$10^{\lg x}=x$

After introducing the number e and the simple way of differentiating exponential functions with base e, it might seem *natural* to introduce logarithms with base e instead of 10. We call these logarithms *natural logarithms*, abbreviated *ln*, and of course they follow the same rules as common logarithms:

$\ln e=1$ because $e^1=e$
$\ln e^4=4$ because $e^4=e^4$
$\ln\frac{1}{e}=-1$ because $e^{-1}=\frac{1}{e^1}=\frac{1}{e}$
$\ln 50\approx 3.912$ because $e^{3.912}\approx 50$
$e^{\ln 4}=4$
$e^{\ln x}=x$

Natural logarithms may be used in solving exponential equations exactly in the same way as we earlier used common logarithms. We need to remember the logarithm rule

$$\ln a^b=b\cdot\ln a$$

Example 3
Solve the equation $1.25^x=10$.
Solution:
$1.25^x=10$
$\ln 1.25^x=\ln 10$
$x\cdot\ln 1.25=\ln 10 \Rightarrow x=\frac{\ln 10}{\ln 1.25}\approx 10.3$

The derivative of a^x

We can now turn to the main purpose of this chapter, how to differentiate an exponential function in general (with any base).

We use the number e (and its logarithms) to make the differentiation easy, in the following way:

Example 4

Differentiate 2^x

Solution:

Use natural logarithms to write

$2^x = (e^{\ln 2})^x$ (since $2 = e^{\ln 2}$)

$(e^{\ln 2})^x = e^{\ln 2 \cdot x}$ (since $(a^x)^y = a^{x \cdot y}$)

$D(e^{\ln 2 \cdot x}) = \ln 2 \cdot e^{\ln 2 \cdot x}$ (since $D(e^{kx}) = k \cdot e^{kx}$)

Finally

$\ln 2 \cdot e^{\ln 2 \cdot x} \approx 0.69 \cdot 2^x$ (since $e^{\ln 2 \cdot x} = 2^x$ and $\ln 2 \approx 0.69$)

So

$D(2^x) = 0.69 \cdot 2^x$

Example 5

Differentiate a^x

Solution:

$D(a^x) = D(e^{\ln a \cdot x}) = \ln a \cdot e^{\ln a \cdot x} = \ln a \cdot a^x$

Thus, the general rule for differentiating an exponential function is

$$D(a^x) = \ln a \cdot a^x$$

Exercises

A

3601 Differentiate

a) $y = 2e^x$ b) $y = e^{3x}$

c) $y = 3e^{5x}$ d) $y = 10e^{0.2x}$

3602 Find $f'(x)$, if

a) $f(x) = e^{\frac{2}{3}x}$ b) $f(x) = e^{\frac{x}{2}}$

c) $f(x) = 5e^{-\frac{2x}{5}}$ d) $f(x) = \dfrac{e^{2x} - e^{-2x}}{2}$

3603 Use natural logarithms to write the following numbers with base e.

a) 23 b) 100

c) 0.07 d) a

3604 Find the derivative of

a) $y = 5^x$ b) $y = 250 \cdot 1.05^x$

3605 Find, correct to three significant figures, $f'(-2)$ if

a) $f(x) = 4e^{-x}$ b) $f(x) = \dfrac{2e^{2x} - 3}{4}$

c) $f(x) = 2e^{0.5x} + e^{-x}$ d) $f(x) = \dfrac{4e^{-2.5x}}{5} - 2e^{\frac{x}{2}}$

3606 Given that $y = 3e^{2x}$,

a) find y' b) show that $y' - 2y = 0$

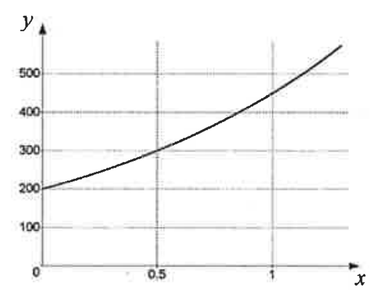
3607 Solve the equation $f'(x) = 0$, if

a) $f(x) = e^x - 2x$ b) $f(x) = 3e^{2x} - 3x + 2$

c) $f(x) = e^{-x}$ d) $f(x) = 100x - 100e^{5x}$

3608 Find the constant C in $f(x) = C \cdot e^{2x}$, if $f(0) = 2$.

3609 Find C and k from the graph of $y = C \cdot 1.5^{kx}$ below.



3610 Find the constants C and k of the function $f(x) = Ce^{kx}$, given that $f(0) = 100$ and $f'(0) = 3$.

B

3611 Differentiate

a) $y = 3e^{ax} - ae^{3x}$ b) $y = \dfrac{e^{4x}}{e^{3x}}$

c) $y = \dfrac{2e^{ax}}{3e^{bx}}$ d) $y = e^2 - e^{2-3x}$

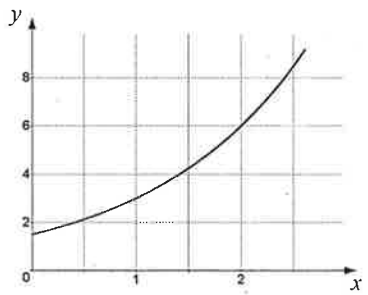
3612 Find

a) De^{x+3} b) $D10^x$

c) $D(5 \cdot 3^x)$ d) $D2^{2x+1}$

3613 $N(t) = N_0 \cdot e^{kt}$. Find N_0 and k, if $N(1) = 2N_0$ and $N'(0) = 69.3$.

3614 Find C and k from the graph of $y = C \cdot 2^{kx}$ in the figure.



Chapter exercises 3

A

1 Find $f'(10)$ if $f(x) = 2{,}1 \cdot 10^2 x - 8x^2$

2 Find $f'(x)$ if

a) $f(x) = \dfrac{x^3 + e^{3x}}{3}$ b) $f(x) = 3x^5 - 2 - 5e^{-x}$

3 Find $f''(x)$ if

a) $f(x) = 3$ b) $f(x) = \dfrac{x^4}{2} - \dfrac{x^5}{4}$

4 a) Find $f''(3)$ if $f(x) = x^3 + 1$ b) Find $f''(-2)$ if $f(x) = 3x^5 - 2x^4$

5 A ball is thrown straight up in the air. The height h metres of the ball after t seconds is given by the formula $h(t) = 2 + 19t - 5t^2$. Find and explain the meaning of

a) $h(2{,}5)$ b) $h'(2{,}5)$

6 Solve the equation $f'(x) = 2$, if $f(x) = e^x$. Answer correct to three significant figures.

7 Find the points on the graph of $y = x^2 + x^3$ at which the tangent line is horizontal.

8 The curve $y = 3x^2 - 4$ has a tangent at the point (2, 8). Find the slope of the tangent.

9 Find the slope of the curve $y = 0.5x^3 - 2x^2 + x$ at the point of the curve where

a) $x = 1$ b) $x = 2.5$

10 The total cost, T kr, of producing x electric pepper-mills is

$T(x) = 40000 + 370x - 0{,}09x^2$ $(0 < x \le 200)$.

Calculate and explain the meaning of

a) $T(120)$ b) $T'(120)$

B

11 Find $f''(x)$ if

a) $f(x) = 2x^3(x + x^2)$ b) $f(x) = (x+3)^2$

12 a) Find $f'(1{,}75)$ if $f(x) = \dfrac{x^2 + e^{2x}}{3}$. Answer with two decimals.

b) Find $f'(-0{,}5)$ if $f(x) = -e^{-x}$. Answer with two decimals.

13 Find the slope of the curve $y = 1.75^x - 3$ at the point where the curve

a) cuts the y-axis b) cuts the x-axis

14 Find the following limits

a) $\lim_{x\to 0}\frac{x(3+x)}{x}$ b) $\lim_{x\to 0}\left(3\cdot 1.5^x-\frac{5x^2-x^3}{2x^2}\right)$

15 Use the definition of the derivative to find $f'(x)$, if

a) $f(x)=\frac{1}{x+2}$ b) $f(x)=\frac{x}{x-1}$

16 An object is moving according to $s(t)=t^3-4t^2+5t$, $(0\le t\le 4)$, where s is the distance, in km, and t is the time, in hours, from start.

a) Find the speed of the object after three hours.
b) Describe how the object is moving.

17 A machine for gilding tweezers is decreasing in value, following $V(x)=119000\cdot e^{-0.16x}$, where x is the age of the machine, in years, and V is the value, in kr. Calculate and explain the meaning of

a) $V(0)$ b) $V(5)$
c) $V'(5)$ d) $\frac{V(5)-V(0)}{5}$

18 The speed of an underground train, travelling between two stations in London, is $v(t)$ m/s t seconds after the departure. Explain the meaning of

a) $v(19)=18$ b) $v'(19)=0$
c) $v''(19)<0$

19 An amount of money was put into a bank at 5% annual interest rate. During the eighth year the money increased by 5980 kr. Find the original amount of money. (No money was taken out during this period)

20 A quadratic curve cuts the y-axis at $y=1$ and the x-axis at $x=1$, and has a minimum point when $x=2$. Find the equation of the curve.

21 Find the points on the graph of $y=\frac{x^2+4\sqrt{x}-1}{x}$ at which the tangent line has slope 1.

Chapter 4

Using The Derivative

4.1 The tangent line equation

If the function is known, we can now find the equation of the tangent to a curve at any point of the curve, using the derivative.

Example 1

Find the equation of the tangent to the curve $y = x^2 + 1$ at the point (1, 2) .

Solution:

The equation of the tangent is $y = mx + c$, but we need to find m (the slope) and c.

The slope of the tangent line is the same as the slope of the curve at (1, 2) which is the same as $y'(1)$.

$$y' = 2x \quad \Rightarrow \quad y'(1) = 2 \cdot 1 = 2$$

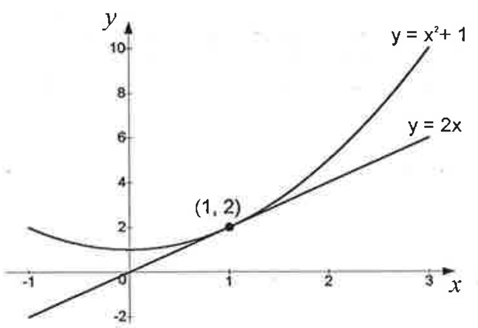
Now we know that $m = 2$.

We also know that the line goes through the point (1, 2) (if $x = 1$ then $y = 2$).

This information put into the equation $y = mx + c$ gives

$$2 = 2 \cdot 1 + c \Rightarrow c = 0$$

The equation of the tangent is $y = 2x$.



Example 2

Find the equation of the line, tangent to the curve $y = 5x^2 + 8x$, at the point of the curve where the x-coordinate is -2 .

Solution:

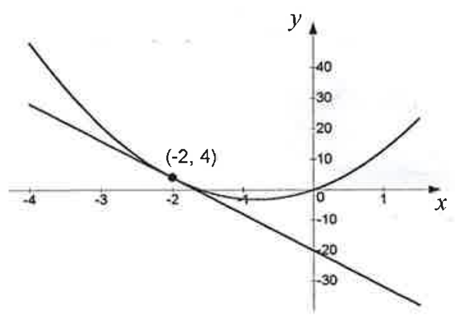
$y(-2) = 4$, so the tangent point is $(-2, 4)$.

$y' = 10x + 8 \quad \Rightarrow \quad y'(-2) = -12 \quad \Rightarrow \quad m = -12$

This put into the equation $y = mx + c$ gives

$4 = -12 \cdot (-2) + c \quad \Rightarrow \quad c = -20$

The equation of the tangent is $y = -12x - 20$.



Example 3

Find the equation of the tangent line to the curve $y = e^x + x$ which has slope 2.

Solution:

We need to find the tangent point, so we have to find the x-value that gives the curve slope 2.

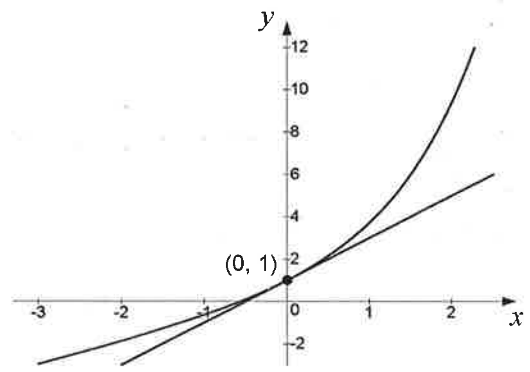
$y' = e^x + 1$

$y' = 2 \quad \Rightarrow \quad e^x = 1 \quad \Rightarrow \quad x = 0$

$y(0) = e^0 + 0 = 1 \quad \Rightarrow$ The tangent point is (0, 1). This gives the equation

$1 = 2 \cdot 0 + c \quad \Rightarrow \quad c = 1$

The equation of the tangent is $y = 2x + 1$.



Example 4

Find the tangent (or tangents) to the curve $y = x^2 - 2x + 9$ that passes through the origin.

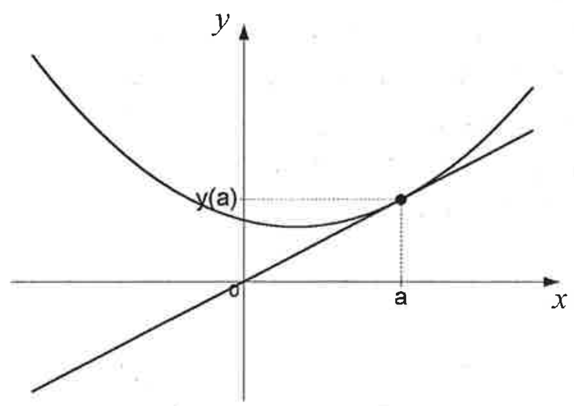
Solution:

We don't know the tangent point, but we can call the x-coordinate of the tangent point $x = a$.

The y-coordinate of the tangent point is then

$$y(a) = a^2 - 2a + 9$$

Graphically, the situation looks as this:



The slope of the curve at $x = a$ is $y'(a)$.

$y' = 2x - 2 \Rightarrow y'(a) = 2a - 2$

We now know that the tangent has slope $m = 2a - 2$. We also know that the line goes through the points $(a,\ a^2 - 2a + 9)$ (the tangent point) and (0, 0) (the origin). From these to points we can find another expression for the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{(a^2 - 2a + 9) - 0}{a - 0}$$

Putting this equal to $m = 2a - 2$ gives

$$\frac{a^2 - 2a + 9}{a} = 2a - 2 \quad \Rightarrow \quad a^2 - 2a + 9 = (2a - 2) \cdot a \quad \Rightarrow$$

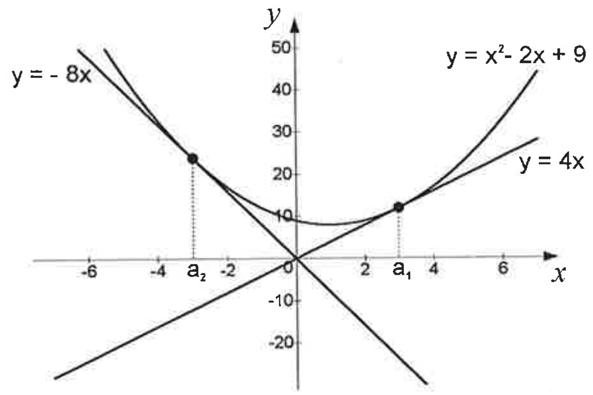
$$a^2 - 2a + 9 = 2a^2 - 2a \quad \Rightarrow$$

$$a^2 = 9 \quad \Rightarrow$$

$$a = \pm 3$$

There are obviously two points on the curve where the tangent line goes through the origin ($a = 3$ and $a = -3$).

$a = 3$	gives	$m = 4$	and the equation	$y = 4x$
$a = -3$	gives	$m = -8$	and the equation	$y = -8x$.



Exercises

A **4101** Find the gradient of the tangent line to the curve $y = 5x - x^2$ at the point (2, 6).

4102 Find the slope of the curve $y = 3x - 2x^2 + 5$ at the point of the curve where $x = 2$.

4103 Find an equation of the tangent line to the curve $y = 3x^2 + x - 1$ at the point (1, 3).

4104 The curve $y = x^3 - x^2 + 4x$ has a tangent line at the point of the curve with x-coordinate 1. Find the equation of the tangent.

4105 Find an equation of the line tangent to the curve $y = e^{3x}$, where $x = 0$.

4106 Find the equation of the tangent line to the curve $y = \frac{x^3}{8} - 1$ at the point where it cuts the x-axis.

B **4107** The curve $y = x^2 + 2x$ has a tangent line with gradient 8. Find the tangential point.

4108 Find the equation(s) of the tangent(s) to the curve $y = x^3 - 1{,}5x^2 - 12x + 2$ which have slope –6.

4109 At what points on the curve $y = x^3 - 3x^2 + x$ is the tangent parallel to the line $y = x + 10$?

4110 Find the equation of the tangent line to the curve $y = \frac{2}{x}$ at the point (2, 1).

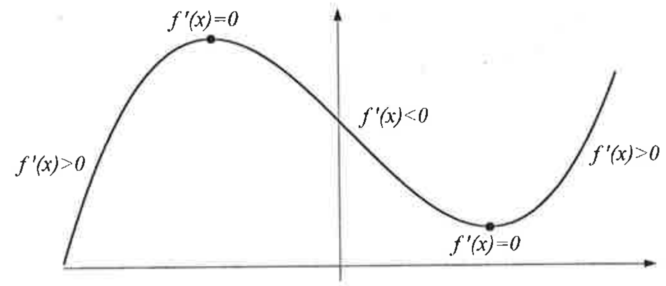
4111 Find the equation of the tangent line to the curve $y = \sqrt{x} - 1$ at the point where it cuts the x-axis.

4112 Find the equation of the tangent line to the curve $y = x^3 - 2x$ that passes through the point (0, –2).

4.2 Finding critical points

By now we can get a good idea of a graph of a function by studying its derivative. We use the fact that the *sign* of $f'(x)$ tells us if the graph of $f(x)$ is pointing up or down:

If $f'(x) > 0$	(positive slope)	$f(x)$ is increasing.
If $f'(x) < 0$	(negative slope)	$f(x)$ is decreasing.
If $f'(x) = 0$	(zero slope)	$f(x)$ is horizontal.



If we need to find out whether a function (or its graph) is increasing or decreasing at a certain x-value, we can simply differentiate the function and check the sign of the derivative at that x-value.

Example 1

Is the function $f(x) = -x^3 - x^2 + x$ increasing or decreasing at

a) $x = 0$ b) $x = -2$

Solution:

$f'(x) = -3x^2 - 2x + 1$

a) $f'(0) = 1 > 0$ (positive) $\Rightarrow$ $f(x)$ is increasing at $x = 0$.

b) $f'(-2) = -5 < 0$ (negative) $\Rightarrow$ $f(x)$ is decreasing at $x = -2$.

Example 2

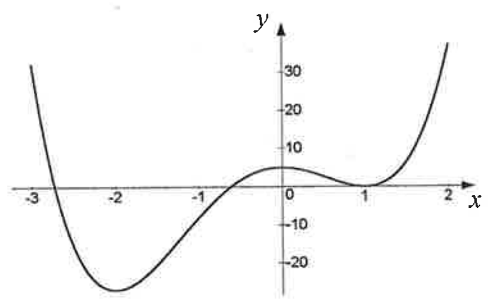
Answer the following questions by studying the graph below:

a) Is $f'(-3)$ positive or negative?

b) Is $f'(-1)$ positive or negative?

c) In which intervals are $f'(x)>0$?

d) Where is $f'(x)<0$?



Solution:

a) $f'(-3)$ is negative (curve is pointing down = negative slope)

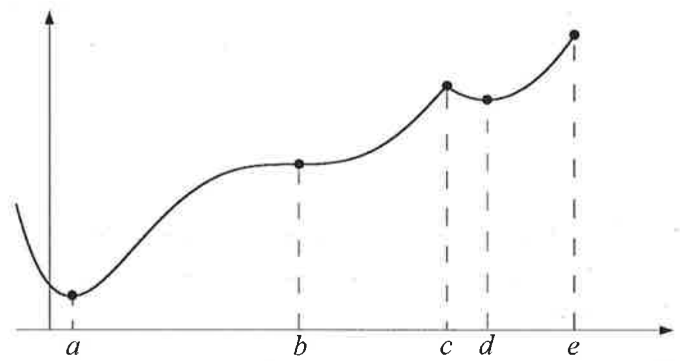
b) $f'(-1)$ is positive (curve is pointing up)

c) When $-2 < x < 0$ and $x > 1$.

d) When $x < -2$ and $0 < x < 1$.

Critical points

Studying the graph of a function, points of certain interest are the points where $f'(x) = 0$ or where $f'(x)$ doesn't exist. Such points are called *critical points*. Consider the graph below:



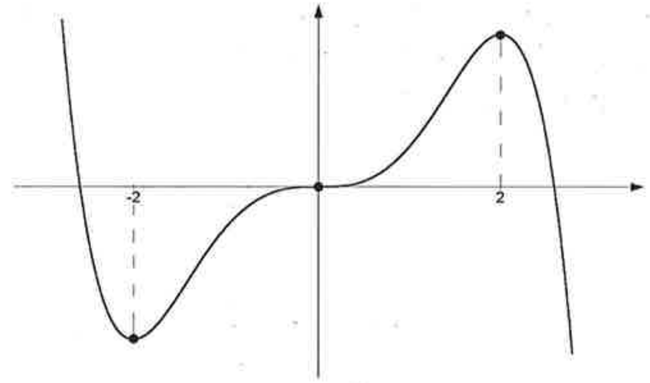
At *a*, *b* and *d*, $f'(x) = 0$ (the curve has a horizontal tangent), but at *c* and *e*, $f'(x)$ doesn't exist.

At endpoints, or "sharp" points of a curve, it is impossible to tell what the slope of the curve is. If $f'(x)$ doesn't exist, we say that $f(x)$ is not *differentiable* at that point. From now on we will only consider functions that are differentiable at every point (except possibly at endpoints), that is, graphs with no "sharp points".

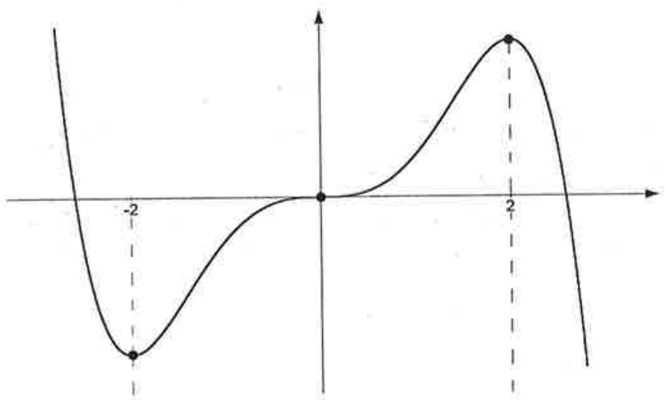
The critical points where $f'(x) = 0$ can be of three different types:

- maximum points
- minimum points
- "terrace" points (from Swedish; "terrasspunkt")

The graph below shows these different types of points, which all have in common that they have horizontal tangent, and therefore $f'(x) = 0$.

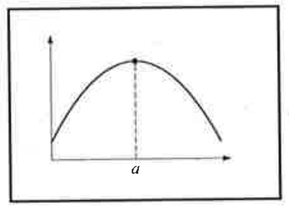


If together with this graph we show the *sign* (+, - or 0) of $f'(x)$ in a table, it becomes obvious how we can use the derivative in an easy way to find the shape of a curve:

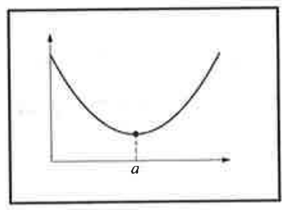


Sign of $f'(x)$: - 0 + 0 + 0 -

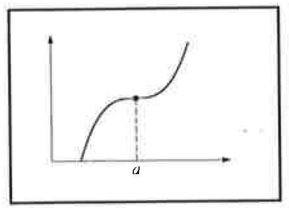
If $f'(a)=0$, we only know that it must be a max., min. or terrace point of the curve at $x=a$, but looking at the sign of $f'(x)$ to the left and right of that point gives us useful information:



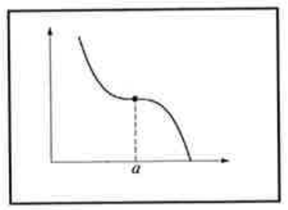
Maximum point if $f'(a)=0$, and $f'(x)$ changes sign like + 0 - around $x=a$.



Minimum point if $f'(a)=0$, and $f'(x)$ changes sign like - 0 + around $x=a$.



Terrace point if $f'(a)=0$, and $f'(x)$ changes sign like + 0 + or - 0 - around $x=a$.



Maximum/minimum points are more precisely called *relative* max/min points, since the value of the function at such a point is greater/smaller than at other points in the neighbourhood. However, the value of the function may be greater/smaller somewhere else along the curve.

The sign table

To get a good picture of the graph of a function we use a *sign table* to put together interesting x-values, the sign of $f'(x)$ and the shape of $f(x)$.

First, we find the x-values where $f'(x)=0$, then we check the sign (+, - or 0) of $f'(x)$ around those x-values, and finally we figure out the shape of $f(x)$.

The method is shown by an example:

Example 3

Use a sign table to find the shape of the graph of $f(x)=4-x^2$.

Solution:

We start by finding the x-values where $f'(x)=0$:

$f'(x)=-2x$

$f'(x)=0 \quad \Rightarrow \quad -2x=0 \quad \Rightarrow \quad x=0$

We know now that the derivative is 0 (the curve is horizontal) at $x=0$, but is this a maximum or minimum point, or perhaps a terrace point? Checking $f'(x)$ on both sides of $x=0$ gives us the answer.

Take an x-value to the left of 0, for example -1, and one to the right of 0, for example 1, and put it into $f'(x)$:

$f'(-1)=2>0$ (positive)

$f'(1)=-2<0$ (negative)

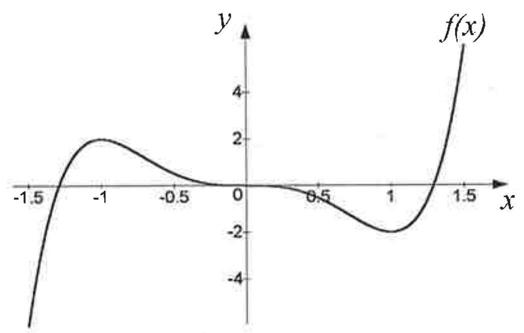
The sign table is then constructed using one row for the x-values, one for the sign of $f'(x)$ and one for the shape of $f(x)$:

x		0	
$f'(x)$	+	0	-
$f(x)$	↗	→	↘

The approximate shape of $f(x)$ now becomes clear, and it is obvious that the function has a <u>maximum</u> point when $x=0$.

Example 4

Construct a sign table for the curve below



Solution:

We see that the curve is horizontal ($f'(x)=0$) at $x=-1$, $x=0$ and $x=1$.
The slope of the curve tells us whether the derivative is positive or negative.
Together this gives the sign table

x		-1		0		1	
$f'(x)$	+	0	-	0	-	0	+
$f(x)$	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

From the sign table we can find that $x=-1$ gives a maximum point, $x=0$ gives a terrace point and $x=1$ a minimum point, without looking at the graph.

Example 5

Find the critical points of $f(x)=-2x^2+8x-9$ and decide whether it is a max., min. or terrace point.

Solution:

$f'(x)=-4x+8$

$f'(x)=0 \;\Rightarrow\; -4x+8=0 \;\Rightarrow\; x=2$

Sign table:

x		2	
$f'(x)$	+	0	-
$f(x)$	$\nearrow$	$\rightarrow$	$\searrow$

$f(x)$ has one critical point; a maximum point when $x=2$.

$f(2)=-1$

The maximum point is $(2,\,-1)$.

Example 6

Find and classify the critical points of $f(x)=x^4-2x^3-2$.

Solution:

$f'(x)=4x^3-6x^2$

$f'(x)=0 \;\Rightarrow\; 4x^3-6x^2=0 \;\Rightarrow\; 2x^2(2x-3)=0 \;\Rightarrow$

$x=0 \quad or \quad x=\frac{3}{2}=1.5$

Sign table:

x		0		1.5	
$f'(x)$	-	0	-	0	+
$f(x)$	$\searrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

$f(0)=-2$

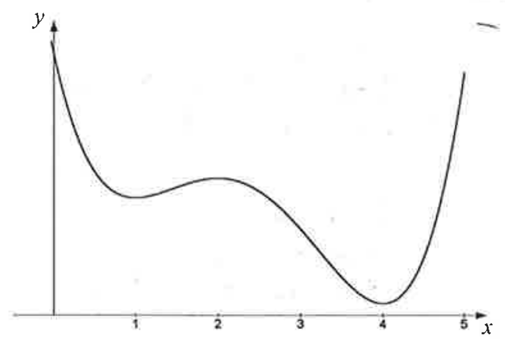
$f(1.5)\approx-3.7$

$f(x)$ has a terrace point $(0,\,-2)$ and a minimum point $(1.5,\,-3.7)$.

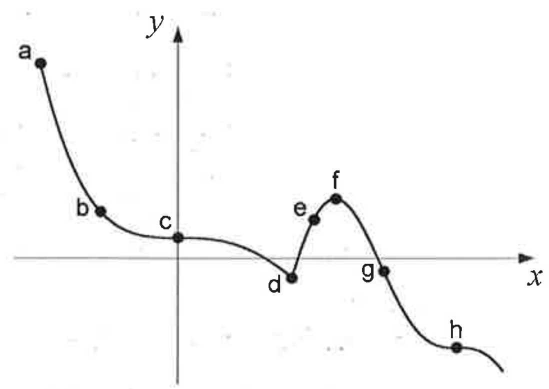
Exercises

A **4201** Study the graph below and find the x-values where

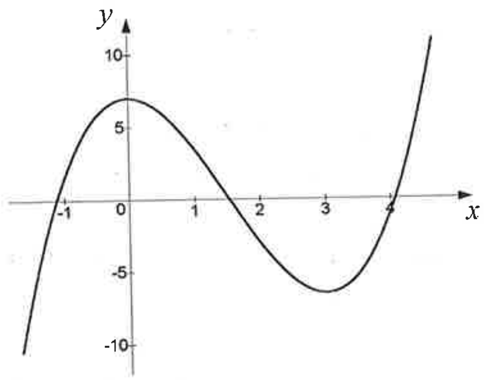
a) $f(x)$ is increasing
b) $f(x)$ is decreasing
c) $f'(x)<0$
d) $f'(x)>0$
e) $f'(x)=0$



4202 Consider the graph below and decide for each point whether the derivative is positive, negative, zero or does not exist.



4203 Study the graph below and answer the following questions
a) On what intervals is the derivative positive?
b) On what intervals is the derivative negative?



4204 Decide for each of the following functions the intervals of increase and decrease.
a) $f(x)=2x-x^2$ b) $f(x)=x^3-3x^2$
c) $f(x)=1-x^3$ d) $f(x)=2x^2-x^4+3$

4205 Are the following functions increasing or decreasing when $x=2$?
a) $f(x)=4x^2-3x^4+79x$ b) $g(x)=e^{3x}-6x^6+1$

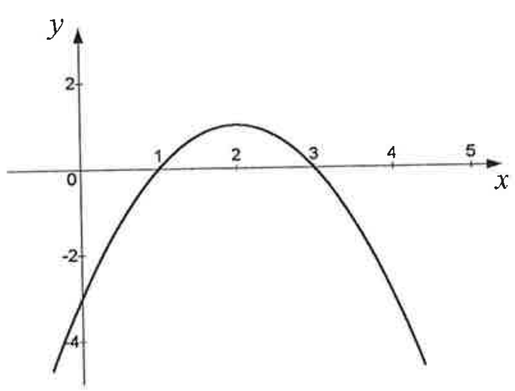
4206 Use the derivative to find the critical points of
a) $f(x)=x^2-2x$ b) $f(x)=-2x^3-3x^2-1$

4207 Which of the following curves has a maximum point? What is the maximum value?
A $y=2x^2-3x-15$ B $y=-2x^2+4x-3$
C $y=1+x-x^2$ D $y=2x-3+5x^2$

4208 Which of the following curves has a minimum point? What is the minimum value?
A $y=3x^2+6x-2$ B $y=2x-4x^2+7$
C $y=1-5x+x^2$ D $y=3-2x-7x^2$

4209 Use a sign table to find the maximum/minimum points of the following functions
a) $f(x)=-x^3+3x^2-4$ b) $f(x)=2x^3-3x^2-12x+12$
c) $f(x)=3x^4-4x^3$ d) $f(x)=4+3x-x^3$

4210 Below is shown the graph of $f'(x)$. At which x-values does $f(x)$ have critical points? Classify the critical points.



B

4211 Are the following functions increasing or decreasing when $x=-2.5$?
a) $y=\frac{3}{x}+x$ b) $y=e^{-x}+0.5x^3-x^2$

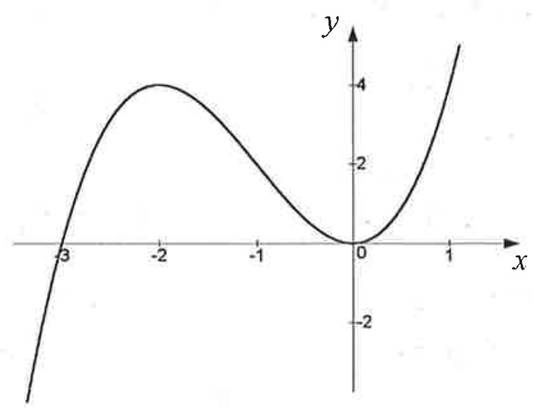
4212 Use the derivative to find the critical points of
a) $f(x)=2x^3+3x^2+1$ b) $g(x)=4x^3-3x^4+2$

4213 Use the derivative and a sign table to find the intervals where the function $f(x)=3x^4+4x^3-12x^2+12$ is increasing or decreasing.

4214 Are the following functions increasing or decreasing when $x=8$?
a) $f(x)=\frac{10}{\sqrt{x}}-\frac{900}{x}-x^2$ b) $f(x)=e^{-\frac{x}{2}}-(x-8)^2$

4215 Find and classify the critical points of the following functions
a) $f(x)=x^4-8x^2$ b) $f(x)=\frac{x^3}{3}-\frac{x^2}{2}-2x+1$
c) $f(x)=2x^4-4x^2-1$ d) $f(x)=2x^3-x^4$

4216 Below is the graph of $f'(x)$. At which x-values does $f(x)$ have critical points? Classify the critical points.

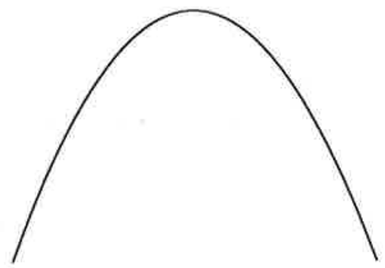


Curve sketching
The sign table gives us the shape of the graph, but we can't draw the graph in a coordinate system without some further information. We need to fix the curve at some strategic point(s).

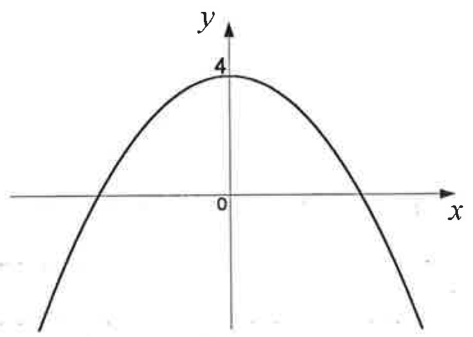
If we look at the sign table of the function $f(x) = 4 - x^2$ from *Example 1* above,

		0	
$f'(x)$	+	0	-
$f(x)$	↗	→	↘

we only know that the curve will have the shape



What we don't know, is where to place the graph in a coordinate system.
The least we can do is to check where the maximum point is, by calculating
$f(0) = 4$.
Now we can get a good sketch of the graph:



It is a good idea as a matter of routine to put the value of the function at the critical points into the sign table, like this:

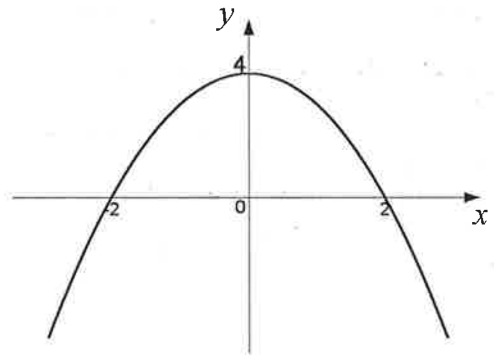
x		0	
$f'(x)$	+	0	-
$f(x)$	↗	4	↘

From a sign table like this, we can sketch the graph of any function in a convenient way, without many calculations involved.

To get a more exact picture of the graph, however, we would need a few more points. If it is possible, and if it is of special interest, we may find the zeroes of the function (the x-values where $f(x) = 0$) by solving the equation $f(x) = 0$.
In our example the equation becomes

$$f(x) = 4 - x^2 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$

Our graph now becomes even more accurate:



Example 7
Sketch the graph of $f(x) = x^3 - 12x + 6$ by using the derivative and a sign table.

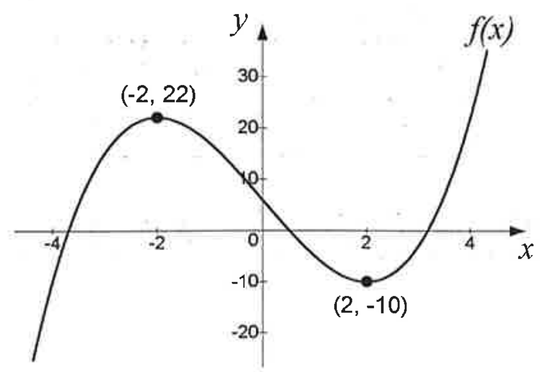
Solution:
We want to find out where $f'(x) = 0$.
$f'(x) = 3x^2 - 12$
$f'(x) = 0 \quad \Rightarrow \quad 3x^2 - 12 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$f(-2) = 22$
$f(2) = -10$

Sign table:

x		-2		2	
$f'(x)$	+	0	-	0	+
$f(x)$	↗	22	↘	-10	↗

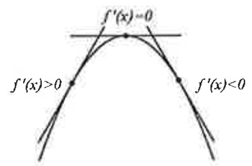
$f(x)$ has a maximum point at $(-2, 22)$ and a minimum point at $(2, -10)$.
A sketch of the graph:



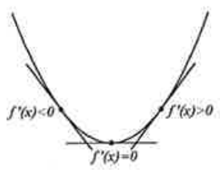
(finding the exact x-values where $f(x)=0$ is a very difficult task here, since we then have to solve the third degree equation $x^3-12x+16=0$, which is far beyond this course)

The second derivative
Another, and sometimes faster, method of classifying the critical points of a function is by using the second derivative, $f''(x)$.

If $f(x)$ has a maximum point:
$\Rightarrow$ the gradient, $f'(x)$, is decreasing
(from positive slope to negative slope)
$\Rightarrow$ the derivative of $f'(x)$ is negative
$\Rightarrow$ $f''(x)<0$



If $f(x)$ has a minimum point:
$\Rightarrow$ the gradient, $f'(x)$, is increasing
(from negative slope to positive slope)
$\Rightarrow$ the derivative of $f'(x)$ is positive
$\Rightarrow$ $f''(x)>0$



We can therefore check the sign of the second derivative and find out what kind of a critical point we have.
However, the method does not always give us the answer. If $f''(x)=0$, we need some further investigation, using a sign table, for example.

Example 8
Find and classify the critical points of $f(x)=x^3-x^2-x+2$.

Solution:
$f'(x)=3x^2-2x-1$
$f'(x)=0 \Rightarrow x^2-\frac{2}{3}x-\frac{1}{3}=0 \Rightarrow x=\frac{1}{3}\pm\sqrt{\frac{1}{9}+\frac{1}{3}}=\frac{1}{3}\pm\frac{2}{3} \Rightarrow$
$x=1$ *or* $x=-\frac{1}{3}$
$f(x)$ has critical points at $x=1$ and $x=-\frac{1}{3}$. The sign of $f''(x)$ tells us what type of critical points:

$f''(x)=6x-2$
$f''(1)=4>0 \Rightarrow f_{\min}$ at $x=1$
$f''(-\frac{1}{3})=-4<0 \Rightarrow f_{\max}$ at $x=-\frac{1}{3}$

Example 9
Find out whether the curve $y=3x^4-4x^3+1$ has any maximum or minimum points.

Solution:
$y'=12x^3-12x^2$
$y'=0 \Rightarrow 12x^3-12x^2=0 \Rightarrow 12x^2(x-1)=0 \Rightarrow$
$x=0$ *or* $x=1$
$f''(x)=36x^2-24x$
$f''(1)=12>0 \Rightarrow f_{\min}$ at $x=1$
But
$f''(0)=0$
which means that this test doesn't give us any information. We have to check the critical point at $x=0$ by looking at the sign of $f'(x)$ around $x=0$. Doing so, we find that $f'(x)$ has the sign changes - 0 - , which means that we have a terrace point at $x=0$.

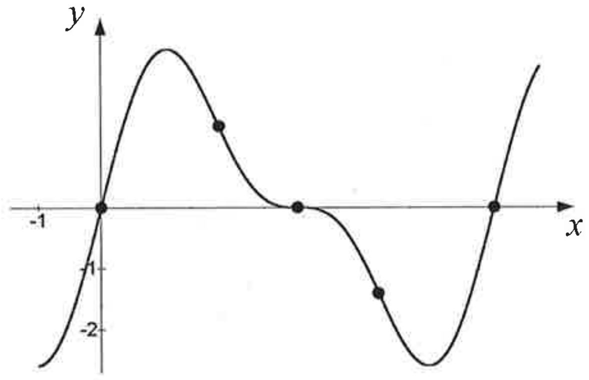
Example 10
Check the critical points of the curve $y=x^4$.

Solution:
$y'=4x^3$
$y''=12x^2$
$y'=0 \Rightarrow x=0$
$y''(0)=0$
A sign table shows:

x		0	
y'	-	0	+
y	↘	→	↗

There is a minimum point of the curve at $x=0$.

Points of inflexion
A point of inflexion (or inflexion point) is a point where the second derivative changes sign. This means that the first derivative, $f'(x)$, is changing from increasing to decreasing or vice versa. This, in turn, means that the graph of $f(x)$ is changing from pointing more and more up to more and more down (or vice versa).



The graph shows the inflexion points of a curve. A terrace point is one kind of an inflexion point.
As $f''(x)$ changes sign, one way of finding possible inflexion points is to check where $f''(x)=0$, but note here that $f''(x)=0$ doesn't always guarantee an inflexion point (see *Example 6*).

Example 11
Find the critical points and inflexion points of the curve $y=x^4-4x^3+5$. Sketch the graph.

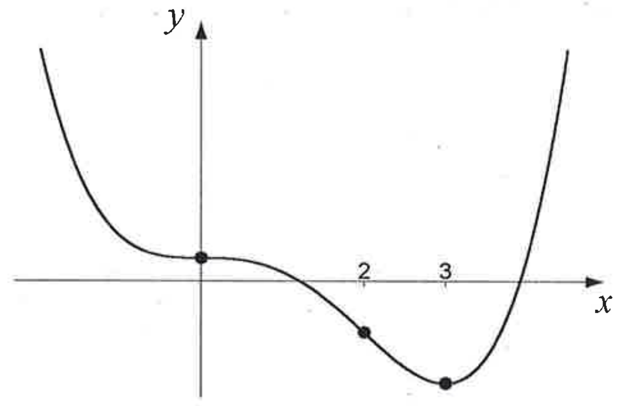
Solution:
$y'=4x^3-12x^2$
$y'=0 \;\Rightarrow\; 4x^2(x-3)=0 \;\Rightarrow\; x=0 \;\; or \;\; x=3$
$y''=12x^2-24x$
$y''=0 \;\Rightarrow\; 12x(x-2)=0 \;\Rightarrow\; x=0 \;\; or \;\; x=2$

We seem to have critical points at $x=0$ and $x=3$, and possible inflexion points at $x=0$ and $x=2$. A sign table gives us further information (with an extra row for y''):

x		0		2		3	
y''	+	0	-	0	+	+	+
y'	-	0	-	-	-	0	+
y	↘	→	↘	↘	↘	→	↗

The sign table now shows that the curve has a terrace point at $x=0$, a minimum point at $x=3$, and inflexion points at $x=0$ (also a terrace point) and $x=2$.

A sketch of the curve:



Exercises

A

4217 Use the derivative and a sign table to sketch the following curves
a) $y=4x-x^2+1$ b) $y=x^3-3x-2$
c) $y=\dfrac{x^3}{3}+x^2-3x+1$ d) $y=2x^3+9x^2+12x-5$

4218 a) Find the critical points of $f(x)=e^{0.5x}-x$. Give an exact answer.
b) Sketch the graph of $f(x)$.
c) How many solutions does the equation $e^{0.5x}-x=0$ have?

B

4219 Sketch the graphs of the following functions
a) $f(x)=x^3+9x^2+27x+25$ b) $f(x)=x^3-x^2-4x+4$
c) $f(x)=0.5x^3-0.25x^4-0.25$ d) $f(x)=\dfrac{2x^3}{27}-\dfrac{x^4}{18}+x^2-2x$

4220 Find the critical points of $f(x)=\sqrt{x}-x^2$. Answer correct to two significant figures. Then sketch the graph of the function between $x=0$ and $x=1.5$.

4221 Find the points of inflexion of the following functions
a) $y=x^3+12x^2$ b) $y=3x^3-2x+12$
c) $y=x^5-x^4$ d) $y=4x^5-5x^4+4$

4222 For the following functions, find the critical points and the inflexion points and sketch the graph.
a) $f(x)=4-12x+6x^2-x^3$ b) $f(x)=0.5x^3-3x^2+2$
c) $f(x)=\dfrac{x^4-18x^2+6}{3}$ d) $f(x)=\dfrac{2x^4-4x^3+3}{2}$

4223 Find the critical points of the function

$$f(x) = \frac{x^2 + 9}{x} \qquad (x > 0)$$

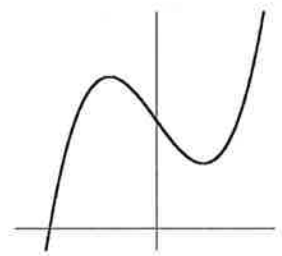
Sketch the graph.

4224 The curve $y = x^3 - ax$ has a critical point at $x = -1$. Find a and determine whether the critical point is a maximum or a minimum.

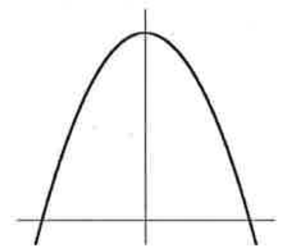
4225 The function $f(x) = ax^3 + bx^2$ has a critical point (1, 1). Find a and b and classify this critical point.

Absolute maximum/minimum

In applications, when we use mathematics to solve some real problem, we are not only interested in finding all max/min points of a function, but rather to find the largest or smallest possible value of the function.
The largest possible value of a function (if it exists) is called the *absolute maximum* of the function. The smallest possible value is called the *absolute minimum.*
A function may have several *relative* max/min points without having an absolute maximum or minimum:

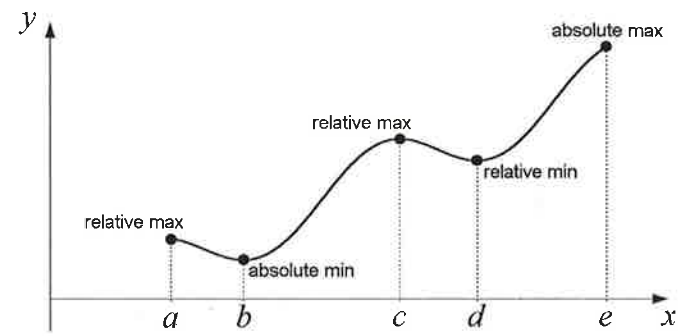


Absolute maximum or minimum doesn't exist



Absolute minimum doesn't exist

In real problems we often meet *restricted domains*, which means that the variable of the function can only take certain values. We often work with *closed intervals*, which means that we study a piece of a curve.
In such a case, the absolute maximum or minimum may occur at the endpoints of the interval:



The figure shows the graph of a function, $f(x)$, in a restricted domain, $a \le x \le e$ (x can take any value from a to e). In this interval, the smallest value occurs at the (relative) minimum point at $x = b$, and the largest value is at $x = e$, which is one endpoint.

If we want to find the largest and smallest value (absolute maximum and minimum) of a function on a closed interval, we must

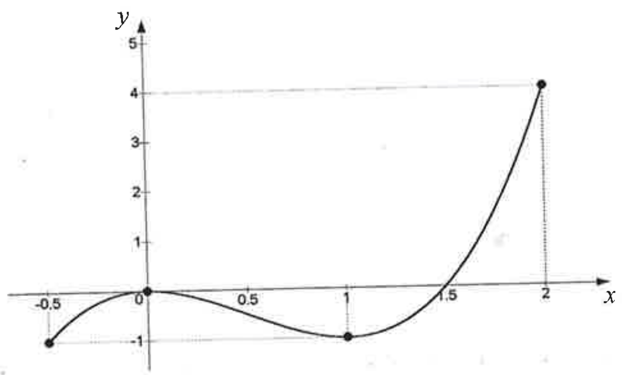
1) check the value at the critical points of the function, within the interval.
2) check the value at the endpoints.

The largest of these values is then the absolute maximum and the smallest value is the absolute minimum.

Example 12
Find the largest and smallest value of the function $f(x)=2x^3-3x^2$ on the interval $-0.5 \le x \le 2$.

Solution:
$f'(x)=6x^2-6x$
$f'(x)=0 \Rightarrow 6x(x-1)=0 \Rightarrow x=0$ *or* $x=1$
$\Rightarrow$ critical points at $x=0$ and $x=1$.

Values at the critical points:
$f(0)=0$
$f(1)=-1$
Values at the endpoints:
$f(-0.5)=-1$
$f(2)=4$



The largest value on the interval is 4.
The smallest value on the interval is -1.
(Note that an absolute maximum or minimum can occur at more than one point)

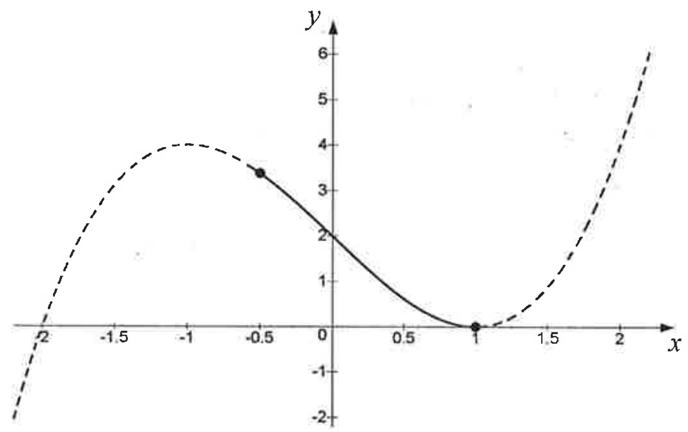
Example 13
Find the absolute maximum and minimum values of $f(x)=x^3-3x+2$ on the interval
a) $-2 \le x \le 1.5$
b) $-0.5 \le x \le 1$

Solution:
$f'(x)=3x^2-3$
$f'(x)=0 \Rightarrow x^2=1 \Rightarrow x=-1$ *or* $x=1$
a) $f(-2)=0$
$f(-1)=4$
$f(1)=0$
$f(1.5)=0.875$
Absolute maximum: 4
Absolute minimum: 0

b) The critical point at $x=-1$ is not in the interval $-0.5 \le x \le 1$, and the critical point at $x=1$ is one of the endpoints, so we only need to check the values at the two endpoints:
$f(-0.5)=3.375$
$f(1)=0$

Absolute maximum: 3.375
Absolute minimum: 0

The figure shows the whole curve as a broken line, with the part corresponding to the interval $-0.5 \le x \le 1$ marked with a thick line.



Exercises

A

4226 Study the function $f(x)=x^2-4x+5$ and find the absolute maximum and minimum of the function (if they exist) on the given interval.
a) $0 \le x \le 5$ b) $x \le 3$

4227 Find the absolute maximum and minimum values of the functions below on the interval $-2 \le x \le 1$.
a) $f(x)=x^3-3x+2$ b) $f(x)=x^2-4x$

4228 Find the absolute maximum and minimum values of the functions below, if they exist, on the indicated interval.
a) $f(x)=x^2-4x+5,\ 0 \le x \le 5$ b) $f(x)=x^2-x+1,\ 1 \le x \le 3$
c) $f(x)=2x^3-3x^2,\ -1 \le x \le 2$ d) $f(x)=3x^2-x^3-5,\ -1 \le x \le 1$

B

4229 Find the largest and smallest value of $f(x) = x^6 - 48x^2$ on the interval $-1 \le x \le 3$.

4230 Find the absolute maximum and minimum values of the functions below, if they exist, on the indicated interval.

a) $f(x) = 5 + x - x^2, \quad 0 \le x \le 2$ b) $f(x) = x^3 - 3x, \quad -5 \le x \le 1$

c) $f(x) = x^3 - 6x^2 + 10, \quad 0 \le x \le 4$ d) $f(x) = x^4 - 2x^2 + 5, \quad -2 \le x \le 2$

4231 Find the largest and smallest value of the functions below, if they exist, when $x > 0$. Answer with two significant figures.

a) $y = 2x + \dfrac{15}{x}$ b) $y = x^2 + \dfrac{250}{x}$

4232 Find the absolute maximum and minimum values of the functions below, if they exist, on the indicated interval. Answer with three significant figures.

a) $f(x) = 0.5e^{-2x} + x, \quad -1 \le x \le 2$ b) $f(x) = e^{3x-1} - 3x, \quad -1 \le x \le 1$

4.3 Applications

In practical situations we often want to solve maximum-minimum problems, that is, to find the absolute maximum or minimum value of some varying quantity and the point at which that maximum or minimum occurs.

The idea is to express the quantity that is to be maximised/minimised as a function of one variable, and then finding the maximum/minimum by using the derivative.

Example 1

An astronaut standing on the moon shoots a stone straight up. The height y metres of the stone after t seconds is given by the formula

$$y = 20t - 0.8t^2 .$$

What is the maximum height of the stone?

Solution:

$y' = 20 - 1.6t$

$y' = 0 \quad \Rightarrow \quad 1.6t = 20 \quad \Rightarrow \quad t = 12.5$

A sign table shows that this is a maximum of the function:

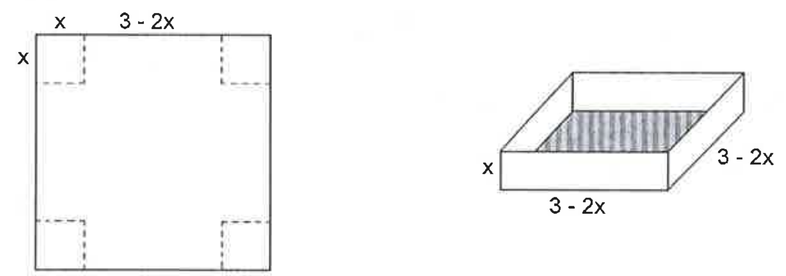
x		12.5	
y'	+	0	-
y	↗	→	↘

The height at $t = 12.5$ is $y(12.5) = 20 \cdot 12.5 - 0.8 \cdot 12.5^2 = 125$

The maximum height is 125 m.

Example 2

From a thin piece of cardboard, 3 dm by 3 dm, square corners are cut out so that the sides can be folded up to make an open box. What dimensions will produce a box of maximum volume? What is the maximum volume?



Solution:

One might think that it doesn't matter what the dimensions are, but studying the volume as a function of x will show that the volume varies and will have a maximum value.

The volume, V, is

$$V(x) = (3 - 2x)(3 - 2x)x = 9x - 12x^2 + 4x^3 .$$

Note here that x can only take values between 0 and 1.5, or $0 \le x \le 1.5$.

$V'(x) = 9 - 24x + 12x^2$

$V'(x) = 0 \quad \Rightarrow \quad x^2 - 2x + \frac{3}{4} = 0 \quad \Rightarrow \quad x = 1 \pm \sqrt{1 - \frac{3}{4}} = 1 \pm \frac{1}{2}$

$x = 0.5 \quad or \quad x = 1.5$

The value $x = 1.5$ is one of the endpoints of the interval and is not interesting, since the volume is 0 (there is no box...).

A sign table shows that $x = 0.5$ gives a maximum:

x		0.5	
V'	+	0	-
V	$\nearrow$	$\rightarrow$	$\searrow$

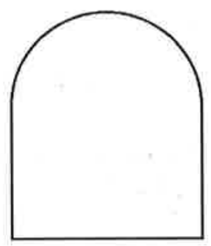
The maximum volume is $V(0.5) = 2 \ (\text{dm}^3)$

The dimensions of the largest box is

height (x): 0.5 dm
length ($3 - 2x$): 2 dm
width ($3 - 2x$): 2 dm

Example 3

A Norman window is a rectangle with a semicircle on top. Suppose that the perimeter of a particular Norman window is to be 24 dm. What should its dimensions be in order to allow the maximum amount of light to enter through the window?



Solution:

The problem is here to maximise the area of the window, with a given perimeter. We need to express the area of the window, A, as a function of some variable, x. Since we are not given any variable, we have to start by putting in some variable in the figure.

Let us denote the radius of the top semicircle by x and the height of the rectangular part by y (temporarily).

The perimeter of the window is

$$\pi x + 2x + 2y = 24 \quad \Rightarrow \quad 2y = 24 - \pi x - 2x \quad \Rightarrow \quad y = \frac{24 - \pi x - 2x}{2}$$

By this we have eliminated the variable y to get only one variable, x.

The area of the window is

$$A(x) = \frac{\pi x^2}{2} + 2xy = \frac{\pi x^2}{2} + \frac{48x - 2\pi x^2 - 4x^2}{2} = \frac{48x - \pi x^2 - 4x^2}{2}$$

$$A'(x) = \frac{48 - 2\pi x - 8x}{2} = 24 - \pi x - 4x = 24 - (\pi + 4)x$$

$$A'(x) = 0 \quad \Rightarrow \quad x = \frac{24}{\pi + 4}$$

A sign table would show that this is a maximum, but we can also use the second derivative (remembering that negative A'' means A_{max}):

$$A''(x) = -(\pi + 4) < 0 \quad \Rightarrow \quad A_{max}$$

The height of the rectangular part of the window, y, is

$$y = \frac{24 - \pi x - 2x}{2} = 12 - \frac{(\pi + 2)x}{2} = 12 - \frac{\frac{(\pi + 2) \cdot 24}{\pi + 4}}{2} =$$

$$= 12 - \frac{12 \cdot (\pi + 2)}{\pi + 4} = \frac{12 \cdot (\pi + 4) - 12 \cdot (\pi + 2)}{\pi + 4} = \frac{24}{\pi + 4}$$

This means that, to get maximum area of this window, x and y should be equal and $\frac{24}{\pi + 4} \approx 3.4 \text{ dm}$.

The three examples above represent problems of different degrees of difficulty:

In *Example 1*, we were given both the variable and the function that was to be maximised. All we had to do was find the maximum of the function.

In *Example 2*, we were given the variable, but had to express the function in terms of that variable before trying to find the maximum point.

In *Example 3*, nothing was given. We had to put in some variables first, then translate the area into a function of two variables, eliminate one of them, and finally calculate the maximum point.

The method of solving maximum-minimum problems could be summarised as follows:

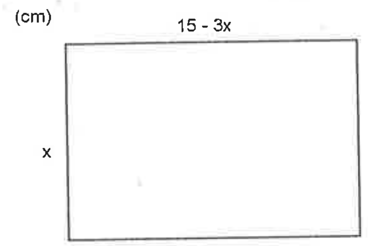
A strategy for solving max/min problems:

1. Read the problem carefully. If it helps, draw a picture.
2. If not given, introduce some appropriate variable(s), noting what varies and what stays fixed.
3. If not given, express the quantity to be maximised or minimised as a function of the variable(s).
4. If necessary, eliminate other variables to get the quantity as a function of *one* variable.
5. Use the methods of sections 4.1 and 4.2 to find the maximum/minimum value of the function.

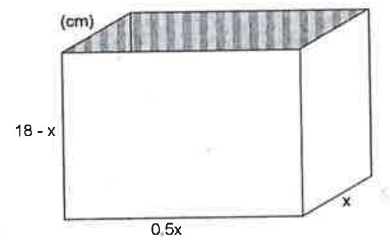
Exercises

A

4301 Calculate the maximum area of the rectangle

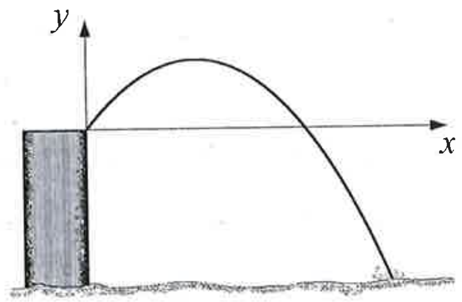


4302 Find the maximum volume of the box



4303 Bill throws a Kiss-record from the top of a 20 *m* high tower in such a way that it follows the curve $y = 1.2x - 0.04x^2$ (see fig.).

a) How far from the tower does the record hit the ground?
b) Find the maximum height above the ground of the record.



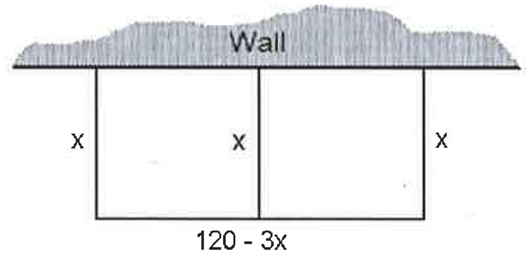
4304 From a piece of sheet-metal, 20 *cm* × 20 *cm*, square corners are cut out so that the sides can be folded up to make a box.
a) What size of the squares will make a box of maximum volume?
b) What is the maximum volume?

4305 When selling a certain tennis ball at "Bjorns Balls" the shop owner has found out that the number of balls sold per month, *y*, is dependent on the price of the ball, *x* kr, following $y = 800 - 20x$.
Find the largest possible revenue per month from these balls, according to this model.

4306 Of all numbers whose sum is 70, find the two that have the maximum product.

4307 Of all numbers whose difference is 4, find the two that have the minimum product.

4308 A gym teacher wants to enclose two rectangular areas next to a wall, one for outdoor wrestling and one for beach hockey. 120 *m* of fencing is available. What is the largest total area that can be enclosed?



4309 A company determines that in order to sell *x* units of a certain product, the price per unit must be $p = 200 - x$.
It also determines that the total cost of producing *x* units is given by $C(x) = 5000 + 8x$.
a) Find the total revenue $R(x)$.
b) Find the total profit $P(x)$.
c) How many units must be produced and sold in order to maximise profit?
d) What is the maximum profit?
e) What price per unit must be charged in order to make this maximum profit?

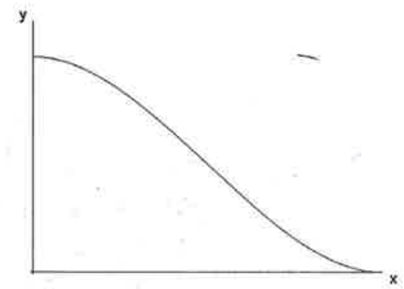
B

4310 The temperature *T*, in °C, during one day in march was given by

$$T(x) = 0.01(171 - 203x + 32.5x^2 - x^3), \quad 0 \le x \le 24$$

where *x* is the number of hours since midnight. Find the minimum and maximum temperature and when it occurs.

4311 The profile of a ski slope follows the function $y = \dfrac{x^3}{600000} - \dfrac{x^2}{800} + \dfrac{625}{6}$,
where *y* is the vertical height and *x* the horizontal distance (in *m*) (see figure).
a) How high is the top of the ski slope above ground level?
b) At what point is the ski slope the steepest?



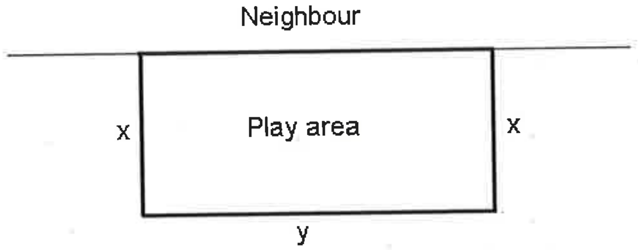
4312 Minimize $x^2 + y^2$ where $x + y = 20$.

4313 A container firm is designing an open top rectangular box, with a square base, that will hold 108 cm^3. What dimensions will give the minimum surface area? What is the minimum surface area?

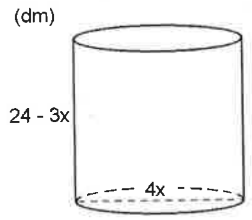
4314 A company, selling advertisement prevention programs through Internet, is generating a profit of $P(t) = 1000(9t^2 - t^3 + 48t + 28)$, where *P* is the profit (in *kr*) after *t* months.
When is the profit at maximum, and how large is it then?

4315 The sum of three sides of a rectangle is 60 *mm*. Find the maximum area of the triangle.

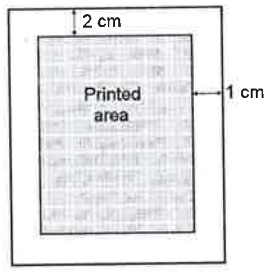
4316 Mr Applecider wants to fence in a play area of 48 m^2 in his backyard. His neighbour agrees to pay half the cost of the fence that lines the garden. What dimensions will minimise the cost of the fence?



4317 Find the maximum volume of the cylinder.



4318 The pages of a best-seller book in mathematics should have an area of 200 cm^2, with a margin at top and bottom of 2 cm and margins on each side of 1 cm (see figure).
What should the dimensions of each page be to get maximum printed area?



4319 A 48 cm piece of string is cut in two pieces. One piece is used to form a circle and the other to form a square. How should the string be cut so that the sum of the areas is
a) a minimum? b) a maximum?

4320 The Artist formerly known as Prippz wants to make a hard case for his personal touring equipment. The case must have a square base and a volume of 300 dm^3. If the cost per dm^2 for material for the top is \$3 , for the bottom is \$2 and for the sides is \$1, What should the dimensions be in order to minimise the cost?

4321 A student who just completed the C-course, is taking part in a contest to find the perfect 50 cl cylinder beer can. He designs a can that conserves temperature the best, but he doesn't win.
a) What is the meaning here of "conserving temperature the best"?
b) Find the dimensions of his beer can.
c) Why didn't he win the contest?

4322 Basil owns a 30-unit hotel. He has seen that if he charges \$20 a day per unit, all units are occupied, and for every increase of x dollars in the daily rate, there are x units vacant. Each occupied room costs \$3 per day and the vacant rooms \$1 per day to service and maintain. What should he charge per unit in order to maximise profit?

Chapter exercises 4

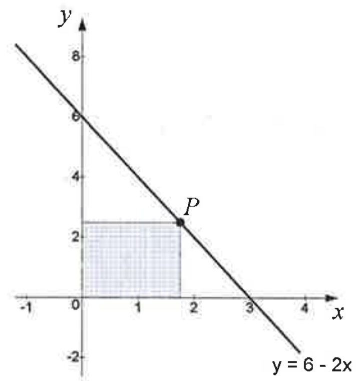
A

1 Solve the equation $f'(x)=0$ if $f(x)=x^3-3x^2-24x-8$.

2 Find $f'(2)$ if $f(x)=x(4-x^2)$.

3 For a linear function $f(x)$, $f(0)=3$ and $f'(2)=-2$. Find the function $f(x)$.

4 Where on the line $y=6-2x$ should the point P be located to give maximum area of the rectangle?



5 Use the derivative to find the critical points of the function $f(x)=3x^2-x^3-5$. Draw a sketch of the graph.

6 The line $y=2x-1$ is tangent to the function $g(x)$ at the point where $x=3$. Find $g(3)$ and $g'(3)$.

7 Determine the value of the constant k in the function $f(x)=k\cdot e^{2x}$, so that $f'(0)=6$.

8 Find the largest and smallest value of the function $f(x)=x^2-4x+5$, if $0\leq x\leq 5$.

9 Find the absolute maximum and minimum values of $f(x)=x^2-4x+5$ on the interval $0\leq x\leq 5$.

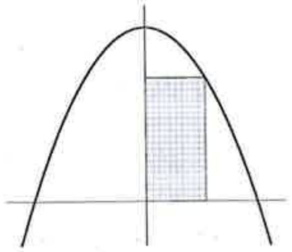
10 Which of the following alternatives means that $f(x)$ has a maximum point at (3, 2) ?

A $f(3)=2$, $f'(3)=0$, $f''(3)=2$
B $f(2)=3$, $f'(2)=0$, $f''(2)=-2$
C $f(3)=2$, $f'(3)=0$, $f''(3)=-2$
D $f(3)=2$, $f'(2)=0$, $f''(2)=0$

11 The temperature (T°C) an early summer morning in Therivergive is expected to change as $T(x)=x^2-4x+19$, where x means the number of hours after 05.00 ($0\leq x\leq 5$). Find the lowest temperature during that period.

12 The revenue from selling a certain product is $R(p) = p(4200 - 35p)$, where p is the price of the product. Use the derivative to find the maximum revenue.

13 A rectangle has one vertex at the origin and the opposite corner on the curve
$y = 8 - \frac{x^2}{2}$.
Find the maximum area of the rectangle.



14 Find and classify all critical points of the curve $y = 48 - x^3 + 12x^2 - 45x$.

15 Find the largest and smallest value of the function $f(x) = 10x^3 - 120x + 40$ on the interval $-3.9 \le x \le 4.1$.

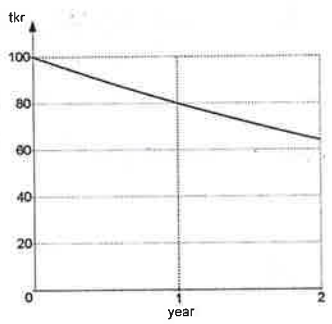
B

16 From 550 cm^2 of sheet-metal, a rectangular box with a square base and open top is to be constructed. Find the maximum volume of such a box. Answer in litres correct to three significant figures.

17 A container is filled with gas through a pipe. The mass, m kg, of the gas after t hours is $m = 5(1 - e^{-kt})$, where $k = 0.125$.
a) How much gas (in g) has been piped into the container after 5.0 h ?
b) How many g gas is streaming through the pipe per second, when $t = 5.0$ h?

18 Two tangent lines to the curve $y = x^2 - 3x + 1$ are drawn, at the points of the curve where $x = 2$ and $x = -3$. Find the point where the tangent lines intersect.

19 The value of a car decreases exponentially with the age of the car, t years, for $0 \le t \le 6$. The graph shows the value on the interval $0 \le t \le 2$.
a) Find the value of the car when $t = 5$.
b) Find the average rate of change of the value during year 1-3 and 4-6, respectively.



20 Find the tangents to the curve $y = x^5 - x$ at the points where the curve cuts the x-axis.

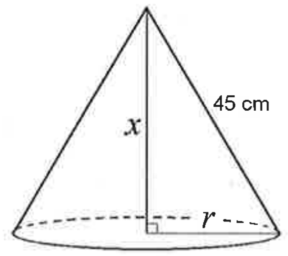
21 The cost, in kr, of producing x kg barbed wire at DeFence & Co is
$K(x) = 500 + 34x - 0{,}02x^2 \quad (0 < x < 500 \text{ kg})$.
If the *marginal cost* is defined as the rate of change of the cost, then
a) find the marginal cost when 200 kg is produced.
b) how much should be produced to give a marginal cost of 28 kr/kg ?

22 Use the derivative to find the minimum point of the function $f(x) = e^x - e^{2x}$. Give an exact answer.

23 Find the equations of all tangent lines to the curve $y = x^3 - 10{,}5x^2 + 32x$, parallel to the line $4x - 2y + 3 = 0$.

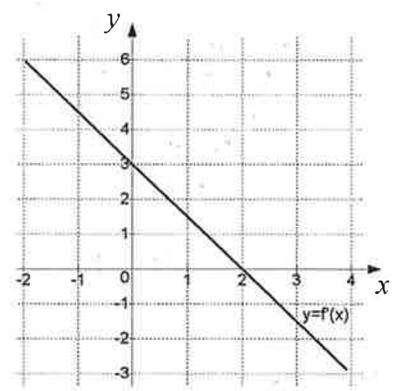
24 Gökböle Chickens is trying to determine what price to charge for football tickets. At a price of 10 kr per ticket, they average 700 people per game. For every increase of 1 kr, they lose 40 people on average. Every person at the game spends an average of 1.50 kr on coffee and hot dogs.
a) What price per ticket should be charged in order to maximize revenue?
b) How many people will attend at that price?

25 Find the maximum volume of the cone.
Answer correct to three significant figures.



26 The volume of a spherical balloon is $V = \frac{4\pi r^3}{3}$.
a) Calculate the rate of change of the volume of a balloon when the radius is changed from 1 dm to 1.5 dm. Answer exact in l/dm.
b) If the radius is increasing by 1 cm/s, how fast is the volume increasing at the moment when $r = 10$ cm ?

27 The figure shows the graph of the derivative of a quadratic function $f(x) = ax^2 + bx + c$.
Find a, b and c if $f(0) = 2$.



28 The function $f(x) = 27x(x - b)^2$, $0 \le x \le b$, has the (relative) maximum value 108. Find b.

29 The curves $y = x^2 - 8x + 12$ and $y = ax^2$ have a common tangent line at one point. Find the exact value of the constant a.