Chapter 5

Integration

5.1 Antidifferentiation

In the previous chapter we introduced the *derivative* and how to use it in problem solving, like finding the slope of a curve or rates of changes of functions. In applications we saw how to interpret the derivative. Some interpretations are listed below.

Function	*Derivative*
Distance	Velocity
Population	Rate of growth of population
Cost	Marginal cost
Work (energy)	Power (work/energy per time unit)

The process of finding the derivative from a function is called *differentiation*. In many problems, however, we know the rate of change of a function and want to know the function. In such a case we need to do the reverse of differentiation, called *antidifferentiation*.

Antiderivatives (primitive functions)

The task of finding $f(x)$ if $f'(x)$ is given means to *antidifferentiate* $f'(x)$ to get $f(x)$. In this process we use the rules of differentiation "backwards", which will be explained below.

Suppose y is a function of x and the derivative of y is constant 5. Can we find the function y ? It is easy to see that one such function is $5x$, since the derivative of $5x$ is 5, but are there others? In fact there are many. Here are some examples:

$$5x+1,\ 5x-7,\ 5x+100,\ 5x+5{,}34,\ \text{etc.}$$

All these functions are of the form $5x + C$, where C is any constant, and all have the derivative 5. They are called *antiderivatives* or *primitive functions* of 5.

We often use $F(x)$ to denote an antiderivative of $f(x)$, $G(x)$ to denote an antiderivative of $g(x)$, and so on.

To sum this up:

> If $F(x)$ is a function where $F'(x) = f(x)$ we say that
>
> the *derivative* of $F(x)$ is $f(x)$, and
> the *antiderivative* of $f(x)$ is $F(x)$.
>
> If $F(x)$ is an antiderivative of $f(x)$, then all antiderivatives of $f(x)$ can be written $F(x)+C$, where C is any constant.

Example 1

Find the antiderivative of

a) $f(x) = 2x$ b) $f(x) = x$

Solution:

a) $F(x) = x^2$, since $F'(x) = 2x = f(x)$

b) $F(x) = \dfrac{x^2}{2}$, since $F'(x) = \dfrac{2x}{2} = x = f(x)$

Rules for antidifferentiation

The previous example may give a hint on how it works. Remember how to differentiate a polynomial, or power of x:

If $f(x) = x^2$ then $f'(x) = 2\cdot x^1 = 2x$

$f(x) = x^3$ then $f'(x) = 3\cdot x^2 = 3x^2$

$f(x) = x^4$ then $f'(x) = 4\cdot x^3 = 4x^3$

.

.

$f(x) = x^n$ then $f'(x) = n\cdot x^{n-1}$

Antidifferentiation means doing this reversely, so

If $f(x) = x$ then $F(x) = \dfrac{x^2}{2}$

$f(x) = x^2$ then $F(x) = \dfrac{x^3}{3}$

$f(x) = x^3$ then $F(x) = \dfrac{x^4}{4}$

and, in general

If $f(x) = x^n$ then $F(x) = \dfrac{x^{n+1}}{n+1}$ $(x \neq -1)$

(the case $x = -1$ requires special attention and is treated in the next course)

Based on the formulas for differentiation, we can construct a list of formulas for *antidifferentiation*, which all may be verified by differentiating the right-hand side:

$f(x)$	$F(x)$
k	$kx + C$
x^n	$\dfrac{x^{n+1}}{n+1} + C$
e^x	$e^x + C$
e^{kx}	$\dfrac{e^{kx}}{k} + C$
a^x	$\dfrac{a^x}{\ln a} + C$

As in differentiation, multiplication or division by any constant does not affect the antidifferentiation; the constants stay where they are.

Moreover, if the function has two (or more) terms, we just antidifferentiate the function term by term.

Example 2
Find all antiderivatives of

a) $f(x)=x^6$ b) $f(x)=x^3+x^2-x-7$

c) $f(x)=5e^x$ d) $f(x)=\dfrac{5x^2}{2}+\dfrac{3e^x}{5}+2e^{3x}$

Solution:

a) $F(x)=\dfrac{x^7}{7}+C$ b) $F(x)=\dfrac{x^4}{4}+\dfrac{x^3}{3}-\dfrac{x^2}{2}-7x+C$

c) $F(x)=5e^x+C$

d) $f(x)=\dfrac{5}{2}\cdot x^2+\dfrac{3}{5}\cdot e^x+2\cdot e^{3x}$, so

$$F(x)=\frac{5}{2}\cdot\frac{x^3}{3}+\frac{3}{5}\cdot e^x+2\cdot\frac{e^{3x}}{3}+C=\frac{5x^3}{6}+\frac{3e^x}{5}+\frac{2e^{3x}}{3}+C$$

Example 3
Sometimes the function needs to be written in another way before antidifferentiation. Here are some examples:

$f(x)$	*...rewritten as*	$F(x)$	*...simplified to*
$(x+3)^2$	$=x^2+6x+9$	$\dfrac{x^3}{3}+\dfrac{6x^2}{2}+9x$	$=\dfrac{x^3}{3}+3x^2+9x$
$\sqrt{x}$	$=x^{\frac{1}{2}}$	$\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}$	$=\dfrac{2x^{\frac{3}{2}}}{3}$
$\dfrac{1}{x^2}$	$=x^{-2}$	$\dfrac{x^{-1}}{-1}$	$=-\dfrac{1}{x}$
$\dfrac{x^2+1}{x^2}$	$=\dfrac{x^2}{x^2}+\dfrac{1}{x^2}=1+x^{-2}$	$x+\dfrac{x^{-1}}{-1}$	$=x-\dfrac{1}{x}$
$\dfrac{e^x-e^{2x}}{e^x}$	$=1-e^x$	$x-e^x$	

Example 4
Find all antiderivatives of

a) $f(x)=\dfrac{3}{x^3}$ b) $f(x)=\dfrac{1-x+3x^4}{5}$

Solution:

a) $f(x)=\dfrac{3}{x^3}=3\cdot x^{-3}\ \rightarrow\ F(x)=3\cdot\dfrac{x^{-2}}{-2}=-\dfrac{3}{2x^2}+C$

b) $F(x)=\dfrac{x-\dfrac{x^2}{2}+\dfrac{3x^5}{5}}{5}+C=\dfrac{x}{5}-\dfrac{x^2}{10}+\dfrac{3x^5}{25}+C$

Initial conditions
We are often given a condition with a function, such as $F(0)=1$, or that the graph of $F(x)$ goes through the point (0, 1). This is called an *initial condition* or *boundary condition*. From this condition we can determine the constant C of the antiderivative and by that the specific antiderivative associated with the given condition.

Example 5
Find the antiderivative of the following functions, such that $F(0)=2$:

a) $f(x)=5x^2+1$ b) $f(x)=e^{3x}+4x$

Solution:

a) $F(x)=\dfrac{5x^3}{3}+x+C$

$F(0)=\dfrac{5\cdot 0^3}{3}+0+C=2\ \rightarrow\ C=2\ \rightarrow\ F(x)=\dfrac{5x^3}{3}+x+2$

b) $F(x)=\dfrac{e^{3x}}{3}+2x^2+C$

$F(0)=\dfrac{e^{3\cdot 0}}{3}+2\cdot 0^2+C=\dfrac{1}{3}+0+C=2\ \rightarrow\ C=\dfrac{5}{3}$

$\rightarrow\ F(x)=\dfrac{e^{3x}}{3}+2x^2+\dfrac{5}{3}$

Exercises

A

5101 Find the antiderivatives of

a) $f(x)=3x$ b) $f(x)=x-1$

c) $f(x)=3e^x$ d) $f(x)=x^{-5}$

e) $f(x)=3-\dfrac{x}{2}+9x^2$ f) $f(x)=3e^x+6e^{2x}-e^{-x}$

5102 Antidifferentiate

a) $f(x)=1+2x+3x^2+4x^3+5x^4$ b) $f(x)=\dfrac{e^x}{5}$

c) $f(x)=\dfrac{3}{x^2}$ d) $f(x)=\dfrac{4x^3}{5}+\dfrac{8x}{3}+7$

e) $f(x)=(x+5)^2$ f) $f(x)=\dfrac{4e^{2x}}{3}$

5103 Find $F(x)$ such that

a) $f(x)=x-3$, $F(2)=9$ b) $f(x)=3e^x-6x^2+1$, $F(0)=5$

c) $f(x)=(1+x)^2$, $F(-1)=1$ d) $f(x)=\sqrt{x}$, $F(4)=1$

5104 Find the function $f(x)$ if $f'(x) = x^2 - 4$ and $f(-1) = 2$.

B

5105 Find the antiderivatives of

a) $f(x) = 2(3x-1)^2$

b) $f(x) = \dfrac{(1-x)^2}{x^4}$

c) $f(x) = 3^x$

d) $f(x) = \dfrac{1}{\sqrt{x}}$

e) $f(x) = \dfrac{3x-1}{x^3}$

f) $f(x) = (\sqrt{x}+1)^2$

g) $f(x) = x\sqrt{x}$

h) $f(x) = \dfrac{1}{e^{2x}}$

5106 Find $F(x)$ such that

a) $f(x) = \dfrac{3x^3 - 6x^2 + 4x}{x}$, $F(1) = 0$

b) $f(x) = \dfrac{3}{2e^x} - \dfrac{2e^x}{3}$, $F(0) = 1$

c) $f(x) = \sqrt[3]{x} + \sqrt{x^3}$, $F(1) = 1$

d) $f(x) = -\dfrac{6000}{x^2}$, $F(-5) = 1000$

5107 The graph of $y = F(x)$ goes through the point (0, 3).
Find $F(x)$ if $f(x) = 3 \cdot e^{-x} + x + 1$.

5108 Find a primitive function of

a) $f(x) = e^{x+2}$

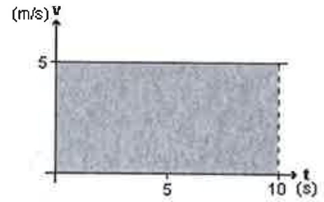
b) $f(x) = (e - e^x)^2$

5109 Find the value of $F(2) - F(1)$ if $f(x) = ax - 6ax^2$

5.2 Integrals

When studying the graph of a function, we have seen how the slope of the curve is telling us how the function changes, which is important in many ways.
In a similar way is the area between the curve and the x axis interesting in many applications. This area is of course depending on the shape of the curve, and thereby depending on the function. It is easy to see that this area has a practical meaning in many contexts.
If an object is moving, we can describe its velocity v at time t in a diagram. Here are three different examples:

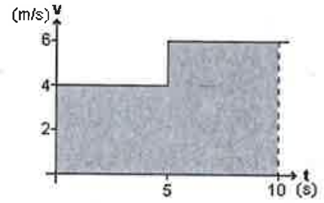
$v(t) = 5$



Distance travelled:

$5 \cdot 10 = 50$ (m)

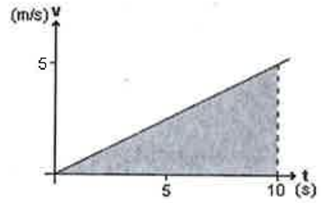
$v(t) = \begin{cases} 4, & 0 \le t < 5 \\ 6, & 5 \le t \le 10 \end{cases}$



Distance travelled:

$5 \cdot 4 + 5 \cdot 6 = 50$ (m)

$v(t) = 0,5t$

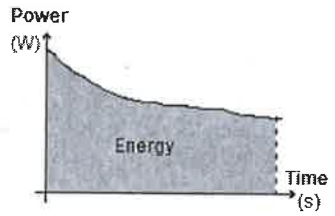


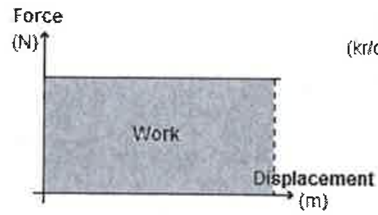
Distance travelled:

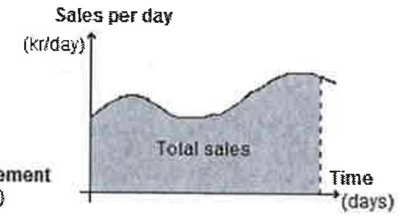
$\dfrac{5 \cdot 10}{2} = 25$ (m)

In all cases we see that the distance travelled is equal to the value of the area under the graph.
Here are some additional examples of the meaning of the area under a curve:

Example 1





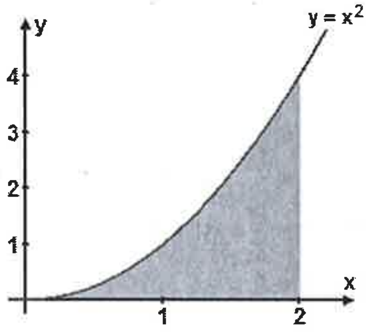


Note that the unit of the area is the product of the units of the axes. Observing this can help us understand the meaning of the area:

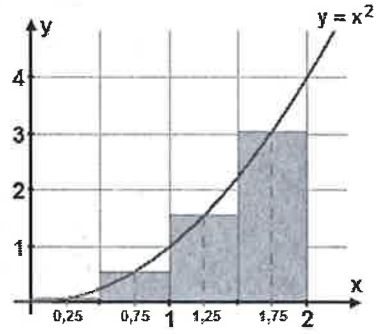
$W \cdot s = J/s \cdot s = J$ $\quad$ $N \cdot m = Nm = J$ $\quad$ $kr/day \cdot days = kr$

($W = Watt = Joule/second$, $N = Newton$, $m = metre$)

The area under a curve
If the graph of the function is not a straight line, there is a difficulty to calculate the exact area. However we can get a good approximation of the area by dividing it into a number of rectangles.

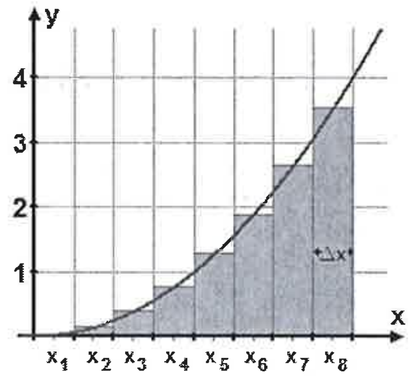


... is almost equal to



If we want to find the area under the curve $y = x^2$ from $x = 0$ to $x = 2$, we can divide it into 4 rectangles and calculate the total area of the rectangles. The width of each rectangle is 0,5 and the heights are $0,25^2$, $0,75^2$, $1,25^2$ and $1,75^2$.
The total area is $0,25^2 \cdot 0,5 + 0,75^2 \cdot 0,5 + 1,25^2 \cdot 0,5 + 1,75^2 \cdot 0,5 = 2,625$.

Clearly, we will get closer to the real area if we use more rectangles:



If the width of each rectangle is Δx, the total area is

$$f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + ... + f(x_7)\cdot\Delta x + f(x_8)\cdot\Delta x$$

By using *summation notation* this sum can be written

$$\sum_{i=1}^{8} f(x_i)\Delta x$$

This is read "the sum of the numbers $f(x_i)\cdot\Delta x$ from $i = 1$ to $i = 8$". The Greek capital letter sigma (Σ) is commonly used to denote sums.

In general, for any curve $y = f(x)$ and with n rectangles, the sum will be

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

We can now obtain the actual area by letting the number of rectangles increase indefinitely and then by taking the limit. The exact area is given by

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

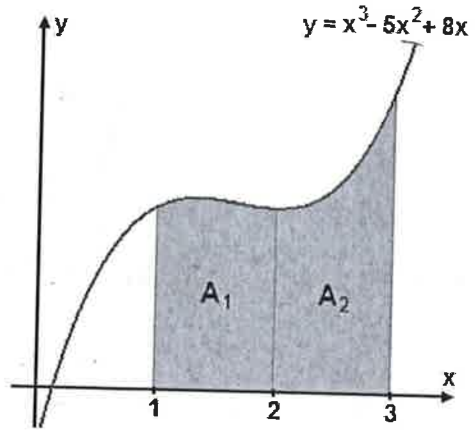
The integral sign
To describe the exact area under a curve in a symbolic way we use the *integral sign* ∫ and make the following definition:

> The area from the curve $y = f(x)$ to the x axis from $x = a$ to $x = b$ is called the *integral* of $f(x)$ from a to b and is written
>
> $$\int_a^b f(x)\,dx$$
>
> a and b are called *lower* and *upper limit of integration*, $f(x)$ is the *integrand* and dx is a symbol, reflecting Δx (see previous page), showing that x is the *variable of integration*.

While $\sum_{i=1}^{n} f(x_i)\Delta x$ is a sum of rectangle areas, giving an approximate value of the actual area, $\int_a^b f(x)\,dx$ is a symbol for the exact area. We will soon learn how to calculate it!

Example 2
a) Use an integral to write an expression for the areas A_1 and A_2 shown in the figure.

b) Write an expression for the total area, $A_1 + A_2$.



Solution:

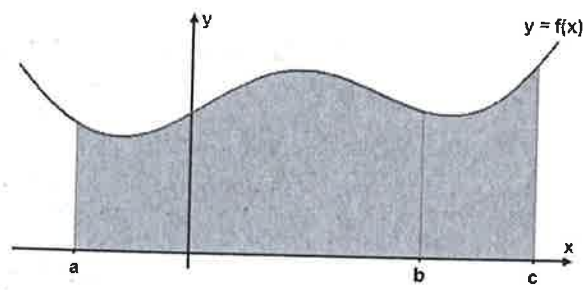
a) $A_1 = \int_1^2 (x^3 - 5x^2 + 8x)\,dx$

$A_2 = \int_2^3 (x^3 - 5x^2 + 8x)\,dx$

b) $A_1 + A_2 = \int_1^3 (x^3 - 5x^2 + 8x)\,dx$

Example 3
The previous example shows that for any function it is clear that

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$



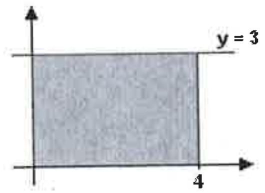
Example 4
Evaluate the following integrals

a) $\int_0^4 3\,dx$ b) $\int_2^5 \left(\frac{x}{2}-1\right)dx$

Solution:

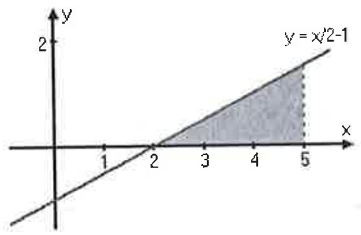
a) The integral can be interpreted as the area under the line $y=3$ from $x=0$ to $x=4$, that is, a rectangle with base 4 and height 3, so

$$\int_0^4 3\,dx = 4\cdot 3 = 12$$



b) The integral can be interpreted as the area under the line $y=\frac{x}{2}-1$ from $x=2$ to $x=5$, that is, a triangle with base 3 and height 1,5 , so

$$\int_2^5 \left(\frac{x}{2}-1\right)dx = \frac{3\cdot 1{,}5}{2} = 2{,}25$$

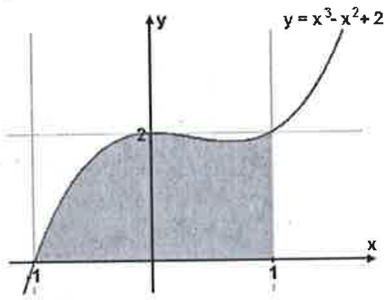


Exercises

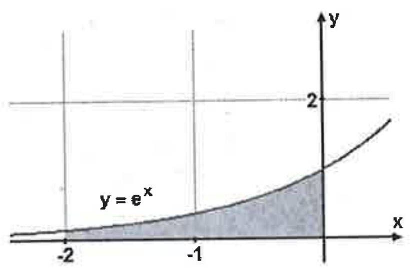
A

5201 Use an integral to write an expression for the shaded area:

a)



b)



5202 Calculate the integral of $f(x)$ from $x=1$ to $x=3$, if

a) $f(x)=5-0{,}5x$ b) $f(x)=2x+5$

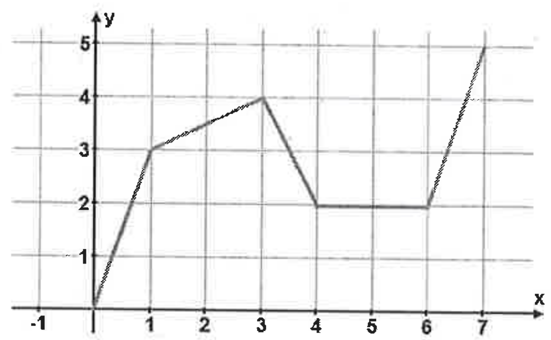
5203 Evaluate the following integrals

a) $\int_1^5 1\,dx$ b) $\int_0^3 2x\,dx$

c) $\int_0^3 (x+1)\,dx$ d) $\int_0^2 (6-3x)\,dx$

5204 The graph of $f(x)$ is shown in the figure.

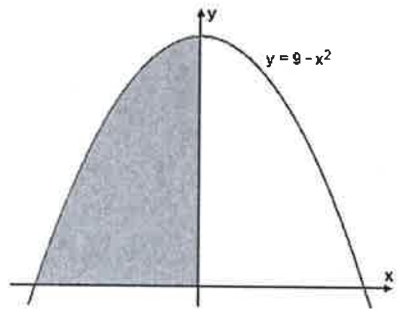
Evaluate the integral $\int_1^7 f(x)\,dx$.



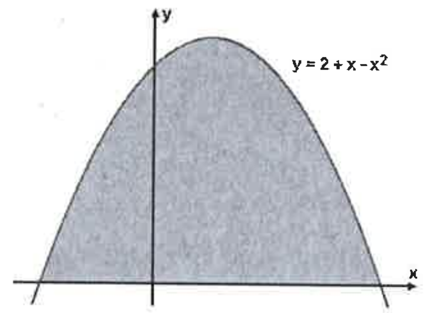
B

5205 Use an integral to write an expression for the shaded area:

a)



b)



5206 Evaluate the following integrals

a) $\int_0^2 (|x-1|)\,dx$ b) $\int_k^{3k} kx\,dx$

5207 The speed of an object is $v(t)$ m/s, where $v(t)=1+0{,}2t$. Find the distance travelled during the fifth second.

5208 Water is leaking from a tank at a rate of $f(t)=2^{-0{,}5t}$ cubic metres per minute.

a) Write down an integral for how many cubic metres that has leaked out from the tank during the first five minutes.

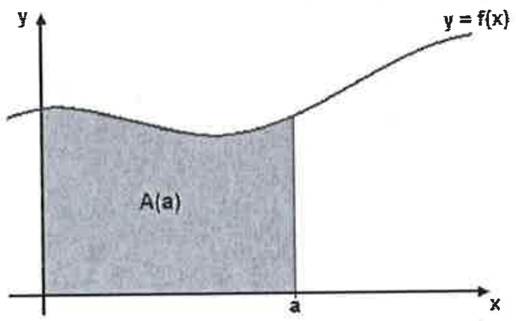
b) Evaluate the integral in a).

5.3 Exact integration

Finding area using antiderivatives
We have seen that the area under a curve is depending on the shape of the curve. We shall now investigate this dependence to see if we can find a way to evaluate an integral exactly.
Suppose *f* is a continuous function (the curve has no holes or gaps). Then the integral

$$\int_0^a f(x)\,dx$$

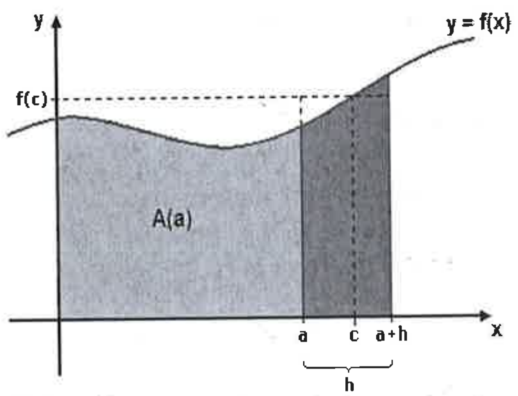
represents the area under the curve $y = f(x)$ from $x = 0$ to $x = a$.



Since the function is given, the integral is now only depending on the upper limit of integration, *a*, so the area is a function of *a*:

$$A(a) = \int_0^a f(x)\,dx$$

Let us now increase the area, by adding a small strip of width *h*:



The area of this strip is $A(a+h) - A(a)$, but is also equal to a rectangle of width *h* and height $f(c)$ for some *c* between *a* and $(a + h)$.
We have

$$A(a+h) - A(a) = h \cdot f(c)$$

which also can be written

$$\frac{A(a+h) - A(a)}{h} = f(c)$$

If *h* approaches 0, the left hand side will approach $A'(a)$ and the right hand side will approach $f(a)$. In mathematical notation:

$$\lim_{h\to 0}\frac{A(a+h) - A(a)}{h} = \lim_{h\to 0} f(c) \quad \text{or} \quad A'(a) = f(a)$$

This means that, for any *a*, the derivative of the area function is equal to the curve function. Antidifferentiation on both sides leads to

$$A(a) = F(a) + C$$

Obviously, the area function is an antiderivative of the function of the curve, $f(x)$.

The fact that $A(0) = \int_0^0 f(x)\,dx = 0$ gives the value of the constant *C*:

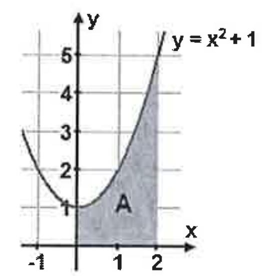
$$A(0) = F(0) + C = 0 \quad \rightarrow \quad C = -F(0)$$

To sum this up, we have found the following, which gives us an opportunity to evaluate an integral by using antiderivatives:

$$A(a) = \int_0^a f(x)\,dx = F(a) - F(0)$$

where $F(x)$ is an antiderivative of $f(x)$.

Example 1
Calculate the area shown in the figure:



Solution:

The area can be written $A = \int_0^2 (x^2+1)\,dx$.

According to the theory above this integral can be calculated like this:

$$A = \int_0^2 (x^2+1)\,dx = F(2) - F(0)$$

where $F(x)$ is an antiderivative of $f(x) = x^2 + 1$.

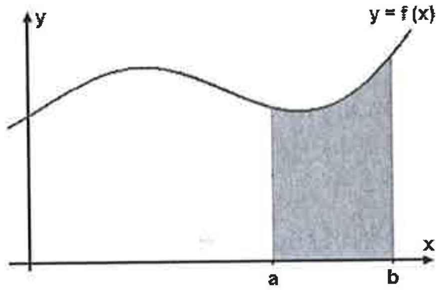
$$F(x) = \frac{x^3}{3} + x + C$$

$$A = \int_0^2 (x^2+1)\,dx = F(2) - F(0) = \left(\frac{2^3}{3} + 2 + C\right) - \left(\frac{0^3}{3} + 0 + C\right) = \frac{8}{3} + 2 = \frac{14}{3}$$

(Notice that the constant *C* is not necessary to include, since it cancels out.)

The definite integral formula

The *definite integral* of *f* from *a* to *b* is the integral $\int_a^b f(x)\,dx$, where $a < b$.



From the above we have

$$\int_0^a f(x)\,dx = F(a) - F(0) \quad \text{and} \quad \int_0^b f(x)\,dx = F(b) - F(0)$$

From the previous section we also know that

$$\int_0^b f(x)\,dx = \int_0^a f(x)\,dx + \int_a^b f(x)\,dx \quad ,$$

which can be rewritten

$$\int_a^b f(x)\,dx = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx$$

which is equal to

$$\int_0^b f(x)\,dx = F(b) - F(0) - \left(F(a) - F(0)\right) =$$
$$= F(b) - F(0) - F(a) + F(0) =$$
$$= F(b) - F(a)$$

This gives the final formula for evaluating the definite integral of *f* from *a* to *b*:

> If $f(x)$ is a continuous function from $x = a$ to $x = b$, and F is any antiderivative of f, then
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

When evaluating integrals, a common way to proceed is to write

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

where the notation $\left[F(x)\right]_a^b$ serves as an aid, showing that the first step is antidifferentiation.

Example 2
Evaluate the following integrals

a) $\int_{-1}^{3} 2x\,dx$ b) $\int_{1}^{2} (x^2 - x)\,dx$

c) $\int_{0}^{1} e^x\,dx$ d) $\int_{1}^{2} (8x^3 - 3x^2 + 4x + 1)\,dx$

Solution:

a) $\int_{-1}^{3} 2x\,dx = \left[x^2\right]_{-1}^{3} = \left(2^2\right) - \left(-1\right)^2 = 4 - 1 = 3$

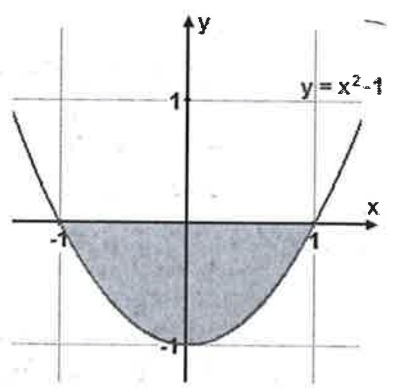
b) $\int_{1}^{2} (x^2 - x)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{1^3}{3} - \frac{1^2}{2}\right) = \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

c) $\int_{0}^{1} e^x\,dx = \left[e^x\right]_0^1 = e^1 - e^0 = e - 1$

d) $\int_{1}^{2} (8x^3 - 3x^2 + 4x + 1)\,dx = \left[2x^4 - x^3 + 2x^2 + x\right]_1^2 =$
$= \left(2\cdot 2^4 - 2^3 + 2\cdot 2^2 + 2\right) - \left(2\cdot 1^4 - 1^3 + 2\cdot 1^2 + 1\right) = 34 - 4 = 30$

Example 3

Evaluate the integral $\int_{-1}^{1} (x^2 - 1)\,dx$.



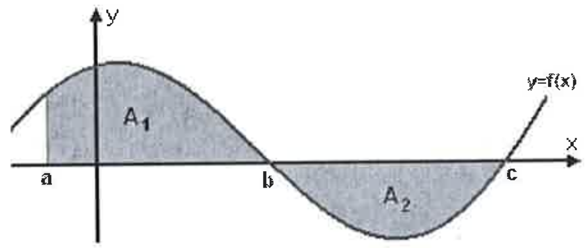
Solution:

$$\int_{-1}^{1} (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_{-1}^{1} =$$
$$= \left(\frac{1^3}{3} - 1\right) - \left(\frac{(-1)^3}{3} - (-1)\right) =$$
$$= \frac{1}{3} - 1 + \frac{1}{3} - 1 = -\frac{4}{3}$$

The value of the integral is negative. The reason is that the function $f(x) = x^2 - 1$ is negative on the given interval; the curve is *below* the x axis.

This illustrates that the value of an integral can be negative, positive or zero. However, if we are asked for the area, we must answer with a positive value. (In that case, the minus sign is omitted)

Example 4
Use integrals to write down an expression for the total area, $A_1 + A_2$, in the figure below:



Solution: $A_1 = \int_a^b f(x)\,dx$, $A_2 = -\int_b^c f(x)\,dx$

Total area: $A_1 + A_2 = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx$

(Since A_2 lies under the *x* axis, the value of the integral will here be negative. To compensate for this, we put a minus sign in front of the integral to get a positive value.)

Indefinite integrals
Without any limits of integration, the integral sign is used to represent all antiderivatives of a function. Such an integral is called an *indefinite integral* and is just another way to symbolize antidifferentiation. This is why we also often call this process *integration.*

$$\int f(x)\,dx = F(x) + C$$

The symbol *dx* indicates that the antidifferentiation should be done with respect to the variable *x*, and the constant *C* must be there to include *all* antiderivatives of *f*.

Basic integration rules
The following properties of integrals can easily be derived from the definition of integrals and holds for both indefinite and definite integrals:

(1) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(2) $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b \big(f(x) \pm g(x)\big)\,dx$

(3) $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$

Example 5
Evaluate

a) $\int (2x^3 - e^{2x} + 5)\,dx$ b) $\int (e^{-x} + \frac{1}{x^2})\,dx$

Solution:

a) $\int (2x^3 - e^{2x} + 5)\,dx = \frac{2x^4}{4} - \frac{e^{2x}}{2} + 5x + C = \frac{x^4}{2} - \frac{e^{2x}}{2} + 5x + C$

b) $\int (e^{-x} + \frac{1}{x^2})\,dx = \int (e^{-x} + x^{-2})\,dx = \frac{e^{-x}}{-1} + \frac{x^{-1}}{-1} + C = -e^{-x} - x^{-1} + C =$
$= -\frac{1}{e^x} - \frac{1}{x} + C$

Example 6
Use the integration rules (above) to simplify and evaluate the following integrals:

a) $\int_{-2}^{3} (x^3 + 2x^2 - 3x + 1)\,dx + \int_{-2}^{3} (x^2 - x^3 + 3x - 1)\,dx$

b) $\int_0^{\ln 2} (6e^{3x} - e^x)\,dx + \int_{\ln 2}^{1} (6e^{3x} - e^x)\,dx$

c) $\int_{-2}^{1} \left(\frac{3x^3}{5}\right) dx$

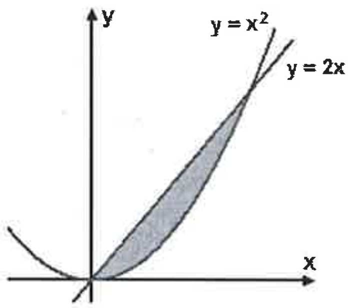
Solution:

a) $\int_{-2}^{3} (x^3 + 2x^2 - 3x + 1)\,dx + \int_{-2}^{3} (x^2 - x^3 + 3x - 1)\,dx =$
$\int_{-2}^{3} (x^3 + 2x^2 - 3x + 1 + x^2 - x^3 + 3x - 1)\,dx = \int_{-2}^{3} (3x^2)\,dx =$
$= \left[x^3\right]_{-2}^{3} = 3^3 - (-2)^3 = 27 + 8 = 35$

b) $\int_0^{\ln 2} (6e^{3x} - e^x)\,dx + \int_{\ln 2}^{1} (6e^{3x} - e^x)\,dx = \int_0^{1} (6e^{3x} - e^x)\,dx =$
$= \int_0^1 (6e^{3x} - e^x)\,dx = \left[2e^{3x} - e^x\right]_0^1 = \left(2e^3 - e^1\right) - \left(2e^0 - e^0\right) = 2e^3 - e - 2 + 1 =$
$= 2e^3 - e - 1$

c) $\int_{-2}^{1} \left(\frac{3x^3}{5}\right) dx = \frac{3}{5}\int_{-2}^{1} x^3\,dx = \frac{3}{5}\left[\frac{x^4}{4}\right]_{-2}^{1} = \frac{3}{5}\left(\frac{1^4}{4} - \frac{(-2)^4}{4}\right) = \frac{3}{5}\left(\frac{1}{4} - 4\right) =$
$\frac{3}{5}\left(-\frac{15}{4}\right) = -\frac{9}{4}$

Example 7
Find the area enclosed by the two curves, as shown in the figure.



Solution:
The two curves intersect when

$x^2 = 2x \quad \rightarrow \quad x^2 - 2x = 0$

$\rightarrow \quad x(x-2) = 0$

$x_1 = 0\ ,\ x_2 = 2$

The shaded area is the difference between the integral of $2x$ and the integral of x^2 from 0 to 2:

$$A = \int_0^2 2x\,dx - \int_0^2 x^2\,dx = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \left(2^2 - \frac{2^3}{3}\right) - (0) =$$
$$= 4 - \frac{8}{3} = \frac{4}{3}$$

Example 8
A population of bacteria in a biological experiment grows at the rate of

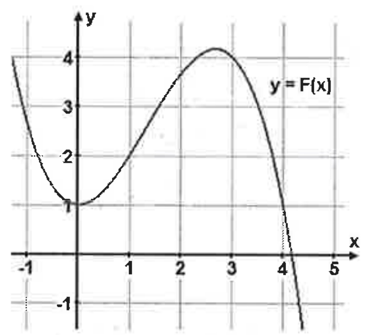
$$P'(t) = 1200e^{0,32t}$$

where $P(t)$ is the total population at time t, in days.
By how many bacteria did the population grow during the first 10 days?

Solution:
The population function $P(t)$ is given by $\int P'(t)\,dt = P(t) + C$.
However, since we don't know $P(0)$, the number of bacteria at start, we cannot find C. But since the question was the increase during the first 10 days, we don't need C and can get the answer by the integral

$$\int_0^{10} P'(t)\,dt = \int_0^{10} 1200e^{0,32t}\,dt = \left[\frac{1200e^{0,32t}}{0,32}\right]_0^{10} = \left[3750e^{0,32t}\right]_0^{10} =$$
$$= \left(3750e^{0,32\cdot 10}\right) - 3750e^0 \approx 92000 - 3750 = 88250$$

Example 9
$F(x)$ is an antiderivative of $f(x)$.
The graph of $F(x)$ is shown in the figure.



Evaluate the integral $\int_1^3 f(x)\,dx$.

Solution:

$\int_1^3 f(x)\,dx = \left[F(x)\right]_1^3 = F(3) - F(1) =$

$=$ (taken from the graph) $= 4 - 2 = 2$

Exercises

A

5301 Evaluate the following integrals

a) $\int_0^2 (x^2 - 1)\,dx$ b) $\int_1^2 (5x^2 + 2x + 1)\,dx$

c) $\int_{-1}^1 (10x^4 + 6x)\,dx$ d) $\int_0^1 (e^{2x} + 6e^{3x})\,dx$

5302 Find the area between the x axis and:
a) $y = x^3$ from $x = 0$ to $x = 1$.
b) $y = x^2 + 3x + 2$ from $x = 1$ to $x = 3$.

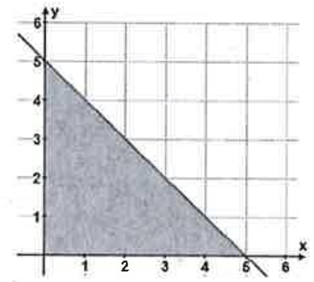
5303 Evaluate

a) $\int_0^3 (2x^4 - x^3 + x - 4)\,dx + \int_0^3 (3x^4 + 5x^3 - x + 4)\,dx$

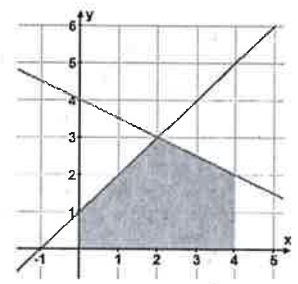
b) $\int_{-1}^0 (2x^3 - x - 1)\,dx + \int_0^1 (2x^3 - x - 1)\,dx$

5304 Find the area of the shaded region:

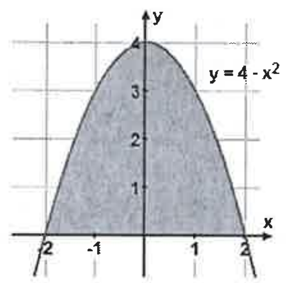
a)



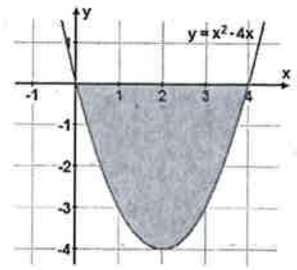
b)



c)



d)



5305 The volume of water in a tank is $V(t)$ cubic meters after t minutes. Explain the meaning of
a) $V(0)$ b) $V'(5)$

c) $\int_0^{10} V'(t)\,dt$ d) $\int_0^1 V'(t)\,dt = 1,5$

B

5306 Explain, without calculations, why $\int_{-1}^1 x\,dx = 0$.

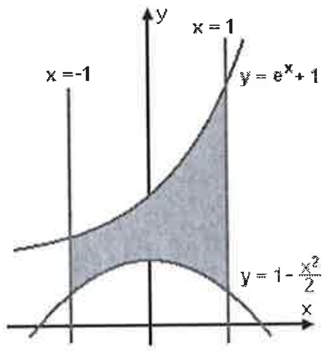
5307 If $F'(x)=f(x)$, $F(5)=7$ and $F(-1)=-2$, evaluate $\int_{-1}^{5} f(x)\,dx$.

5308 Find the area under the graph $f(x)$ from $x=-2$ to $x=3$, if

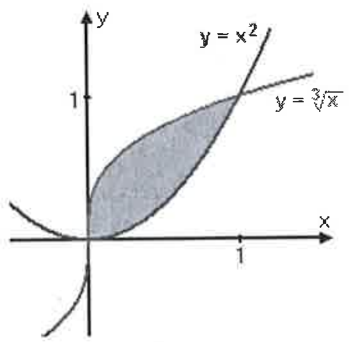
$$f(x)=\begin{cases} x^2 & \text{if} \quad x<1 \\ 1 & \text{if} \quad x\geq 1 \end{cases}$$

5309 Find the area of the shaded region

a)



b)



5310 Evaluate the following integrals

a) $\int_{1}^{4} \sqrt{x}\,dx$ b) $\int_{-2}^{-1} \frac{x^2+1}{x^2}\,dx$

c) $\int_{-1}^{1} (1+x^2)^2\,dx$ d) $\int_{0}^{1} \frac{1+e^{2x}}{e^x}\,dx$

5311 a) Evaluate $\int_{0}^{1} 2^x\,dx$ b) Evaluate $\int_{0}^{1} \frac{3}{2^x}\,dx$

5312 A company estimates that its sales will grow continuously at a rate given by the function

$$S'(t)=10e^{\frac{t}{2}}$$

where $S'(t)$ is the sales rate, in dollars per day, at time t, in days.

a) Find the total sales for the first 5 days.

b) On what day will the total sales exceed \$10000 ?

5313 An object is moving with increasing speed. Its velocity at time t is given by

$$v(t)=\frac{3t^2}{10}+\frac{2t}{5}$$

where $v(t)$ is the speed in m/s after t seconds.

a) How far does it travel during the first 10 seconds?

b) How far does it travel during the tenth second?

Chapter exercises 5

A

1 Find an antiderivative of

a) $f(x)=x^4$ b) $f(x)=5x$

c) $f(x)=e^{2x}$ d) $f(x)=6x^2-16x+e^{5x}+1$

2 Find the antiderivative where $F(0)=2$ of

a) $f(x)=5x^4+\frac{9x^2}{2}$ b) $f(x)=4e^x-x^3+2$

3 Evaluate

a) $\int (3x^2-5)\,dx$ b) $\int (x+3)^2\,dx$

c) $\int \left(\frac{3x^5}{2}\right)dx$ d) $\int \left(\frac{3x^5+2x^3-x^2}{x^2}\right)dx$

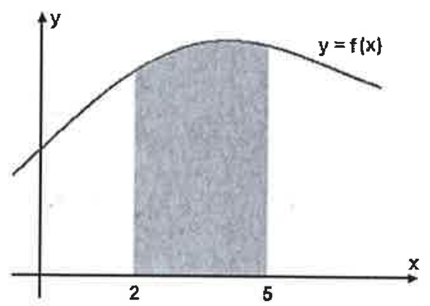
4 Find the exact value of

a) $\int_{1}^{3} 2x\,dx$ b) $\int_{0}^{2} x(2-x)\,dx$

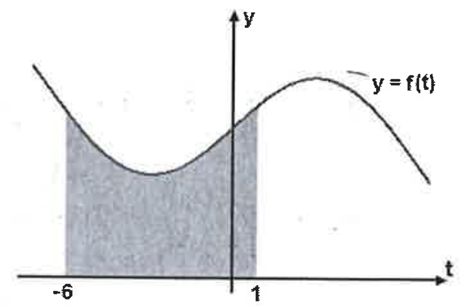
5 Explain why $\int_{-3}^{3} 5x\,dx=0$.

6 Write down an integral corresponding to the shaded area:

a)

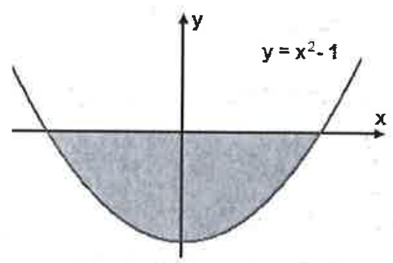


b)

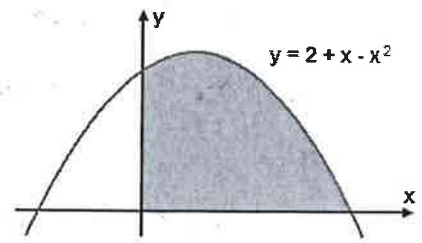


7 Find the shaded area:

a)



b)



8 Water is leaking out from an aquarium at the speed of V l/s, according to the formula $V(t)=3e^{-0,1t}$, where t is the time in seconds from when the leak started. How much water will leak out in 60 seconds?

B **9** Find an antiderivative of

a) $f(x) = \dfrac{3}{x^2}$ b) $f(x) = 5^x$

10 Evaluate

a) $\int \left(\dfrac{2 - 3x^2 + 4x^5}{x^2} \right) dx$ b) $\int (e^x + 3)^2\, dx$

c) $\int \sqrt{x}\, dx$ d) $\int \left(\dfrac{1}{e^{3x}} \right) dx$

11 Find the exact value of

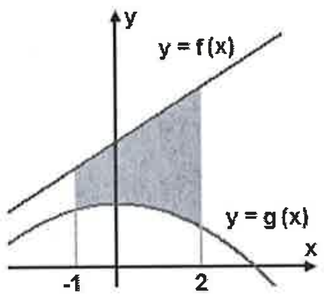
a) $\int_1^2 \dfrac{1}{x^3}\, dx$ b) $\int_0^1 (e^x - e^{-x})^2\, dx$

12 Evaluate

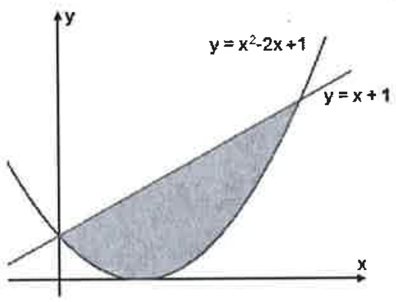
a) $\int t^x\, dt$ b) $\int t^x\, dx$

13 Write down an integral corresponding to the shaded area:

a)



b)



14 Find the area between the positive part of the x-axis and the curve $y = x^3 - 2x^2 - 3x$

15 Find $f(x)$ if $f''(x) = 3x^2 + 6x - 1$ and $f(1) = f'(1) = 0$.

16 Hot water in a cup cools down at a rate of $-5e^{-0{,}08t}$ °C/min. How much is the temperature changing during the first 5 minutes?

Chapter 6

Trigonometry