**B**

**9** Find an antiderivative of

a) $f(x) = \dfrac{3}{x^2}$ b) $f(x) = 5^x$

**10** Evaluate

a) $\displaystyle\int\left(\frac{2-3x^2+4x^5}{x^2}\right)dx$ b) $\displaystyle\int\left(e^x+3\right)^2 dx$

c) $\displaystyle\int\sqrt{x}\,dx$ d) $\displaystyle\int\left(\frac{1}{e^{3x}}\right)dx$

**11** Find the exact value of

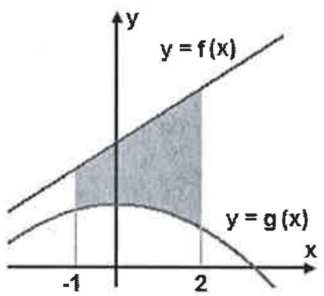
a) $\displaystyle\int_1^2 \frac{1}{x^3}dx$ b) $\displaystyle\int_0^1 (e^x - e^{-x})^2\,dx$

**12** Evaluate

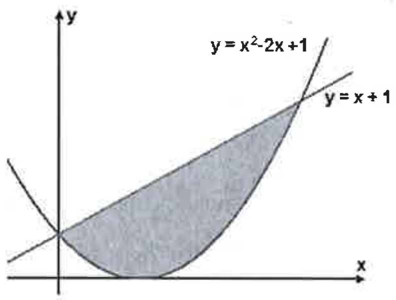
a) $\displaystyle\int t^x\,dt$ b) $\displaystyle\int t^x\,dx$

**13** Write down an integral corresponding to the shaded area:

a)




b)




**14** Find the area between the positive part of the $x$-axis and the curve $y = x^3 - 2x^2 - 3x$

**15** Find $f(x)$ if $f''(x) = 3x^2 + 6x - 1$ and $f(1) = f'(1) = 0$ .

**16** Hot water in a cup cools down at a rate of $-5e^{-0{,}08t}$ °C/min. How much is the temperature changing during the first 5 minutes?



# Chapter 6

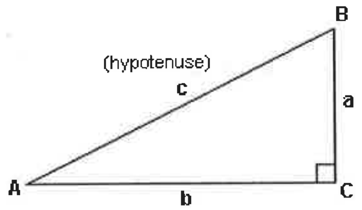
# Trigonometry

## 6.1 Basic trigonometry

In **Ma1c**, *trigonometry* was introduced as the study of relationships between sides and angles in right-angled triangles, and the three ratios *sine, cosine* and *tangent* were defined. In this chapter we will develop this concept.

**Notation**

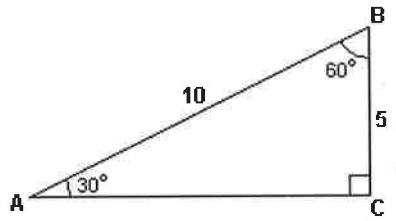
In a right-angled triangle, the longest side is called the *hypotenuse*. The two other sides are called *opposite side* and *adjacent side* (adjacent = next to) depending on which angle we start from. Since one angle is always a right angle, we are only interested in the other two angles.




In the figure, $c$ is the hypotenuse, $a$ is opposite side to the angle A, but adjacent to B. Similarly, $b$ is opposite to B, but adjacent to A.

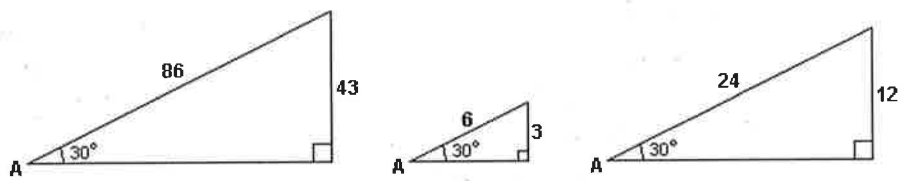
**Revision of sine, cosine and tangent**

The figure shows a right-angled triangle with angles 30°, 60° and 90° (a half equilateral triangle). In such a triangle, one side is always half as long as the hypotenuse.




This means that the ratio of the side opposite to the angle A and the hypotenuse is $\frac{5}{10}$ or $\frac{1}{2}$ (or 0,5).

Actually, it does not matter if the triangle is large or small. As long as the angle A is 30°, this ratio will always be $\frac{1}{2}$:




This ratio is connected to the angle 30° and is called the *sine* of 30°. It can be found on a calculator by using the sine-button. The calculator will give the ratio as a decimal number. We write this: $\sin 30° = 0,5$.

Similarly, the ratio of the side adjacent to A and the hypotenuse will always be the same, as long as the angle A is 30°. We call this ratio the *cosine* of A. It can be found on a calculator to be approximately 0,866. We write $\cos 30° \approx 0,866$.

Finally, the ratio of the side opposite to A and the side adjacent to A is called the *tangent* of A and is abbreviated "tan".

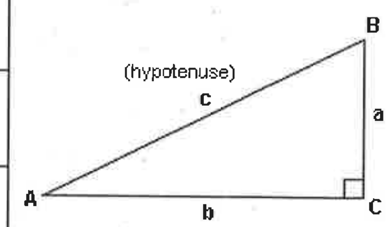
All these ratios are only depending on the size of the angle. Therefore, to each angle there are specific *sine, cosine* and *tangent* values.

We can summarize this in the following way:





| | |
|---|---|
| *The sine of A* | $\sin A = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c}$ |
| *The cosine of A* | $\cos A = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c}$ |
| *The tangent of A* | $\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b}$ |




(Note that most sine, cosine or tangent values have to be rounded. Unless instructed otherwise, we normally round off to 3 significant figures.)

---

*Example 1*

a) On a calculator, you may find (check!) that

$\sin 67° \approx 0,921$

$\tan 15° \approx 0,268$

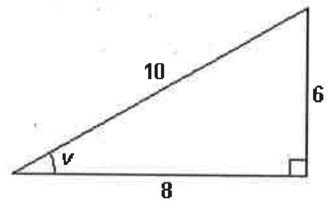
$\cos 60° = 0,5$

b) From the figure we can find that

$\sin v = \dfrac{6}{10} = 0,6$

$\cos v = \dfrac{8}{10} = 0,8$

$\tan v = \dfrac{6}{8} = 0,75$

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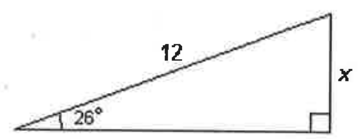
**Finding a side**

The ratios sine, cosine and tangent can be used to calculate an unknown side in a right-angled triangle.

For example, if we want to find $x$ in the triangle, we can use the sine ratio:




$$\sin 26° = \frac{x}{12}$$

Since sin 26° ≈ 0,43837… can be found on the calculator, we get

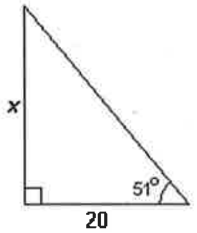
$$0,43837... = \frac{x}{12} \quad \rightarrow \quad x = 12 \cdot 0,43837... \approx 5,3$$

(A good rule is to keep all decimals until the calculation is finished, and then round off)

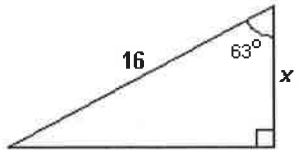
In problems like this, we can always use one (or more) of the ratios sine, cosine or tangent. Which one to use is depending on which two sides and what angle are involved in the problem. If the hypotenuse and the side opposite to the angle in question (like in the example above) are involved, we use the sine ratio, since $sine = \frac{\text{opposite side}}{\text{hypotenuse}}$. The following examples illustrate this:

*Example 2*
Find the side marked $x$ in the following triangles:

a)




b)




*Solution*:

a) We know one angle (except for the right angle), we are looking for the side opposite to that angle, and know the adjacent side. The most suitable ratio to use is the *tangent*, since $tan = \frac{\text{opposite side}}{\text{adjacent side}}$.

We get the equation

$$\tan 51° = \frac{x}{20} \quad \rightarrow \quad x = 20 \cdot \tan 51° \approx 24,7$$

b) We know the hypotenuse and the side adjacent to the known angle. We use the *cosine* ratio, since $cos = \frac{\text{adjacent side}}{\text{hypotenuse}}$.

The equation becomes

$$\cos 63° = \frac{x}{16} \quad \rightarrow \quad x = 16 \cdot \cos 63° \approx 7,3$$

---

**Finding an angle**

We can use the calculator to find the angle of a certain trigonometric ratio. For example, if we know the sine ratio of an angle, we can find the angle by going "backwards", using a button on the calculator marked $sin^{-1}$ (alt. *arcsin* or INV*sin*). It is often hidden "behind" the sine button.

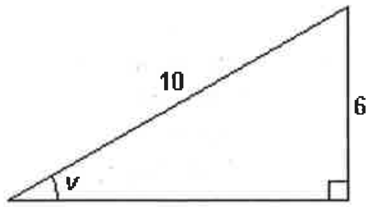
So, to find the angle $v$ in the triangle, we first calculate the sine ratio for $v$,

$$\sin v = \frac{6}{10} = 0,6$$

and then get the angle with a sine of 0,6 by writing and receiving

$$\sin^{-1} 0,6 \approx 36,9$$

on the calculator. The angle $v \approx 36,9°$.




Similarly, if we know the cosine of an angle, we use the $cos^{-1}$ button, and if the tangent value is known, we use the $tan^{-1}$ button.

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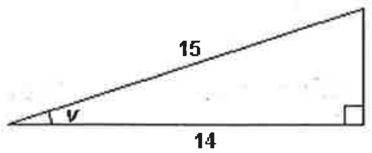
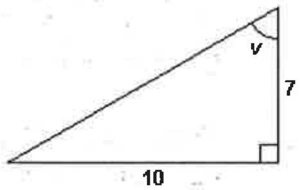
*Example 3*
Find the angle $v$ in the following triangles:

a)




b)








*Solution*:

a) We can calculate the cosine of $v$: $\cos v = \frac{14}{15} \approx 0,933$

The angle with that cosine ratio is $v = \cos^{-1} 0,933 \approx 21,0°$

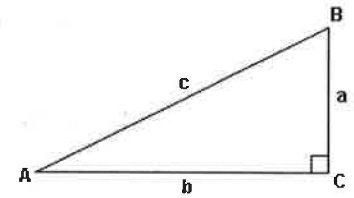
b) $\tan v = \frac{10}{7} \quad \rightarrow \quad v = \tan^{-1}\left(\frac{10}{7}\right) \approx 55,0°$

---

***Exercises***

**A**

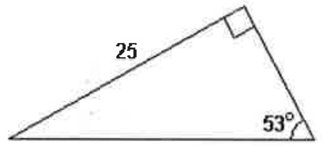
**6101** In the triangle, find
a) $a$, if $c$ = 10 cm and A = 28°.
b) $b$, if $c$ = 6 cm and A = 34°.
c) $b$, if $c$ = 28 cm and B = 57°.
d) $b$, if $a$ = 40 cm and B = 68°.



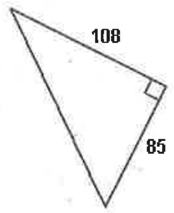

**6102** In the triangle of 4801, find
a) A, if $a$ = 8 cm and $b$ = 18 cm.
b) A, if $a$ = 56 cm and $c$ = 67 cm.
c) B, if $a$ = 2,34 cm and $b$ = 4,2 cm.
d) B, if $a$ = 45 cm m and $c$ = 0,5 m

**6103** Find all unknown sides and angles of the following triangles:

a)

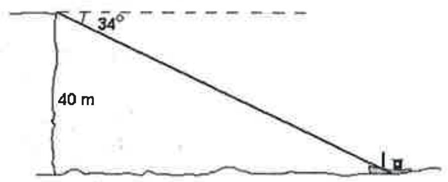



b)



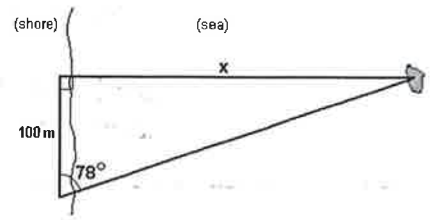

**6104** The foot of a ladder is placed 3,5 m from the wall of a house. The ladder makes an angle of 70° with the ground. How far up the wall does it reach?

**6105** From the top of a vertical cliff of height 40 m, the angle from the horizontal plane to a boat is 34°.
Calculate the distance to the boat from the foot of the cliff.




**B**

**6106** Arne is trying to find out the distance from the shore to an island. He has done some measuring according to the figure.
Can you help him to find out the distance $x$?

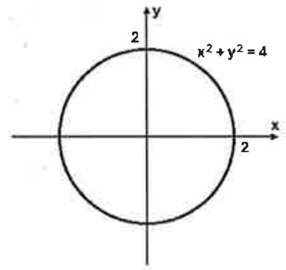



**6107** A rhombus has four sides of 15 cm and two angles of 70°. Calculate
a) the length of the longer diagonal.
b) the length of the shorter diagonal.

**6108** From a point 12 m from the base of a flag pole, the angle of elevation of the top is 42°. Find the height of the flag pole.

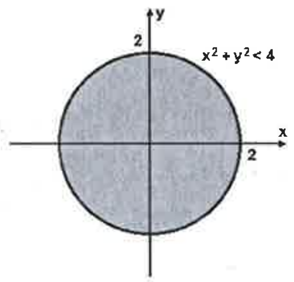
## 6.2 The equation of a circle

**Circles with centre at the origin**
A circle is a set of points which have in common that they are at the same distance from one given point; *the centre of the circle*. This distance is called the *radius* of the circle.




If we draw a circle with centre at the origin and with radius 2, at each point $(x, y)$ on the circle Pythagoras' Theorem will give

$$x^2 + y^2 = 2^2$$



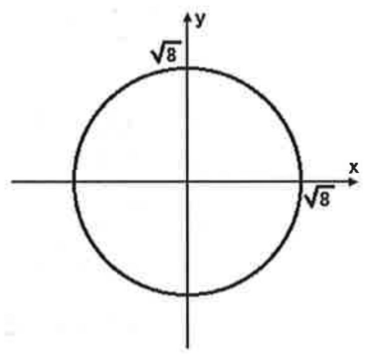

We say that $x^2 + y^2 = 4$ is the equation of a circle with centre at the origin and radius 2, because all points on this circle (but no others) satisfy this equation.

All points inside this circle are at a distance from the origin less than 2, so the graph of the inequality $x^2 + y^2 < 4$ is the region inside the circle $x^2 + y^2 = 4$.

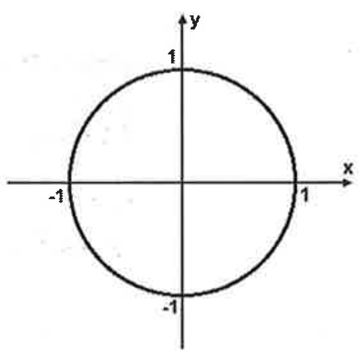
---

*Example 1*
a) Draw a graph of the circle $x^2 + y^2 = 1$

b) Find the equation of the circle in the figure.




*Solution*:
a) All points $(x, y)$ must satisfy Pythagoras' Theorem, $x^2 + y^2 = 1$, so this must be a circle with centre at the origin and with radius 1.
(This special circle is called a *unit circle*.)




b) The circle has centre at the origin and radius $\sqrt{8}$, so the equation of the circle is $x^2 + y^2 = \sqrt{8}^2$ or $x^2 + y^2 = 8$.

---

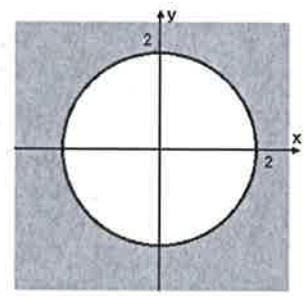




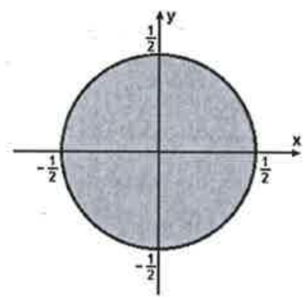
*Example 2*
Graph the inequalities

a) $x^2 + y^2 > 4$ b) $x^2 + y^2 \leq \frac{1}{4}$

*Solution*:
a) The graph consists of all points $(x, y)$ more than 2 units from the origin, that is, the region outside the circle $x^2 + y^2 = 2^2$.




b) All points on and inside the circle $x^2 + y^2 = \left(\frac{1}{2}\right)^2$, which is a circle with centre at the origin and with radius $\frac{1}{2}$.

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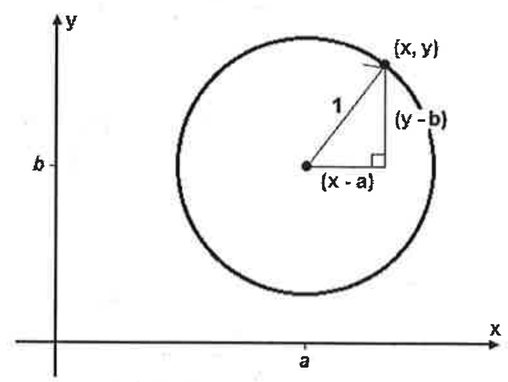
**Circles with centre at $(a, b)$**
If the centre of the circle is located at the point $(a, b)$ and the radius of the circle is 1, then for any point $(x, y)$ on the circle, Pythagoras' Theorem will hold:

$$(x-a)^2 + (y-b)^2 = 1$$




So, for any real numbers $a$, $b$, and $r$ we can state that

$$(x-a)^2 + (y-b)^2 = r^2$$

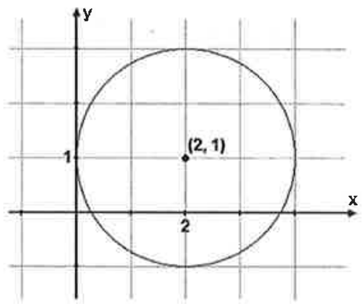
is the equation of a circle with centre at $(a, b)$ and with radius $r$.

As shown earlier, the corresponding inequalities

$(x-a)^2 + (y-b)^2 < r^2$ refers to all points inside the circle, and
$(x-a)^2 + (y-b)^2 > r^2$ all points outside the circle.

*Example 3*
Sketch the graph of the following circles:
a) $(x-2)^2+(y-1)^2=4$ b) $(x+3)^2+(y-2)^2=1$

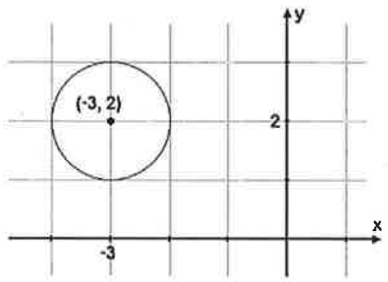
*Solution*:
a) According to the general equation of a circle, this is a circle with centre at (2, 1) and with radius 2.




b) The equation can be written

$(x-(-3))^2+(y-2)^2=1$ ,

which shows that this is the equation of a circle with centre at (–3, 2) and with radius 1.

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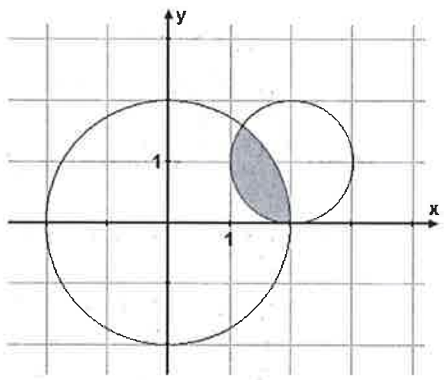
*Example 4*
Find the region enclosed by the two inequalities

$$\begin{cases} x^2+y^2\le 4 \\ (x-2)^2+(y-1)^2\le 1 \end{cases}$$

*Solution*:
The region consists of all points on and inside both circles,
$x^2+y^2=4$ , and
$(x-2)^2+(y-1)^2=1$.

This is the shaded area in the figure.

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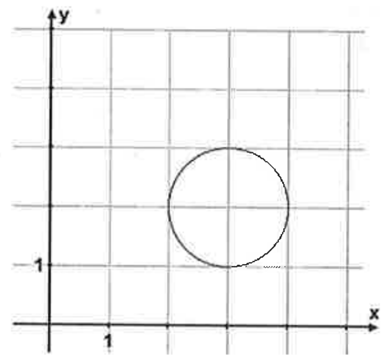
***Exercises***

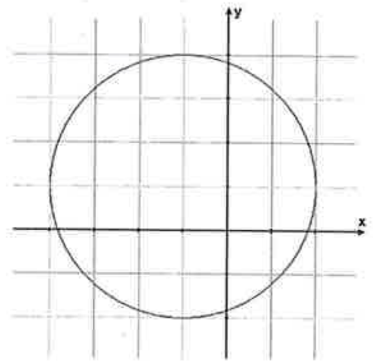
**A**

**6201** Describe in words all points that satisfy the equation
a) $x^2+y^2=9$ b) $(x-4)^2+y^2=9$
c) $x^2+(y-2)^2=5$ d) $(x-7)^2+(y-3)^2=25$

**6202** Find the equation of the following circles:
a)




b)




**6203** Sketch a graph of the following circles
a) $x^2+(y-4)^2=4$ b) $(x+2)^2+(y-1)^2=1$

**B**

**6204** Decide where the following circles intersect the coordinate axes.
a) $x^2+(y-2)^2=4$ b) $(x-9)^2+(y-13)^2=196$

**6205** Describe in words all points that satisfy the equation
a) $(x-\frac{1}{2})^2+(y+\frac{3}{2})^2=3$ b) $(x+1,2)^2+(y-3,2)^2=6,25$
c) $x^2+y^2=-1$ d) $y=\sqrt{1-x^2}$

**6206** How many points do the following circles have in common?
a) $(x-3)^2+y^2=4$ and $(x+2)^2+y^2=9$
b) $(x+2)^2+(y-3)^2=4$ and $(x+1)^2+(y+1)^2=1$
c) $(x+3)^2+(y-1)^2=36$ and $x^2+(y-1)^2=4$

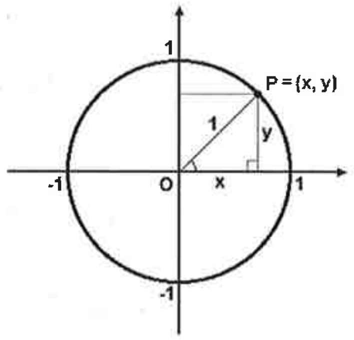
**6207** The point $(a, b)$ lies on the circle $(x+4)^2+(y-2)^2=169$.
a) Find $b$ , if $a=1$ .
b) Find $a$ , if $b=-3$ .
c) Find the largest possible value of $b$.

**6208** The centre of a circle is located at the point (3, 6). Give the equation of the circle, if
a) the point (–1, 3) lies on the circle.
b) the circle is tangent to the $x$ axis.
c) the circle is tangent to the $y$ axis.

## 6.3 The unit circle

The trigonometric ratios $\sin v$, $\cos v$ and $\tan v$ have been defined for angles in a right-angled triangle, which means that the angle $v$ must be between $0°$ and $90°$. However, we shall see that it also makes sense to define sine, cosine and tangent for angles greater than $90°$. To do this we use a *unit circle*, a circle centred at the origin with radius 1.

If $P=(x,\ y)$ is a point on this circle and $v$ is the angle between the line $OP$ and the $x$ axis, then from the right-angled triangle inside the circle we get




$$\sin v = \frac{y}{1} = y \quad , \quad \text{and}$$

$$\cos v = \frac{x}{1} = x$$

This means that $\cos v$ the $x$-coordinate of $P$ and $\sin v$ is the $y$-coordinate.

From this it seems reasonable to make the following definitions:

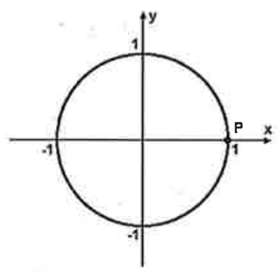
> As the point $P$ moves anywhere on the unit circle,
>
> $\cos v$ is the $x$-coordinate of $P$
> $\sin v$ is the $y$-coordinate of $P$
>
> where $v$ is the angle made by the line $OP$ with the positive $x$ axis.

---

*Example 1*

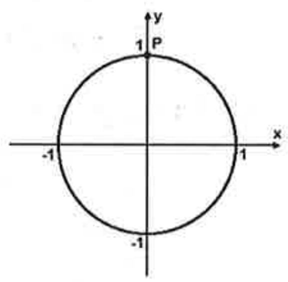
The values of $\sin 0°$, $\cos 0°$, $\sin 90°$ and $\cos 90°$ can now easily be found by using the unit circle:

If $P=(1,\ 0)$, then $v=0°$ and by definition




$$\sin 0° = 0 \quad , \quad \text{and}$$
$$\cos 0° = 1$$

If $P=(0,\ 1)$, then $v=90°$ and




$$\sin 90° = 1$$
$$\cos 90° = 0$$

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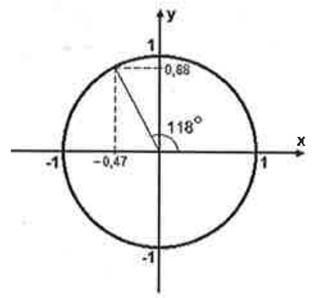
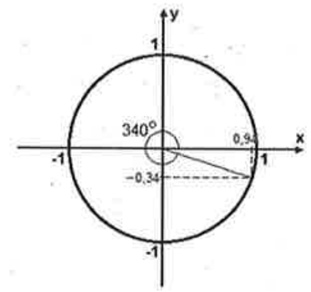
**Sine, cosine and tangent for any angle**

Using a calculator we can now find and understand the trigonometric ratios for any angle:

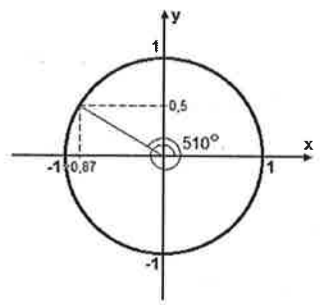



$\sin 118° \approx 0{,}88$
$\cos 118° \approx -0{,}47$

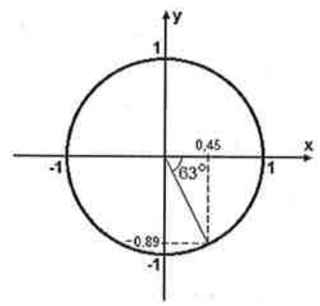



$\sin 340° \approx -0{,}34$
$\cos 340° \approx 0{,}94$

We can also allow $v$ to be more than $360°$ or less than $0°$:




$\sin 510° = 0{,}5$
$\cos 510° \approx -0{,}87$




$\sin(-63°) \approx -0{,}89$
$\cos(-63°) \approx 0{,}45$

From the definitions of $\sin v$ and $\cos v$ and the unit circle we can now state the following important facts:

For all points on the unit circle, $-1 \le x \le 1$ and $-1 \le y \le 1$, so

- $-1 \le \cos v \le 1$ and $-1 \le \sin v \le 1$

The equation of the unit circle is $x^2 + y^2 = 1$, which leads to the formula

- $(\cos v)^2 + (\sin v)^2 = 1$ for all angles $v$.
  (usually written $\cos^2 v + \sin^2 v = 1$)

The tangent ratio is defined as

- $\tan v = \dfrac{y}{x} = \dfrac{\sin v}{\cos v}$

---

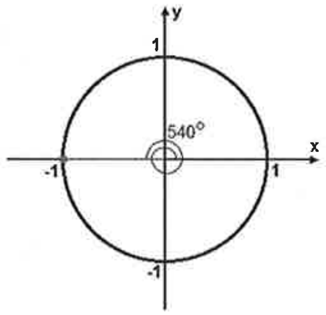
*Example 2*

Find $\sin v$, $\cos v$ and $\tan v$ , if $v$ is

a) 540°   b) −90°

*Solution*:

a)



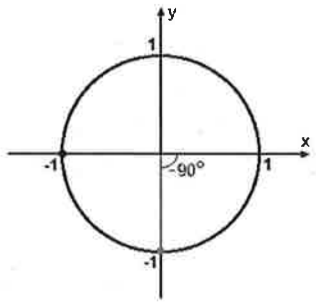

$\sin 540° = 0$

$\cos 540° = -1$

$\tan 540° = \dfrac{\sin 540°}{\cos 540°} = \dfrac{0}{-1} = 0$

( $\tan v = 0$ wherever $\sin v = 0$ )

b)




$\sin(-90°) = -1$

$\cos(-90°) = 0$

$\tan(-90°) = \dfrac{\sin(-90°)}{\cos(-90°)} = \dfrac{-1}{0} =$

$=$ *undefined*

( $\tan v$ is undefined wherever $\cos v = 0$ )

---

**Periodicity of sine and cosine**

As we move the point $P$ along the unit circle we will after each revolution return to a previous location. For example, when $v = 30°$, $P$ will be located in the same position as when $v = 30° + 360° = 390°$.

It is easy to see that, for any angle $v$,

$$\sin v = \sin(v + 360°) \quad \text{, and}$$
$$\cos v = \cos(v + 360°)$$

Since this will hold for any number of revolutions, forward or backwards, we can write more generally

$$\sin v = \sin(v + n \cdot 360°) \quad \text{, and}$$
$$\cos v = \cos(v + n \cdot 360°) \quad \text{, for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

This *periodic* feature is an important property of the trigonometric functions.

---

*Example 3*

Find the lowest possible positive angle $v$ that gives the same sines or cosines as

a) $\sin 740°$   b) $\cos(-600°)$

*Solution*:

a) $\sin 740° = \sin(740° - 2 \cdot 360°) = \sin 20°$

b) $\cos(-600°) = \cos(-600° + 2 \cdot 360°) = \cos 120°$

---

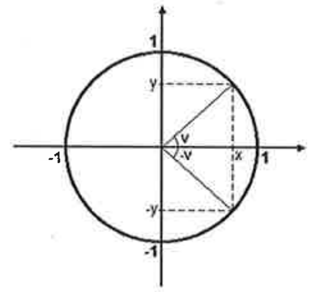




**Trigonometric identities**

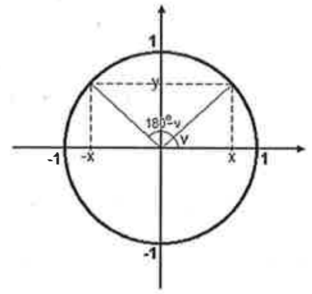
We saw above that for example $\sin v = \sin(v + 360°)$. Such an equality (equation) is called an *identity*, since the two expressions are equal for all values of $v$. There are many similar relationships in trigonometry, of which some can be found directly through symmetry in the unit circle. Here we shall investigate a few.

The angles $v$ and $-v$ are located symmetrically on opposite sides of the $x$-axis, and are related in the following way:



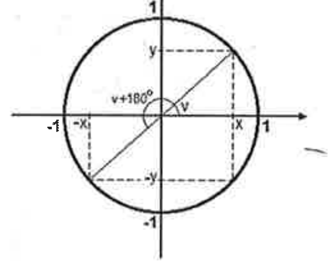

$$\cos(-v) = \cos v$$
$$\sin(-v) = -\sin v$$

Similarly, the angles $(180° - v)$ and $v$ are located on opposite sides of the $y$-axis, and are related in the following way:




$$\sin(180° - v) = \sin v$$
$$\cos(180° - v) = -\cos v$$

The relationships between $(180° + v)$ and $v$ are:




$$\sin(180° + v) = -\sin v$$
$$\cos(180° + v) = -\cos v$$

Investigations of the angles $(v + 90°)$ and $(90° - v)$ are left as exercises.

---

*Example 4*

If $\sin v = 0{,}4$ , find

a) $\sin(-v)$   b) $\sin(180° - v)$

c) $\sin(v + 180°)$   d) $\sin(v + 360°)$

*Solution*:

a) $\sin(-v) = -\sin v = -0{,}4$   b) $\sin(180° - v) = \sin v = 0{,}4$

c) $\sin(v + 180°) = -\sin v = -0{,}4$   d) $\sin(v + 360°) = \sin v = 0{,}4$

---

*Example 5*
Simplify
a) $\cos(-v)+\cos(180°-v)$  b) $\sin(v+540°)-\sin(-v)$

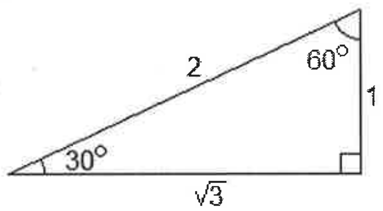
*Solution*:
a) $\cos(-v)=\cos v$ , and $\cos(180°-v)=-\cos v$ , which gives
$\cos(-v)+\cos(180°-v)=\cos v+(-\cos v)=0$

b) $\sin(v+540°)=\sin(v+180°+360°)=\sin(v+180°)=-\sin v$ , and
$\sin(-v)=-\sin v$ , so
$\sin(v+540°)-\sin(-v)=-\sin v-(-\sin v)=0$

---

**Exact values of sin$v$ and cos$v$**
For some angles it is easy to find the exact values of $\sin v$ and $\cos v$. This can be useful when working with trigonometric expressions.

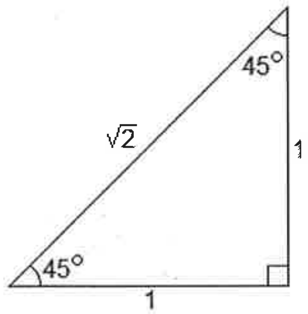
If we start with an equilateral triangle with side length 2 and cut it in half, we get half an equilateral triangle with sides 2, 1 and (by Pythagoras' Theorem) $\sqrt{3}$. Since the angles are $90°, 60°$ and $30°$, we can now obtain the following exact values:




$\sin 30° = \frac{1}{2}$ , $\cos 30° = \frac{\sqrt{3}}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$ , $\cos 60° = \frac{1}{2}$

In a similar way we can use half a square to get the values of $\sin 45°$ and $\cos 45°$:




$\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

Now we can construct a table of some exact values of $\sin v$, $\cos v$ and $\tan v$:

| $v$ | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| $\sin v$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos v$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan v$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | *undef.* |

From this table and the unit circle it is now easy to get exact values for some angles greater than 90° or less than 0°.





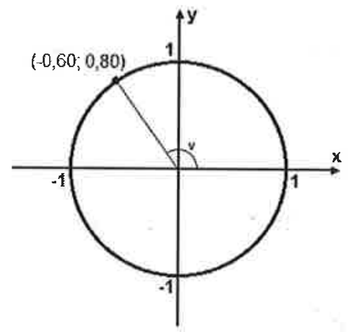
***Exercises***

**A**

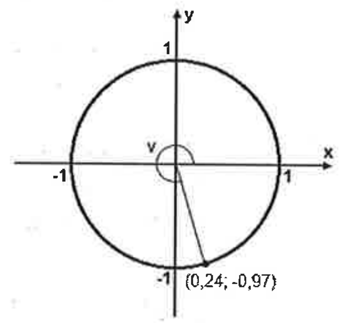
**6301** Draw a unit circle and mark the points on the circle corresponding to the angle
a) 45°  b) 270°
c) −450°  d) 900°

**6302** Without using your calculator, find
a) $\sin 137°$ if $\sin 43° \approx 0{,}6820$  b) $\cos 143°$ if $\cos 37° \approx 0{,}7986$
c) $\sin 59°$ if $\sin 121° \approx 0{,}8572$  d) $\cos 125°$ if $\cos 235° \approx -0{,}574$

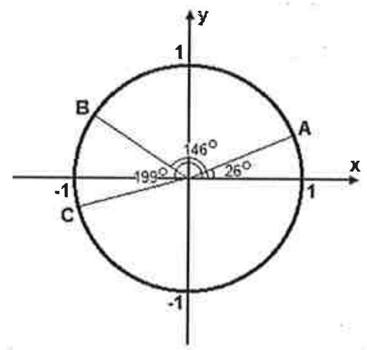
**6303** Find $\sin v$, $\cos v$, $\tan v$ and $v$:
a)



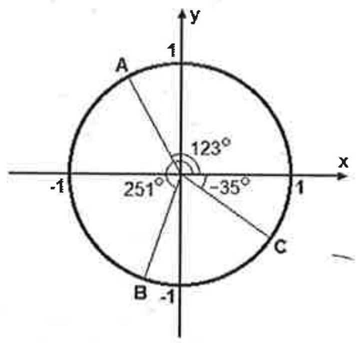

b)




**6304** For each angle illustrated, give the coordinates of A, B and C:
a)




b)




**6305** Find the smallest positive angle giving the same position in the unit circle as
a) 743°  b) −638°
c) 3000°  d) −1270°

**6306** Find two angles, for which
a) $\sin v = 1$  b) $\sin v = -1$
c) $\cos v = 1$  d) $\cos v = -1$

**6307** If $\cos v = 0{,}6$ , find without a calculator
a) $\cos(-v)$  b) $\cos(v+180°)$

**6308** Find the exact value of
a) $\sin 150°$  b) $\cos(-120°)$

**6309** The angle $v$ is in the first quadrant ($0° \le v \le 90°$). Use the trigonometric formula $\sin^2 v + \cos^2 v = 1$ to find $\cos v$, if
a) $\sin v = 0{,}8$  b) $\sin v = 0{,}2$

**B**

**6310** The angle $v$ is in the third quadrant ($180° \le v \le 270°$). Use the trigonometric formula $\sin^2 v + \cos^2 v = 1$ to find $\sin v$, if
a) $\cos v = -0,6$  b) $\cos v = -0,707$

**6311** Use the unit circle to find the relationships between $\sin(v+90°)$, $\cos(v+90°)$, $\sin v$ and $\cos v$.

**6312** Use the unit circle to find the relationships between $\sin(90°-v)$, $\cos(90°-v)$, $\sin v$ and $\cos v$.

**6313** Find the exact values of
a) $\sin 135°$ and $\cos 135°$  b) $\sin 225°$ and $\cos 225°$
c) $\sin 300°$ and $\cos 300°$  d) $\sin(-210)°$ and $\cos(-210)°$

**6314** a) If $\sin v = \frac{1}{3}$, use a triangle to find the exact value of $\cos v$, if $0° \le v \le 90°$.
b) Find the exact values of $\sin(v+180°)$ and $\cos(v+180°)$.

**6315** Express in terms of the angle $v$:
a) $\sin(v+180°) + 2\sin(-v) - \sin(180°-v)$
b) $\cos(180°-v) + \cos v + \cos(v+180°)$

**6316** Show that
a) $\tan(v+180°) = \tan v$  b) $\tan(-v) = -\tan v$





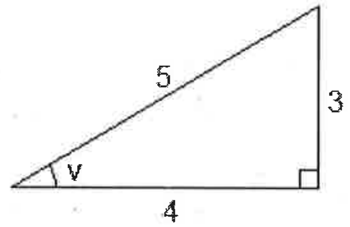
## 6.4 Trigonometric equations

**The equation $\sin x = a$**
You should already be familiar with how to find the angle $v$ if the sine (or cosine) value is given.
In the right-angled triangle given, we know the sides but not the angle $v$.
However, since we know that

$$\sin v = \frac{3}{5} = 0,6$$

we can find $v$ with the calculator, using

$$v = \sin^{-1} 0,6 \approx 37°$$

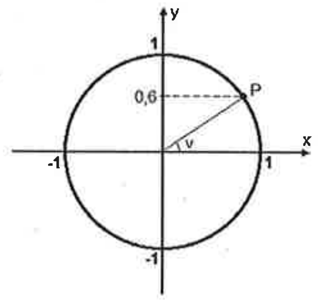



In this triangle, this is the only possible solution, but in general the equation $\sin v = 0,6$ has infinitely many solutions, since from the previous section we know that

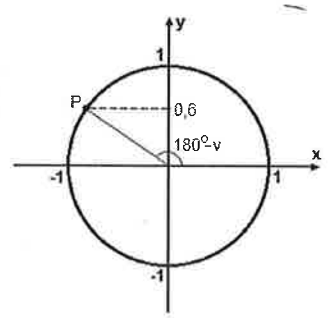
(1) $\sin v = \sin(180° - v)$ , and
(2) $\sin v = \sin(v + 360°)$

Let us look at this in the unit circle:

Given that $\sin v = 0,6$, we want to find a point $P$ that gives the correct angle $v$. Working backwards and using the calculator gives $v \approx 37°$, which will solve the equation.



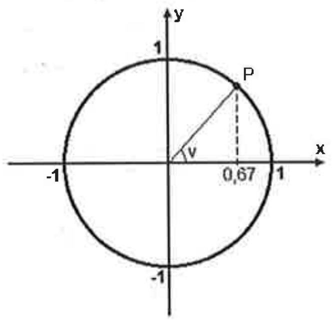

But there is another possibility.
From (2) above we know that $P$ could be located at the opposite side of the $y$-axis, which will give the angle $180° - 37° = 143°$, another solution to the equation.




Furthermore, from these two points, we may rotate infinitely many revolutions as long as we stop at either of these positions, giving infinitely many solutions to the equation. In this way, for example the angles $37° + 360° = 397°$, $37° + 5 \cdot 360° = 1837°$, $143° + 360° = 503°$ and $143° - 360° = -217°$ are all solutions to the equation $\sin v = 0,6$.

However, in this chapter we will only allow solutions to be less than $360°$, since most of the problems here are taken from geometry.

**The equation $\cos v = a$**
Similarly, if the value of $\cos v$ is given, we can find $v$ with a calculator, using the $\cos^{-1}$ function.




If $\cos v = 0,67$
we get
$v = \cos^{-1} 0,67 \approx 48°$

As with $\sin v$ there is however another solution. Remember that

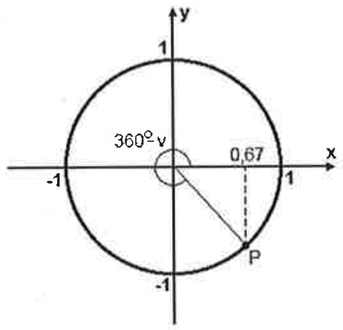
(3) $\cos v = \cos(-v)$

From this we understand that $P$ could be located on the opposite side of the $x$-axis, giving the same value of $\cos v$.
The angle of this solution is by symmetry

$-v = -48°$

or, if a positive solution is required,

$360° - 48° = 312°$




If we rotate in either direction we may of course find infinitely many solutions to the equation $\cos v = 0,67$.

The method of solving equations of the type $\sin v = a$ (or $\cos v = a$) within $0° \le v \le 360°$ is, if the angle $v$ is not obvious, as follows:

1) Use a calculator to find first possible $v$, as
   $v_1 = \sin^{-1} a$ (or $v_1 = \cos^{-1} a$)

2) The other possible angle $v_2$ is found by taking
   $v_2 = 180° - v_1$ for $\sin v = a$, or
   $v_2 = 360° - v_1$ for $\cos v = a$

---

*Example 1*
Solve the following equations, where $0° \le x \le 360°$.
a) $\sin x = 0,83$  b) $2\sin x + 3 = 4$

*Solution*:
a) $x_1 = \sin^{-1} 0,83 \approx 56°$, or
$x_2 = 180° - 56° = 124°$

b) $2\sin x = 4 - 3 = 1$
$\sin x = \frac{1}{2} = 0,5$
$x_1 = \sin^{-1} 0,5 = 30°$, or
$x_2 = 180° - 30° = 150°$

---





*Example 2*
Solve the following equations, where $0° \le x \le 360°$.
a) $\cos x = 0,85$  b) $\cos x = -0,6$

*Solution*:
a) $x_1 = \cos^{-1} 0,85 \approx 32°$
$x_2 = 360° - 32° = 328°$

b) $x_1 = \cos^{-1}(-0,6) \approx 127°$
$x_2 = 360° - 127° = 233°$

---

***Exercises***

**A**

**6401** Find an angle in the first quadrant ($0° \le v \le 90°$) for which
a) $\sin v = 0,31$  b) $\sin v = 0,99$
c) $\cos v = 0,78$  d) $\cos v = 0,83$

**6402** Solve the following equations, where ($0° \le x \le 180°$):
a) $\cos x = 0,559$  b) $\cos x = -0,951$
c) $\sin x = 0,122$  d) $\sin x = 0,978$

**6403** Solve the following equations, where ($0° \le x \le 360°$):
a) $\sin x = -0,35$  b) $\cos x = -0,940$
c) $\cos x = 1,105$  d) $\sin x = -0,766$

**6404** Solve without using a calculator the following equations ($0° \le x \le 360°$):
a) $\sin x = \frac{1}{2}$  b) $\sin x = 0$
c) $\cos x = -1$  d) $\cos x = \frac{1}{2}$
e) $\sin x = -1$  f) $\cos x = 0$

**6405** Solve the following equations, where $0° \le x \le 360°$:
a) $3\sin x - 2 = 1$  b) $1 - 2\cos x = 0$

**B**

**6406** Solve without using a calculator the following equations ($0° \le x \le 360°$):
a) $\cos x = -\frac{1}{2}$  b) $\sin x = \frac{\sqrt{3}}{2}$
c) $\cos x = -\frac{\sqrt{3}}{2}$  d) $\sin x = -\frac{1}{2}$

**6407** Use a unit circle to find out the number of solutions between 180° and 270° to the following equations:
a) $\sin x = -0,99$  b) $\cos x = -0,99$
c) $\cos x = 0,33$  d) $\sin x = 0,78$

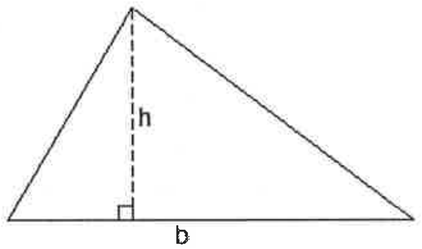
**6408** Solve the following equations, where $0° \le x \le 180°$:
a) $\sin(x + 90°) = -1$  b) $4\sin^2 x = 1$
c) $\sin 2x = 1$  d) $\sin x \cdot \cos x = 0$

## 6.5 Triangle formulas

In geometrical problems we often meet triangles of many kind. These triangles may not be right-angled, why the trigonometric ratios *sine*, *cosine* and *tangent* cannot be used directly.
However, by splitting any triangle into right-angled triangles we can work out some formulas that allow us to use trigonometry even if the triangle is non right angled.

**The area of a triangle**
The area of any triangle is $\frac{b \cdot h}{2}$, where $b$ is the base (one side) of the triangle and $h$ is the height from the opposite top, *perpendicular* (at right angle) to the base.

To calculate the area of the triangle we need to know one side of the triangle and the corresponding height.



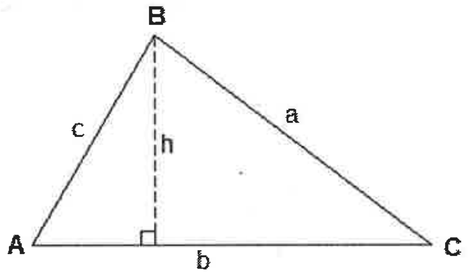

Using trigonometry we can develop an alternative formula for the area, where we don't need to know the height of the triangle.

Since

$$\sin A = \frac{h}{c} \quad \rightarrow \quad h = c \cdot \sin A$$

we get the area $T$ of the triangle:

$$T = \frac{b \cdot h}{2} = \frac{bc \sin A}{2}$$

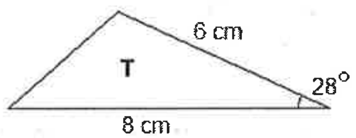



where $b$ and $c$ are two sides of a triangle and $A$ is the angle between them.

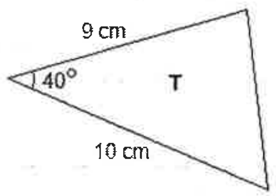
(Starting with angle $B$ or $C$, we might as well write $T = \dfrac{ac \sin B}{2}$ or $T = \dfrac{ab \sin C}{2}$)

---

*Example 1*
Find the area of the following triangles
a)




b)




*Solution*:

a) $T = \dfrac{6 \cdot 8 \cdot \sin 28°}{2} = 24 \sin 28° \approx$
$\approx 11{,}3 \ (\text{cm}^2)$

b) $T = \dfrac{9 \cdot 10 \cdot \sin 40°}{2} = 45 \sin 40° \approx$
$\approx 28{,}9 \ (\text{cm}^2)$

---

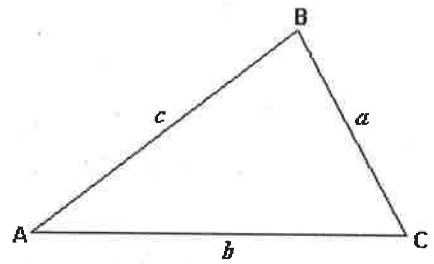




**The sine rule**
The *sine rule* is a set of equations which connects the sides of a triangle with the sines of the angles of the triangle. The triangle does not have to be right-angled for the sine rule to be used.

Using the area formula, the area of a triangle may be calculated in three ways:

$$T = \frac{bc \sin A}{2} = \frac{ac \sin B}{2} = \frac{ab \sin C}{2}$$




Multiplying by 2 gives:

$$bc \sin A = ac \sin B = ab \sin C$$

Dividing by $abc$ gives *the sine rule*:

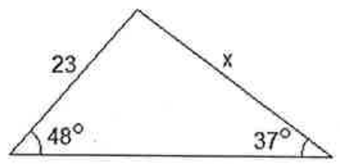
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

(which also may be written $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$)

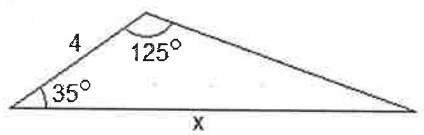
The sine rule is often used to solve problems involving triangles, given either
- two angles and one side
- two sides and one angle

---

*Example 2*
Find the value of $x$.
a)




b)




*Solution*:

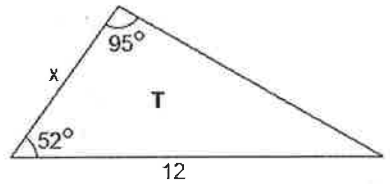
a) $\dfrac{x}{\sin 48°} = \dfrac{23}{\sin 37°} \quad \rightarrow \quad x = \dfrac{23 \cdot \sin 48°}{\sin 37°} \approx 28{,}4$

b) The third angle is $180° - 125° - 35° = 20°$

$\dfrac{x}{\sin 125°} = \dfrac{4}{\sin 20°} \quad \rightarrow \quad x = \dfrac{4 \cdot \sin 125°}{\sin 20°} \approx 9{,}6$

---

*Example 3*
Find the area $T$ of the triangle.




*Solution*:
The third angle is $180° - 95° - 52° = 33°$

According to the sine rule,

$$\frac{x}{\sin 33°} = \frac{12}{\sin 95°} \quad \rightarrow \quad x = \frac{12 \cdot \sin 33°}{\sin 95°} \approx 6,56$$

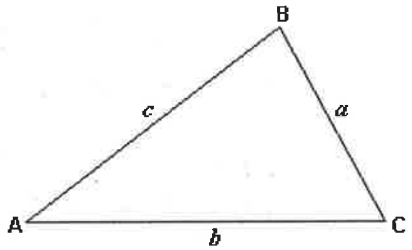
The area formula then gives

$$T = \frac{12 \cdot 6,56 \cdot \sin 52°}{2} \approx 31$$

---

**The ambiguous case**
The problem of finding angles using the sine rule can be complicated because there may be two possible answers. Depending on the circumstances we might get two possible triangles, one triangle, or it may be impossible to draw any triangle from the information given.

We will here illustrate the different cases that may occur.
In each case we use the standard notation of a triangle (see fig.)



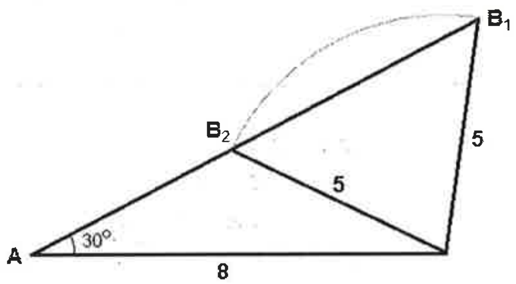

***Case 1***
Given $b = 8$, $a = 5$, $A = 30°$
Using the sine rule to find $B$:

$$\frac{\sin B}{8} = \frac{\sin 30°}{5}$$

$$\sin B = \frac{8 \cdot \sin 30°}{5} = 0,8$$

$$B_1 = \sin^{-1} 0,8 \approx 53°$$

or, $B_2 = 180° - B_1 = 127°$



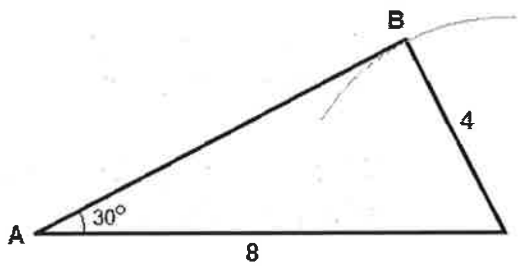

The equation $\sin C = 0,8$ has two solutions less than 180°, which both are relevant and give two possible triangles.

***Case 2***
Given $b = 8$, $a = 4$, $A = 30°$
Finding $B$:

$$\frac{\sin B}{8} = \frac{\sin 30°}{4}$$

$$\sin B = \frac{8 \cdot \sin 30°}{4} = 1$$

$$B = \sin^{-1} 1 = 90°$$




The equation $\sin B = 1$ has only one solution, $B = 90°$. Only one triangle is therefore possible.

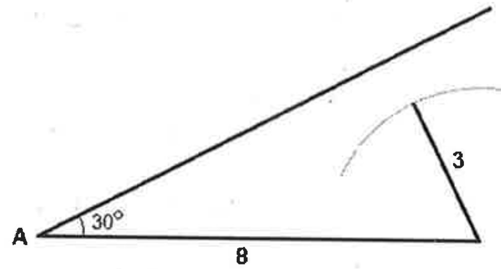




***Case 3***
Given $b = 8$, $a = 3$, $A = 30°$
Finding $B$:

$$\frac{\sin B}{8} = \frac{\sin 30°}{3}$$

$$\sin B = \frac{8 \cdot \sin 30°}{3} \approx 1,33$$




The equation $\sin B = 1,33$ has no solution, $B = 90°$. There is no possible solution for this given data.

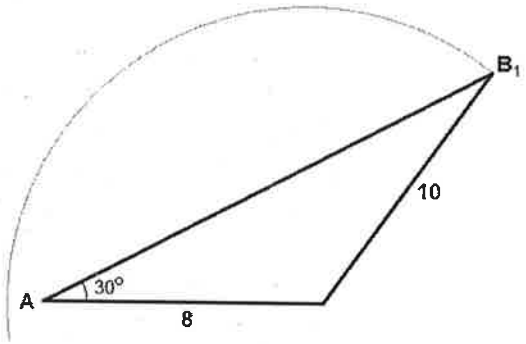
***Case 4***
Given $b = 8$, $a = 10$, $A = 30°$
Finding $B$:

$$\frac{\sin B}{8} = \frac{\sin 30°}{10}$$

$$\sin B = \frac{8 \cdot \sin 30°}{10} \approx 0,4$$

$$B_1 = \sin^{-1} 0,4 \approx 24°$$

or, $B_2 = 180° - B_1 = 156°$




However, only $B = 24°$ is possible (if $A = 30°$, $B$ cannot be 156° because $30° + 156° > 180°$)

The conclusion here is that, when using the sine rule, one must examine the situation carefully.

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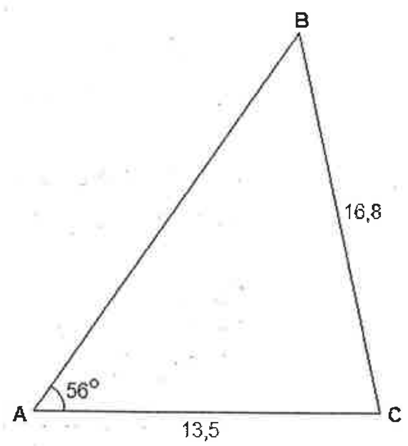
*Example 4*
In the triangle $ABC$, the angle $A = 56°$, $AC = 16,8$ cm and $AB = 13,5$ m.
Find the measure of angle $B$.




*Solution*:

$$\frac{\sin B}{13,5} = \frac{\sin 56°}{16,8}$$

$$\sin B = \frac{13,5 \cdot \sin 56°}{16,8} \approx 0,666$$

$$B_1 = \sin^{-1} 0,666 \approx 42°$$

$$B_2 = 180° - B_1 = 138°$$

We can reject $B_2 = 138°$ as $138° + 56° > 180°$ which is impossible, so the only solution is $B = 42°$.

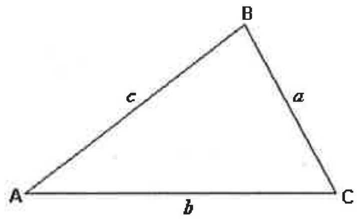
---

**The cosine rule**
For any non-right-angled triangle, the sine rule may help to find an angle or side, depending on the given data. However, there are two cases where the sine rule will not help:

- Three sides are given, but no angle
- Two sides and the angle in between are given

The *cosine rule* involves the three sides of a triangle and one of the angles, and will help us in these situations:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$




(Note that if $A = 90°$, then $\cos A = 0$ and $a^2 = b^2 + c^2 - 2bc \cdot \cos A$ reduces to $a^2 = b^2 + c^2$, which is Pythagoras' Theorem)

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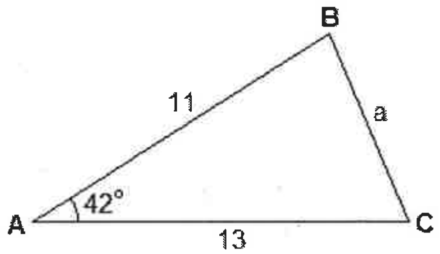
*Example 5*
Find the length of $a$ in the triangle.

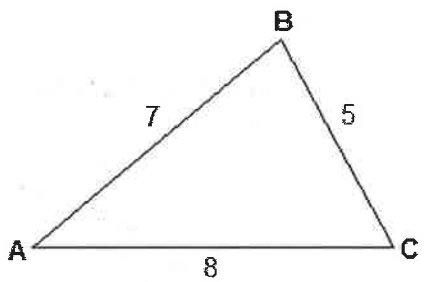



*Solution*:
By the cosine rule:

$$a^2 = 11^2 + 13^2 - 2 \cdot 11 \cdot 13 \cdot \cos 42° \approx$$
$$\approx 77{,}46$$
$$a = \sqrt{77{,}46} \approx 8{,}8$$

---

*Example 6*
Find the angle $A$ in the triangle.




*Solution*:
By the cosine rule:

$$5^2 = 7^2 + 8^2 - 2 \cdot 7 \cdot 8 \cdot \cos A$$
$$25 = 113 - 112\cos A$$
$$112\cos A = 113 - 25 = 88$$
$$\cos A = \frac{88}{112} \approx 0{,}786$$
$$A = \cos^{-1} 0{,}786 \approx 38°$$

---





**Proof of the cosine rule**
By drawing $h$ perpendicular to $b$ we obtain two right-angled triangles.

Applying Pythagoras' Theorem on both right-angled triangles:

$$a^2 = h^2 + (b-x)^2 = h^2 + b^2 - 2bx + x^2$$
$$c^2 = h^2 + x^2 \quad \to \quad h^2 = c^2 - x^2$$ , substitute this into the equation above:

$$a^2 = c^2 - x^2 + b^2 - 2bx + x^2 = b^2 + c^2 - 2bx$$

Since $\cos A = \dfrac{x}{c} \quad \to \quad x = c \cdot \cos A$ , we finally get:

$$\boxed{a^2 = b^2 + c^2 - 2bc \cdot \cos A}$$

Starting from a different angle, the cosine rule may also be written

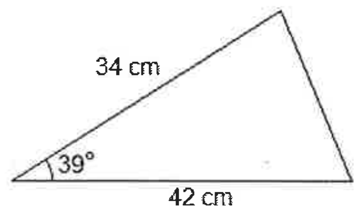
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B \quad \text{, or} \quad c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

---

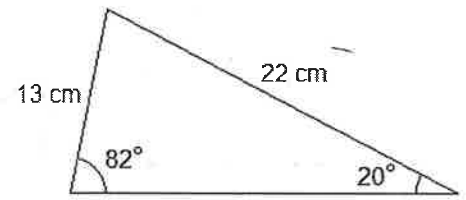
***Exercises***

**A**

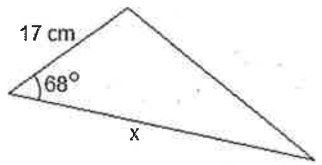
**6501** Find the area of the triangles
a)



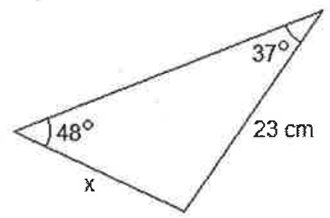

b)



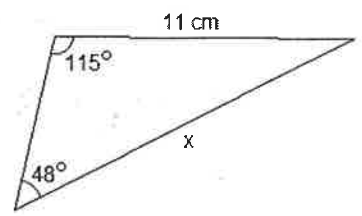

**6502** If the triangle has area 150 cm$^2$, find the value of $x$.




**6503** Find the length $x$ :
a)




b)

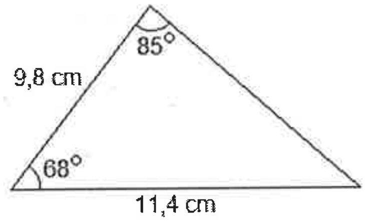



**6504** A parallelogram has side lengths 4 cm and 6 cm. One angle inside the parallelogram measures 52°. Find the area of the parallelogram.

**6505** A rhombus has sides of length 12 cm and an angle of 72°. Find its area.

**6506** Triangle ABC has angle $B = 40°$, $b = 8$ cm and $c = 11$ cm. Find the two possible values for angle $C$.

**6507** Is it possible to have a triangle with measurements as shown? Explain!

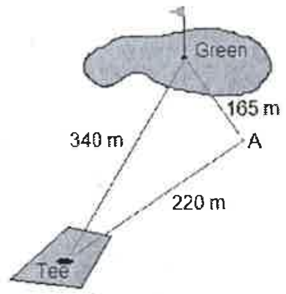



**6508** In the triangle ABC, $a = 5$ cm, $b = 7$ cm and $c = 10$ cm. Find the angles $A$, $B$ and $C$

**6509** Find the distance from P to R, if PQ = 200 m, QR = 180 m and the angle PQR is 108°.


B

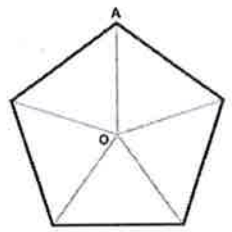
**6510** A golfer played his tee shot a distance of 220 m to a point $A$. He then played a 165 m six iron to reach the green. If the distance from tee to green is 340 m, determine the number of degrees the golfer was off line with his tee shot.




**6511** For a triangle ABC, decide if the following measurements are possible, and if so, find all angles.
a) $A = 50°$, $b = 20$ cm and $a = 15$ cm
b) $B = 35°$, $b = 10$ cm and $a = 8$ cm

**6512** Find the area of a regular hexagon with sides of length 12 cm.

**6513** A regular pentagonal garden plot has its centre at $O$ and an area of 338 $m^2$. Find the distance $OA$.




**6514** From the foot of a building I have to look upwards at an angle of 22° to sight the top of a tree. From the top of the building, 150 metres above ground level, I have to look down at an angle of 50° below the horizontal to sight the tree top.
a) How high is the tree?
b) How far from the building is this tree?

**6515** Two observation posts are 12 km apart at A and B. A third observation post C s located such that angle CAB is 42° and angle CBA s 67°. Find the distance of C from both A and B.





## Chapter exercises 6

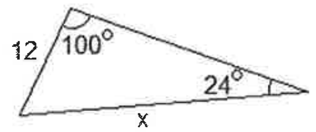
A

**1** Evaluate
a) $\sin 134°$ b) $\cos(-65°)$
c) $\sin 490°$ d) $\cos 800°$

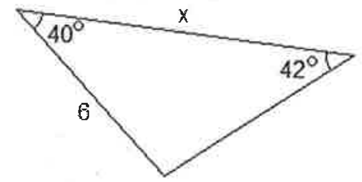
**2** Draw a right-angled triangle where
a) $\tan v = \frac{2}{7}$ b) $\sin v = \frac{4}{5}$

**3** Find the exact value of
a) $\sin 270°$ b) $\cos(-180)°$
c) $\sin 30° \cdot \cos 60°$ d) $(\sin 135°)^2$

**4** Find the value of $x$:
a)




b)



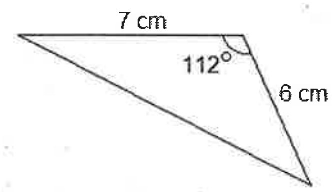

**5** For which angles between 0° and 360° is it true that
a) $\sin v = \sin 40°$ b) $\cos v = \cos 170°$

**6** Describe the circle with equation
a) $x^2 + (y-2)^2 = 9$ b) $(x+1)^2 + (y-7)^2 = 100$

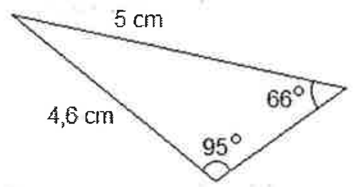
**7** Solve the following equations, if $0° \le v \le 360°$:
a) $\sin v = 0{,}97$ b) $\cos v = -0{,}94$

**8** A football goal is 5 metres wide. When a player is 26 m from one goal post and 23 m from the other, he shoots for goal. What is the angle of view of the goal that the player sees?

**9** Find the area of the triangle:
a)




b)




**10** Put the following in order of size, without using a calculator.
$a = \sin 24°$, $b = \cos 100°$, $c = \sin 165°$ and $d = \sin 185°$

**11** Aaron is standing 22 m from Betty and 18 m from Caesar. The angle of view from Aaron to Betty and Caesar is 38°. What is the distance between Betty and Caesar?

**12** Find all sides and angles of a triangle where two sides are 14 cm and 23 cm and the angle between these two sides is 73°.

**13** Draw two different triangles where $A = 35°$, $a = 10$ and $b = 15$.

**B**

**14** Find the exact value of $\cos v$, if $0° \le v \le 90°$ and

a) $\sin v = \frac{2}{3}$

b) $\tan v = 1$

**15** Find the exact value of

a) $\sin(-210)°$

b) $\sin 315°$

c) $\cos 765°$

d) $\cos(-120°)$

**16** Find the exact value of

a) $\sin 120° \cdot \cos 30°$

b) $\cos 210° \cdot \sin(-60°)$

c) $\sin^2(-225°)$

d) $\sin^2 34° + \cos^2 34°$

**17** For which angles between 0° and 360° is it true that

a) $\sin v = \cos v$

b) $\cos(-v) = \cos v$

c) $\sin v = \cos 230°$

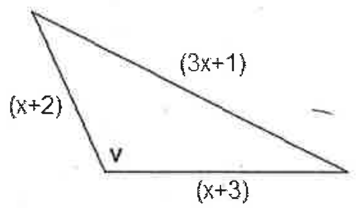
d) $\cos v = \sin(-176°)$

**18** Victoria and Daniel start walking at one point A. They each walk in a straight line at an angle of 120° to each other. Daniel walks at 6 km/h and Victoria walks at 8 km/h. How far apart are they after 45 minutes?

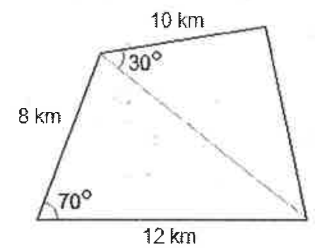
**19** Given that $\cos v = -\frac{1}{5}$,

a) Find $x$

b) Find the exact area of the triangle




**20** Glen and Gwen are considering buying a sheep farm. A surveyor has supplied them with the given sketch.
Find the area of the property, giving your answer in $km^2$.




**21** Find the measure of

a) angle ABC

b) angle ACB