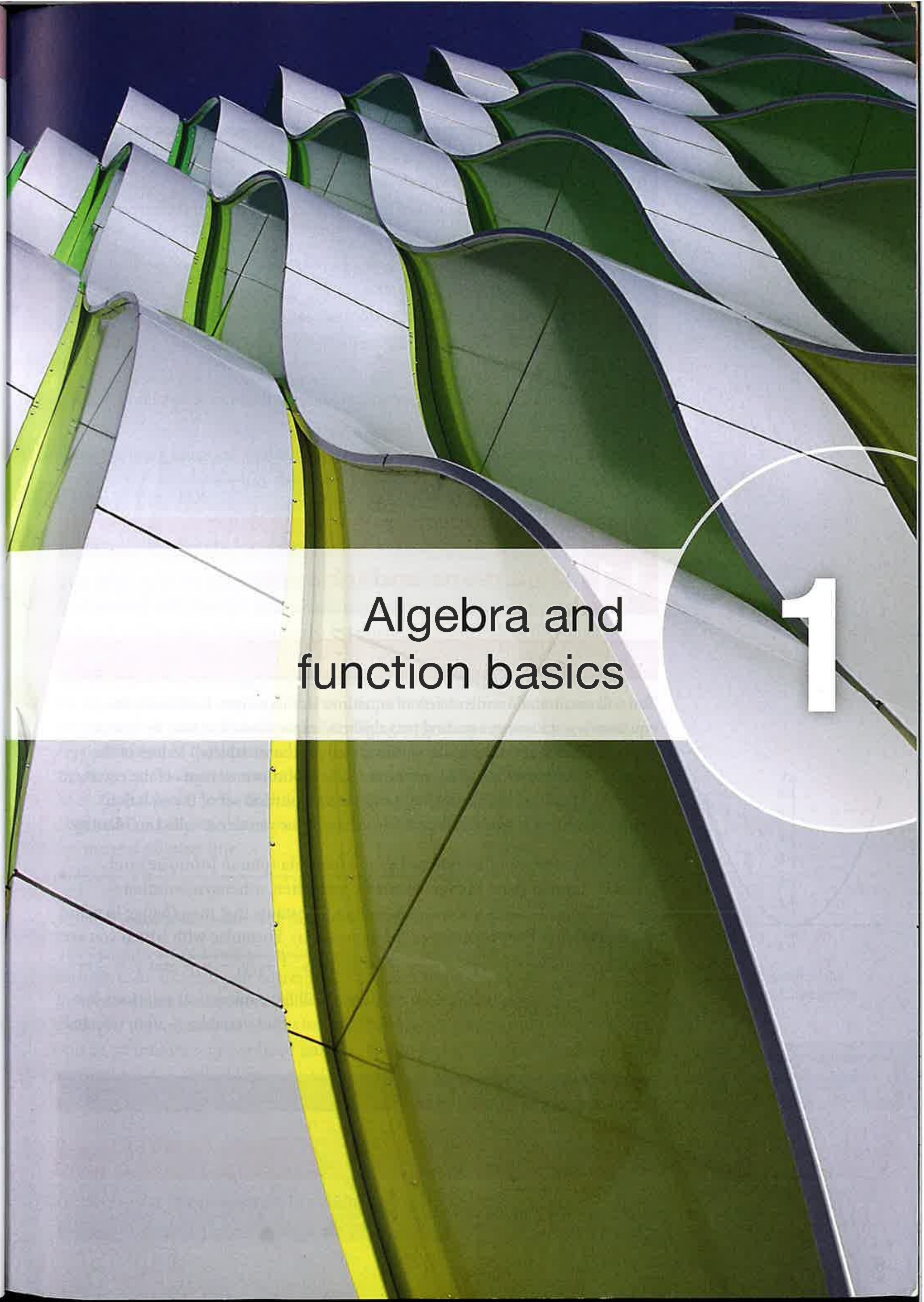
How to use this book

This book is designed to be read by you – the student. It is very important that you read this book carefully. We have strived to write a readable book – and we hope that your teacher will routinely give you reading assignments from this textbook, thus giving you valuable time for productive explanations and discussions in the classroom. Developing your ability to read and understand mathematical explanations will prove to be valuable to your long-term intellectual development, while also helping you to comprehend mathematical ideas and acquire vital skills to be successful in the *Analysis and Approaches* HL course. Your goal should be understanding, not just remembering. You should always read a chapter section thoroughly before attempting any of the exercises at the end of the section.

Our aim is to support genuine inquiry into mathematical concepts while maintaining a coherent and engaging approach. We have included material to help you gain insight into appropriate and wise use of your GDC and an appreciation of the importance of proof as an essential skill in mathematics. We endeavoured to write clear and thorough explanations supported by suitable worked examples, with the overall goal of presenting sound mathematics with sufficient rigour and detail at a level appropriate for a student of HL mathematics.

For over 10 years, we have been writing successful textbooks for IB mathematics courses. During that time, we have received many useful comments from both teachers and students. If you have suggestions for improving this textbook, please feel free to write to us at globalschools@pearson.com. We wish you all the best in your mathematical endeavours.

Ibrahim Wazir and Tim Garry



1 Algebra and function basics

Learning objectives

By the end of this chapter, you should be familiar with...

- different forms of equations of lines and their gradients and intercepts
- parallel and perpendicular lines
- different methods to solve a system of linear equations (maximum of three equations in three unknowns)
- the concept of a function and its domain, range and graph
- mathematical notation for functions
- composite functions
- characteristics of an inverse function and finding the inverse function $f^{-1}(x)$
- self-inverse functions
- transformations of graphs and composite transformations of graphs
- the graphs of the functions $y = |f(x)|$, $y = f(|x|)$ and $y = \frac{1}{f(x)}$

1.1 Equations and formulae

Equations, identities and formulae

You will encounter a wide variety of equations in this course. Essentially, an equation is a statement equating two algebraic expressions that may be true or false depending upon the value(s) substituted for the variable(s). Values of the variables that make the equation true are called **solutions** or **roots** of the equation. All of the solutions to an equation comprise the **solution set** of the equation. An equation that is true for all possible values of the variable is called an **identity**.

Many equations are often referred to as a **formula** (plural: formulae) and typically contain more than one variable and, often, other symbols that represent specific constants or **parameters** (constants that may change in value but do not alter the properties of the expression). Formulae with which you are familiar include: $A = \pi r^2$, $d = rt$, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $V = \frac{4}{3}\pi r^3$.

Whereas most equations that we encounter will have numerical solutions, we can solve a formula for one variable in terms of other variables – often referred to as changing the subject of a formula.

Example 1.1

(a) Solve for b in the formula $a^2 + b^2 = c^2$

(b) Solve for l in the formula $T = 2\pi\sqrt{\frac{l}{g}}$

(c) Solve for R in the formula $M = \frac{nR}{R + r}$

Solution

(a) $a^2 + b^2 = c^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b = \pm\sqrt{c^2 - a^2}$

If b is a length then $b = \sqrt{c^2 - a^2}$

(b) $T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow \sqrt{\frac{l}{g}} = \frac{T}{2\pi} \Rightarrow \frac{l}{g} = \frac{T^2}{4\pi^2} \Rightarrow l = \frac{T^2 g}{4\pi^2}$

(c) $I = \frac{nR}{R + r} \Rightarrow I(R + r) = nR \Rightarrow IR + Ir = nR$

$\Rightarrow IR - nR = -Ir \Rightarrow R(I - n) = -Ir$

$\Rightarrow R = \frac{Ir}{n - I}$

Note that factorisation was required in solving for R in part (c).

Equations and graphs

Two important characteristics of any equation are the number of variables (unknowns) and the type of algebraic expressions it contains (e.g. polynomials, rational expressions, trigonometric, exponential). Nearly all of the equations in this course will have either one or two variables. In this chapter we will only discuss equations with algebraic expressions that are polynomials. Solutions for equations with a single variable consist of individual numbers that can be graphed as points on a number line. The **graph** of an equation is a visual representation of the equation's solution set. For example, the solution set of the one-variable equation containing quadratic and linear polynomials $x^2 = 2x + 8$ is $x \in \{-2, 4\}$. The graph of this one-variable equation (Figure 1.1) is depicted on a one-dimensional coordinate system, i.e. the real number line.

Figure 1.1 Graph of the solution set for the equation $x^2 = 2x + 8$

The solution set of a two-variable equation will be an **ordered pair** of numbers. An ordered pair corresponds to a location indicated by a point on a two-dimensional coordinate system, i.e. a **coordinate plane**. For example, the solution set of the two-variable **quadratic equation** $y = x^2$ will be an infinite set of ordered pairs (x, y) that satisfy the equation. Four ordered pairs in the solution set are shown in red in Figure 1.2. The graph of all the ordered pairs in the solution set forms a curve as shown in blue.

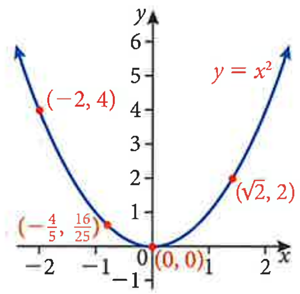


Figure 1.2 Graph of the solution set of the equation $y = x^2$

Quadratic equations will be covered in detail in Chapter 2.

Equations of lines

A one-variable **linear equation** in x can always be written in the form $ax + b = 0$, with $a \neq 0$, and it will have exactly one solution, namely $x = -\frac{b}{a}$. An example

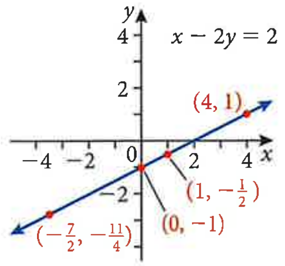


Figure 1.3 The graph of $x - 2y = 2$

of a two-variable **linear equation in *x* and *y*** is $x - 2y = 2$. The graph of this equation's solution set (an infinite set of ordered pairs) is a **line** (Figure 1.3).

The **slope** or **gradient** m, of a non-vertical line is defined by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}}$$

Because division by zero is undefined, the slope of a vertical line is undefined.

Using the two points $\left(1, -\frac{1}{2}\right)$ and $(4, 1)$ we compute the slope of the line with equation $x - 2y = 2$ to be $m = \dfrac{1 - \left(-\frac{1}{2}\right)}{4 - 1} = \dfrac{\frac{3}{2}}{3} = \dfrac{1}{2}$

If we solve for y we can rewrite the equation in the form $y = \frac{1}{2}x - 1$

Note that the coefficient of x is the slope of the line and the constant term is the y-coordinate of the point at which the line intersects the y-axis, that is, the y-intercept. There are several forms for writing linear equations.

general form	$ax + by + c = 0$	every line has an equation in this form if both a and $b \neq 0$
slope-intercept form	$y = mx + c$	m is the slope; $(0, c)$ is the y-intercept
point-slope form	$y - y_1 = m(x - x_1)$	m is the slope; (x_1, y_1) is a known point on the line
horizontal line	$y = c$	slope is zero; $(0, c)$ is the y-intercept
vertical line	$x = c$	slope is undefined; unless the line is the y-axis, no y-intercept

Table 1.1 Forms for equations of lines

Most problems involving linear equations and their graphs fall into two categories: (1) given an equation, determine its graph; and (2) given a graph, or some information about it, find its equation.

For lines, the first type of problem is often best solved by using the slope-intercept form. For the second type of problem, the point-slope form is usually most useful.

Example 1.2

Without using a GDC, sketch the line that is the graph of each linear equation written in general form.

(a) $5x + 3y - 6 = 0$ (b) $y - 4 = 0$ (c) $x + 3 = 0$

Solution

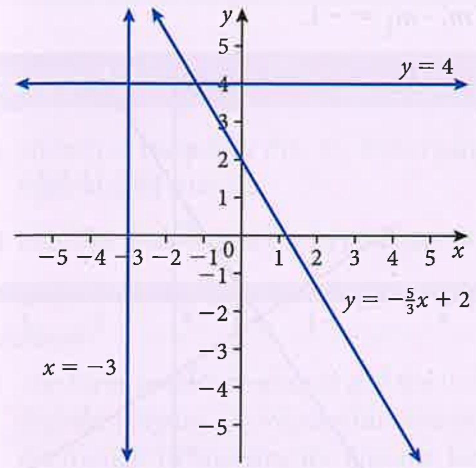
(a) Solve for y to write the equation in slope-intercept form.

$$5x + 3y - 6 = 0 \Rightarrow 3y = -5x + 6 \Rightarrow y = -\frac{5}{3}x + 2$$

The line has a y-intercept of $(0, 2)$ and a slope of $-\frac{5}{3}$

(b) The equation $y - 4 = 0$ is equivalent to $y = 4$, the graph of which is a horizontal line with a y-intercept of $(0, 4)$

(c) The equation $x + 3 = 0$ is equivalent to $x = -3$, the graph of which is a vertical line with no y-intercept; but, it has an x-intercept of $(-3, 0)$



Example 1.3

(a) Find the equation of the line that passes through the point $(3, 31)$ and has a slope of 12. Write the equation in slope-intercept form.

(b) Find the linear equation in C and F knowing that $C = 10$ when $F = 50$, and $C = 100$ when $F = 212$. Solve for F in terms of C.

Solution

(a) Substitute $x_1 = 3$, $y_1 = 31$ and $m = 12$ into the point-slope form:

$$y - y_1 = m(x - x_1) \Rightarrow y - 31 = 12(x - 3) \Rightarrow y = 12x - 36 + 31$$
$$\Rightarrow y = 12x - 5$$

(b) The two points, ordered pairs (C, F), that are known to be on the line are $(10, 50)$ and $(100, 212)$. The variable C corresponds to the x variable and F corresponds to y in the definitions and forms stated above.

The slope of the line is $m = \dfrac{F_2 - F_1}{C_2 - C_1} = \dfrac{212 - 50}{100 - 10} = \dfrac{162}{90} = \dfrac{9}{5}$

Choose one of the points on the line, say $(10, 50)$, and substitute it and the slope into the point-slope form:

$$F - F_1 = m(C - C_1) \Rightarrow F - 50 = \frac{9}{5}(C - 10) \Rightarrow F = \frac{9}{5}C - 18 + 50$$
$$\Rightarrow F = \frac{9}{5}C + 32$$

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular. The two lines shown in Figure 1.4 suggest that two distinct non-vertical lines are **parallel** if and only if their slopes are equal, $m_1 = m_2$.

The two lines shown in Figure 1.5 suggest that two non-vertical lines are perpendicular if and only if their slopes are negative reciprocals – that is, $m_1 = -\frac{1}{m_2}$, which is equivalent to $m_1 \cdot m_2 = -1$.

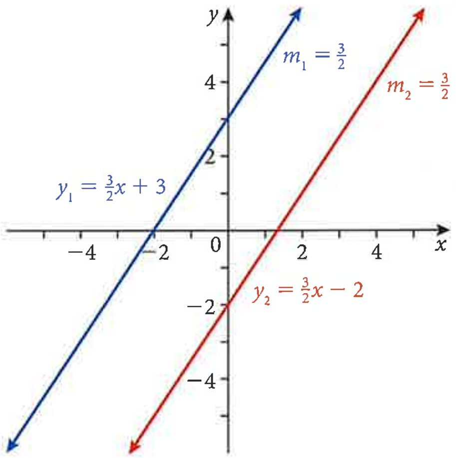


Figure 1.4 Parallel lines

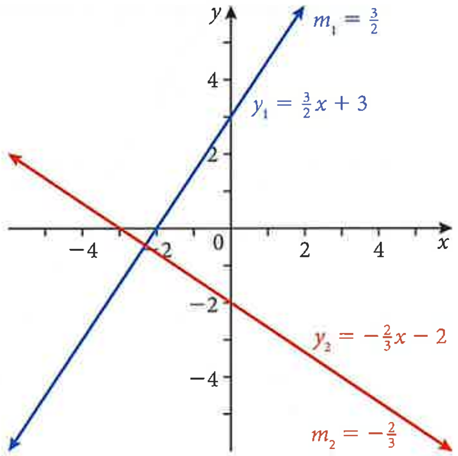


Figure 1.5 Perpendicular lines

Distances and midpoints

Recall that absolute value is used to define the **distance** (always non-negative) between two points on the real number line. The distance between the points A and B on the real number line is $|B - A|$, which is equivalent to $|A - B|$.

The points A and B are the endpoints of a line segment that is denoted with the notation $[AB]$ and the length of the line segment is denoted AB. In Figure 1.6, the distance between A and B is $AB = |4 - (-2)| = |-2 - 4| = 6$.

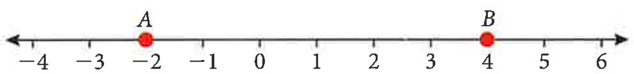


Figure 1.6 The length of the line segment $[AB]$ is denoted by AB

We can find the distance between two general points (x_1, y_1) and (x_2, y_2) on a coordinate plane using the definition for distance on a number line and Pythagoras' theorem. For the points (x_1, y_1) and (x_2, y_2), the horizontal distance between them is $|x_1 - x_2|$ and the vertical distance is $|y_1 - y_2|$. As illustrated in Figure 1.7, these distances are the lengths of two legs of a right-angled triangle whose hypotenuse is the distance between the points. If d represents the distance between (x_1, y_1) and (x_2, y_2), then by Pythagoras' theorem $d^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$. Because the square of any number is positive, the absolute value is not necessary to give us the **distance formula** for two-dimensional coordinates.

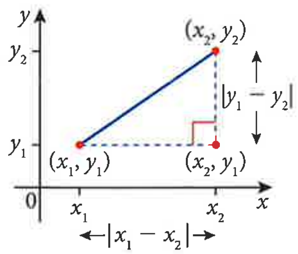


Figure 1.7 Distance between two points on a coordinate plane

The distance d between the two points (x_1, y_1) and (x_2, y_2) in the coordinate plane is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

The coordinates of the **midpoint** of a line segment are the average values of the corresponding coordinates of the two endpoints.

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example 1.4

(a) Show that the points $P(1, 2)$, $Q(3, 1)$ and $R(4, 8)$ are the vertices of a right-angled triangle.

(b) Find the midpoint of the hypotenuse of the triangle PQR.

Solution

(a) The three points are plotted and the line segments joining them are drawn in Figure 1.8. We can find the exact lengths of the three sides of the triangle by applying the distance formula.

$$PQ = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$QR = \sqrt{(3-4)^2 + (1-8)^2} = \sqrt{1+49} = \sqrt{50}$$

$$PR = \sqrt{(1-4)^2 + (2-8)^2} = \sqrt{9+36} = \sqrt{45}$$

$(PQ)^2 + (PR)^2 = (QR)^2$ because $(\sqrt{5})^2 + (\sqrt{45})^2 = 5 + 45 = 50 = (\sqrt{50})^2$

The lengths of the three sides of the triangle satisfy Pythagoras' theorem, confirming that the triangle is a right-angled triangle.

(b) QR is the hypotenuse. Let the midpoint of QR be point M. Using the midpoint formula, $M = \left(\frac{3+4}{2}, \frac{1+8}{2}\right) = \left(\frac{7}{2}, \frac{9}{2}\right)$. This point is plotted in Figure 1.8

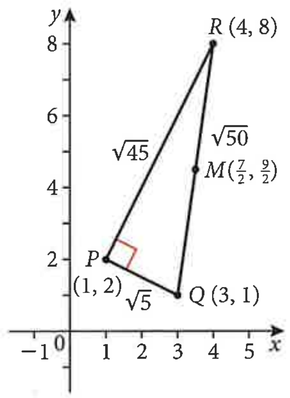


Figure 1.8 Diagram for Example 1.4

Example 1.5

Find x such that the distance between points $(1, 2)$ and $(x, -10)$ is 13 units.

Solution

$$d = 13 = \sqrt{(x-1)^2 + (-10-2)^2} \Rightarrow 13^2 = (x-1)^2 + (-12)^2$$

$$\Rightarrow 169 = x^2 - 2x + 1 + 144 \Rightarrow x^2 - 2x - 24 = 0$$

$$\Rightarrow (x-6)(x+4) = 0 \Rightarrow x - 6 = 0 \text{ or } x + 4 = 0$$

$$\Rightarrow x = 6 \text{ or } x = -4 \text{ (see Figure 1.9)}$$

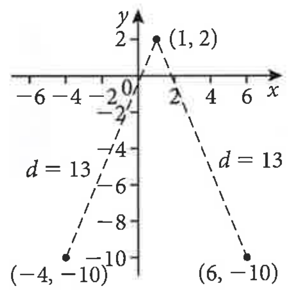


Figure 1.9 Graph for Example 1.5 showing the two points that are 13 units from (1, 2)

Systems of linear equations

Many problems involve sets of equations with several variables, rather than just a single equation with one or two variables. Such a set of equations is often called a set, or system, of **simultaneous equations** because we find the values for the variables that solve all of the equations simultaneously. In this section, we only consider sets of simultaneous equations containing linear equations; that is, **systems of linear equations**. We will take a brief look at four solution methods:

- graphical method (with technology)
- substitution method
- elimination method
- technology (without graphing)

We will first consider systems of two linear equations in two unknowns and then extend our methods to systems of three linear equations in three unknowns.

Although the systems of two equations in two unknowns considered in this chapter contain only linear equations, it is important to mention that the graphical and substitution methods are effective for solving systems of two equations in two unknowns where not all of the equations are linear, e.g. one linear and one quadratic equation.

Graphical method

The graph of each equation in a **system of two linear equations in two unknowns** is a line. The graphical interpretation of such a system of equations corresponds to determining the point or points that lie on both lines. Two lines in a coordinate plane relate to one another in only one of three ways: (1) intersect at exactly one point, (2) intersect at all points on each line (i.e. the lines are identical, or coincident), or (3) the two lines do not intersect (i.e. the lines are parallel). These three possibilities are illustrated in Figure 1.10.

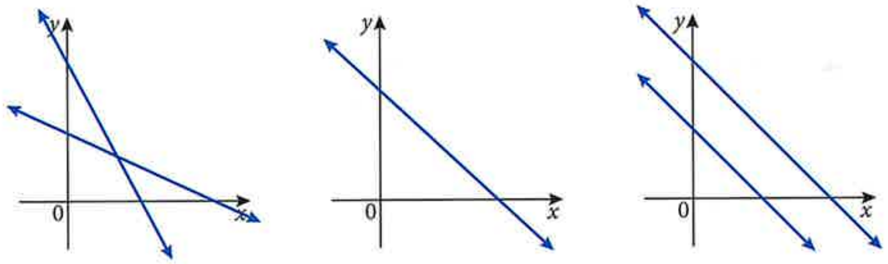


Figure 1.10 Possible relationship between two lines in a coordinate plane

Although a graphical approach to solving a system of linear equations provides a helpful visual picture of the number and location of solutions, it can be tedious and inaccurate if done by hand. The graphical method is far more efficient and accurate when performed on a GDC.

A system of equations that has no solution is **inconsistent**. A system of equations that has at least one solution is **consistent**.

Example 1.6

Use the graphical features of a GDC to solve each system of linear equations.

(a) $2x + 3y = 6$
$2x - y = -10$

(b) $7x - 5y = 20$
$3x + y = 2$

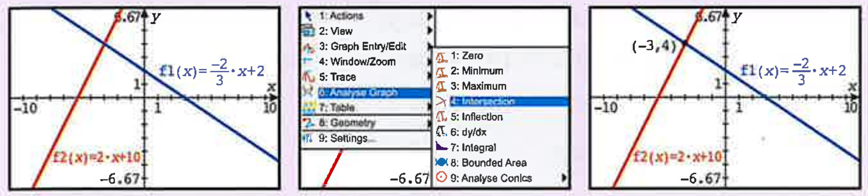
Solution

(a) Rewrite each equation in slope-intercept form, i.e. $y = mx + c$. This is a necessity if we use our GDC and is also very useful for graphing by hand.

$$2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2$$

and

$$2x - y = -10 \Rightarrow y = 2x + 10$$

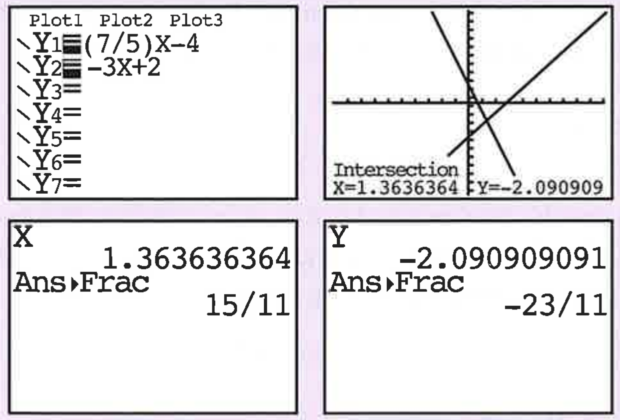


The intersection point and solution to the system of equations is $x = -3$ and $y = 4$, or $(-3, 4)$.

(b) $7x - 5y = 20 \Rightarrow 5y = 7x - 20 \Rightarrow y = \frac{7}{5}x - 4$

and

$3x + y = 2 \Rightarrow y = -3x + 2$



The solution to the system of equations is $x = \frac{15}{11}$ and $y = -\frac{23}{11}$, or $\left(\frac{15}{11}, -\frac{23}{11}\right)$.

Elimination method

To solve a system using the **elimination method**, we try to combine the two linear equations using sums or differences in order to eliminate one of the variables. Before combining the equations, we often need to multiply one or both of the equations by a suitable constant to produce coefficients for one of the variables that are equal (then subtract the equations), or that differ only in sign (then add the equations).

Example 1.7

Use the elimination method to solve each system of linear equations.

(a) $5x + 3y = 9$
$2x - 4y = 14$

(b) $x - 2y = 7$
$2x - 4y = 5$

Solution

(a) We can obtain coefficients for y that differ only in sign by multiplying the first equation by 4 and the second equation by 3. Then add the equations to eliminate the variable y.

$$\begin{aligned} 5x + 3y &= 9 \rightarrow & 20x + 12y &= 36 \\ 2x - 4y &= 14 \rightarrow & 6x - 12y &= 42 \\ & & 26x &= 78 \\ & & x &= \frac{78}{26} \\ & & x &= 3 \end{aligned}$$

By substituting the value of 3 for x in either of the original equations we can solve for y.

$$5x + 3y = 9 \Rightarrow 5(3) + 3y = 9 \Rightarrow 3y = -6 \Rightarrow y = -2$$

The solution is $(3, -2)$

(b) To obtain coefficients for x that are equal, multiply the first equation by 2 and then subtract the equations to eliminate the variable x.

$$\begin{aligned} x - 2y &= 7 \rightarrow & 2x - 4y &= 14 \\ 2x - 4y &= 5 \rightarrow & 2x - 4y &= 5 \\ & & 0 &= 9 \end{aligned}$$

Because it is not possible for 0 to equal 9, there is no solution. The lines on the graphs of the two equations are parallel. To confirm this, we can rewrite each of the equations in the form $y = mx + c$.

$x - 2y = 7 \Rightarrow 2y = x - 7 \Rightarrow y = \frac{1}{2}x - \frac{7}{2}$ and

$2x - 4y = 5 \Rightarrow 4y = 2x - 5 \Rightarrow y = \frac{1}{2}x - \frac{5}{2}$

Both equations have a slope of $\frac{1}{2}$, but different y-intercepts.

Therefore, the lines are parallel. This confirms that this system of linear equations has no solution.

Substitution method

The algebraic method that can be applied effectively to the widest variety of simultaneous equations, including non-linear equations, is the substitution method. Using this method, we choose one of the equations and solve for one of the variables in terms of the other variable. We then substitute this expression into the other equation to produce an equation with only one variable, which we can solve directly.

Example 1.8

Use the substitution method to solve each system of linear equations.

(a) $3x - y = -9$
$6x + 2y = 2$

(b) $-2x + 6y = 4$
$3x - 9y = -6$

Solution

(a) Solve for y in the top equation:

$$3x - y = -9 \Rightarrow y = 3x + 9$$

Substitute $3x + 9$ for y in the bottom equation:

$$6x + 2(3x + 9) = 2 \Rightarrow 6x + 6x + 18 = 2 \Rightarrow 12x = -16$$
$$\Rightarrow x = -\frac{16}{12} = -\frac{4}{3}$$

Now substitute $-\frac{4}{3}$ for x in either equation to solve for y.

$$3\left(-\frac{4}{3}\right) - y = -9 \Rightarrow y = -4 + 9 \Rightarrow y = 5$$

The solution is $x = -\frac{4}{3}, y = 5$; or $\left(-\frac{4}{3}, 5\right)$

(b) Solve for x in the top equation: $-2x + 6y = 4 \Rightarrow 2x = 6y - 4$

$$\Rightarrow x = 3y - 2$$

Substitute $3y - 2$ for x in the bottom equation:

$$3(3y - 2) - 9y = -6 \Rightarrow 9y - 6 - 9y = -6 \Rightarrow 0 = 0$$

This resulting equation $0 = 0$ is true for any values of x and y. The two equations are equivalent, and their graphs will produce identical lines, that is coincident lines. Therefore, the solution set consists of all points (x, y) lying on the line $-2x + 6y = 4$ $\left(\text{or } y = \frac{1}{3}x + \frac{2}{3}\right)$

Technology

As shown in Example 1.6 (a), we can use our GDC to graph the two lines whose equations constitute a system of two linear equations and then apply an 'intersection' command to find the ordered pair that solves the system. Alternatively, your GDC should have a simultaneous equation solver (or systems of equations solver) that can be used to solve a system of linear equations without graphing.

If you use a graph to answer a question on an IB mathematics exam, you must include a clear and well-labelled sketch in your working. Thus, on an exam there is often more effort involved in solving a system of linear equations by graphing with your GDC compared to solving the system using a simultaneous equation solver (or systems of linear equations solver) on your GDC.

Example 1.9

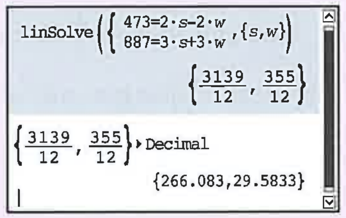
When flying in calm air (with no headwind or tailwind) with its propellers rotating at a particular rate, a small aeroplane has a speed of s kilometres per hour. With the propeller rotating at the same rate, the aeroplane flies 473 km in 2 hours as it flies against a headwind, and 887 km in 3 hours flying with the same wind as a tailwind. Find s and find the speed of the wind.

Solution

Let w represent the speed of the wind. Applying distance = rate × time gives the following two equations:

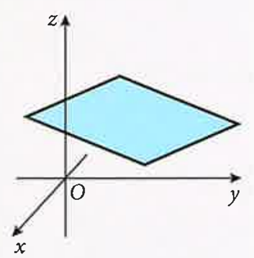
$$\begin{cases} 473 = 2(s - w) \\ 887 = 3(s + w) \end{cases} \Rightarrow \begin{cases} 473 = 2s - 2w \\ 887 = 3s + 3w \end{cases}$$

This system of equations can be solved with a simultaneous equation solver on a GDC.



Therefore, $s \approx 266\,\text{km}\,\text{h}^{-1}$ and the wind speed is $29.6\,\text{km}\,\text{h}^{-1}$, accurate to 3 significant figures.

The graph of a linear equation in x, y and z, $ax + by + cz = d$, is a plane in a three-dimensional coordinate system. 'Linear' refers to the fact that the equation is a first-degree equation.



As with solving systems of two linear equations in two unknowns, **systems of three linear equations in three unknowns** also have three possible outcomes: a unique solution, an infinite number of solutions or no solution (inconsistent system). However, the graph of a linear equation in three unknowns is not a line, but a plane in three-dimensional space (covered in detail in Chapter 9). Hence, using a graphical approach is impractical for solving systems of three linear equations in three unknowns. Applying an elimination method is feasible, but the algebra required can be very tedious. So, in general, when solving a system of three linear equations in three unknowns it is best to use your GDC's simultaneous equation solver if a GDC is allowed, and an elimination method when a GDC is not allowed. The elimination method demonstrated in Example 1.10 (a) using **elementary row operations** is essentially the same as the elimination method shown earlier for systems of two linear equations, where equations are combined with each other, or each side of an equation is multiplied by a constant in order to isolate one of the variables. However, when using row operations to solve systems with more than two equations, we normally do not write the variables if all of the equations have the variables in the same order and the constant is on the other side of the equals sign.

Example 1.10

Solve the following system of equations in two ways:

$$\begin{cases} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ 2x + 4y + 6z = 6 \end{cases}$$

(a) using row operations (an elimination method), and

(b) using a simultaneous equation solver on a GDC.

Solution

(a) There are three types of elementary row operations: (1) multiply any row by a non-zero real number; (2) interchange any two rows; (3) add or subtract a multiple of one row to another row. The objective is to apply a sequence of row operations so that both the x and y terms are eliminated from one equation and the x term is eliminated from another of the three equations.

The sequence of row operations needed to produce one equation with both the x and y terms eliminated and another one of the three equations with the x term eliminated is not unique. Two people can carry out a different sequence of row operations while obtaining the correct solution.

1. Subtract row 1 from row 3:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 0 & 3 & 7 & 4 \end{array}\right]$$

2. Multiply row 2 by 2 and subtract it from row 1:

$$\left[\begin{array}{ccc|c} 0 & -5 & -5 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 3 & 7 & 4 \end{array}\right]$$

3. Multiply row 3 by 5:

$$\left[\begin{array}{ccc|c} 0 & -5 & -5 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 15 & 35 & 20 \end{array}\right]$$

4. Multiply row 1 by 3 and add to row 3:

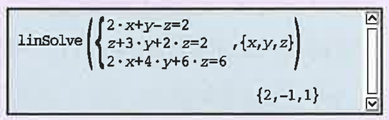
$$\left[\begin{array}{ccc|c} 0 & -5 & -5 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 0 & 20 & 20 \end{array}\right]$$

At this stage, we have eliminated both of the x and y terms from one equation and eliminated the x term from another equation. We can simplify the resulting system by one last step where we multiply row 1 by $-\frac{1}{5}$ and multiply row 3 by $\frac{1}{20}$, producing $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$ which is equivalent to the original system of equations. Writing each equation (row) with the variables we can solve for z directly, and for x and y by a process of substitution.

$$\begin{cases} 0x + y + z = 0 \\ x + 3y + 2z = 1 \\ 0x + 0y + z = 1 \end{cases}$$

Clearly, $z = 1$. Substituting this into the first equation gives $y + 1 = 0$. Thus, $y = -1$. And, substituting into the second equation gives $x + 3(-1) + 2(1) = 1$. So, $x = 2$. Therefore, the system has the unique solution of $x = 2$, $y = -1$, $z = 1$.

(b) Using the systems of linear equations solver on a GDC.



Don't be confused when row operations on a system of three equations produces something like

$$\left[\begin{array}{ccc|c} 2 & -1 & 5 & 1 \\ 1 & 4 & -2 & -3 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

The bottom equation is $3z = 0$. So, $z = 0$. Thus, the system has the unique solution $\left(\frac{1}{8}, -\frac{3}{4}, 0\right)$

When using row operations to solve a system of three linear equations that has **no solution**, the equation (row) in which x and y have been eliminated will be an equation that is false for all values of z. For example, if after a sequence of row operations, we obtain $\left[\begin{array}{ccc|c} 2 & -1 & 5 & 1 \\ 1 & 4 & -2 & -3 \\ 0 & 0 & 0 & 2 \end{array}\right]$ then the bottom equation is $0x + 0y + 0z = 2$ which is always false. Hence, the system has no solution.

If row operations on a system produces $\left[\begin{array}{ccc|c} 2 & -1 & 5 & 1 \\ 1 & 4 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$, then there are **infinite solutions** because the bottom equation is always true.

Exercise 1.1

1. Solve for the indicated variable in each formula.

(a) $m(h - x) = n$, solve for x

(b) $v = \sqrt{ab - t}$, solve for a

(c) $A = \frac{h}{2}(b_1 + b_2)$, solve for b_1

(d) $A = \frac{1}{2}r^2\theta$, solve for r

(e) $\frac{f}{g} = \frac{h}{k}$, solve for k

(f) $at = x - bt$, solve for t

(g) $V = \frac{1}{3}\pi r^3 h$, solve for r

(h) $F = \frac{g}{m_1 k + m_2 k}$, solve for k

2. Find the equation of the line that passes through the two given points. Write the line in slope-intercept form ($y = mx + c$), if possible.

(a) $(-9, 1)$ and $(3, -7)$

(b) $(3, -4)$ and $(10, -4)$

(c) $(-12, -9)$ and $(4, 11)$

(d) $\left(\frac{7}{3}, -\frac{1}{2}\right)$ and $\left(\frac{7}{3}, \frac{5}{2}\right)$

(e) Find the equation of the line that passes through the point $(7, -17)$ and is parallel to the line with equation $4x + y - 3 = 0$. Write the line in slope-intercept form ($y = mx + c$).

(f) Find the equation of the line that passes through the point $\left(-5, \frac{11}{2}\right)$ and is perpendicular to the line with equation $2x - 5y - 35 = 0$. Write the line in slope-intercept form ($y = mx + c$).

3. Find the exact distance between each pair of points and then find the midpoint of the line segment joining the two points.

(a) $(-4, 10)$ and $(4, -5)$

(b) $(-1, 2)$ and $(5, 4)$

(c) $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{5}{2}, \frac{4}{3}\right)$

(d) $(12, 2)$ and $(-10, 9)$

4. Find the value(s) of k so that the distance between the points is 5 units.

(a) $(5, -1)$ and $(k, 2)$

(b) $(-2, -7)$ and $(1, k)$

5. Show that the given points form the vertices of the indicated polygon.

(a) Right-angled triangle: $(4, 0)$, $(2, 1)$ and $(-1, -5)$

(b) Isosceles triangle: $(1, -3)$, $(3, 2)$ and $(-2, 4)$

(c) Parallelogram: $(0, 1)$, $(3, 7)$, $(4, 4)$ and $(1, -2)$

6. Use the elimination method to solve each system of linear equations.

(a) $x + 3y = 8$, $x - 2y = 3$

(b) $x - 6y = 1$, $3x + 2y = 13$

(c) $6x + 3y = 6$, $5x + 4y = -1$

(d) $8x - 12y = 4$, $-2x + 3y = 2$

(e) $5x + 7y = 9$, $-11x - 5y = 1$

7. Use the substitution method to solve each system of linear equations.

(a) $2x + y = 1$, $3x + 2y = 3$

(b) $3x - 2y = 7$, $5x - y = -7$

(c) $2x + 8y = -6$, $-5x - 20y = 15$

(d) $\frac{x}{5} + \frac{y}{2} = 8$, $x + y = 20$

(e) $2x - y = -2$, $4x + y = 5$

(f) $0.4x + 0.3y = 1$, $0.25x + 0.1y = -0.25$

8. Use your GDC to solve each system of two linear equations.

(a) $3x + 2y = 9$, $7x + 11y = 2$

(b) $3.62x - 5.88y = -10.11$, $0.08x - 0.02y = 0.92$

(c) $2x - 3y = 4$, $5x + 2y = 1$

9. Use row operations to solve each system of three linear equations. If the system has infinite solutions, simply write 'infinite solutions'.

(a) $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

(b) $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

(c) $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$

(d) $\begin{cases} 3x - 2y + z = -29 \\ -4x + y - 3z = 37 \\ x - 5y + z = -24 \end{cases}$

10. Use your GDC to solve each system of three linear equations. If the system has infinite solutions, simply write 'infinite solutions'.

(a) $\begin{cases} x - 3y - 2z = 8 \\ -2x + 7y + 3z = -19 \\ x - y - 3z = 3 \end{cases}$

(b) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 1 \end{cases}$

(c) $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

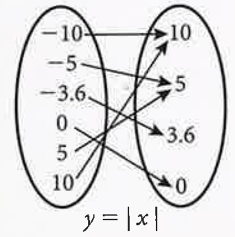
(d) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

11. Find the value(s) of k such that the following system of equations has no solution.

$x + y + (k - 1) = 2$
$kx - z = -3$
$6x + 2y - 3z = 1$

1.2 Definition of a function

A mapping illustrates how some values in the domain of a function are paired with values in the range of the function. Here is a mapping for the function $y = |x|$



Many mathematical relationships concern how the value of one variable determines the value of a second variable. In general, suppose that the values of a particular **independent variable**, for example x, determine the values of a **dependent variable** y in such a way that for a specific value of x, a single value of y is determined. Then we say that y is a **function** of x and we write $y = f(x)$ (read y equals f of x) or $y = g(x)$, and so on, where the letter f or g, etc. represents the name of the function. For example:

- Period T is a function of length L: $T = 2\pi\sqrt{\frac{L}{g}}$
- Area A is a function of radius r: $A = \pi r^2$
- °F (degrees Fahrenheit) is a function of °C: $F = \frac{9}{5}C + 32$
- Distance d from the origin is a function of x: $d = |x|$

Other useful ways of representing a function include a graph of the equation on a **Cartesian coordinate system** (also called a rectangular coordinate system), a **table**, a **set of ordered pairs**, or a **mapping**.

The coordinate system for the graph of an equation has the independent variable on the horizontal axis and the dependent variable on the vertical axis.

The largest possible set of values for the independent variable (the **input** set) is called the **domain**, and the set of resulting values for the dependent variable (the **output** set) is called the **range**. In the context of a mapping, each value in the domain is mapped to its **image** in the range. All of the various ways of representing a mathematical function illustrate that its defining characteristic is that it is a rule by which each number in the domain determines a unique number in the range.

A **function** is a correspondence (mapping) between two sets X and Y in which each element of set X corresponds to (maps to) exactly one element of set Y. The domain is set X (independent variable) and the range is set Y (dependent variable).

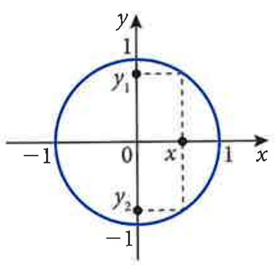


Figure 1.11 Graph of $x^2 + y^2 = 1$

Not all equations represent a function. The solution set for the equation $x^2 + y^2 = 1$ is the set of ordered pairs (x, y) on the circle of radius equal to 1 and centre at the origin (see Figure 1.11). If we solve the equation for y, we get $y = \pm\sqrt{1 - x^2}$. It is clear that any value of x between -1 and 1 will produce two different values of y (opposites). Since at least one value in the domain (x) determines more than one value in the range (y), the equation does not represent a function. A correspondence between two sets that does not satisfy the definition of a function is called a **relation**.

Alternative definition of a function

A function is a relation in which no two different ordered pairs have the same first coordinate. A vertical line intersects the graph of a function at no more than one point (vertical line test).

For many physical phenomena, we observe that one quantity depends on another. The word function is used to describe this dependence of one quantity on another – that is, how the value of an independent variable determines the value of a dependent variable. A common mathematical task is to find how to express one variable as a function of another variable.

Example 1.11

(a) Express the volume V of a cube as a function of the length e of each edge.

(b) Express the volume V of a cube as a function of its surface area S.

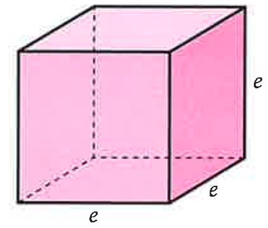


Figure 1.12 Cube for Example 1.11

Solution

(a) V as a function of e is $V = e^3$

(b) The surface area of the cube consists of six squares each with an area of e^2. Hence, the surface area is $6e^2$; that is, $S = 6e^2$. We need to write V in terms of S. We can do this by first expressing e in terms of S, and then substituting this expression for e in the equation $V = e^3$.

$$S = 6e^2 \Rightarrow e^2 = \frac{S}{6} \Rightarrow e = \sqrt{\frac{S}{6}}. \text{ Substituting, } V = \left(\sqrt{\frac{S}{6}}\right)^3 = \frac{(S^{\frac{1}{2}})^3}{(6^{\frac{1}{2}})^3}$$

$$= \frac{S^{\frac{3}{2}}}{6^{\frac{3}{2}}} = \frac{S \cdot S^{\frac{1}{2}}}{6 \cdot 6^{\frac{1}{2}}} = \frac{S}{6}\sqrt{\frac{S}{6}}$$

V as a function of S is $V = \frac{S}{6}\sqrt{\frac{S}{6}}$

Example 1.12

An offshore wind turbine is located at point W, 4 km offshore from the nearest point P on a straight coastline. A maintenance station is at point M, 3 km down the coast from P. An engineer is returning by a small boat from the wind turbine. He sails to point D that is located between P and M at an unknown distance x km from point P. From there, he walks to the maintenance station. The boat sails at 3 km hr^{-1} and the engineer can walk at 6 km hr^{-1}. Express the total time T (hours) for the trip from the wind turbine to the maintenance station as a function of x (km).

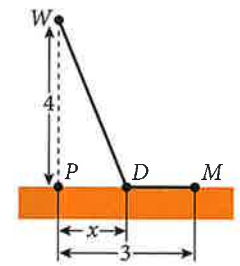


Figure 1.13 Diagram for Example 1.12

Solution

To get an equation for T in terms of x, use the fact that time $= \frac{\text{distance}}{\text{rate}}$

We then have

$$T = \frac{\text{distance } WD}{3} + \frac{\text{distance } DM}{6}$$

The distance WD can be expressed in terms of x using Pythagoras' theorem.

$$WD^2 = x^2 + 4^2 \Rightarrow WD = \sqrt{x^2 + 16}$$

To express T in terms of only the single variable x, note that $DM = 3 - x$

Then the total time T can be written in terms of x by the equation

$$T = \frac{\sqrt{x^2 + 16}}{3} + \frac{3 - x}{6} \text{ or } T = \frac{1}{3}\sqrt{x^2 + 16} + \frac{1}{2} - \frac{x}{6}$$

Domain and range of a function

The domain of a function may be stated explicitly, or it may be implied by the expression that defines the function. For most of this course, we can assume that functions are real-valued functions of a real variable. The domain and range will contain only real numbers or some subset of the real numbers. The domain of a function is the set of all real numbers for which the expression is defined as a real number, if not explicitly stated otherwise. For example, if a certain value of x is substituted into the algebraic expression defining a function and it causes division by zero or the square root of a negative number (both undefined in the real numbers) to occur, that value of x cannot be in the domain.

The domain of a function may also be implied by the physical context or limitations that exist in a problem. For example, in both functions derived in Example 1.11 the domain is the set of positive real numbers (denoted by $\mathbb{R}^+$) because neither a length (edge of a cube) nor a surface area (face of a cube) can have a value that is negative or zero. In Example 1.12 the domain for the function is $0 < x < 3$ because of the constraints given in the problem. Usually the range of a function is not given explicitly and is determined by analysing the output of the function for all values of the input (domain). The range of a function is often more difficult to find than the domain, and analysing the graph of a function is very helpful in determining it. A combination of algebraic and graphical analysis is very useful in determining the domain and range of a function.

In Chapter 8, we learn about functions where the variables can have values that are imaginary numbers.

Example 1.13

Find the domain of each function.

(a) $\{(-6, -3), (-1, 0), (2, 3), (3, 0), (5, 4)\}$

(b) Volume of a sphere: $V = \frac{4}{3}\pi r^3$

(c) $y = \frac{5}{2x - 6}$

(d) $y = \sqrt{3 - x}$

Solution

(a) The function consists of a set of ordered pairs. The domain of the function consists of all first coordinates of the ordered pairs. Therefore, the domain is the set $x \in \{-6, -1, 2, 3, 5\}$.

(b) The physical context tells us that a sphere cannot have a radius that is negative or zero. Therefore, the domain is the set of all real numbers r such that $r > 0$.

(c) Since division by zero is not defined for real numbers then $2x - 6 \neq 0$. Therefore, the domain is the set of all real numbers x such that $x \in \mathbb{R}$, $x \neq 3$.

(d) Since the square root of a negative number is not real, then $3 - x > 0$. Therefore, the domain is all real numbers x such that $x < 3$.

Example 1.14

Find the domain and range for the function $y = x^2$

Solution

Using algebraic analysis: Squaring any real number produces another real number. Therefore, the domain of $y = x^2$ is the set of all real numbers ($\mathbb{R}$). Since the square of any positive or negative number will be positive and the square of zero is zero, then the range is the set of all real numbers greater than or equal to zero.

Using graphical analysis: For the domain, focus on the x-axis and scan the graph from $-\infty$ to $+\infty$. There are no gaps or blank regions in the graph and the parabola will continue to get wider as x goes to either $-\infty$ or $+\infty$. Therefore, the domain is all real numbers. For the range, focus on the y-axis and scan from $-\infty$ to $+\infty$. The parabola will continue to increase as y goes to $+\infty$, but the graph does not go below the x-axis. The parabola has no points with negative y coordinates. Therefore, the range is the set of real numbers greater than or equal to zero.

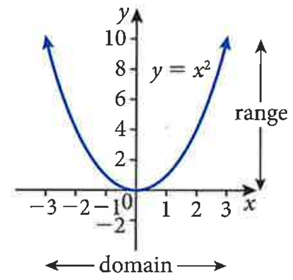


Figure 1.14 Graphical solution to Example 1.14

Description in words	Interval notation
Domain is any real number	Domain is $\{x : x \in \mathbb{R}\}$, or Domain is $x \in]-\infty, \infty[$
Range is any number greater than or equal to zero	Range is $\{y : y \geqslant 0\}$, or Range is $y \in [0, \infty[$

Table 1.2 Different ways of expressing the domain and range of $y = x^2$

The inequality $2 \leqslant x < 5$ can also be written as $[2, 5[$. The number 2 is included, but 5 is not. When determining the domain and range of a function, use both algebraic and graphical analysis. Do not rely too much on using just one approach. For graphical analysis of a function, producing a graph on your GDC that shows all the important features is essential.

Function notation

It is common practice to name a function using a single letter, with f, g and h commonly used. Given that the domain variable is x and the range variable is y, the symbol $f(x)$ denotes the unique value of y that is generated by the value of x.

Another notation – sometimes referred to as mapping notation – is based on the idea that the function f is the rule that maps x to $f(x)$ and is written $f : x \mapsto f(x)$. For each value of x in the domain, the corresponding unique value of y in the range is called the function value at x, or the image of x under f. The image of x may be written as $f(x)$ or as y. For example, for the function $f(x) = x^2$: '$f(3) = 9$', or 'if $x = 3$, then $y = 9$.'

Notation	Description in words
$f(x) = x^2$	The function f, in terms of x, is x^2; or, simply f of x equals x^2
$f : x \mapsto x^2$	The function f maps x to x^2
$f(3) = 9$	The value of the function f when $x = 3$ is 9; or, simply f of 3 equals 9
$f : 3 \mapsto 9$	The image of 3 under the function f is 9

Table 1.3 Function notation

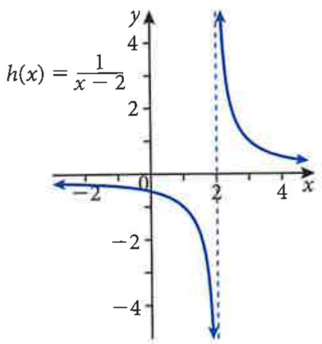


Figure 1.15 Diagram for Example 1.15

Example 1.15

Find the domain and range of the function $h: x \mapsto \frac{1}{x-2}$

Solution

Using algebraic analysis: The function produces a real number for all x, except for $x = 2$ when division by zero occurs. Hence, $x = 2$ is the only real number not in the domain. Since the numerator of $\frac{1}{x-2}$ can never be zero, the value of y cannot be zero. Hence, $y = 0$ is the only real number not in the range.

Using graphical analysis: A horizontal scan shows a gap at $x = 2$ dividing the graph of the equation into two branches that both continue indefinitely with no other gaps as $x \to \pm\infty$. Both branches are **asymptotic** (approach but do not intersect) to the vertical line $x = 2$. This line is a **vertical asymptote** and is drawn as a dashed line (it is not part of the graph of the equation). A vertical scan reveals a gap at $y = 0$ (x-axis) with both branches of the graph continuing indefinitely with no other gaps as $y \to \pm\infty$. Both branches are also asymptotic to the x-axis. The x-axis is a **horizontal asymptote**.

Both approaches confirm that the domain and range for $h: x \mapsto \frac{1}{x-2}$ are:

domain: $\{x : x \in \mathbb{R}, x \neq 2\}$ or $x \in]-\infty, 2[\cup]2, \infty[$

range: $\{y : y \in \mathbb{R}, y \neq 0\}$ or $y \in]-\infty, 0[\cup]0, \infty[$

Example 1.16

Consider the function $f(x) = \sqrt{x+4}$

(a) Find:

(i) $f(7)$ (ii) $f(32)$ (iii) $f(-4)$

(b) Find the values of x for which f is undefined.

(c) State the domain and range of f.

Solution

(a) (i) $f(7) = \sqrt{7+4} = \sqrt{11} \approx 3.32$ (3 s.f.)

(ii) $f(32) = \sqrt{32+4} = \sqrt{36} = 6$

(iii) $f(-4) = \sqrt{-4+4} = \sqrt{0} = 0$

(b) $f(x)$ will be undefined (square root of a negative) when $x + 4 < 0$.

Therefore, $f(x)$ is undefined when $x < -4$.

(c) It follows from the result in (b) that the domain of f is $\{x : x \geq -4\}$.

The symbol $\sqrt{}$ stands for the **principal square root** that, by definition, can only give a result that is positive or zero. Therefore, the range of f is $\{y : y \geq 0\}$. The domain and range are confirmed by analysing the graph of the function.

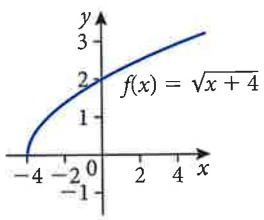
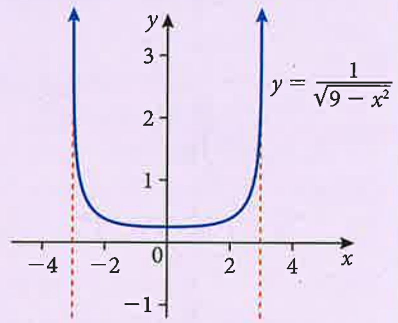


Figure 1.16 Graph for the solution to Example 1.16 (c)

Example 1.17

Find the domain and range of the function $f(x) = \frac{1}{\sqrt{9-x^2}}$

Solution



The graph of $y = \frac{1}{\sqrt{9-x^2}}$ shown here, agrees with algebraic analysis indicating that the expression $\frac{1}{\sqrt{9-x^2}}$ will be positive for all x, and is defined only for $-3 < x < 3$. Further analysis and tracing the graph reveals that $f(x)$ has a minimum at $\left(0, \frac{1}{3}\right)$. The graph on the GDC is misleading in that it appears to show that the function has a maximum value of approximately $y \approx 2.8037849$. Can this be correct? A lack of algebraic thinking and over-reliance on a GDC could easily lead to a mistake. The graph abruptly stops its curve upwards because of low screen resolution. Function values should get quite large for values of x a little less than 3, because the value of $\sqrt{9-x^2}$ will be small, making the fraction $\frac{1}{\sqrt{9-x^2}}$ large.

Using a GDC to make a table for $f(x)$ or evaluating the function for values of x very close to -3 or 3 confirms that as x approaches -3 or 3, y increases without bound, i.e. y goes to $+\infty$. Hence, $f(x)$ has vertical asymptotes of $x = -3$ and $x = 3$. This combination of graphical and algebraic analysis leads to the conclusion that the domain of $f(x)$ is $\{x : -3 < x < 3\}$, and the range of $f(x)$ is $\left\{y : y \geq \frac{1}{3}\right\}$

As Example 1.17 illustrates, it is dangerous to completely trust graphs produced on a GDC without also doing some algebraic thinking. It is important to check that the graph shown is comprehensive (shows all important features), and that the graph agrees with algebraic analysis of the function, for example, where the function should be zero, positive, negative, undefined, or increasing/decreasing without bound.

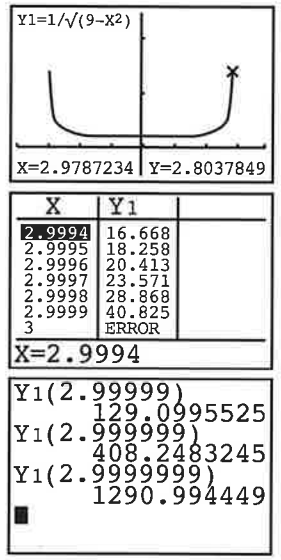


Figure 1.17 GDC screens for solution to Example 1.17

Exercise 1.2

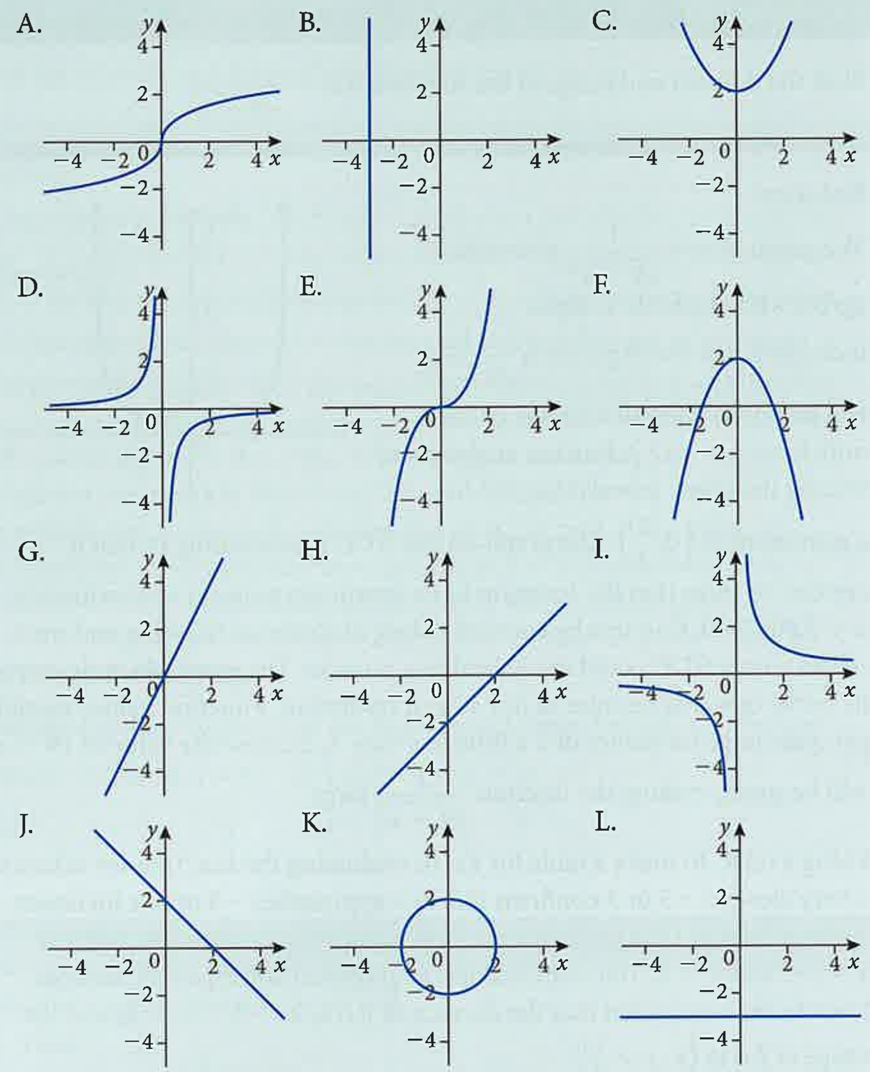
1. **(i)** Match each equation to one of the graphs.

(ii) State whether or not the equation represents any of the functions shown. Justify your answer. Assume that x is the independent variable and y is the dependent variable.

(a) $y = 2x$ **(b)** $y = -3$ **(c)** $x - y = 2$

(d) $x^2 + y^2 = 4$ **(e)** $y = 2 - x$ **(f)** $y = x^2 + 2$

(g) $y^3 = x$ **(h)** $y = \frac{2}{x}$ **(i)** $x^2 + y = 2$



2. Express the area, A, of a circle as a function of its circumference, C.

3. Express the area, A, of an equilateral triangle as a function of the length, ℓ, of each of its sides.

4. A rectangular swimming pool with dimensions 12 metres by 18 metres is surrounded by a pavement of uniform width x metres. Find the area of the pavement, A, as a function of x.

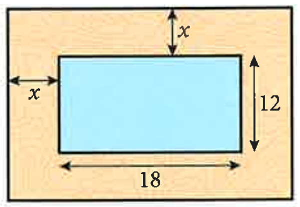


Figure 1.18 Diagram for question 4

5. In a right-angled isosceles triangle, the two equal sides have length x units and the hypotenuse has length h units. Write h as a function of x.

6. The pressure P (measured in kilopascals, kPa) for a particular sample of gas is directly proportional to the temperature T (measured in degrees kelvin, K) and inversely proportional to the volume V (measured in litres, L). With k representing the constant of proportionality, this relationship can be written in the form of the equation $P = k\frac{T}{V}$

   (a) Find the constant of proportionality, k, if 150 L of gas exerts a pressure of 23.5 kPa at a temperature of 375 K.

   (b) Using the value of k from part **(a)** and assuming that the temperature is held constant at 375 K, write the volume V as a function of pressure P for this sample of gas.

7. In physics, Hooke's law states that the force F (measured in newtons, N) needed to extend a spring by x units beyond its natural length is directly proportional to the extension x. Assume that the constant of proportionality is k (known as the spring constant for a particular spring).

   (a) Write F as a function of x.

   (b) A spring has a natural length of 12 cm and a force of 25 N stretches the spring to a length of 16 cm. Work out the spring constant k.

   (c) What force is needed to stretch the spring to a length of 18 cm?

8. Find the domain of each of the following functions.

   (a) $\{(-6.2, -7), (-1.5, -2), (0.7, 0), (3.2, 3), (3.8, 3)\}$

   (b) Surface area of a sphere: $S = 4\pi r^2$

   (c) $f(x) = \frac{2}{5}x - 7$ **(d)** $h: x \mapsto x^2 - 4$ **(e)** $g(t) = \sqrt{3 - t}$

   (f) $h(t) = \sqrt[3]{t}$ **(g)** $f: x \mapsto \frac{6}{x^2 - 9}$ **(h)** $f(x) = \sqrt{\frac{1}{x^2} - 1}$

9. Do all linear equations represent a function? Explain.

10. Consider the function $h(x) = \sqrt{x - 4}$

   (a) Find: **(i)** $h(21)$ **(ii)** $h(53)$ **(iii)** $h(4)$

   (b) Find the values of x for which h is undefined.

   (c) State the domain and range of h.

11. For each function below:

   (i) find the domain and range of the function

   (ii) sketch a comprehensive graph of the function, clearly indicating any intercepts or asymptotes.

   (a) $f: x \mapsto \frac{1}{x - 5}$ **(b)** $g(x) = \frac{1}{\sqrt{x^2 - 9}}$ **(c)** $h(x) = \frac{2x - 1}{x + 2}$

   (d) $p: x \mapsto \sqrt{5 - 2x^2}$ **(e)** $f(x) = \frac{1}{x} - 4$

1.3 Composite functions

Composition of functions

The **argument** of a function is the variable or expression on which a function operates. For example, the argument of $f(x) = x^3$ is x, the argument of $g(x) = \sqrt{x-3}$ is $x - 3$, and the argument of $y = 10^{2x}$ is $2x$.

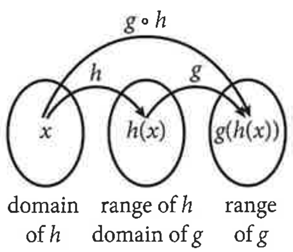


Figure 1.19 Mapping for composite function $g(h(x))$

The composition of two functions, g and h, such that h is applied first and g second is given by $(g \circ h)(x) = g(h(x))$. The domain of the composite function $g \circ h$ is the set of all x in the domain of h such that $h(x)$ is in the domain of g.

Consider the function in Example 1.16, $f(x) = \sqrt{x+4}$. When we evaluate $f(x)$ for a certain value of x in the domain, for example, $x = 5$, it is necessary to perform computations in two separate steps in a certain order.

$f(5) = \sqrt{5+4} \Rightarrow f(5) = \sqrt{9}$ Step 1: compute the sum of 5 + 4

$\Rightarrow f(5) = 3$ Step 2: compute the square root of 9

Given that the function has two separate evaluation steps, $f(x)$ can be seen as a combination of two simpler functions that are performed in a specified order. According to how $f(x)$ is evaluated, the simpler function to be performed first is the rule of adding 4 and the second is the rule of taking the square root. If $h(x) = x + 4$ and $g(x) = \sqrt{x}$, then we can create (compose) the function $f(x)$ from a combination of $h(x)$ and $g(x)$ as follows:

$f(x) = g(h(x))$

$= g(x + 4)$ Step 1: substitute $x + 4$ for $h(x)$ making $x + 4$ the argument of $g(x)$

$= \sqrt{x+4}$ Step 2: apply the function $g(x)$ on the argument $x + 4$

We obtain the rule $\sqrt{x+4}$ by first applying the rule $x + 4$ and then applying the rule $\sqrt{x}$. A function that is obtained from simpler functions by applying one after another in this way is called a **composite function**. $f(x) = \sqrt{x+4}$ is the **composition** of $h(x) = x + 4$ followed by $g(x) = \sqrt{x}$. In other words, f is obtained by substituting h into g, and can be denoted in function notation by $g(h(x))$ – read 'g of h of x.'

Start with a number x in the domain of h and find its image $h(x)$. If this number $h(x)$ is in the domain of g, we then compute the value of $g(h(x))$. The resulting composite function is denoted as $(g \circ h(x))$. See Figure 1.19.

Example 1.18

If $f(x) = 3x$ and $g(x) = 2x - 6$, find:

(a) (i) $(f \circ g)(5)$ (ii) Express $(f \circ g)(x)$ as a single function rule (expression).

(b) (i) $(g \circ f)(5)$ (ii) Express $(g \circ f)(x)$ as a single function rule (expression).

(c) (i) $(g \circ g)(5)$ (ii) Express $(g \circ g)(x)$ as a single function rule (expression).

Solution

(a) (i) $(f \circ g)(5) = f(g(5)) = f(2 \cdot 5 - 6) = f(4) = 3 \cdot 4 = 12$

(ii) $(f \circ g)(x) = f(g(x)) = f(2x - 6) = 3(2x - 6) = 6x - 18$

Therefore, $(f \circ g)(x) = 6x - 18$

Check with result from (i): $(f \circ g)(5) = 6 \cdot 5 - 18 = 30 - 18 = 12$

(b) (i) $(g \circ f)(5) = g(f(5)) = g(3 \cdot 5) = g(15) = 2 \cdot 15 - 6 = 24$

(ii) $(g \circ f)(x) = g(f(x)) = g(3x) = 2(3x) - 6 = 6x - 6$

Therefore, $(g \circ f)(x) = 6x - 6$

Check with result from (i): $(g \circ f)(5) = 6 \cdot 5 - 6 = 30 - 6 = 24$

(c) (i) $(g \circ g)(5) = g(g(5)) = g(2 \cdot 5 - 6) = g(4) = 2 \cdot 4 - 6 = 2$

(ii) $(g \circ g)(x) = g(g(x)) = g(2x - 6) = 2(2x - 6) - 6 = 4x - 18$

Therefore, $(g \circ g)(x) = 4x - 18$

Check with result from (i): $(g \circ g)(5) = 4 \cdot 5 - 18 = 20 - 18 = 2$

The notations $(g \circ h)(x)$ and $g(h(x))$ are both commonly used to denote a composite function where h is applied first then followed by applying g. Since you are reading this from left to right, it is easy to apply the functions in the incorrect order. It may be helpful to read $g \circ h$ as 'g following h' to highlight the order in which the functions are applied. Also, in either notation, $(g \circ h)(x)$ or $g(h(x))$, the function applied first is closest to the variable x.

It is important to notice that in parts (a)(ii) and (b)(ii) in Example 1.18, $f \circ g$ is not equal to $g \circ f$. At the start of this section, it was shown how the two functions $h(x) = x + 4$ and $g(x) = \sqrt{x}$ could be combined into the composite function $(g \circ h)(x)$ to create the single function $f(x) = \sqrt{x+4}$. However, the composite function $(h \circ g)(x)$ (the functions applied in reverse order) creates a different function: $(h \circ g)(x) = h(g(x)) = h(\sqrt{x}) = \sqrt{x} + 4$. Since, $\sqrt{x} + 4 \neq \sqrt{x+4}$ then $f \circ g$ is not equal to $g \circ f$. Is it always true that $f \circ g \neq g \circ f$? The next example will answer that question.

Example 1.19

Given $f: x \mapsto 3x - 6$ and $g: x \mapsto \frac{1}{3}x + 2$, find:

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution

(a) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3}x + 2\right) = 3\left(\frac{1}{3}x + 2\right) - 6 = x + 6 - 6 = x$

(b) $(g \circ f)(x) = g(f(x)) = g(3x - 6) = \frac{1}{3}(3x - 6) + 2 = x - 2 + 2 = x$

Example 1.19 shows that it is possible for $f \circ g$ to be equal to $g \circ f$. You will learn in the next section that this occurs in some cases where there is a special relationship between the pair of functions. However, in general $f \circ g \neq g \circ f$.

Decomposing a composite function

In examples 1.18 and 1.19, we created a single function by forming the composite of two functions. As with the function $f(x) = \sqrt{x + 4}$, it is also important for us to be able to identify two functions that make up a composite function, in other words, to decompose a function into two simpler functions. When we are doing this it is very useful to think of the function that is applied first as the inside function, and the function that is applied second as the outside function. In the function $f(x) = \sqrt{x + 4}$, the inside function is $h(x) = x + 4$ and the outside function is $g(x) = \sqrt{x}$.

Example 1.20

Each of these functions is a composite function of the form $(f \circ g)(x)$. For each, find the two component functions f and g.

(a) $h: x \mapsto \dfrac{1}{x + 3}$ (b) $k: x \mapsto 2^{4x+1}$ (c) $p(x) = \sqrt[3]{x^2 - 4}$

Solution

(a) When we evaluate the function $h(x)$ for a certain x in the domain, we first evaluate the expression $x + 3$, and then evaluate the expression $\frac{1}{x}$. Hence, the inside function (applied first) is $y = x + 3$, and the outside function (applied second) is $y = \frac{1}{x}$. So the two component functions are $g(x) = x + 3$ and $f(x) = \frac{1}{x}$

(b) Evaluating $k(x)$ requires us to first evaluate the expression $4x + 1$, and then evaluate the expression 2^x. Hence, the inside function is $y = 4x + 1$, and the outside function is $y = 2^x$. The two composite functions are $g(x) = 4x + 1$ and $f(x) = 2^x$.

(c) Evaluating $p(x)$ requires us to perform three separate evaluation steps: squaring a number, subtracting four, and then taking the cube root. Hence, it is possible to decompose $p(x)$ into three component functions: $h(x) = x^2$, $g(x) = x - 4$ and $f(x) = \sqrt[3]{x}$. However, for our purposes it is best to decompose the composite function into only two component functions: $g(x) = x^2 - 4$, and $f(x) = \sqrt[3]{x}$.

Finding the domain of a composite function

It is important to note that in order for a value of x to be in the domain of the composite function $g \circ h$, two conditions must be met: (1) x must be in the domain of h, and (2) $h(x)$ must be in the domain of g. Likewise, it is also worth noting that $g(h(x))$ is in the range of $g \circ h$ only if x is in the domain of $g \circ h$. The next example illustrates these points – and also that, in general, the domains of $g \circ h$ and $h \circ g$ are not the same.

Example 1.21

Let $g(x) = x^2 - 4$ and $h(x) = \sqrt{x}$. Find:

(a) $(g \circ h)(x)$ and its domain and range

(b) $(h \circ g)(x)$ and its domain and range.

Solution

First, establish the domain and range for both g and h. For $g(x) = x^2 - 4$, the domain is $x \in \mathbb{R}$ and the range is $y \geqslant -4$. For $h(x) = \sqrt{x}$, the domain is $x \geqslant 0$ and the range is $y \geqslant 0$.

(a) $(g \circ h)(x) = g(h(x))$

$= g(\sqrt{x})$ — To be in the domain of $g \circ h$, $\sqrt{x}$ must be defined for $x \Rightarrow x \geqslant 0$

$= (\sqrt{x^2}) - 4$ — Therefore, the domain of $g \circ h$ is $x \geqslant 0$

$= x - 4$ — Since $x \geqslant 0$, then the range for $y = x - 4$ is $y \geqslant -4$.

Therefore, $(g \circ h)(x) = x - 4$, and its domain is $x \geqslant 0$, and its range is $y \geqslant -4$

(b) $(h \circ g)(x) = h(g(x))$ — $g(x) = x^2 - 4$ must be in the domain of $h \Rightarrow x^2 - 4 \geqslant 0 \Rightarrow x^2 \geqslant 4$

$= h(x^2 - 4)$ — Therefore, the domain of $h \circ g$ is $x \leqslant -2$ or $x \geqslant 2$ and

$= \sqrt{x^2 - 4}$ — with $x \leqslant -2$ or $x \geqslant 2$, the range for $y = \sqrt{x^2 - 4}$ is $y \geqslant 0$

Therefore, $(h \circ g)(x) = \sqrt{x^2 - 4}$, and its domain is $x \leqslant -2$ or $x \geqslant 2$, and its range is $y \geqslant 0$

Exercise 1.3

1. Let $f(x) = 2x$ and $g(x) = \dfrac{1}{x - 3}$, $x \neq 0$

Find the value of:

(a) $(f \circ g)(5)$ **(b)** $(g \circ f)(5)$

Find the function rule (expression) for:

(c) $(f \circ g)(x)$ **(d)** $(g \circ f)(x)$

2. Let $f: x \mapsto 2x - 3$ and $g: x \mapsto 2 - x^2$

Evaluate:

(a) $(f \circ g)(0)$ **(b)** $(g \circ f)(0)$ **(c)** $(f \circ f)(4)$

(d) $(g \circ g)(-3)$ **(e)** $(f \circ g)(-1)$ **(f)** $(g \circ f)(-3)$

Find the expression for:

(g) $(f \circ g)(x)$ **(h)** $(g \circ f)(x)$ **(i)** $(f \circ f)(x)$ **(j)** $(g \circ g)(x)$

3. For each pair of functions, find $(f \circ g)(x)$ and $(g \circ f)(x)$ and state the domain for each.
 (a) $f(x) = 4x - 1, g(x) = 2 + 3x$ (b) $f(x) = x^2 + 1, g(x) = -2x$
 (c) $f(x) = \sqrt{x+1}, g(x) = 1 + x^2$ (d) $f(x) = \frac{2}{x+4}, g(x) = x - 1$
 (e) $f(x) = 3x + 5, g(x) = \frac{x-5}{3}$ (f) $f(x) = 2 - x^3, g(x) = \sqrt[3]{1-x^2}$
 (g) $f(x) = \frac{2x}{4-x}, g(x) = \frac{1}{x^2}$
 (h) $f(x) = \frac{2}{x+3} - 3, g(x) = \frac{2}{x+3} - 3$
 (i) $f(x) = \frac{x}{x-1}, g(x) = x^2 - 1$

4. Let $g(x) = \sqrt{x-1}$ and $h(x) = 10 - x^2$. Find:
 (a) $(g \circ h)(x)$ and its domain and range
 (b) $(h \circ g)(x)$ and its domain and range.

5. Let $f(x) = \frac{1}{x}$ and $g(x) = 10 - x^2$. Find:
 (a) $(f \circ g)(x)$ and its domain and range
 (b) $(g \circ f)(x)$ and its domain and range.

6. Determine functions g and h so that $f(x) = g(h(x))$
 (a) $f(x) = (x+3)^2$ (b) $f(x) = \sqrt{x-5}$ (c) $f(x) = 7 - \sqrt{x}$
 (d) $f(x) = \frac{1}{x+3}$ (e) $f(x) = 10^{x+1}$ (f) $f(x) = \sqrt[3]{x-9}$
 (g) $f(x) = |x^2 - 9|$ (h) $f(x) = \frac{1}{\sqrt{x-5}}$

7. Find the domain for:
 (i) the function f (ii) the function g (iii) the composite function $f \circ g$
 (a) $f(x) = \sqrt{x}, g(x) = x^2 + 1$ (b) $f(x) = \frac{1}{x}, g(x) = x + 3$
 (c) $f(x) = \frac{3}{x^2 - 1}, g(x) = x + 1$ (d) $f(x) = 2x + 3, g(x) = \frac{x}{2}$

1.4 Inverse functions

Pairs of inverse functions

If we choose a number and cube it (raise it to the power of 3), and then take the cube root of the result, the answer is the original number. The same result would occur if we applied the two rules in the reverse order. That is, first take the cube root of a number and then cube the result; again, the answer is the original number.

Write each of these rules as a function with function notation. Write the cubing function as $f(x) = x^3$, and the cube root function as $g(x) = \sqrt[3]{x}$. Now, using what we know about composite functions and operations with radicals and powers, we can write what was described above in symbolic form.

Cube a number and then take the cube root of the result:

$$g(f(x)) = \sqrt[3]{x^3} = (x^3)^{\frac{1}{3}} = x^1 = x$$

For example, $g(f(-2)) = \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$

Take the cube root of a number and then cube the result:

$$f(g(x)) = \left(\sqrt[3]{x}\right)^3 = \left(x^{\frac{1}{3}}\right)^3 = x^1 = x$$

For example, $f(g(27)) = \left(\sqrt[3]{27}\right)^3 = (3)^3 = 27$

Because function g has this reverse (inverse) effect on function f, we call function g the **inverse** of function f. Function f has the same inverse effect on function g $[g(27) = 3$ and then $f(3) = 27]$, making f the inverse function of g. The functions f and g are inverses of each other. The cubing and cube root functions are an example of a pair of **inverse functions**. The mapping diagram for functions f and g (Figure 1.20) illustrates the relationship for a pair of inverse functions where the domain of one is the range for the other.

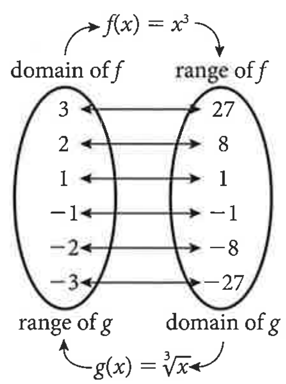


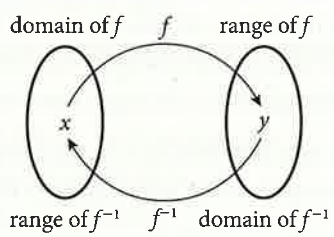
Figure 1.20 A mapping diagram for the cubing and cube root functions

You should already be familiar with pairs of **inverse operations**. Addition and subtraction are inverse operations. For example, the rule of adding six $(x + 6)$, and the rule of subtracting six $(x - 6)$, undo each other. Accordingly, the functions $f(x) = x + 6$ and $g(x) = x - 6$ are a pair of inverse functions. Multiplication and division are also inverse operations.

The composite of two inverse functions is the function that always produces the same number that was first substituted into the function. This function is called the **identity function** because it assigns each number in its domain to itself and is denoted by $I(x) = x$.

If f and g are two functions such that $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f, then the function g is the **inverse** of the function f. The notation to indicate the function that is the inverse of function f is f^{-1}. Therefore, $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.



Remember that the notation $(f \circ g)(x)$ is equivalent to $f(g(x))$. It follows from the definition that if g is the inverse of f, then it must also be true that f is the inverse of g.

Do not mistake the -1 in the notation f^{-1} for a power. It is not a power. If a superscript of -1 is applied to the name of a function, as in f^{-1} or $\sin^{-1}$, then it denotes the function that is the inverse of the named function (e.g. f or sin). If a superscript of -1 is applied to an expression, as in 7^{-1} or $(2x+5)^{-1}$, then it is a power and denotes the reciprocal of the expression.

In general, the functions $f(x)$ and $g(x)$ are a pair of inverse functions if the following two statements are true:

1 $g(f(x)) = x$ for all x in the domain of f

2 $f(g(x)) = x$ for all x in the domain of g

For a pair of inverse functions, f and g, the composite functions $f(g(x))$ and $g(f(x))$ are equal. Remember that this is not generally true for an arbitrary pair of functions.

Example 1.22

Given $h(x) = \frac{x-3}{2}$ and $p(x) = 2x + 3$, show that h and p are inverse functions.

Solution

Since the domain and range of both $h(x)$ and $p(x)$ is the set of all real numbers, then:

For any real number x, $p(h(x)) = p\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3$

$= x - 3 + 3 = x$

For any real number x, $h(p(x)) = h(2x+3) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x$

Since $p(h(x)) = h(p(x)) = x$ then h and p are a pair of inverse functions.

A function is **one-to-one** if each element y in the range is the image of exactly one element x in the domain.

It is clear that both $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ satisfy the definition of a function because for both f and g every number in its domain determines exactly one number in its range. Since they are a pair of inverse functions then the reverse is also true for both; that is, every number in its range is determined by exactly one number in its domain. Such a function is called a **one-to-one function**. The phrase one-to-one is appropriate because each value in the domain corresponds to exactly **one** value in the range, and each value in the range corresponds to exactly **one** value in the domain.

The existence of an inverse function

Determining whether a function is one-to-one is very useful because the inverse of a one-to-one function will also be a function. Analysing the graph of a function is the most effective way to determine if a function is one-to-one. Let's look at the graph of the one-to-one function $f(x) = x^3$ shown in Figure 1.21. It is clear that as the values of x increase over the domain (from $-\infty$ to ∞), the function values are always increasing. A function that is always increasing, or always decreasing, throughout its domain is one-to-one and has an inverse function.

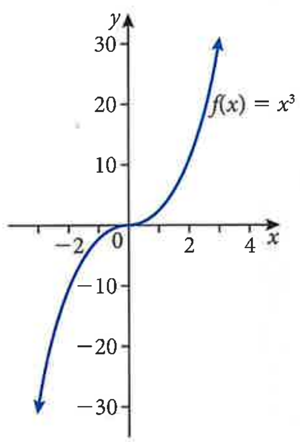


Figure 1.21 Graph of $f(x) = x^3$ which is increasing as x goes from $-\infty$ to ∞

A function that is not one-to-one (always increasing or always decreasing) can be made so by restricting its domain.

The function $f(x) = x^2$ (Figure 1.22) is not one-to-one for all real numbers. However, the function $g(x) = x^2$ with domain $x \geqslant 0$ (Figure 1.23) is always increasing (one-to-one), and the function $h(x) = x^2$ with domain $x \leqslant 0$ (Figure 1.24) is always decreasing (one-to-one).

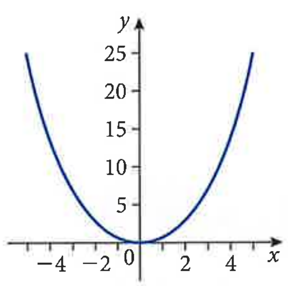


Figure 1.22 $f(x) = x^2$

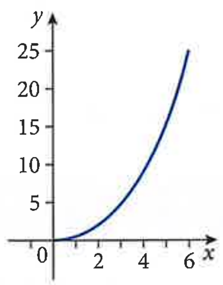


Figure 1.23 $g(x) = x^2, x \geqslant 0$

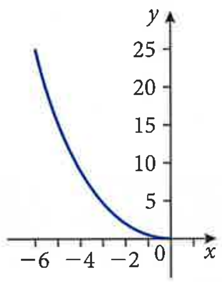


Figure 1.24 $h(x) = x^2, x \leqslant 0$

A function for which at least one element y in the range is the image of more than one element x in the domain is called a **many-to-one function**. Examples of many-to-one functions that we have already encountered are $y = x^2, x \in \mathbb{R}$ and $y = |x|, x \in \mathbb{R}$. As Figure 1.25 illustrates for $y = |x|$, a horizontal line exists that intersects a many-to-one function at more than one point. Thus, the inverse of a many-to-one function will not be a function.

If a function f is always increasing or always decreasing in its domain (i.e. it is monotonic), then f has an inverse f^{-1}.

No horizontal line can pass through the graph of a one-to-one function at more than one point.

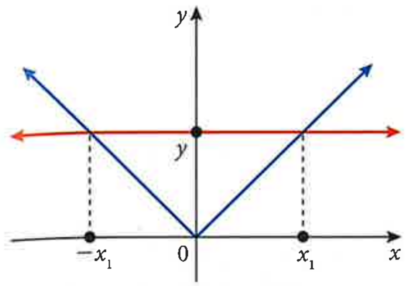


Figure 1.25 Graph of $y = |x|$; an example of a many-to-one function

Finding the inverse of a function

Example 1.23

The function f is defined for $x \in \mathbb{R}$ by $f(x) = 4x - 8$

(a) Determine if f has an inverse f^{-1}. If not, restrict the domain of f in order to find an inverse function f^{-1}

(b) Verify the result by showing that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$

(c) Graph f and its inverse function f^{-1} on the same set of axes.

Solution

(a) Recognise that f is an increasing function for $(-\infty, \infty)$ because the graph of $f(x) = 4x - 8$ is a straight line with a constant slope of 4. Therefore, f is a one-to-one function and it has an inverse f^{-1}

(b) To find the equation for f^{-1}, start by switching the domain (x) and range (y) since the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}, as stated in the definition. Also, recall that $y = f(x)$.

$f(x) = 4x - 8$

$y = 4x - 8$ write $y = f(x)$

$x = 4y - 8$ interchange x and y (switch the domain and range)

$4y = x + 8$ solve for y (dependent variable) in terms of x (independent variable)

$y = \frac{1}{4}x + 2$

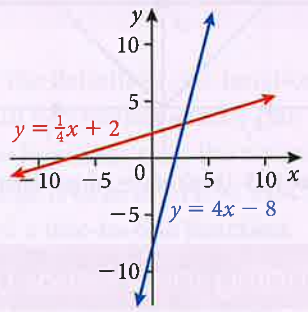
$f^{-1}(x) = \frac{1}{4}x + 2$ resulting equation is $y = f^{-1}(x)$

Verify that f and f^{-1} are inverses by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

$f\left(\frac{1}{4}x + 2\right) = 4\left(\frac{1}{4}x + 2\right) - 8 = x + 8 - 8 = x$

$f^{-1}(4x - 8) = \frac{1}{4}(4x - 8) + 2 = x - 2 + 2 = x$

This confirms that $y = 4x - 8$ and $y = \frac{1}{4}x + 2$ are inverses of each other. Here is a graph of this pair of inverse functions.



The method of interchanging domain (x) and range (y) to find the inverse function used in Example 1.23 also gives us a way for obtaining the graph of f^{-1} from the graph of f. Given the reversing effect that a pair of inverse functions have on each other, if $f(a) = b$ then $f^{-1}(b) = a$. Hence, if the ordered pair (a, b) is a point on the graph of $y = f(x)$, then the reversed ordered pair (b, a) must be on the graph of $y = f^{-1}(x)$. Figure 1.26 shows that the point (b, a) can be found by reflecting the point (a, b) about the line $y = x$. Therefore, the following statement can be made about the graphs of a pair of inverse functions.

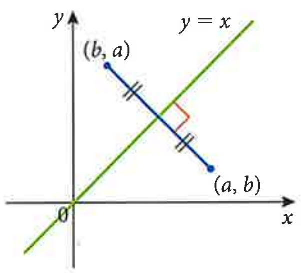


Figure 1.26 The point (b, a) is a reflection about the line $y = x$ of the point (a, b)

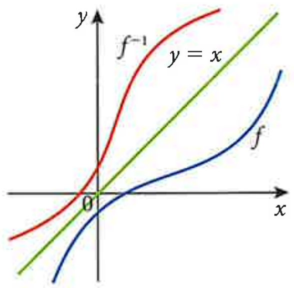


Figure 1.27 Graphs of f and f^{-1} are symmetric about the line $y = x$

The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

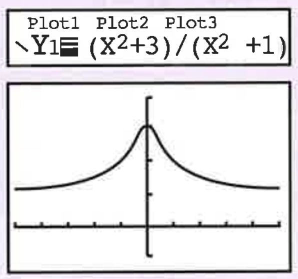
Example 1.24

The function f is defined for $x \in \mathbb{R}$ by $f: x \mapsto \frac{x^2 + 3}{x^2 + 1}$

(a) Determine if f has an inverse f^{-1}. If not, restrict the domain of f in order to find an inverse function f^{-1}.

(b) Graph f and its inverse f^{-1} on the same set of axes.

Solution



A graph of f produced on a GDC reveals that it is not monotonic over its domain $(-\infty, \infty)$. It is increasing for $(-\infty, 0]$, and decreasing for $[0, \infty)$. Therefore, f does not have an inverse f^{-1} for $x \in \mathbb{R}$. It is customary to restrict the domain to the 'largest' set possible. Hence, we can choose to restrict the domain to either $x \in (-\infty, 0]$ (making f an increasing function), or $x \in [0, \infty)$ (making f a decreasing function). Change the domain from $x \in \mathbb{R}$ to $x \in [0, \infty)$.

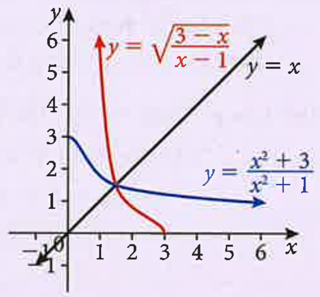
We use a method similar to that in Example 1.23 to find the equation for f^{-1}. First solve for x in terms of y and then interchange the domain (x) and range (y).

$$f: x \mapsto \frac{x^2 + 3}{x^2 + 1} \Rightarrow y = \frac{x^2 + 3}{x^2 + 1} \Rightarrow x^2y + y = x^2 + 3 \Rightarrow x^2y - x^2 = 3 - y$$

$$\Rightarrow x^2(y - 1) = 3 - y \Rightarrow x^2 = \frac{3 - y}{y - 1} \Rightarrow x = \pm\sqrt{\frac{3 - y}{y - 1}}$$

$$\Rightarrow y = \pm\sqrt{\frac{3 - x}{x - 1}}$$

Since we chose to restrict the domain of f to $x \in [0, \infty)$, the range of f^{-1} will be $y \in [0, \infty)$. Therefore, from the working above the resulting inverse function is $f^{-1}(x) = \sqrt{\frac{3 - x}{x - 1}}$. The graphs of f and f^{-1} show symmetry about the line $y = x$.



To find the inverse of a function f:

1. Determine if the function is one-to-one; if not, restrict the domain so that it is.
2. Replace $f(x)$ with y.
3. Solve for x in terms of y.
4. Interchange x and y.
5. Replace y with $f^{-1}(x)$.
6. The domain of f^{-1} is equal to the range of f; and the range of f^{-1} is equal to the domain of f.

Example 1.25

Consider the function $f: x \mapsto \sqrt{x + 3}, x \geqslant -3$

(a) Determine the inverse function f^{-1}

(b) Find the domain of f^{-1}

Solution

(a) Following the steps for finding the inverse of a function gives:

$y = \sqrt{x + 3}$	replace $f(x)$ with y
$y^2 = x + 3$	solve for x in terms of y; squaring both sides
$x = y^2 - 3$	solve for x
$y = x^2 - 3$	interchange x and y
Therefore, $f^{-1}: x \mapsto x^2 - 3$	replace y with $f^{-1}(x)$

(b) The domain explicitly defined for f is $x \geqslant -3$ and since the $\sqrt{\ }$ symbol stands for the principal square root (positive), then the range of f is all positive real numbers, i.e. $y \geqslant 0$. The domain of f^{-1} is equal to the range of f, therefore the domain of f^{-1} is $x \geqslant 0$.

Example 1.26

Consider the functions $f(x) = 2(x + 4)$ and $g(x) = \frac{1 - x}{3}$

(a) Find g^{-1} and state its domain and range.

(b) Solve the equation $(f \circ g^{-1})(x) = 2$

Solution

(a) $y = \frac{1-x}{3}$ replace $f(x)$ with y

$x = \frac{1-y}{3}$ interchange x and y

$3x = 1 - y$ solve for y

$y = -3x + 1$ solved for y

Therefore, $g^{-1}(x) = -3x + 1$ replace y with $g^{-1}(x)$

g is a linear function and its domain is $x \in \mathbb{R}$ and its range is $y \in \mathbb{R}$; therefore, for g^{-1} the domain is $x \in \mathbb{R}$ and the range is $y \in \mathbb{R}$.

(b) $(f \circ g^{-1})(x) = f(g^{-1}(x)) = f(-3x + 1) = 2$

$2[(-3x + 1) + 4] = 2$

$-6x + 2 + 8 = 2$

$-6x = -8$

$x = \frac{4}{3}$

Exercise 1.4

In questions 1–4, assume that f is a one-to-one function.

1. **(a)** If $f(2) = -5$, then what is $f^{-1}(-5)$?

(b) If $f^{-1}(6) = 10$, then what is $f(10)$?

2. **(a)** If $f(-1) = 13$, then what is $f^{-1}(13)$?

(b) If $f^{-1}(b) = a$, then what is $f(a)$?

3. If $g(x) = 3x - 7$, then what is $g^{-1}(5)$?

4. If $h(x) = x^2 - 8x$, with $x \geqslant 4$, then what is $h^{-1}(-12)$?

5. For each pair of functions, show algebraically and graphically that f and g are inverse functions by:

(i) verifying that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$

(ii) sketching the graphs of f and g on the same set of axes with equal scales on the x-axis and y-axis.

Use your GDC to assist in making your sketches on paper.

(a) $f: x \mapsto x + 6$; $g: x \mapsto x - 6$

(b) $f: x \mapsto 4x$; $g: x \mapsto \frac{x}{4}$

(c) $f: x \mapsto 3x + 9$; $g: x \mapsto \frac{1}{3}x - 3$

(d) $f: x \mapsto \frac{1}{x}$; $g: x \mapsto \frac{1}{x}$

(e) $f: x \mapsto x^2 - 2, x \geqslant 0$; $g: x \mapsto \sqrt{x + 2}, x \geqslant -2$

(f) $f: x \mapsto 5 - 7x$; $g: x \mapsto \frac{5 - x}{7}$

(g) $f: x \mapsto \frac{1}{1 + x}$; $g: x \mapsto \frac{1 - x}{x}$

(h) $f: x \mapsto (6 - x)^{\frac{1}{2}}$; $g: x \mapsto 6 - x^2, x \geqslant 0$

(i) $f: x \mapsto x^2 - 2x + 3, x \geqslant 1$; $g: x \mapsto 1 + \sqrt{x - 2}, x \geqslant 2$

(j) $f: x \mapsto \sqrt[3]{\frac{x + 6}{2}}$; $g: x \mapsto 2x^3 - 6$

6. Find the inverse function f^{-1} and state its domain.

(a) $f(x) = 2x - 3$ **(b)** $f(x) = \frac{x + 7}{4}$

(c) $f(x) = \sqrt{x}$ **(d)** $f(x) = \frac{1}{x + 2}$

(e) $f(x) = 4 - x^2, x \geqslant 0$ **(f)** $f(x) = \sqrt{x - 5}$

(g) $f(x) = ax + b, a \neq 0$ **(h)** $f(x) = x^2 + 2x, x \geqslant -1$

(i) $f(x) = \frac{x^2 - 1}{x^2 + 1}, x \leqslant 0$ **(j)** $f(x) = x^3 + 1$

7. Determine if f has an inverse f^{-1}. If not, restrict the domain of f in order to find an inverse function. Graph f and its inverse f^{-1} on the same set of axes.

(a) $f(x) = \frac{2x + 3}{x - 1}$ **(b)** $f(x) = (x - 2)^2$

(c) $f(x) = \frac{1}{x^2}$ **(d)** $f(x) = 2 - x^4$

8. Use your GDC to graph the function $f(x) = \frac{2x}{1 + x^2}, x \in \mathbb{R}$. Find three intervals for which f is a one-to-one function (monotonic) and hence, will have an inverse f^{-1} on the interval. The union of all three intervals is all real numbers.

9. Use the functions $g(x) = x + 3$ and $h(x) = 2x - 4$ to find the indicated value or the indicated function.

(a) $(g^{-1} \circ h^{-1})(5)$ **(b)** $(h^{-1} \circ g^{-1})(9)$

(c) $(g^{-1} \circ g^{-1})(2)$ **(d)** $(h^{-1} \circ h^{-1})(2)$

(e) $g^{-1} \circ h^{-1}$ **(f)** $h^{-1} \circ g^{-1}$

(g) $(g \circ h)^{-1}$ **(h)** $(h \circ g)^{-1}$

10. The reciprocal function, $f(x) = \frac{1}{x}$, is its own inverse (self-inverse). Show that any function in the form $f(x) = \frac{a}{x + b} - b, a \neq 0$ is its own inverse.

When analysing the graph of a function it is often convenient to express a function in the form $y = f(x)$. As we have done throughout this chapter, we can refer to a function such as $f(x) = x^2$ by the equation $y = x^2$.

1.5 Transformations of functions

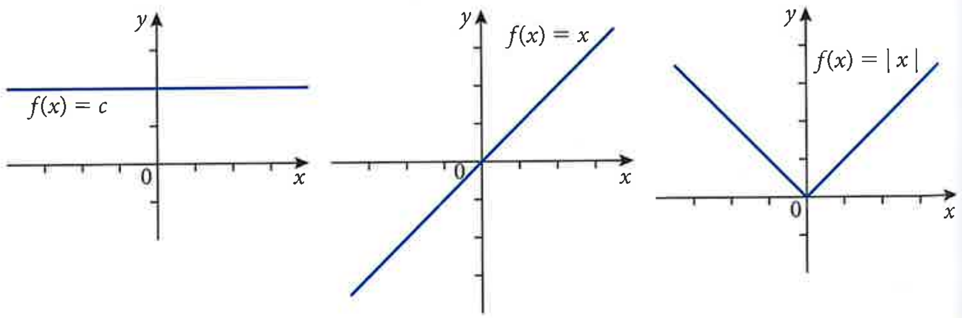
Even when you use your GDC to sketch the graph of a function, it is helpful to know what to expect in terms of the location and shape of the graph – and even more so, if you're not allowed to use your GDC for a particular question. In this section, you will look at how certain changes to the equation of a function can affect, or **transform**, the location and shape of its graph. You will investigate three different types of **transformations** of functions that include how the graph of a function can be **translated**, **reflected** and **stretched** (or **shrunk**). Studying graphical transformations will help you to sketch and visualise many different functions efficiently. You will also take a closer look at two specific functions: the absolute value function, $y = |x|$, and the reciprocal function, $y = \frac{1}{x}$

Graphs of common functions

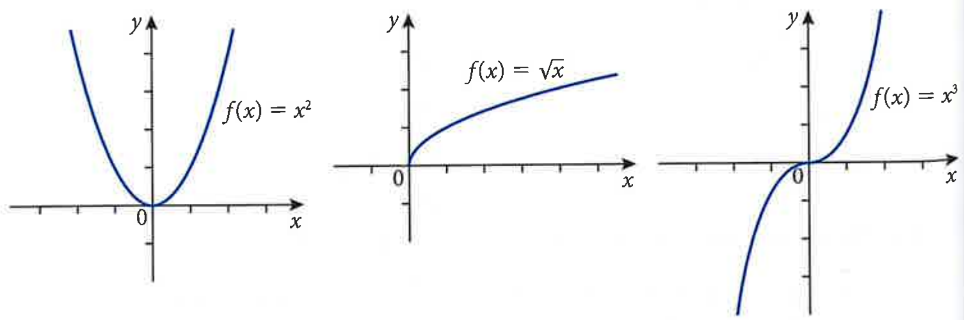
It is important to be familiar with the location and shape of a certain set of common functions. For example, from our previous knowledge about linear equations, we can determine the location of the linear function $f(x) = ax + b$. We know that the graph of this function is a line whose slope is a and whose y-intercept is $(0, b)$.

There are other important basic functions with which you should be familiar, for example, logarithmic and exponential functions, but you will learn about these in later chapters.

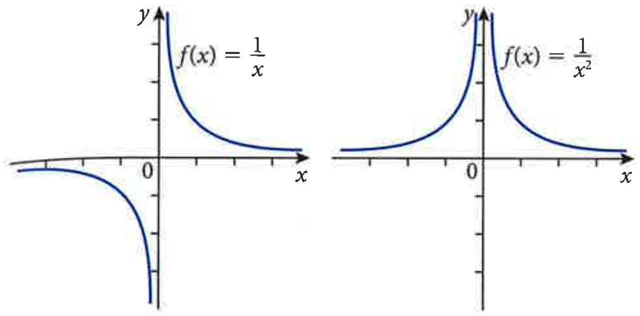
The eight graphs in Figure 1.28 represent some of the most commonly used functions in algebra. You should be familiar with the characteristics of the graphs of these common functions. This will help you predict and analyse the graphs of more complicated functions that are derived from applying one or more transformations to these simple functions.



(a) Constant function (b) Identity function (c) Absolute value function



(d) Squaring function (e) Square root function (f) Cubing function



(g) Reciprocal function (h) Inverse square function

Figure 1.28 Graphs of common functions

We will see that many functions have graphs that are a transformation (translation, reflection or stretch), or a combination of transformations, of one of these common functions.

The word *inverse* can have different meanings in mathematics depending on the context. In section 1.4, 'inverse' is used to describe operations or functions that undo each other. However, 'inverse' is sometimes used to denote the **multiplicative inverse** (or **reciprocal**) of a number or function. This is how it is used in the names for the functions g and h shown above. The function in g is sometimes called the **reciprocal function**.

Vertical and horizontal translations

Use your GDC to graph each of these functions: $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 2$. How do the graphs of g and h compare with the graph of f? The graphs of g and h appear to have the same shape – it's only the location, or position, that has changed compared to f. Although the curves (parabolas) appear to be getting closer together, their vertical separation at every value of x is constant.

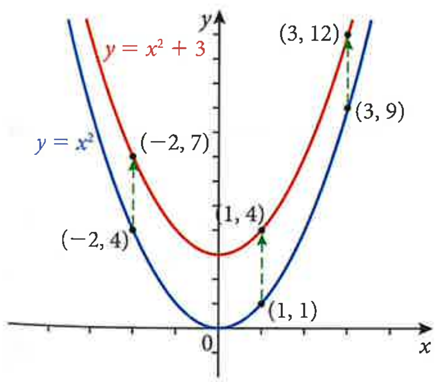


Figure 1.29 Translating $f(x) = x^2$ up 3 units.

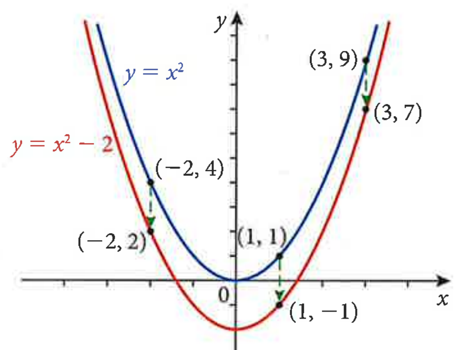


Figure 1.30 Translating $f(x) = x^2$ down 2 units.

Given $k > 0$:

- The graph of $y = f(x) + k$ is obtained by translating the graph of $y = f(x)$ up by k units.
- The graph of $y = f(x) - k$ is obtained by translating the graph of $y = f(x)$ down by k units.

As Figures 1.29 and 1.30 show, we can obtain the graph of $g(x) = x^2 + 3$ by translating (shifting) the graph of $f(x) = x^2$ **up** three units, and we can obtain the graph of $h(x) = x^2 - 2$ by translating the graph of $f(x) = x^2$ **down** two units.

Change function g to $g(x) = (x + 3)^2$ and change function h to $h(x) = (x - 2)^2$. Graph these two functions along with the original function $f(x) = x^2$ on your GDC. This time you can observe that the functions g and h can be obtained by a horizontal translation of f.

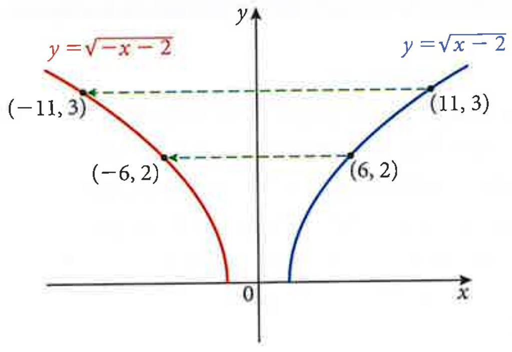


Figure 1.36 Reflecting $y=\sqrt{x-2}$ in the y-axis

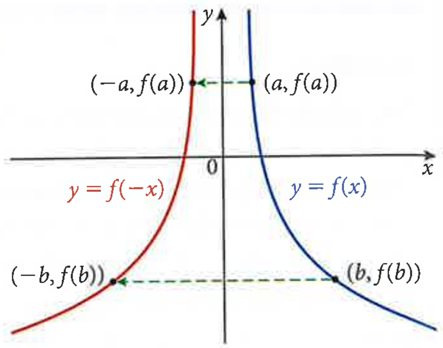


Figure 1.37 Reflecting $y = f(x)$ in the y-axis

The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.
The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

Figures 1.36 and 1.37 illustrate how the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

Example 1.29

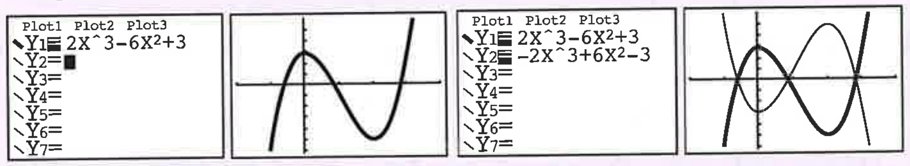
For $g(x) = 2x^3 - 6x^2 + 3$, find:

(a) the function $h(x)$ that is the reflection of $g(x)$ in the x-axis

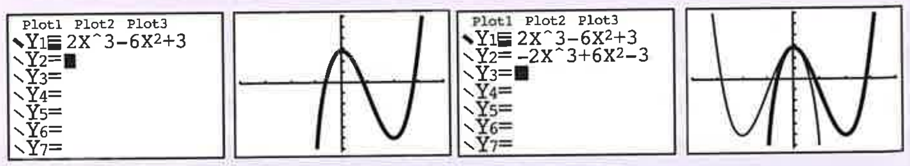
(b) the function $p(x)$ that is the reflection of $g(x)$ in the y-axis.

Solution

(a) Knowing that $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis, then $h(x) = -g(x) = -(2x^3 - 6x^2 + 3) \Rightarrow h(x) = -2x^3 + 6x^2 - 3$ will be the reflection of $g(x)$ in the x-axis. We can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.



(b) Knowing that $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis, we need to substitute $-x$ in for x in $y = g(x)$.
Thus, $p(x) = g(-x) = 2(-x)^3 - 6(-x)^2 + 3 \Rightarrow p(x) = -2x^3 - 6x + 3$ will be the reflection of $g(x)$ in the y-axis. Again, we can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.



Non-rigid transformations: stretching and shrinking

Horizontal and vertical translations, and reflections in the x- and y-axes are called **rigid transformations** because the shape of the graph does not change – only its position is changed. **Non-rigid transformations** cause the shape of the original graph to change. The non-rigid transformations that you will study cause the shape of a graph to stretch or shrink in either the vertical or horizontal direction.

Vertical stretch or shrink

Graph the functions: $f(x) = x^2$, $g(x) = 3x^2$ and $h(x) = \frac{1}{3}x^2$. How do the graphs of g and h compare to the graph of f? Refer to figures 1.38 and 1.40. Clearly, the shape of the graphs of g and h is not the same as the graph of f. Multiplying the function f by a positive number greater than one, or less than one, has distorted the shape of the graph. For a certain value of x, the y-coordinate of $y = 3x^2$ is three times the y-coordinate of $y = x^2$. Therefore, the graph of $y = 3x^2$ can be obtained by **vertically stretching** the graph of $y = x^2$ by a factor of 3 (**scale factor** 3).

Likewise, the graph of $y = \frac{1}{3}x^2$ can be obtained by **vertically shrinking** the graph of $y = x^2$ by scale factor $\frac{1}{3}$

Figures 1.38 and 1.39 below show how multiplying a function by a positive number, a, greater than 1 causes a transformation in which the function stretches vertically by scale factor a. A point (x, y) on the graph of $y = f(x)$ is transformed to the point (x, ay) on the graph of $y = af(x)$.

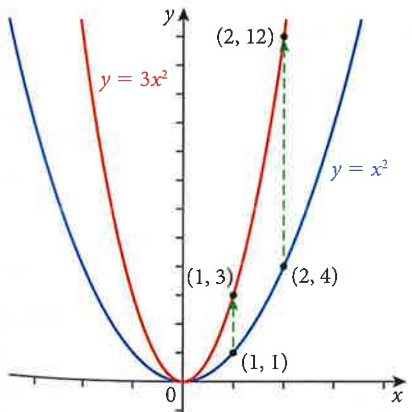


Figure 1.38 Vertical stretch of $y = x^2$ by scale factor 3

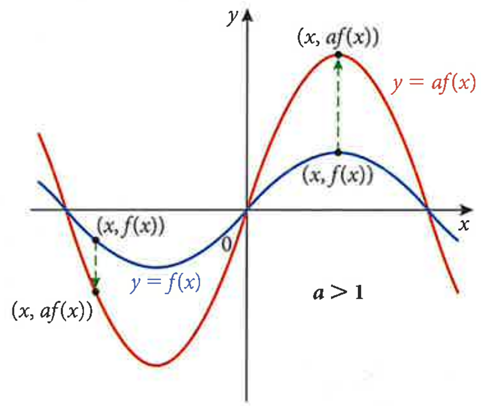


Figure 1.39 Vertical stretch of $y = f(x)$ by scale factor a when $a > 1$

Figures 1.40 and 1.41 below show how multiplying a function by a positive number, a, greater than 0 and less than 1 causes the function to shrink vertically by scale factor a. A point (x, y) on the graph of $y = f(x)$ is transformed to the point (x, ay) on the graph of $y = af(x)$.

If $a > 1$, then the graph of $y = af(x)$ is obtained by vertically stretching the graph of $y = f(x)$.
If $0 < a < 1$, then the graph of $y = af(x)$ is obtained by vertically shrinking the graph of $y = f(x)$.

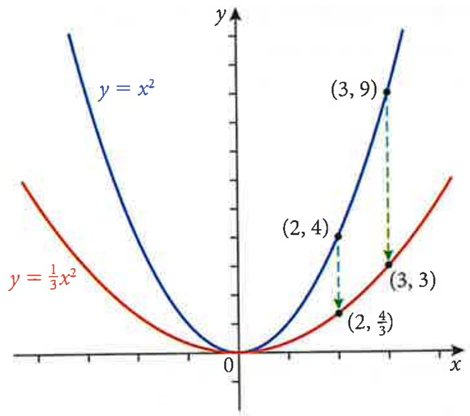


Figure 1.40 Vertical shrink of $y = x^2$ by scale factor $\frac{1}{3}$

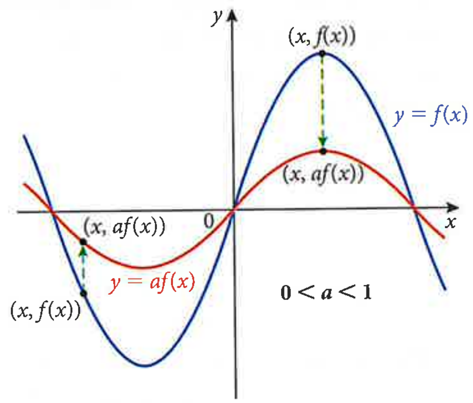


Figure 1.41 Vertical shrink of $y = f(x)$ by scale factor a, when $0 < a < 1$

Horizontal stretch or shrink

We will investigate how the graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$. Given $f(x) = x^2 - 4x$, find another function, $g(x)$, such that $g(x) = f(2x)$. We substitute $2x$ for x in the function f, giving $g(x) = (2x)^2 - 4(2x)$. For the purposes of our investigation, leave $g(x)$ in this form. On your GDC, graph these two functions, $f(x) = x^2 - 4x$ and $g(x) = (2x)^2 - 4(2x)$, using the indicated viewing window and graphing f in bold style (Figure 1.42).

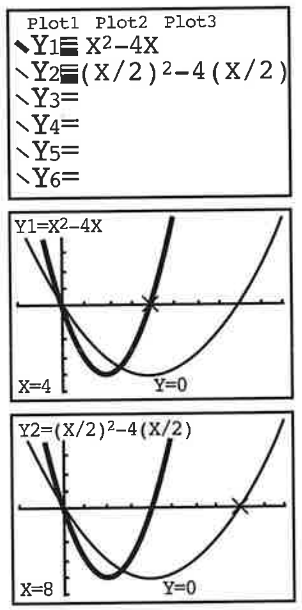


Figure 1.42 Graphs of $y = x^2 - 4x$ (in bold) and $y = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$

Comparing the graphs of the two equations, we can see that $y = g(x)$ is not a translation or a reflection of $y = f(x)$. It is similar to the shrinking effect that occurs for $y = af(x)$ when $0 < a < 1$, except, instead of a vertical shrinking, the graph of $y = g(x) = f(2x)$ is obtained by horizontally shrinking the graph of $y = f(x)$. Given that it is a shrinking, the scale factor must be less than 1.

Consider the point $(4, 0)$ on the graph of $y = f(x)$. The point on the graph of $y = g(x) = f(2x)$ with the same y-coordinate and on the same side of the parabola is $(2, 0)$. The x-coordinate of the point on $y = f(2x)$ is the x-coordinate of the point on $y = f(x)$ multiplied by $\frac{1}{2}$. Use your GDC to confirm this for other pairs of corresponding points on $y = x^2 - 4x$ and $y = (2x)^2 - 4(2x)$ that have the same y-coordinate. The graph of $y = f(2x)$ can be obtained by horizontally shrinking the graph of $y = f(x)$ with scale factor $\frac{1}{2}$. This makes sense because if $f(2x_2) = (2x_2)^2 - 4(2x_2)$ and $f(x_1) = x_1{}^2 - 4x_1$ are to produce the same y-value, then $2x_2 = x_1$, and thus $x_2 = \frac{1}{2}x_1$. Figures 1.43 and 1.44 show how multiplying the x variable of a function by a positive number, a, greater than 1, causes the function to shrink horizontally by scale factor $\frac{1}{a}$. A point (x, y) on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

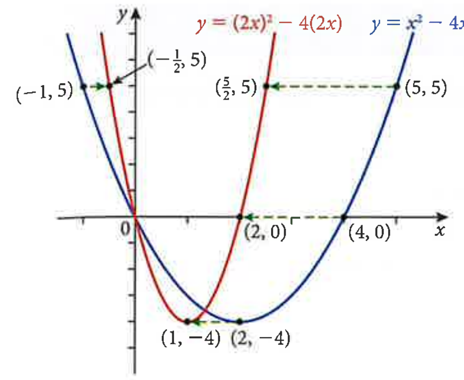


Figure 1.43 Horizontal shrink of $y = x^2 - 4x$ by scale factor $\frac{1}{2}$

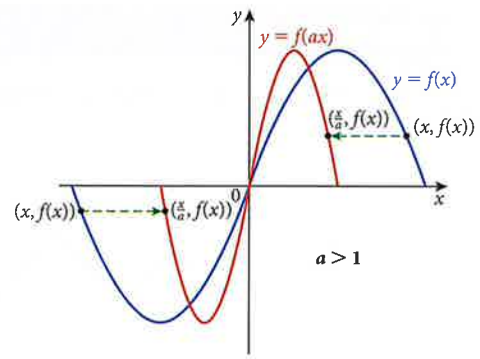


Figure 1.44 Horizontal shrink of $y = f(x)$ by scale factor $\frac{1}{a}$, $a > 1$

If $0 < a < 1$, then the graph of the function $y = f(ax)$ is obtained by a horizontal stretching – rather than a shrinking – of the graph of $y = f(x)$ because the scale factor $\frac{1}{a}$ will be a value between 0 and 1 if $0 < a < 1$. Now, letting $a = \frac{1}{2}$ and, again using the function $f(x) = x^2 - 4x$, find $g(x)$, such that $g(x) = f\left(\frac{1}{2}x\right)$. Substitute $\frac{x}{2}$ for x in f, giving $g(x) = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$. On your GDC, graph the functions f and g using the indicated viewing window with f in bold.

The graph of $y = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$ is a horizontal stretching of the graph of $y = x^2 - 4x$ by scale factor $\frac{1}{a} = \frac{1}{\frac{1}{2}} = 2$. For example, the point $(4, 0)$ on $y = f(x)$ has been moved horizontally to the point $(8, 0)$ on $y = g(x) = f\left(\frac{x}{2}\right)$.

If $a > 1$, then the graph of $y = f(ax)$ is obtained by horizontally shrinking the graph of $y = f(x)$.
If $0 < a < 1$, then the graph of $y = f(ax)$ is obtained by horizontally stretching the graph of $y = f(x)$.

Figures 1.45 and 1.46 below show how multiplying the x variable of a function by a positive number, a, greater than 0 and less than 1, causes the function to stretch horizontally by scale factor $\frac{1}{a}$. A point (x, y) on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

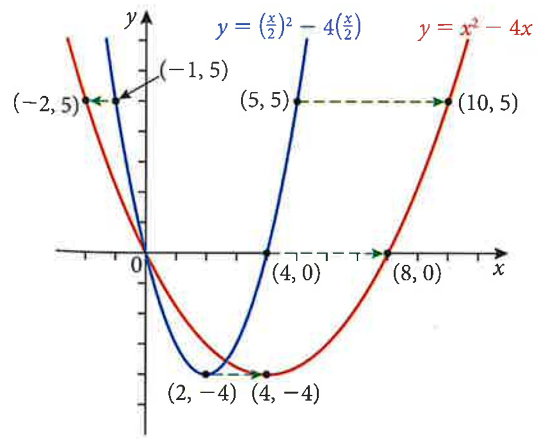


Figure 1.45 Horizontal stretch of $y = x^2 - 4x$ by scale factor 2

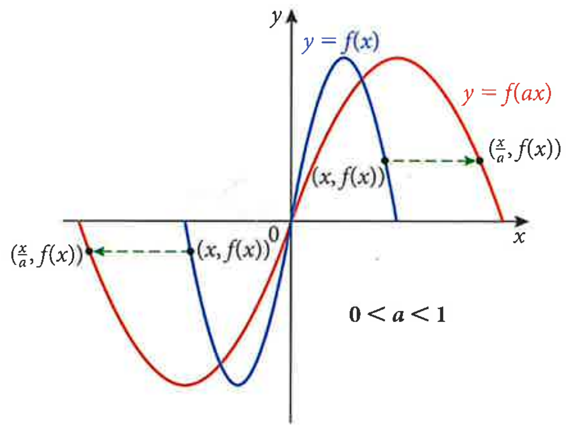
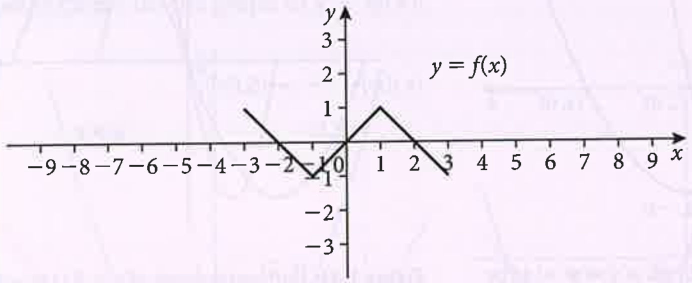


Figure 1.46 Horizontal stretch of $y = f(x)$ by scale factor $\frac{1}{a}$, $0 < a < 1$

Example 1.30

The graph of $y = f(x)$ is shown. Sketch the graph of each transformation.



(a) $y = 3f(x)$

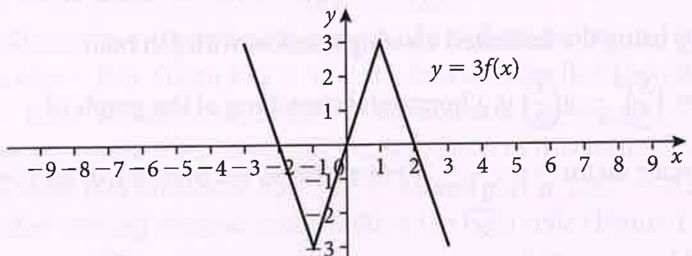
(b) $y = \frac{1}{3}f(x)$

(c) $y = f(3x)$

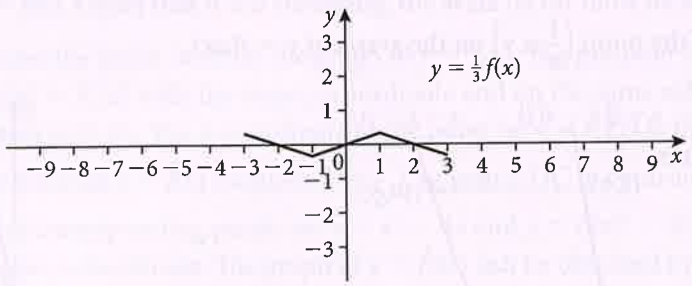
(d) $y = f\left(\frac{1}{3}x\right)$

Solution

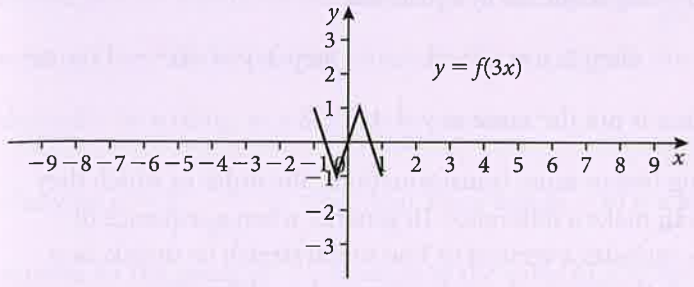
(a) The graph of $y = 3f(x)$ is obtained by vertically stretching the graph of $y = f(x)$ with scale factor 3.



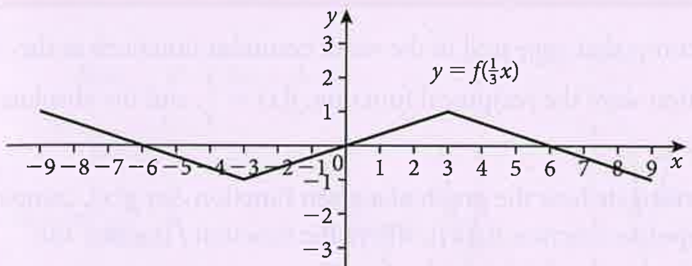
(b) The graph of $y = \frac{1}{3}f(x)$ is obtained by vertically shrinking the graph of $y = f(x)$ with scale factor $\frac{1}{3}$.



(c) The graph of $y = f(3x)$ is obtained by horizontally shrinking the graph of $y = f(x)$ with scale factor $\frac{1}{3}$.



(d) The graph of $y = f\left(\frac{1}{3}\right)x$ is obtained by horizontally stretching the graph of $y = f(x)$ with scale factor 3.



Example 1.31

Describe the sequence of transformations performed on the graph of $y = x^2$ to obtain the graph of $y = 4x^2 - 3$

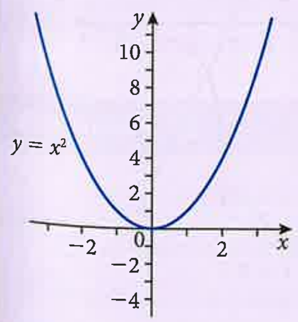
Solution

Step 1: Start with the graph of $y = x^2$

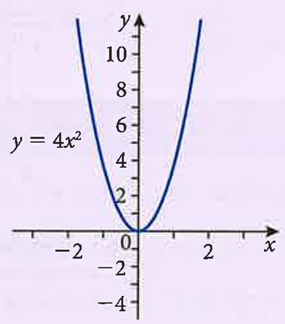
Step 2: Vertically stretch $y = x^2$ by scale factor 4

Step 3: Vertically translate $y = 4x^2$ three units down

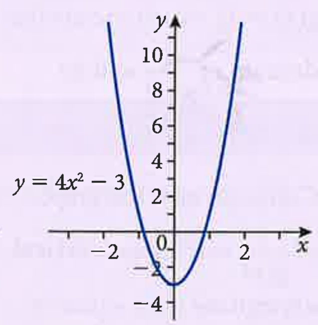
Step 1:



Step 2:



Step 3:



Note that in Example 1.31, a vertical stretch followed by a vertical translation does not produce the same graph if the two transformations are performed in reverse order. A vertical translation followed by a vertical stretch would generate the following sequence of equations:

Step1: $y = x^2$ Step 2: $y = x^2 - 3$ Step 3: $y = 4(x^2 - 3) = 4x^2 - 12$

This final equation is not the same as $y = 4x^2 - 3$

When combining two or more transformations, the order in which they are performed can make a difference. In general, when a sequence of transformations includes a vertical or horizontal stretch or shrink, or a reflection through the x-axis, the order may make a difference.

Reciprocal and absolute value graphs

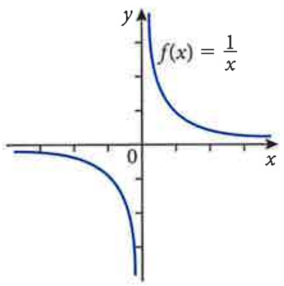


Figure 1.47 Reciprocal function

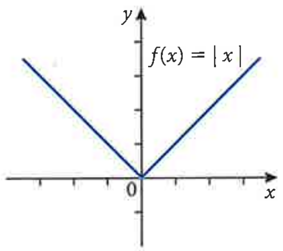


Figure 1.48 Absolute value function

Two of the functions that appeared in the set of common functions at the start of this section were the reciprocal function, $f(x) = \frac{1}{x}$, and the absolute value function.

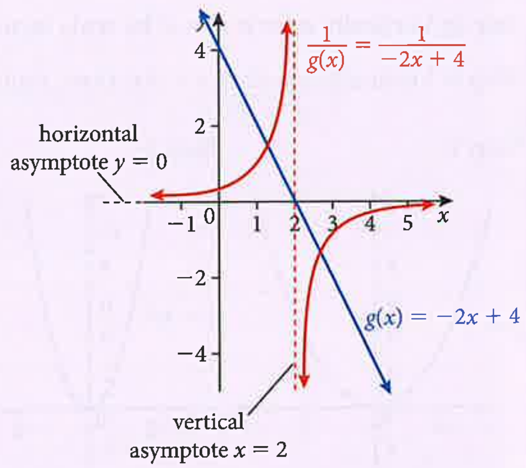
We will now investigate how the graph of a given function, say $g(x)$, compares to that of a composite function $f(g(x))$, where the function f is either the reciprocal function or the absolute value function.

Example 1.32

Given that $f(x) = \frac{1}{x}$, $g(x) = -2x + 4$ and $h(x) = x^2 + 2x - 3$, sketch the graphs of the composite functions $f(g(x))$ and $f(h(x))$. Discuss the characteristics of the graphs of $f \circ g$ and $g \circ f$.

Solution

$f(g(x)) = \frac{1}{g(x)} \Rightarrow y = \frac{1}{-2x + 4}$



Clearly the reciprocal of g will be undefined wherever $g(x) = 0$, which means the domain of $\frac{1}{g(x)}$ will be $\{x : x \in \mathbb{R}, x \neq 2\}$.

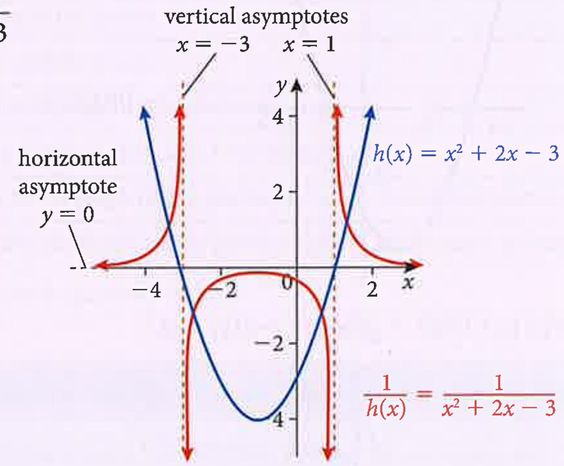
Consequently, the graph of $\frac{1}{g(x)}$ will have a **vertical asymptote** with equation $x = 2$

The graph of g illustrates that as x approaches the value of 2 ($x \to 2$) from the left side, the value of $g(x)$ is always positive but is converging to zero.

Therefore, as $x \to 2$ from the left (or, $x \to 2^-$), the values of $\frac{1}{g(x)}$ become increasingly large in the positive direction. We can express this behaviour symbolically by writing, as $x \to 2^-$, $\frac{1}{g(x)} \to +\infty$.

Similarly, as $x \to 2^+$, $\frac{1}{g(x)} \to -\infty$. Also, the x-axis ($y = 0$) is a **horizontal asymptote** for the graph of $\frac{1}{g(x)}$ because as the value of $g(x)$ becomes very large (either positive or negative), the value of $\frac{1}{g(x)}$ converges to zero; or, in symbols, as $x \to \pm\infty$, $\frac{1}{g(x)} \to 0$.

$$f(h(x)) = \frac{1}{h(x)} = \frac{1}{x^2 + 2x - 3}$$
$$= \frac{1}{(x + 3)(x - 1)}$$



Domain for $\frac{1}{h(x)}$ is $\{x : x \in \mathbb{R}, x \neq -3, x \neq 1\}$

Since $h(x) = 0$ for $x = -3$ and $x = 1$ we anticipate that the graph of its reciprocal, $\frac{1}{h(x)}$, will have vertical asymptotes of $x = -3$ and $x = 1$.

This is confirmed by the fact that as $x \to -3^-$, $\frac{1}{h(x)} \to +\infty$; as $x \to -3^+$, $\frac{1}{h(x)} \to -\infty$; and as $x^- \to 1$, $\frac{1}{h(x)} \to -\infty$; as $x \to 1^+$, $\frac{1}{h(x)} \to +\infty$.

The graph of $\frac{1}{h(x)}$ will also have a horizontal asymptote of $y = 0$ (x-axis) because as $x \to \pm\infty$, $\frac{1}{h(x)} \to 0$

In general, the line $x = c$ is a vertical asymptote of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as x approaches c from either the left or from the right. The line $y = c$ is a horizontal asymptote of the graph of f if $f(x)$ approaches c as $x \to \infty$ or $x \to -\infty$

Example 1.33

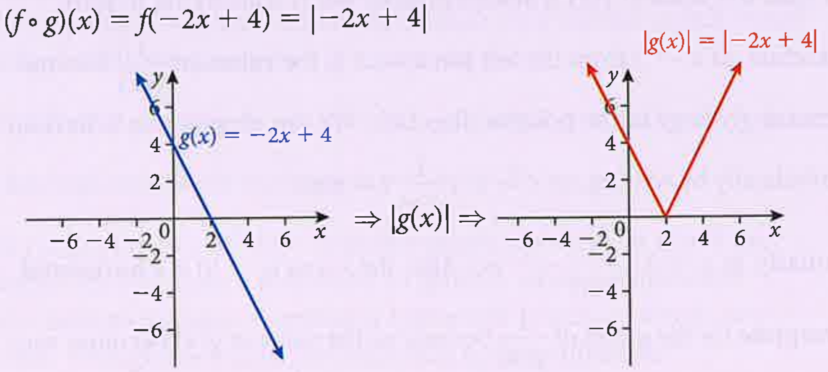
Given $f(x) = |x|$ and using the same functions g and h from Example 1.32,

(a) graph the composite functions $f \circ g$ and $f \circ h$

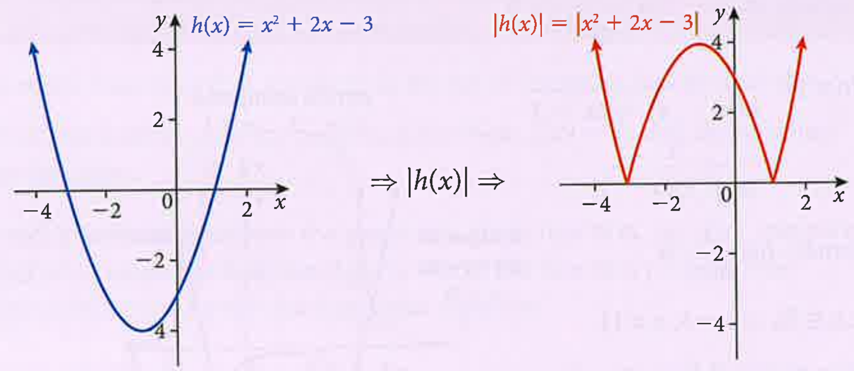
(b) graph the composite functions $g \circ f$ and $h \circ f$

Solution

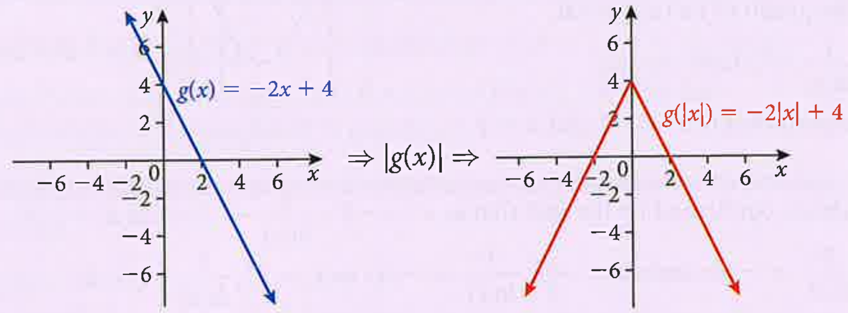
(a) $(f \circ g)(x) = f(-2x + 4) = |-2x + 4|$



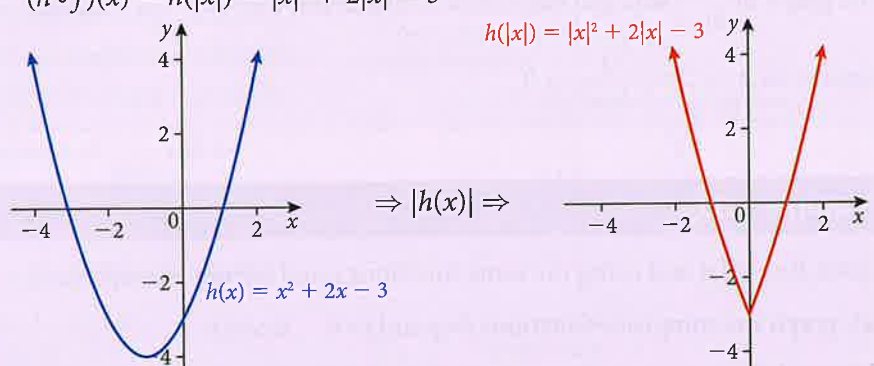
$(f \circ h)(x) = f(x^2 + 2x - 3) = |x^2 + 2x - 3|$



(b) $(g \circ f)(x) = g(|x|) = -2|x| + 4$



$(h \circ f)(x) = h(|x|) = |x|^2 + 2|x| - 3$



From part (a) of Example 1.33, you can see the change that occurs from the graph of a function to the graph of the **absolute value of the function**. Any portion of the graph of $g(x)$ or $h(x)$ that was below the x-axis gets reflected above the x-axis.

In part (b), you can see a change from the graph of a function to the graph of the **function of the absolute value**. Any portion of the graph of $g(x)$ or $h(x)$ that was left of the y-axis is eliminated, and any portion that was to the right of the y-axis is reflected to the left of the y-axis. Since the portion that was right of the y-axis remains, the resulting graph is always symmetric about the y-axis.

Assume that a, h and k are positive real numbers.

Transformed function	Transformation performed on $y = f(x)$
$y = f(x) + k$	vertical translation k units up
$y = f(x) - k$	vertical translation k units down
$y = f(x - h)$	horizontal translation h units right
$y = f(x + h)$	horizontal translation h units left
$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis
$y = af(x)$	vertical stretch ($a > 1$) or shrink ($0 < a < 1$)
$y = f(ax)$	horizontal stretch ($0 < a < 1$) or shrink ($a > 1$)
$y = \|f(x)\|$	portion of the graph of $y = f(x)$ below x-axis is reflected above the x-axis
$y = f(\|x\|)$	symmetric about the y-axis; portion right of the y-axis is reflected in the y-axis

Table 1.4 Summary of transformations on the graphs of functions

Exercise 1.5

1. Sketch the graph of f, without a GDC or plotting points, by using your knowledge of some of the basic functions shown at the start of the Section 1.5.

(a) $f: x \mapsto x^2 - 6$

(b) $f: x \mapsto (x - 6)^2$

(c) $f: x \mapsto |x| + 4$

(d) $f: x \mapsto |x + 4|$

(e) $f: x \mapsto 5 + \sqrt{x - 2}$

(f) $f: x \mapsto \dfrac{1}{x - 3}$

(g) $f: x \mapsto \dfrac{1}{(x + 5)^2} + 2$

(h) $f: x \mapsto -x^3 - 4$

(i) $f: x \mapsto -|x - 1| + 6$

(j) $f: x \mapsto \sqrt{-x + 3}$

(k) $f: x \mapsto 3\sqrt{x}$

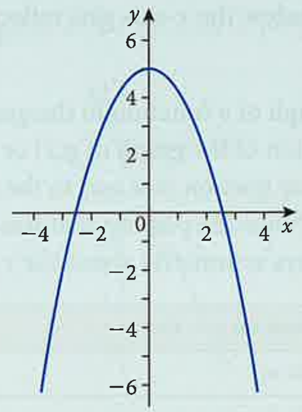
(l) $f: x \mapsto \dfrac{1}{2}x^2$

(m) $f: x \mapsto \left(\dfrac{1}{2}x\right)^2$

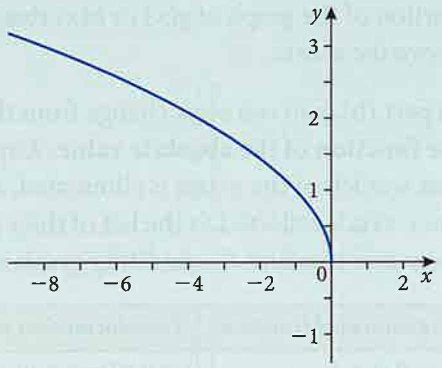
(n) $f: x \mapsto (-x)^3$

2. Write the equation for each graph.

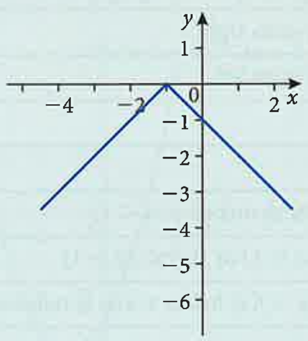
(a)



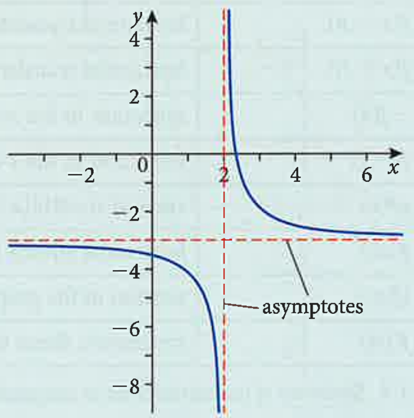
(b)



(c)



(d)



3. The graph of f is given. Sketch the graph of each transformed function.

(a) $y = f(x) - 3$

(b) $y = f(x - 3)$

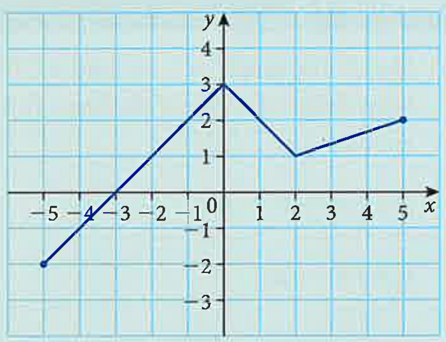
(c) $y = 2f(x)$

(d) $y = f(2x)$

(e) $y = -f(x)$

(f) $y = f(-x)$

(g) $y = 2f(x) + 4$



4. Specify a sequence of transformations to perform on the graph of $y = x^2$ to obtain the graph of the given function.

(a) $g: x \mapsto (x - 3)^2 + 5$

(b) $h: x \mapsto -x^2 + 2$

(c) $p: x \mapsto \frac{1}{2}(x + 4)^2$

(d) $f: x \mapsto [3(x - 1)]^2 - 6$

5. Without using your GDC, for each function $f(x)$, sketch the graph of:

(i) $\dfrac{1}{f(x)}$ **(ii)** $|f(x)|$ **(iii)** $f(|x|)$

Clearly label any intercepts or asymptotes.

(a) $f(x) = \frac{1}{2}x - 4$

(b) $f(x) = (x - 4)(x + 2)$

(c) $f(x) = x^3$

Chapter 1 practice questions

1. The functions f and g are defined as $f: x \mapsto \sqrt{x - 3}$ and $g: x \mapsto x^2 + 2x$. The function $(f \circ g)(x)$ is defined for all $x \in \mathbb{R}$, except for the interval $]a, b[$.

(a) Calculate the values of a and b

(b) Find the range of $f \circ g$

2. Two functions g and h are defined as $g(x) = 2x - 7$ and $h(x) = 3(2 - x)$

Find: **(a)** $g^{-1}(3)$

(b) $(h \circ g)(6)$

3. Consider the functions $f(x) = 5x - 2$ and $g(x) = \dfrac{4 - x}{3}$

(a) Find g^{-1}

(b) Solve the equation $(f \circ g^{-1})(x) = 8$

4. The functions g and h are defined by $g: x \mapsto x - 3$ and $h: x \mapsto 2x$

(a) Find an expression for $(g \circ h)(x)$

(b) Show that $g^{-1}(14) + h^{-1}(14) = 24$

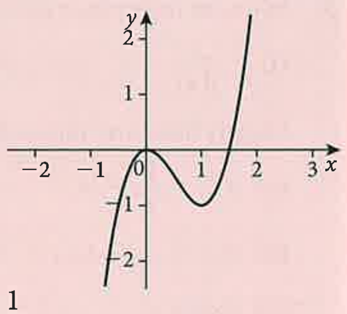
5. Use row operations to show that the following system of three linear equations has an infinite number of solutions.

$$\begin{cases} 2x - y + 3z = 2 \\ 3x + y + 2z = -2 \\ -x + 2y - 3z = -4 \end{cases}$$

6. Use your GDC to solve the following system of three linear equations.

$$\begin{cases} 3x - 2y + 5z = -3 \\ 2x + 6y - 4z = 20 \\ 4x - 3y + 5z = -5 \end{cases}$$

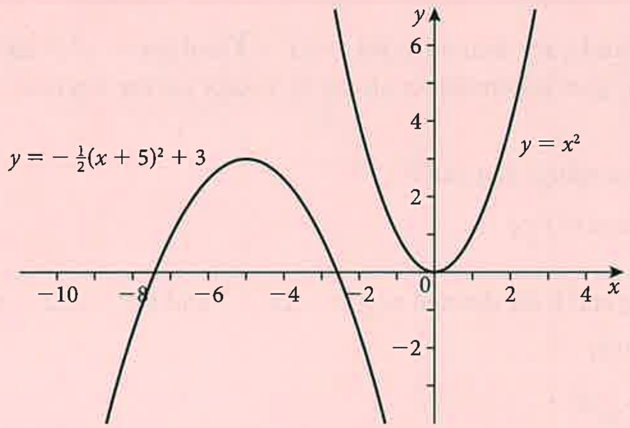
7. The diagram shows the graph of $y = f(x)$. It has maximum and minimum points at $(0, 0)$ and $(1, -1)$, respectively.



(a) Copy the diagram, and then add in the graph of $y = f(x+1) - \frac{1}{2}$

(b) Find the coordinates of the minimum and maximum points of $y = f(x+1) - \frac{1}{2}$

8. The diagram shows parts of the graphs of $y = x^2$ and $y = -\frac{1}{2}(x+5)^2 + 3$



The graph of $y = x^2$ may be transformed into the graph of $y = -\frac{1}{2}(x+5)^2 + 3$ by these transformations:

A reflection in the line $y = 0$, followed by a vertical stretch with scale factor k, followed by a horizontal translation of p units, followed by a vertical translation of q units.

Write down the value of:

(a) k **(b)** p **(c)** q

9. The function f is defined by $f(x) = \frac{4}{\sqrt{16 - x^2}}$, for $-4 < x < 4$

(a) Without using a GDC, sketch the graph of f.

(b) Write down the equation of each vertical asymptote.

(c) Write down the range of the function f.

10. Let $g: x \mapsto \frac{1}{x}, x \neq 0$

(a) Without using a GDC, sketch the graph of g.

The graph of g is transformed to the graph of h by a translation of 4 units to the left and 2 units down.

(b) Find an expression for the function h.

(c) **(i)** Find the x- and y-intercepts of h.

(ii) Write down the equations of the asymptotes of h.

(iii) Sketch the graph of h.

11. Consider $f(x) = \sqrt{x+3}$

(a) Find:

(i) $f(8)$ **(ii)** $f(46)$ **(iii)** $f(-3)$

(b) Find the values of x for which f is undefined.

(c) Let $g: x \mapsto x^2 - 5$. Find $(g \circ f)(x)$.

12. Let $g(x) = \frac{x-8}{2}$ and $h(x) = x^2 - 1$

(a) Find $g^{-1}(-2)$

(b) Find an expression for $(g^{-1} \circ h)(x)$

(c) Solve $(g^{-1} \circ h)(x) = 22$

13. Given the functions $f: x \mapsto 3x - 1$ and $g: x \mapsto \frac{4}{x}$, find the following:

(a) f^{-1} **(b)** $f \circ g$ **(c)** $(f \circ g)^{-1}$ **(d)** $g \circ g$

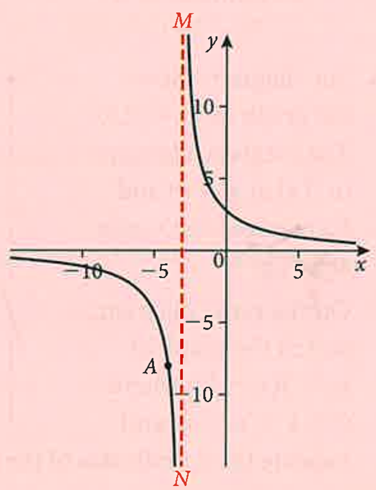
14. **(a)** The diagram shows part of the graph of the function $h(x) = \frac{a}{x-b}$

The curve passes through the point $A(-4, -8)$.

The vertical line MN is an asymptote.

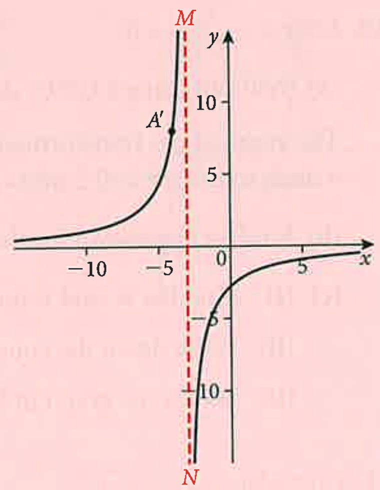
Find the value of:

(i) a **(ii)** b

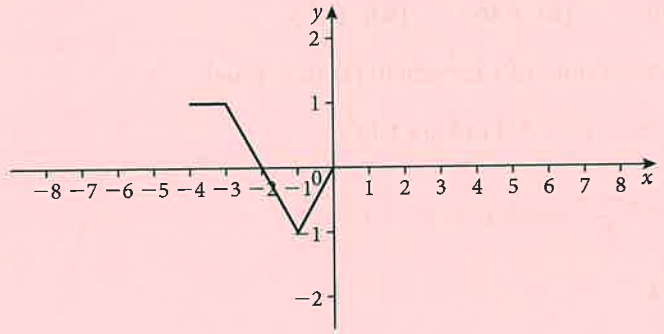


(b) The graph of $h(x)$ is transformed as shown in the diagram. The point A is transformed to $A'(-4, 8)$.

Give a full geometric description of the transformation.



15. The graph of $y = f(x)$ is shown in the diagram.

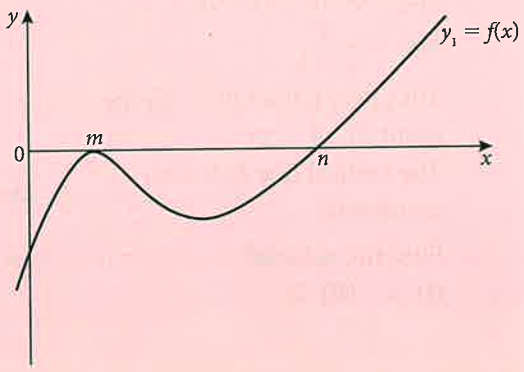


(a) Make two copies of the coordinate system as shown in the diagram but without the graph of $y = f(x)$. On the first diagram sketch a graph of $y = 2f(x)$, and on the second diagram sketch a graph of $y = f(x - 4)$

(b) The point $A(-3, 1)$ is on the graph of $y = f(x)$. The point A' is the corresponding point on the graph of $y = -f(x) - 1$. Find the coordinates of A'.

16. The diagram shows the graph of $y_1 = f(x)$. The x-axis is a tangent to $f(x)$ at $x = m$ and $f(x)$ crosses the x-axis at $x = n$.

On the same diagram, sketch the graph of $y_2 = f(x - k)$, where $0 < k < n - m$ and indicate the coordinates of the points of intersection of y_2 with the x-axis.



17. Given functions $f: x \mapsto x + 1$ and $g: x \mapsto x^3$, find the function $(f \circ g)^{-1}$

18. If $f(x) = \dfrac{x}{x+1}$, for $x \neq -1$ and $g(x) = (f \circ f)(x)$, find:

(a) $g(x)$ **(b)** $(g \circ g)(2)$

19. Let $f: x \mapsto \sqrt{\dfrac{1}{x^2} - 2}$. Find:

(a) the set of real values of x for which f is real and finite

(b) the range of f.

20. The function $f: x \mapsto \dfrac{2x+1}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. Find the inverse function, f^{-1}, clearly stating its domain.

21. The one-to-one function f is defined on the domain $x > 0$ by

$$f(x) = \frac{2x-1}{x+2}$$

(a) State the range, A, of f.

(b) Obtain an expression for $f^{-1}(x)$, for $x \in A$.

22. The function f is defined by $f: x \mapsto x^3$

Find an expression for $g(x)$ in terms of x in each of the following cases:

(a) $(f \circ g)(x) = x + 1$ **(b)** $(g \circ f)(x) = x + 1$

23. **(a)** Find the largest set S of values of x such that the function $f(x) = \dfrac{1}{\sqrt{3 - x^2}}$ takes real values.

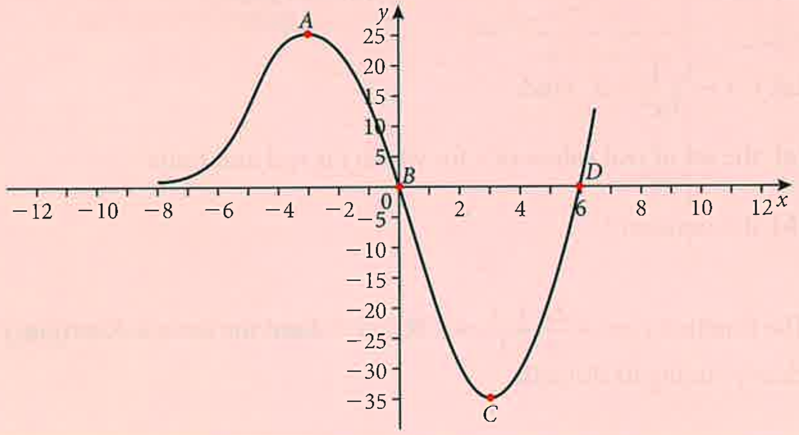
(b) Find the range of the function f defined on the domain S.

24. Let f and g be two functions. Given that $(f \circ g)(x) = \dfrac{x+1}{2}$ and $g(x) = 2x - 1$, find $f(x - 3)$.

25. The diagram shows part of the graph of $y = f(x)$ that passes through the points A, B, C and D.

Sketch, indicating clearly the images of A, B, C and D, the graphs of

(a) $y = f(x - 4)$ **(b)** $y = f(-3x)$



Functions 2