22. Find the values of x for which $|5 - 3x| \leqslant |x + 1|$

23. Solve the inequality $x^2 - 4 + \frac{3}{x} < 0$

24. Solve the inequality $|x - 2| \geqslant |2x + 1|$

25. Let $f(x) = \frac{x+4}{x+1}, x \neq -1$ and $g(x) = \frac{x-2}{x-4}, x \neq 4$

Find the set of values of x such that $f(x) \leqslant g(x)$.

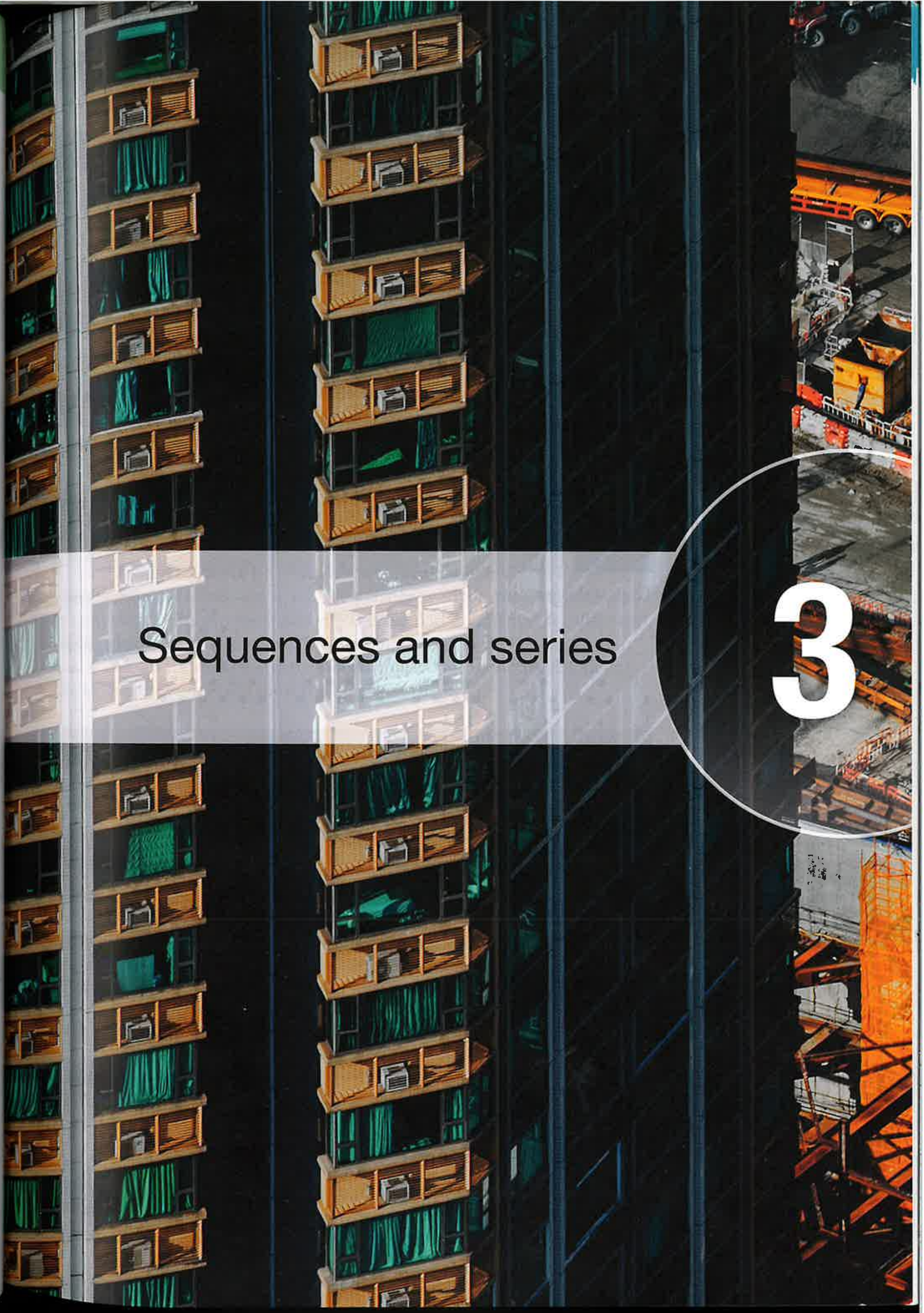
26. Solve the inequality $\left|\frac{x+9}{x-9}\right| \leqslant 2$

27. Find all values of x that satisfy the inequality $\frac{2x}{|x-1|} < 1$

28. Express the fraction $\frac{2x-5}{x^2+x-2}$ as the sum of two fractions.

29. Given that $\frac{2x-48}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}$, find the values of A and B.

30. Write $\frac{a-b}{(x-a)(x-b)}$ as the sum of partial fractions.



3 Sequences and series

Learning objectives

By the end of this chapter, you should be familiar with...

- working with arithmetic and geometric sequences; finding the sum of finite arithmetic and geometric sequences, and finding the sum of infinite geometric series
- using sigma notation
- using the binomial theorem for the expansion of $(a + b)^n$, $n \in \mathbb{N}$
- working with counting principles, including permutations and combinations.

The heights of consecutive bounces of a ball, compound interest, population growth, and Fibonacci numbers are only a few of the applications of sequences and series that we have seen in previous courses. In this chapter we will review these concepts, consolidate understanding, and take them one step further.

3.1 Sequences

Look at this pattern:

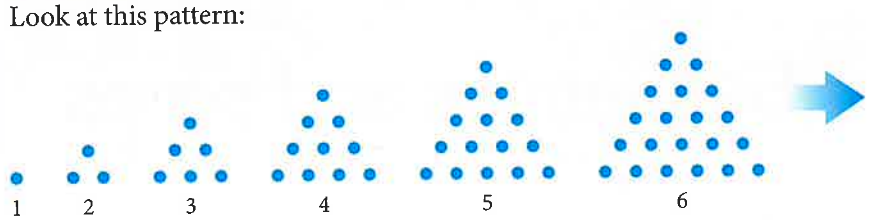


Figure 3.1 Sequence of dots in triangular arrays

The first term represents 1 dot, the second represents 3 dots, etc...
This pattern can be represented as a list of numbers written in a definite order:

$a_1 = 1, a_2 = 3, a_3 = 6, \ldots$

The number a_1 is the first term, a_2 the second term, a_3 the third term, and so on.
The nth term is a_n.

> If a_n is the nth term of a sequence, then a_{n-1} is the term before it and a_{n+1} is the term after.

While the idea of a sequence of numbers, $a_1, a_2, a_3, \ldots$ is straightforward, it is useful to think of a sequence as a function. The sequence in Figure 3.1 can also be described in function notation as:

$f(1) = 1, f(2) = 3, f(3) = 6$, and so on, where the domain is $\mathbb{Z}^+$

Here are some more examples of sequences:

1 6, 12, 18, 24, 30

2 $3, 9, 27, \ldots, 3^k, \ldots$

3 $\left\{\frac{1}{i^2}; i = 1, 2, 3, \ldots, 10\right\}$

4 $\{b_1, b_2, \ldots, b_n, \ldots\}$, sometimes used with an abbreviation $\{b_n\}$

The first and third sequences are **finite** and the second and fourth are **infinite**. In the second and fourth sequences, we were able to define a rule that yields the nth number in the sequence (called the nth term) as a function of n, the term's number. In this sense you can think of a sequence, as a **function** that assigns a **unique** number (a_n) to each positive integer n.

Example 3.1

Find the first 5 terms and the 50th term of the sequence $\{b_n\}$ such that $b_n = 2 - \frac{1}{n^2}$

Solution

Since we are given an explicit expression for the nth term as a function of its term number n, we only need to find the value of that function for the required terms:

$b_1 = 2 - \frac{1}{1^2} = 1$ $\quad b_2 = 2 - \frac{1}{2^2} = 1\frac{3}{4}$ $\quad b_3 = 2 - \frac{1}{3^2} = 1\frac{8}{9}$

$b_4 = 2 - \frac{1}{4^2} = 1\frac{15}{16}$ $\quad b_5 = 2 - \frac{1}{5^2} = 1\frac{24}{25}$ $\quad b_{50} = 2 - \frac{1}{50^2} = 1\frac{2499}{2500}$

So, informally, a **sequence** is an **ordered set** of **real numbers**. That is, there is a first number, a second, and so on. The notation used for these sets is shown in Example 3.1. The way the function was defined in Example 3.1 is called the **explicit** definition of a sequence. There are other ways to define sequences, one of which is the **recursive** definition (also called the **inductive** definition). The following example will show you how this is used.

Example 3.2

Find the first 5 terms and the 20th term of the sequence $\{b_n\}$ such that $b_1 = 5$ and $b_n = 2(b_{n-1} + 3)$

Solution

The defining formula for this sequence is recursive. It allows us to find the nth term b_n if we know the preceding term b_{n-1}. Thus, we can find the second term from the first, the third from the second, and so on. Since we know the first term $b_1 = 5$, we can calculate the rest:

$b_2 = 2(b_1 + 3) = 2(5 + 3) = 16$

$b_3 = 2(b_2 + 3) = 2(16 + 3) = 38$

$b_4 = 2(b_3 + 3) = 2(38 + 3) = 82$

$b_5 = 2(b_4 + 3) = 2(82 + 3) = 170$

So, the first 5 terms of this sequence are 5, 16, 38, 82, and 170. However, to find the 20th term, we must first find all 19 preceding terms. This is one of the drawbacks of this type of definition, unless we can change the definition into explicit form. This can easily be done using a GDC (Figure 3.2).

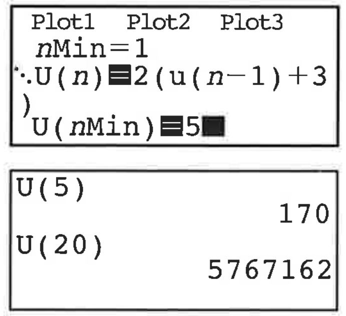


Figure 3.2 GDC screens for Example 3.2

Example 3.3

A Fibonacci sequence is defined recursively as:

$$F_n = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ F_{n-1} + F_{n-2} & n > 2 \end{cases}$$

(a) Find the first 10 terms of the sequence.

(b) Find the sum of the first 10 terms of the sequence.

(c) By observing that $F_1 = F_3 - F_2, F_2 = F_4 - F_3$, and so on, derive a formula for the sum of the first n Fibonacci numbers.

Solution

(a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b) $S_1 = 1, S_2 = 2, S_3 = 4, S_4 = 7, S_5 = 12, S_6 = 20, S_7 = 33, S_8 = 54, S_9 = 88, S_{10} = 143$

(c) Since $F_3 = F_2 + F_1$, then:

$$F_1 = F_3 - F_2$$
$$F_2 = F_4 - F_3$$
$$F_3 = F_5 - F_4$$
$$F_4 = F_6 - F_5$$
$$\vdots \quad \vdots \quad \vdots$$
$$F_n = F_{n+2} - F_{n+1}$$

$$S_n = F_{n+2} - F_2$$

Notice that $S_5 = 12 = F_7 - F_2 = 13 - 1$ and $S_8 = 54 = F_{10} - F_2 = 55 - 1$

Note: parts (a) and (b) can be made easy by using a spreadsheet:

	A	B	C	D
1	F(n)	S(n)		
2	1	1		
3	1	2		
4	2	4		
5	3	7		
6	5	12		
7	8	20		
8	13	33		
9	21	54		
10	34	88		
11	55	143		Let this cell be A2 + A3 Then copy it down
12	89	232		
13	144	376		
14	233	609		
15	377	986		Let this cell be B10 + A11 Then copy it down
16	610	1596		
17	987	2583		

Notice that not all sequences have formulae, either recursive or explicit. Some sequences are given only by listing their terms. There are two types that we will look at here: arithmetic and geometric sequences. We will look at them in the next two sections.

Exercise 3.1

1. Find the first 5 terms of each infinite sequence.

(a) $s(n) = 2n - 3$ **(b)** $g(k) = 2^k - 3$

(c) $f(n) = 3 \times 2^{-n}$ **(d)** $a_n = (-1)^n(2^n) + 3$

(e) $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 3; \quad \text{for } n > 1 \end{cases}$ **(f)** $\begin{cases} b_1 = 3 \\ b_n = b_{n-1} + 2n; \quad \text{for } n \geqslant 2 \end{cases}$

2. Find the first 5 terms and the 50th term of each sequence.

(a) $a_n = 2n - 3$ **(b)** $b_n = 2 \times 3^{n-1}$

(c) $u_n = (-1)^{n-1}\left(\frac{2n}{n^2+2}\right)$ **(d)** $a_n = n^{n-1}$

(e) $a_n = 2a_{n-1} + 5$ and $a_1 = 3$ **(f)** $u_{n+1} = \frac{3}{2u_n + 1}$ and $u_1 = 0$

(g) $b_n = 3b_{n-1}$ and $b_1 = 2$ **(h)** $a_n = a_{n-1} + 2$ and $a_1 = -1$

3. Suggest a recursive definition for each sequence.

(a) $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \frac{1}{192}, \ldots$

(b) $\frac{1}{2}a, \frac{2}{3}a^3, \frac{8}{9}a^5, \frac{32}{27}a^7, \ldots$

(c) $a - 5k, 2a - 4k, 3a - 3k, 4a - 2k, 5a - k, \cdots$

4. Write down a possible formula that gives the nth term of each sequence.

(a) 4, 7, 12, 19, … **(b)** 2, 5, 8, 11, …

(c) $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$ **(d)** $\frac{1}{4}, \frac{3}{5}, \frac{5}{6}, 1, \frac{9}{8}, \ldots$

5. A sequence is defined as $a_n = \frac{F_{n+1}}{F_n}, n > 1$, where F_n is a member of the Fibonacci sequence.

(a) Write down the first 10 terms of a_n

(b) Show that $a_n = 1 + \frac{1}{a_{n-1}}$

6. A sequence is defined as:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n}\right)$$

(a) Find the first 10 terms of this sequence and compare them to Fibonacci numbers.

(b) Show that $3 \pm \sqrt{5} = \frac{(1 \pm \sqrt{5})^2}{2}$

(c) Use the result in **(b)** to verify that F_n satisfies the recursive definition of the Fibonacci sequence.

3.2 Arithmetic sequences

Here are some sequences and a recursive formula for each of them.

$7, 14, 21, 28, 35, 42, \ldots$ $\quad a_1 = 7$ and $a_n = a_{n-1} + 7$, for $n > 1$

$2, 11, 20, 29, 38, 47, \ldots$ $\quad a_1 = 2$ and $a_n = a_{n-1} + 9$, for $n > 1$

$39, 30, 21, 12, 3, -6, \ldots$ $\quad a_1 = 39$ and $a_n = a_{n-1} - 9$, for $n > 1$

Definition of an arithmetic sequence

A sequence $a_1, a_2, a_3, \ldots$ is an arithmetic sequence if there is a constant d for which $a_n = a_{n-1} + d$ for all integers $n > 1$, where d is called the common difference of the sequence, and $d = a_n - a_{n-1}$ for all integers $n > 1$.

Note that in each case, every term is formed by adding a constant number to the preceding term. Sequences formed in this manner are called **arithmetic sequences**.

So, for the sequences above, 7 is the common difference for the first, 9 is the common difference for the second, and -9 is the common difference for the third.

This description gives us the recursive definition of the arithmetic sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable us to see the expression we are seeking:

$$a_2 = a_1 + d$$
$$a_3 = a_2 + d = a_1 + d + d = a_1 + 2d$$
$$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d$$

So, we get to the nth term by adding d to a_1, $(n - 1)$ times.

The general (nth) term of an arithmetic sequence, a_n with first term a_1 and common difference d may be expressed explicitly as $a_n = a_1 + (n - 1)d$.

This result is useful in finding any term of a sequence without knowing the previous terms.

The arithmetic sequence can be looked at as a linear function as explained in the introduction to this chapter. In other words, for every increase of one unit in n, the value of the sequence will increase by d units. As the first term is a_1, the point $(1, a_1)$ belongs to this function. The constant increase d can be considered to be the gradient (slope) of this linear model, hence the nth term, the dependent variable in this case, can be found by using the point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$
$$a_n - a_1 = d(n - 1) \Leftrightarrow a_n = a_1 + (n - 1)d$$

This agrees with our definition of an arithmetic sequence.

Example 3.4

Find the nth term and the 50th term of the sequence 2, 11, 20, 29, 38, 47, …

Solution

This is an arithmetic sequence with first term 2 and common difference 9. Therefore:

$$a_n = a_1 + (n - 1)d = 2 + (n - 1) \times 9 = 9n - 7$$
$$\Rightarrow a_{50} = 9 \times 50 - 7 = 443$$

Example 3.5

(a) Find the recursive and the explicit forms of the definition of the sequence:

$13, 8, 3, -2, \ldots$

(b) Calculate the value of the 25th term.

Solution

(a) This is clearly an arithmetic sequence, with common difference -5.

Recursive definition: $\quad a_1 = 13$, $\quad a_n = a_{n-1} - 5$

Explicit definition: $\quad a_n = 13 - 5(n - 1) = 18 - 5n$

(b) $a_{25} = 18 - 5 \times 25 = -107$

Example 3.6

Find a definition for the arithmetic sequence whose first term is 5 and fifth term is 11.

Solution

Since the fifth term is given, using the explicit form, we have:

$$a_5 = a_1 + (5 - 1)d \Rightarrow 11 = 5 + 4d \Rightarrow d = \frac{3}{2}$$

This leads to the general term:

$$a_n = 5 + \frac{3}{2}(n - 1), \text{ or equivalently}$$

$$\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + \frac{3}{2}, n > 1 \end{cases}$$

Example 3.7

Insert four arithmetic means between 3 and 7.

Solution

Since there are four means between 3 and 7, the problem can be reduced to a situation similar to Example 3.6, by considering the first term to be 3 and the sixth term to be 7. The rest is left as an exercise for you.

In a finite arithmetic sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3, \ldots, a_{k-1}$ are called **arithmetic means** between a_1 and a_k.

Exercise 3.2

1. Insert four arithmetic means between 3 and 7.

2. State whether or not each sequence is an arithmetic sequence. If it is, find the common difference and the 50th term. If it is not, say why not.
   (a) $a_n = 2n - 3$ (b) $b_n = n + 2$
   (c) $c_n = c_{n-1} + 2$, and $c_1 = -1$ (d) $u_n = 3u_{n-1} + 2$
   (e) 2, 5, 7, 12, 19, ... (f) 2, −5, −12, −19, ...

3. For each arithmetic sequence find:
   (i) the 8th term
   (ii) an explicit formula for the nth term
   (iii) a recursive formula for the nth term.
   (a) −2, 2, 6, 10, ... (b) 29, 25, 21, 17, ...
   (c) −6, 3, 12, 21, ... (d) 10.07, 9.95, 9.83, 9.71, ...
   (e) 100, 97, 94, 91, ... (f) $2, \frac{3}{4}, -\frac{1}{2}, -\frac{7}{4}, \ldots$

4. Find five arithmetic means between 13 and −23.

5. Find three arithmetic means between 299 and 300.

6. In an arithmetic sequence, $a_5 = 16$ and $a_{14} = 42$.
   Find an explicit formula for the nth term of this sequence.

7. In an arithmetic sequence, $a_3 = -40$ and $a_9 = -18$.
   Find an explicit formula for the nth term of this sequence.

8. The first three terms and the last term are given for each sequence. Find the number of terms.
   (a) 3, 9, 15, ..., 525 (b) 9, 3, −3, ..., −201
   (c) $3\frac{1}{8}, 4\frac{1}{4}, 5\frac{3}{8}, \ldots, 14\frac{3}{8}$ (d) $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots, 2\frac{5}{6}$
   (e) $1 - k, 1 + k, 1 + 3k, \ldots, 1 + 19k$

9. Find five arithmetic means between 15 and −21.

10. Find three arithmetic means between 99 and 100.

11. In an arithmetic sequence, $a_3 = 11$ and $a_{12} = 47$.
    Find an explicit formula for the nth term of this sequence.

12. In an arithmetic sequence, $a_7 = -48$ and $a_{13} = -10$.
    Find an explicit formula for the nth term of this sequence.

13. The 30th term of an arithmetic sequence is 147 and the common difference is 4. Find a formula for the nth term.

14. The first term of an arithmetic sequence is −7 and the common difference is 3. Is 9803 a term of this sequence? If so, which term?

15. The first term of an arithmetic sequence is 9689 and the 100th term is 8996. Show that the 110th term is 8926. Is 1 a term of this sequence? If so, which term?

16. The first term of an arithmetic sequence is 2 and the 30th term is 147. Is 995 a term of this sequence? If so, which term?

3.3 Geometric sequences

Examine the following sequences and the most likely recursive formula for each of them.

7, 14, 28, 56, 112, 224, ... $a_1 = 7$ and $a_n = a_{n-1} \times 2$, for $n > 1$

2, 18, 162, 1458, 13 122, ... $a_1 = 2$ and $a_n = a_{n-1} \times 9$, for $n > 1$

48, −24, 12, −6, 3, −1.5, ... $a_1 = 48$ and $a_n = a_{n-1} \times (-0.5)$, for $n > 1$

Note that in each case, every term is formed by multiplying a constant number with the preceding term. Sequences formed in this manner are called **geometric sequences.**

Thus, for the preceding sequences, 2 is the common ratio for the first, 9 is the common ratio for the second and −0.5 is the common ratio for the third.

This description gives us the recursive definition of the geometric sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable us to see the expression we are seeking:

$$a_2 = a_1 \times r$$

$$a_3 = a_2 \times r = a_1 \times r \times r = a_1 \times r^2$$

$$a_4 = a_3 \times r = a_1 \times r^2 \times r = a_1 \times r^3$$

We can see that we get to the nth term by multiplying r by a_1, $(n - 1)$ times.

This result is useful in finding any term of a sequence without knowing the previous terms.

Definition of a geometric sequence

A sequence a_1, a_2, a_3, is a **geometric sequence** if there is a constant r for which

$a_n = a_{n-1} \times r$

for all integers $n > 1$, where r is the **common ratio** of the sequence, and $r = a_n \div a_{n-1}$ for all integers $n > 1$.

nth term of a geometric sequence

The general (nth) term of a geometric sequence, a_n with common ratio r and first term a_1 may be expressed explicitly as

$a_n = a_1 \times r^{(n-1)}$

Example 3.8

(a) Find the geometric sequence with $a_1 = 2$ and $r = 3$

(b) Describe the sequence 3, −12, 48, −192, 768, …

(c) Describe the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

(d) Graph the sequence $a_n = \frac{1}{4} \cdot 3^{n-1}$

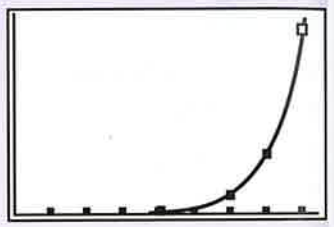
Solution

(a) The geometric sequence is 2, 6, 18, 54, …, $2 \times 3^{n-1}$. Notice that the ratio of any two consecutive terms is 3.

(b) This is a geometric sequence with $a_1 = 3$ and $r = -4$. The nth term is $a_n = 3 \times (-4)^{n-1}$. Notice that when the common ratio is negative, the terms of the sequence alternate in sign.

(c) The nth term of this sequence is $a_n = 1 \times \left(\frac{1}{2}\right)^{n-1}$. Notice that the ratio of any two consecutive terms is $\frac{1}{2}$. Also, notice that the terms decrease in value.

(d) The terms of the sequence lie on the graph of the exponential function $y = \frac{1}{4} \cdot 3^{x-1}$



Example 3.9

At 8:00 a.m., 1000 mg of medicine is given to a patient. At the end of each hour, the amount of medicine in the patient's bloodstream is 60% of the amount present at the beginning of the hour.

(a) What portion of the medicine remains in the patient's bloodstream at 12 noon if no additional medication had been given?

(b) If a second dose of 1000 mg is given at 10:00 a.m., what is the total amount of the medication in the patient's bloodstream at 12 noon?

Solution

(a) Use the geometric model, as there is a constant multiple at the end of each hour. Hence, the amount at the end of any hour after giving the medicine is:

$a_n = a_1 \times r^{n-1}$, where n is the number of hours.

So, at 12 noon, $n = 5$ and $a_5 = 1000 \times 0.6^{(5-1)} = 129.6$ mg

(b) For the second dose, the amount of medicine at noon corresponds to $n = 3$:

$a_3 = 1000 \times 0.6^{(3-1)} = 360$

So, the amount of medicine is 129.6 + 360 = 489.6 mg

Compound interest

Compound interest is an example of a geometric sequence.

Interest compounded annually

When we borrow money we pay interest, and when we invest money we receive interest. Suppose an amount of €1000 is put into a savings account that has an annual interest rate of 6%. How much money will we have in the bank at the end of 4 years?

It is important to note that the 6% interest is given annually and is added to the savings account, so that in the following year it will also earn interest, and so on.

Time in years	Amount in the account (€)
0	1000
1	$1000 + 1000 \times 0.06 = 1000(1 + 0.06)$
2	$1000(1 + 0.06) + (1000(1 + 0.06)) \times 0.06 = 1000(1 + 0.06)(1 + 0.06) = 1000(1 + 0.06)^2$
3	$1000(1 + 0.06)^2 + (1000(1 + 0.06)^2) \times 0.06 = 1000(1 + 0.06)^2(1 + 0.06) = 1000(1 + 0.06)^3$
4	$1000(1 + 0.06)^3 + (1000(1 + 0.06)^3) \times 0.06 = 1000(1 + 0.06)^3(1 + 0.06) = \mathbf{1000(1 + 0.06)^4}$

Table 3.1 Compound interest

This appears to be a geometric sequence with five terms. You will notice that the number of terms is five, as both the beginning and the end of the first year are counted. (Initial value, when time = 0, is the first term.)

In general, if a **principal** of P euros is invested in an account that yields an annual interest rate r (expressed as a decimal), and this interest is added at the end of every year to the principal, then we can use the geometric sequence formula to calculate the **future value** A, which is accumulated after t years.

If we repeat the steps above, with

$A_0 = P$ = initial amount

r = annual interest rate

t = number of years

it becomes easier to develop the formula:

Time in years	Amount in the account
0	$A_0 = P$
1	$A_1 = P + Pr = P(1 + r)$
2	$A_2 = A_1(1 + r) = P(1 + r)^2$
⋮	⋮
t	$A_t = P(1 + r)^t$

Table 3.2 Compound interest formula

Notice that since we are counting from 0 to t, we have $t + 1$ terms, so we are using the geometric sequence formula:

$$a_n = a_1 \times r^{n-1} \Rightarrow A_t = A_1 \times (1 + r)^{t+1-1}$$

Interest compounded n times per year

Suppose that the principal P is invested as before but the interest is paid n times per year. Then $\frac{r}{n}$ is the interest paid every compounding period. Since every year we have n periods, for t years we have nt periods. The amount A in the account after t years is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 3.10

€1000 is invested in an account paying compound interest at a rate of 6% per annum. Calculate the amount of money in the account after 10 years if the compounding is:

(a) annual (b) quarterly (c) monthly.

Solution

(a) The amount after 10 years is:

$$A = 1000(1 + 0.06)^{10} = €1790.85$$

(b) The amount after 10 years quarterly compounding is:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{40} = €1814.02$$

(c) The amount after 10 years monthly compounding is:

$$A = 1000\left(1 + \frac{0.06}{12}\right)^{120} = €1819.40$$

Example 3.11

You invest €1000 at 6% per annum, compounded quarterly. How long will it take for this investment to increase to €2000?

Solution

Let $P = 1000$, $r = 0.06$, $n = 4$, and $A = 2000$ in the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

and then solve for t.

$$2000 = 1000\left(1 + \frac{0.06}{4}\right)^{4t} \Rightarrow 2 = 1.015^{4t}$$

Using a GDC, we can graph the functions $y = 2$ and $y = 1.015^{4t}$ and then find the intersection between their graphs.

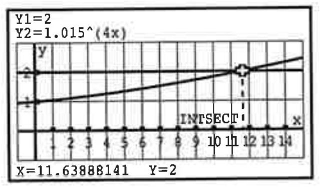


Figure 3.3 GDC screen for the solution to Example 3.11

It will take the €1000 investment 11.64 years to double to €2000. This translates into approximately 47 quarters.

We can check our work to see that this is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{47} = €2013.28$$

In the next chapter you will learn how to solve Example 3.11 algebraically, using logarithms.

Example 3.12

You want to invest €1000. What annual interest rate is needed to make this investment grow to €2000 in 10 years if interest is compounded quarterly?

Solution

Let $P = 1000$, $n = 4$, $t = 10$ and $A = 2000$ in the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

and solve for r:

$$2000 = 1000\left(1 + \frac{r}{4}\right)^{40} \Rightarrow 2 = \left(1 + \frac{r}{4}\right)^{40}$$

$$\Rightarrow 1 + \frac{r}{4} = \sqrt[40]{2} \Rightarrow r = 4\left(\sqrt[40]{2} - 1\right) = 0.0699$$

So, at an annual rate of 7% compounded quarterly, the €1000 investment will grow to at least €2000 in 10 years.

We can check to see if our work is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.07}{4}\right)^{40} = €2001.60$$

Population growth

The same formulae can be applied when dealing with population growth.

Example 3.13

The population of Baden in Austria grows at an annual rate of 0.35%. The population of Baden in 1981 was 23 140. What is the estimate of the population of Baden for 2025?

Solution

This situation can be modelled by a geometric sequence whose first term is 23 140 and common ratio 1.0035. Since we count the population of 1981 among the terms, the number of terms is 45.

In Chapter 4, more realistic population growth models will be explored and more efficient methods will be developed, including the ability to calculate interest that is continuously compounded.

2025 is equivalent to the 45th term in this sequence. The estimated population for Baden is therefore:

Population (2025) $= a_{45} = 23\,140(1.0035)^{44} = 26\,985$

Exercise 3.3

1. For each sequence:
   (i) determine whether the sequence is arithmetic, geometric, or neither.
   (ii) find the common difference for the arithmetic ones and the common ratio for the geometric ones.
   (iii) find the 10th term for each arithmetic or geometric sequence.

   (a) $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots$
   (b) $a_n = 3n - 3$
   (c) $b_n = 2^{n+2}$
   (d) $c_n = 2c_{n-1} - 2$, and $c_1 = -1$
   (e) $u_n = 3u_{n-1}$, and $u_1 = 4$
   (f) 2, 5, 12.5, 31.25, 78.125, …
   (g) 2, −5, 12.5, −31.25, 78.125, …
   (h) 2, 2.75, 3.5, 4.25, 5, …
   (i) $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$
   (j) 52, 55, 58, 61, …
   (k) −1, 3, −9, 27, −81, …
   (l) 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, …
   (m) 3, 6, 12, 18, 21, 27, …
   (n) 6, 14, 20, 28, 34, …
   (o) 2.4, 3.7, 5, 6.3, 7.6, …

2. For each arithmetic or geometric sequence, find:
   (i) the 8th term
   (ii) an explicit formula for the nth term
   (iii) a recursive formula for the nth term.

   (a) −3, 2, 7, 12, …
   (b) 19, 15, 11, 7, …
   (c) −8, 3, 14, 25, …
   (d) 10.05, 9.95, 9.85, 9.75, …
   (e) 100, 99, 98, 97, …
   (f) $2, \frac{1}{2}, -1, -\frac{5}{2}, \ldots$
   (g) 3, 6, 12, 24, …
   (h) 4, 12, 36, 108, …
   (i) 5, −5, 5, −5, …
   (j) 3, −6, 12, −24, …
   (k) 972, −324, 108, −36, …
   (l) $-2, 3, -\frac{9}{2}, \frac{27}{4}, \ldots$
   (m) $35, 25, \frac{125}{7}, \frac{625}{49}, \ldots$
   (n) $-6, -3, -\frac{3}{2}, -\frac{3}{4}, \ldots$
   (o) 9.5, 19, 38, 76, …
   (p) 100, 95, 90.25, …
   (q) $2, \frac{3}{4}, \frac{9}{32}, \frac{27}{256}, \ldots$

3. Find four geometric means between 3 and 96.
4. Find three geometric means between 7 and 4375.
5. Find a geometric mean between 16 and 81.
6. Find four geometric means between 7 and 1701.
7. Find a geometric mean between 9 and 64.
8. The first term of a geometric sequence is 24 and the fourth term is 3. Find the fifth term and an expression for the nth term.
9. The first term of a geometric sequence is 24 and the third term is 6. Find the fourth term and an expression for the nth term.
10. The common ratio in a geometric sequence is $\frac{2}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.
11. Which term of the geometric sequence 6, 18, 54, … is 118 098?
12. The fourth term and the seventh term of a geometric sequence are 18 and $\frac{729}{8}$. Is $\frac{59\,049}{128}$ a term of this sequence? If so, which term is it?
13. The third term and the sixth term of a geometric sequence are 18 and $\frac{243}{4}$. Is $\frac{19\,683}{64}$ a term of this sequence? If so, which term is it?
14. Vitoria put €1500 into a savings account that pays 4% annual interest compounded semiannually. How much will her account hold 10 years later if she does not make any additional investments in this account?
15. At the birth of her daughter Jane, Charlotte deposited £500 into a savings account. The annual interest rate was 4% compounded quarterly. How much money will Jane have on her 16th birthday?
16. How much money should you invest now if you wish to have an amount of €4000 in your account after 6 years if interest is compounded quarterly at an annual rate of 5%?
17. In 2017, the population of a town in Switzerland was estimated to be 7554. How large would the town's population be in 2022 if it grows at a rate of 0.5% annually?
18. The common ratio in a geometric sequence is $\frac{3}{7}$ and the fourth term is $\frac{14}{3}$. Find the second term.

In a finite geometric sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3, \ldots a_k - 1$ are called **geometric means** between a_1 and a_k

In questions 5 and 7, this is also called the **mean proportional**.

19. Which term of the geometric sequence 7, 21, 63, … is 137 781?

20. At her son Erik's birth, Astrid deposited £1000 into a savings account. The annual interest rate was 6% compounded quarterly. How much money will Erik have on his 18th birthday?

3.4 Series

In common usage, the word series is the same thing as sequence. But in mathematics, a series is the sum of terms in a sequence. For a sequence of values a_n, the corresponding series is the sequence of S_n with:

$$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$$

If the terms are in an arithmetic sequence, then the sum is an **arithmetic series**.

Sigma notation

Most of the series we consider in mathematics are **infinite** series. This is to emphasise that the series contain an infinite number of terms. Any sum in the series S_k will be called a partial sum and is given by:

$$S_k = a_1 + a_2 + \ldots + a_{k-1} + a_k$$

For convenience, this partial sum is written using sigma notation:

$$S_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_{k-1} + a_k$$

Sigma notation is a concise and convenient way to represent long sums. The symbol $\sum$ is the Greek capital letter *Sigma* that refers to the initial letter of the word 'sum'. So this expression means the sum of all the terms a_i where i takes the values from 1 to k. We can also write $\sum_{i=m}^{n} a_i$ to mean the sum of the terms a_i where i takes the values from m to n. In such a sum, m is called the lower limit and n the upper limit.

This indicates ending with $i = n$
This indicates addition → $\sum_{i=m}^{n} a_i$
This indicates starting with $i = m$

For example, suppose we measure the heights of six children. We will denote their heights by x_1, x_2, x_3, x_4, x_5 and x_6.

The sum of their heights $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ is written more neatly, by using sigma notation, as $\sum_{i=1}^{6} x_i$.

The symbol $\sum$ means 'add up'. Underneath $\sum$ we see $i = 1$ and on top of it 6. This means that i is replaced by whole numbers starting at the bottom number, 1, until the top number, 6, is reached.

Thus $\sum_{i=3}^{6} x_i = x_3 + x_4 + x_5 + x_6$ and $\sum_{i=3}^{5} x_i = x_3 + x_4 + x_5$

So, the notation $\sum_{i=1}^{n} x_i$ tells us:

- to add the scores x_i
- where to start: x_1
- where to stop: x_n (where n is some integer)

Now take the heights of the children to be $x_1 = 112$ cm, $x_2 = 96$ cm, $x_3 = 120$ cm, $x_4 = 132$ cm, $x_5 = 106$ cm, and $x_6 = 120$ cm.

Then the total height (in cm) is

$$\sum_{k=1}^{6} x_k = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$
$$= 112 + 96 + 120 + 132 + 106 + 120 = 686 \text{ cm}$$

Notice that we have used k instead of i in the formula above. The i is what we call a dummy variable – any letter can be used.

$$\sum_{k=1}^{n} x_k = \sum_{i=1}^{n} x_i$$

Example 3.14

Write each series in full:

(a) $\sum_{i=1}^{5} i^4$ (b) $\sum_{r=3}^{7} 3^r$ (c) $\sum_{j=1}^{n} x_j p(x_j)$

Solution

(a) $\sum_{i=1}^{5} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4$

(b) $\sum_{r=3}^{7} 3^r = 3^3 + 3^4 + 3^5 + 3^6 + 3^7$

(c) $\sum_{j=1}^{n} x_j p(x_j) = x_1 p(x_1) + x_2 p(x_2) + \ldots + x_n p(x_n)$

Example 3.15

Evaluate $\sum_{n=0}^{5} 2^n$

Solution

$\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

Example 3.16

Write the sum $\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\ldots+\frac{99}{100}$ using sigma notation.

Solution

The numerator and denominator in each term are consecutive integers, so they take on the absolute value of $\frac{k}{k+1}$ or any equivalent form. The signs of the terms alternate and there are 99 terms. To take care of the sign, we use some power of (-1) that will start with a positive value. If we use $(-1)^k$, then the first term will be negative, hence we can use $(-1)^{k+1}$ instead. We can therefore write the sum as

$$(-1)^{1+1}\left(\frac{1}{2}\right)+(-1)^{2+1}\left(\frac{2}{3}\right)+(-1)^{3+1}\left(\frac{3}{4}\right)+\ldots+(-1)^{99+1}\left(\frac{99}{100}\right)$$

$$=\sum_{k=1}^{99}(-1)^{k+1}\left(\frac{k}{k+1}\right)$$

Properties of sigma notation

There are a number of useful results we can obtain when we use sigma notation.

1. For example, suppose we have a sum of constant terms:

$$\sum_{i=1}^{5}2$$

What does this mean? If we write this out in full, we get:

$$\sum_{i=1}^{5}2=2+2+2+2+2=5\times 2=10$$

In general, if we sum a constant n times then we can write:

$$\sum_{i=1}^{n}k=k+k+\ldots+k=n\times k=nk$$

2. Suppose we have the sum of a constant multiplied by i. For example:

$$\sum_{i=1}^{5}5i=5\times 1+5\times 2+5\times 3+5\times 4+5\times 5$$
$$=5\times(1+2+3+4+5)=75$$

However, this can also be interpreted as:

$$\sum_{i=1}^{5}5i=5\times 1+5\times 2+5\times 3+5\times 4+5\times 5$$
$$=5\times(1+2+3+4+5)=5\sum_{i=1}^{5}i$$

which implies that:

$$\sum_{i=1}^{5}5i=5\sum_{i=1}^{5}i$$

In general, we can say:

$$\sum_{i=1}^{n}ki=k\times 1+k\times 2+\ldots+k\times n$$
$$=k\times(1+2+\ldots+n)$$
$$=k\sum_{i=1}^{n}i$$

3. Suppose that we need to consider the summation of two different functions, such as:

$$\sum_{k=1}^{n}(k^2+k^3)=(1^2+1^3)+(2^2+2^3)+\ldots+(n^2+n^3)$$
$$=(1^2+2^2+\ldots+n^2)+(1^3+2^3+\ldots+n^3)$$
$$=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k^3$$

In general

$$\sum_{k=1}^{n}(f(k)+g(k))=\sum_{k=1}^{n}f(k)+\sum_{k=1}^{n}g(k)$$

4. At times it is convenient to change powers, for example:

(i) $\sum_{i=1}^{k}a_i=a_1+a_2+\ldots+a_{k-1}+a_k$ is the same as $\sum_{i=0}^{k-1}a_{i+1}$

$$=a_1+a_2+\ldots+a_{k-1}+a_k$$

(ii) $\sum_{i=1}^{n}f(i)=\sum_{i=1}^{k}f(i)+\sum_{i=k+1}^{n}f(i)$ or equivalently $\sum_{i=k+1}^{n}f(i)=\sum_{i=1}^{n}f(i)-\sum_{i=1}^{k}f(i)$

Arithmetic series

In arithmetic series, we are concerned with adding the terms of arithmetic sequences. It is very helpful to be able to find an easy expression for the partial sums of such a series.

Let's start with an example:

Find the partial sum for the first 50 terms of the series

$3+8+13+18+\ldots$

Write the 50 terms in ascending order and then in descending order underneath. Add the terms together as shown.

$$S_{50}=3+8+13+\ldots+248$$
$$S_{50}=248+243+238+\ldots+3$$
$$2S_{50}=251+251+251+\ldots+251$$

There are 50 terms in this sum, and hence

$$2S_{50}=50\times 251\Rightarrow S_{50}=6275$$

This reasoning can be extended to any arithmetic series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. We can construct the series in two ways: Forwards, by adding d to a_1 repeatedly, and backwards by subtracting d from a_n repeatedly. We get the following two expressions for the sum:

$$S_n=a_1+a_2+a_3+\ldots+a_n=a_1+(a_1+d)+(a_1+2d)+\ldots+(a_1+(n-1)d)$$

$$S_n=a_n+a_{n-1}+a_{n-2}+\ldots+a_1=a_n+(a_n-d)+(a_n-2d)+\ldots+(a_n-(n-1)d)$$

By adding, term by term vertically, we get:

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d) \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d) \\ 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) \end{aligned}$$

Since there are n terms, we can reduce the expression above to:

$$2S_n = n(a_1 + a_n)$$

which can be reduced to:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

which in turn can be changed to give an interesting perspective of the sum:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

which is n times the average of the first and last terms!

If we substitute $a_1 + (n-1)d$ for a_n then we get an alternative formula for the sum:

$$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d) = \frac{n}{2}(2a_1 + (n-1)d)$$

The partial sum S_n of an arithmetic series is given by one of the following:
$S_n = \frac{n}{2}(a_1 + a_n)$, or $S_n = n\left(\frac{a_1 + a_n}{2}\right)$, or $S_n = \frac{n}{2}(2a_1 + (n-1)d)$

Example 3.17

Find the partial sum for the first 50 terms of the series $3 + 8 + 13 + 18 + \ldots$

Solution

Using the second formula for the sum we get:

$$S_{50} = \frac{50}{2}(2 \times 3 + (50-1)5) = 25 \times 251 = 6275$$

Using the first formula requires that we know the nth term.
So, $a_{50} = 3 + 49 \times 5 = 248$ which now can be used:

$$S_{50} = 25(3 + 248) = 6275$$

Example 3.18

You are given a sequence of figures as shown in the diagram.

P1 P2 P3

(a) P1 has six line segments. How many line segments are in P20?

(b) Is there a figure with 4401 segments? If so, which one? If not, why not?

(c) Find the total number of line segments in the first 880 figures.

Solution

(a) In each new figure, 5 line segments are added. This is an arithmetic sequence with first term 6, and common difference 5.
Using the nth term form:

$$P20 = 6 + 5(20 - 1) = 101$$

(b) The term whose value is 4401 satisfies the nth term form:

$$4401 = 6 + 5(n-1), \text{ thus } n = \frac{4401 - 6}{5} + 1 = 880$$

Therefore, 4401 is the 880th figure.

(c) We use one of the formulae for the arithmetic series:

$$S_{880} = \frac{880}{2}(6 + 4401) = 1\,939\,080$$

$$\text{or } S_{880} = \frac{880}{2}(2 \times 6 + 5(880 - 1)) = 1\,939\,080$$

Geometric series

As is the case with arithmetic series, in several cases it is desirable to find a general expression for the nth partial sum of a geometric series.

Let us start with an example:

Find the partial sum for the first 20 terms of the series $3 + 6 + 12 + 24 + \ldots$

We express S_{20} in two different ways and subtract them:

$$\begin{aligned} S_{20} &= 3 + 6 + 12 + \ldots + 1572864 \\ 2S_{20} &= \phantom{3 +{}} 6 + 12 + \ldots + 1572864 + 3\,145\,728 \\ -S_{20} &= 3 - 3\,145\,728 \\ \Rightarrow S_{20} &= 3\,145\,725 \end{aligned}$$

This reasoning can be extended to any geometric series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio $r \neq 1$. We can construct the series in two ways as before, and, using the definition of the geometric sequence, $a_n = a_{n-1} \times r$, then:

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \\ rS_n &= ra_1 + ra_2 + ra_3 + \ldots + ra_{n-1} + ra_n \\ &= a_2 + a_3 + \ldots + a_{n-1} + a_n + ra_n \end{aligned}$$

Now, we subtract the second expression from the first to get:

$$S_n - rS_n = a_1 - ra_n \Rightarrow S_n(1 - r) = a_1 - ra_n \Rightarrow S_n = \frac{a_1 - ra_n}{1 - r}, r \neq 1$$

This expression, however, requires that r, a_1, and a_n be known to find the sum. But, using the nth term expression developed earlier, we can simplify this sum formula to:

$$S_n = \frac{a_1 - ra_n}{1 - r} = \frac{a_1 - ra_1r^{n-1}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}, r \neq 1$$

Partial sum of a geometric series
The partial sum, S_n, of n terms of a geometric sequence with common ratio r $(r \neq 1)$ and first term a_1 is:

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \left[\text{equivalent to } S_n = \frac{a_1(r^n - 1)}{r - 1}\right]$$

Example 3.19

Find the partial sum for the first 20 terms of the series $3 + 6 + 12 + 24 + \ldots$

Solution

$$S_{20} = 3 \times \frac{(1 - 2^{20})}{1 - 2} = \frac{3(1 - 1\,048\,576)}{-1} = 3\,145\,725$$

Infinite geometric series

Consider the series $\sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Consider also finding the partial sums for 10, 20, and 100 terms.
We are looking for are the partial sums of a geometric series:

$$\sum_{k=1}^{10} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}} \approx 3.996$$

$$\sum_{k=1}^{20} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} \approx 3.999\,996$$

$$\sum_{k=1}^{100} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{100}}{1 - \frac{1}{2}} \approx 4$$

As the number of terms increases, the partial sum appears to be approaching the number 4. This is no coincidence. In the language of limits,

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = \lim_{n\to\infty} 2 \times \frac{1 - \left(\frac{1}{2}\right)^{k}}{1 - \frac{1}{2}} = 2 \times \frac{1 - 0}{\frac{1}{2}} = 4, \text{ since } \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n} = 0$$

This type of problem allows us to extend the usual concept of a sum of a **finite** number of terms to make sense of sums in which an **infinite** number of terms are involved. Such series are called **infinite series.**

One thing to be made clear about infinite series is that they are not true sums! The associative property of addition of real numbers allows us to extend the definition of the sum of two numbers, such as $a + b$, to three or four, or n numbers, but not to an infinite number of numbers. For example, you can add any specific number of 5s together and get a real number, but if you add an infinite number of 5s together, you cannot get a real number. The remarkable thing about infinite series though, is that in some cases, such as the example above, the sequence of partial sums (which are true sums) approaches a finite limit L. The limit in our example is 4.

We write this as $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty}(a_1 + a_2 + \ldots + a_n) = L$

We say that the series **converges** to L, and it is convenient to define L as the **sum of the infinite series.** We use the notation:

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L$$

We can therefore write the limit in the previous example:

$$\sum_{k=1}^{\infty} 2\left(\frac{1}{2}\right)^{k-1} = \lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = 4$$

If the series does not have a limit, then it **diverges.**

We are now ready to develop a general rule for **infinite geometric series.** As we know, the sum of a geometric sequence is given by:

$$S_n = \frac{a_1 - ra_n}{1 - r} = \frac{a_1 - ra_1r^{n-1}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}, r \neq 1$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and hence:

$$\lim_{n\to\infty} S_n = S = \lim_{n\to\infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}$$

We will call this **the sum of an infinite convergent geometric sequence.** In all other cases the series diverges. The proof is left as an exercise.

In the case of $\sum_{k=1}^{\infty} 2\left(\frac{1}{2}\right)^{k-1} = \frac{2}{1 - \frac{1}{2}} = 4$ as already shown.

Sum of an infinite convergent geometric sequence
The sum, S_∞, of an infinite convergent geometric sequence with first term a_1 such that the common ratio, r, satisfies the condition $|r| < 1$ is given by:

$$S_\infty = \frac{a_1}{1 - r}$$

Example 3.20

A rational number is a number that can be expressed as a quotient of two integers. Show that $0.\bar{6} = 0.666\ldots$ is a rational number.

Solution

$$0.\bar{6} = 0.666\ldots = 0.6 + 0.06 + 0.006 + 0.0006 + \ldots$$

$$= \frac{6}{10} + \frac{6}{10}\cdot\frac{1}{10} + \frac{6}{10}\cdot\left(\frac{1}{10}\right)^2 + \frac{6}{10}\cdot\left(\frac{1}{10}\right)^3 + \ldots$$

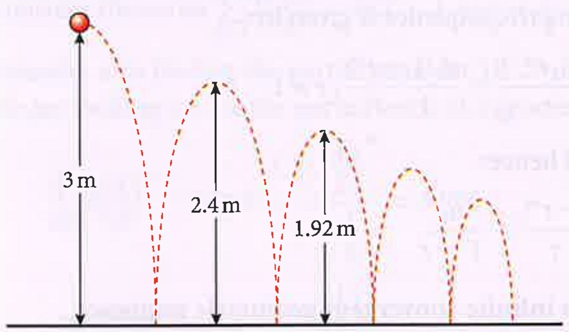
This is an infinite geometric series with $a_1 = \frac{6}{10}$ and $r = \frac{1}{10}$, therefore:

$$0.\bar{6} = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{10}\cdot\frac{10}{9} = \frac{2}{3}$$

Example 3.21

A ball has elasticity such that, on each bounce, it bounces to 80% of its previous height. Find the total vertical distance travelled down and up by this ball when it is dropped from a height of 3 m and is allowed to keep bouncing until it comes to rest. Ignore friction and air resistance.

Solution



After the ball is dropped, the initial 3 m, it bounces up and down a distance of 2.4 m. On each bounce after the first bounce, the ball travels 0.8 times the previous height twice – once upwards and once downwards. So, the total vertical distance is given by

$$h = 3 + 2[2.4 + (2.4 \times 0.8) + (2.4 \times 0.8^2) + \ldots] = 3 + 2 \times l$$

The terms inside the square brackets form an infinite geometric series with $a_1 = 2.4$ and $r = 0.8$. The value of that quantity is:

$$l = \frac{2.4}{1 - 0.8} = 12$$

Hence the total distance required is $h = 3 + 2(12) = 27$ m

Applications of series to compound interest calculations

Annuities

An **annuity** is a sequence of equal periodic payments. If you are saving money by depositing the same amount at the end of each compounding period, the annuity is called an **ordinary annuity**. Using geometric series, you can calculate the **future value (FV)** of this annuity, which is the amount of money you have after making the last payment.

You invest €1000 at the end of each year for 10 years at a fixed annual interest rate of 6%, as shown in Table 3.3.

Year	Amount invested (€)	Future value (€)
10	1000	1000
9	1000	$1000(1 + 0.06)$
8	1000	$1000(1 + 0.06)^2$
⋮		
1	1000	$1000(1 + 0.06)^9$

Table 3.3 Calculating the future value

The future value *FV* of this investment is the sum of the entries in the last column:

$$FV = 1000 + 1000(1 + 0.06) + 1000(1 + 0.06)^2 + \ldots + 1000(1 + 0.06)^9$$

This sum is a partial sum of a geometric series with $n = 10$ and $r = 1 + 0.06$

Hence:

$$FV = \frac{1000(1 - (1 + 0.06)^{10})}{1 - (1 + 0.06)} = \frac{1000(1 - (1 + 0.06)^{10})}{-0.06} = €13\,180.79$$

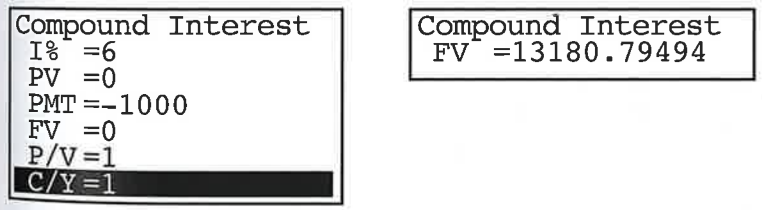


Figure 3.4 This result can also be produced with a GDC

We can generalise the previous formula in the same manner. Let the periodic payment be R, and the periodic interest rate be i – that is, $i = \frac{r}{n}$. Let the number of periodic payments be m.

Period	Amount invested	Future value
m	R	R
$m - 1$	R	$R(1 + i)$
$m - 2$	R	$R(1 + i)^2$
⋮		
1	R	$R(1 + i)^{m-1}$

Table 3.4 Formula for calculating the future value

The future value FV is the sum of the entries in the last column:

$$FV = R + R(1+i) + R(1+i)^2 + \ldots + R(1+i)^{m-1}$$

This is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence:

$$FV = \frac{R(1-(1+i)^m)}{1-(1+i)} = \frac{R(1-(1+i)^m)}{-i} = R\left(\frac{(1+i)^m - 1}{i}\right)$$

If the payment is made at the beginning of the period rather than at the end, then the annuity is called an **annuity due** and the future value after m periods will be slightly different.

Period	Amount invested	Future value
m	R	$R(1+i)$
$m-1$	R	$R(1+i)^2$
$m-2$	R	$R(1+i)^3$
1	R	$R(1+i)^m$

The future value of this investment is the sum of the entries in the last column:

$$FV = R(1+i) + R(1+i)^2 + \ldots + R(1+i)^{m-1} + R(1+i)^m$$

This is a partial sum of a geometric series with m terms and $r = 1 + i$.

$$FV = \frac{R(1+i)(1-(1+i)^m)}{1-(1+i)} = \frac{R(1+i-(1+i)^{m+1})}{-i} = R\left(\frac{(1+i)^{m+1}-1}{i} - 1\right)$$

If the previous investment is made at the beginning of the year rather than at the end, then in 10 years we have:

$$FV = R\left(\frac{(1+i)^{m+1}-1}{i}\right) - 1 = 1000\left(\frac{(1+0.06)^{10+1}-1}{0.06} - 1\right) = 13\,971.64$$

Exercise 3.4

1. Find the sum of the arithmetic sequence $11 + 17 + \ldots + 365$

2. Find the sum of this sequence:

$$2 - 3 + \frac{9}{2} - \frac{27}{4} + \ldots - \frac{177\,147}{1024}$$

3. Evaluate $\sum_{k=0}^{13}(2 - 0.3k)$

4. Evaluate $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \ldots$

5. Evaluate $\frac{1}{3} + \frac{\sqrt{3}}{12} + \frac{1}{16} + \frac{\sqrt{3}}{64} + \frac{3}{256} + \ldots$

6. Express each repeating decimal as a fraction:

   (a) $0.\overline{52}$ (b) $0.4\overline{53}$ (c) $3.01\overline{37}$

7. At the beginning of every month, Maggie invests \$150 in an account that pays a 6% annual rate of interest. How much money will there be in the account after six years?

8. Find the sum of each series.

   (a) $9 + 13 + 17 + \ldots + 85$ (b) $8 + 14 + 20 + \ldots + 278$

   (c) $155 + 158 + 161 + \ldots + 527$

9. The kth term of an arithmetic sequence is $2 + 3k$. Find, in terms of n, the sum of the first n terms of this sequence.

10. For the arithmetic sequence that begins $17 + 20 + 23 + \ldots$, for what value of n will the partial sum S_n of the sequence exceed 678?

11. For the arithmetic sequence that begins $-18 - 11 - 4 \ldots$, for what value of n will the partial sum S_n of the sequence exceed 2335?

12. An arithmetic sequence has a as first term and $2d$ as common difference, i.e., $a, a + 2d, a + 4d, \ldots$. The sum of the first 50 terms is T. Another sequence, with first term $a + d$ and common difference $2d$, is combined with the first one to produce a new arithmetic sequence. Let the sum of the first 100 terms of the new combined sequence be S. If $2T + 200 = S$, find d.

13. Consider the arithmetic sequence $3, 7, 11, \ldots, 999$.

    (a) Find the number of terms and the sum of this sequence.

    (b) Create a new sequence by removing every third term, i.e., $11, 23, \ldots$. Find the sum of the terms of the remaining sequence.

14. The sum of the first 10 terms of an arithmetic sequence is 235 and the sum of the second 10 terms is 735. Find the first term and the common difference.

15. Use your GDC or a spreadsheet to evaluate each sum.

    (a) $\sum_{k=1}^{20}(k^2 + 1)$ (b) $\sum_{i=3}^{17}\left(\frac{1}{i^2+3}\right)$ (c) $\sum_{n=1}^{100}(-1)^n\frac{3}{n}$

16. Find the sum of the arithmetic series $13 + 19 + \ldots + 367$

17. Find the sum of the arithmetic series:

$$2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots - \frac{4096}{177\,147}$$

18. Evaluate $\sum_{k=0}^{11}(3 + 0.2k)$

19. Evaluate $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots$

20. Evaluate $\frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{3}} + \frac{1}{3} + \frac{\sqrt{2}}{3\sqrt{3}} + \frac{2}{9} + \ldots$

21. Find the first four partial sums and the nth partial sum of each sequence.

    (a) $u_n = \frac{3}{5^n}$ (b) $v_n = \frac{1}{n^2 + 3n + 2}$ (c) $u_n = \sqrt{n+1} - \sqrt{n}$

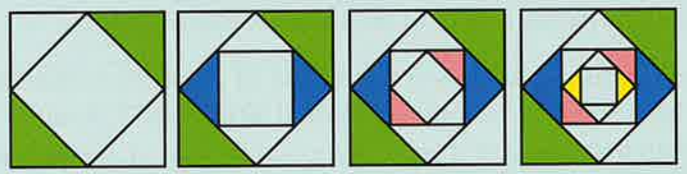
For question 21 part (b), show that $v_n = \frac{1}{n+1} - \frac{1}{n+2}$

22. A ball is dropped from a height of 16 m. Every time it hits the ground it bounces to 81% of its previous height.

(a) Find the maximum height it reaches after the 10th bounce.

(b) Find the total distance travelled by the ball until it comes to rest. (Assume no friction and no loss of elasticity.)

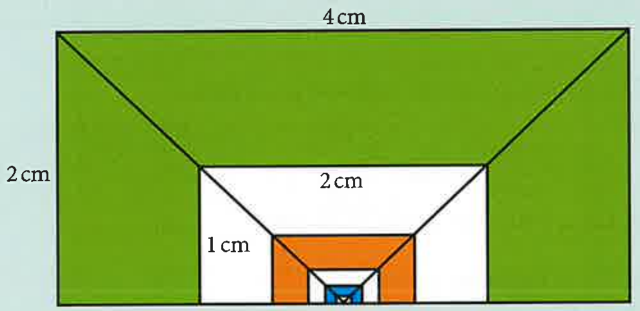
23. The sides of a square are 16 cm in length. A new square is formed by joining the midpoints of the adjacent sides and then two of the resulting triangles are coloured, as shown.



(a) If the process is repeated six more times, determine the total area of the shaded region.

(b) If the process were to be repeated indefinitely, find the total area of the shaded region.

24. The largest rectangle in the diagram below measures 4 cm by 2 cm. Another rectangle is constructed inside it, measuring 2 cm by 1 cm. The process is repeated. The region surrounding every other inner rectangle is shaded, as shown.



(a) Find the total area for the three regions shaded already.

(b) If the process were to be repeated indefinitely, find the total area of the shaded regions.

25. Find each sum.

(a) $7 + 12 + 17 + 22 + \ldots + 337 + 342$

(b) $9486 + 9479 + 9472 + 9465 + \ldots + 8919 + 8912$

(c) $2 + 6 + 18 + 54 + \ldots + 3\,188\,646 + 9\,565\,938$

(d) $120 + 24 + \frac{24}{5} + \frac{24}{25} + \ldots + \frac{24}{78\,125}$

3.5 The binomial theorem

A binomial is a polynomial with two terms. For example, $x + y$ is a binomial. In principle, it is easy to raise $x + y$ to any power, but raising it to high powers would be tedious. In this chapter, we will find a formula that gives the expansion of $(x + y)^n$ for any positive integer n. (The proof of the binomial theorem is given in Chapter 5 as optional material.)

Let's look at some particular cases of the expansion of $(x + y)^n$

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

There are several things that we notice after looking at the expansion:

- There are $n + 1$ terms in the expansion of $(x + y)^n$
- The degree of each term is n.
- The powers on x begin with n and decrease to 0.
- The powers on y begin with 0 and increase to n.
- The coefficients are symmetric.

For instance, notice how the powers of x and y behave in the expansion of $(x + y)^5$

The powers of x decrease:

$(x + y)^5 = x^{\boxed{5}} + 5x^{\boxed{4}}y + 10x^{\boxed{3}}y^2 + 10x^{\boxed{2}}y^3 + 5x^{\boxed{1}}y^4 + y^5$

The powers of y increase:

$(x + y)^5 = x^5 + 5x^4y^{\boxed{1}} + 10x^3y^{\boxed{2}} + 10x^2y^{\boxed{3}} + 5xy^{\boxed{4}} + y^{\boxed{5}}$

With these observations, we can now proceed to expand any binomial raised to power n: $(x + y)^n$. For example, leaving a blank for the missing coefficients, the expansion for $(x + y)^7$ can be written as:

$(x + y)^7 = \square x^7 + \square x^6y + \square x^5y^2 + \square x^4y^3 + \square x^3y^4 + \square x^2y^5 + \square xy^6 + \square y^7$

To finish the expansion, we need to determine these coefficients. In order to see the pattern, look at the coefficients of the expansion at the start of the section.

	column 0	column 1	column 2	column 3	column 4	column 5	column 6	
$(x+y)^0$	1							row 0
$(x+y)^1$	1	1						row 1
$(x+y)^2$	1	2	1					row 2
$(x+y)^3$	1	3	3	1				row 3
$(x+y)^4$	1	4	6	4	1			row 4
$(x+y)^5$	1	5	10	10	5	1		row 5
$(x+y)^6$	1	6	15	20	15	6	1	row 6

Pascal's triangle
Every entry in a row is the sum of the term directly above it and the entry to the left of it. When there is no entry, the value is considered zero.

A triangle like the one above is known as Pascal's triangle. The first and second terms in row 3 give you the second term in row 4, the third and fourth terms in row 3 give you the fourth term of row 4, the second and third terms in row 5 give the third term in row 6, and the fifth and sixth terms in row 5 give you the sixth term in row 6. So now we can state a key property of Pascal's triangle.

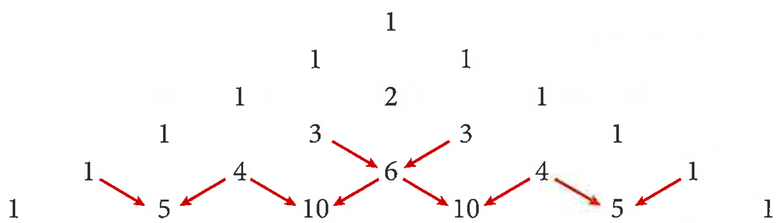
Take the last entry in row 5, for example; there is no entry directly above it, so its value is $0 + 1 = 1$.

From this property it is easy to find all the terms in any row of Pascal's triangle from the row above it. So, for the expansion of $(x + y)^7$, the terms are found from row 6 as follows:

$$0 \to 1 \to 6 \to 15 \to 20 \to 15 \to 6 \to 1 \to 0$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

So, $(x+y)^7 = x^7 + \boxed{7}x^6y + \boxed{21}x^5y^2 + \boxed{35}x^4y^3 + \boxed{35}x^3y^4 + \boxed{21}x^2y^5 + \boxed{7}xy^6 + y^7$

Several sources use a slightly different arrangement for Pascal's triangle. The common usage considers the triangle as isosceles and uses the principle that every two entries add up to give the entry diagonally below them, as shown in the diagram.



Example 3.22

Use Pascal's triangle to expand $(2k - 3)^5$

Solution

We can find the expansion by replacing x by $2k$ and y by -3 in the binomial expansion of $(x + y)^5$.

Using the fifth row of Pascal's triangle for the coefficients will give:

$$1(2k)^5 + 5(2k)^4(-3) + 10(2k)^3(-3)^2 + 10(2k)^2(-3)^3 + 5(2k)(-3)^4 + 1(-3)^5$$
$$= 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243$$

Pascal's triangle is a useful tool for finding the coefficients of the binomial expansion for relatively small values of n. It is not very efficient for large values of n. Imagine we want to evaluate $(x + y)^{20}$. Using Pascal's triangle, we would need the terms in the 19th row, and the 18th row and so on. This makes the process tedious and not practical.

Luckily, there is a formula we can use to find the coefficients of any Pascal's triangle row. This formula is the binomial formula, whose proof will appear in Chapter 5. Every entry in Pascal's triangle is denoted by $\binom{n}{r}$ or $_nC_r$ – this is also known as the binomial coefficient. In $_nC_r$, n is the row number and r is the column number. To understand the binomial coefficient, we need to understand what factorial notation means.

Factorial notation
The product of the first n positive integers is denoted by $n!$ and is called n **factorial**:
$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$
We also define $0! = 1$.

This definition of the factorial makes many formulae involving the multiplication of consecutive positive integers shorter and easier to write. That includes the binomial coefficient.

The binomial coefficient
With n and r as non-negative integers such that $n \geqslant r$, the **binomial coefficient** $_nC_r$ $\left[\text{or } \binom{n}{r}\right]$ is defined by

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$_nC_r$ is also written as ^nC_r

When simplified, $_nC_r$ can be written as $_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.
For example $_nC_3 = \frac{n(n-1)(n-2)}{3!}$.

Example 3.23

Find the value of:

(a) $_7C_3$ (b) $_7C_4$ (c) $\binom{7}{0}$ (d) $\binom{7}{7}$

Solution

(a) $_7C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{\cancel{1\cdot2\cdot3\cdot4}\cdot5\cdot6\cdot7}{(1\cdot2\cdot3)(\cancel{1\cdot2\cdot3\cdot4})} = \frac{5\cdot6\cdot7}{1\cdot2\cdot3} = 35,$

or using the other form of the expression for the binomial coefficient

$_7C_3 = \frac{7\cdot6\cdot5}{3!} = \frac{210}{6} = 35$

(b) $_7C_4 = \frac{7!}{4!(7-4)!} = \frac{7!}{4!3!} = \frac{\cancel{1\cdot2\cdot3\cdot4}\cdot5\cdot6\cdot7}{(\cancel{1\cdot2\cdot3\cdot4})(1\cdot2\cdot3)} = \frac{5\cdot6\cdot7}{1\cdot2\cdot3} = 35$

(c) $\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{\cancel{7!}}{0!\cancel{7!}} = \frac{1}{1} = 1$

(d) $\binom{7}{7} = \frac{7!}{7!(7-7)!} = \frac{\cancel{7!}}{\cancel{7!}0!} = \frac{1}{1} = 1$

Your calculator can do the tedious work of evaluating the binomial coefficient. On most calculators, the binomial coefficient appears as $_nC_r$.

```
7 nCr 3
                35
7 nCr 4
                35
7 nCr 0
                 1
```

Although the binomial coefficient $\binom{n}{r}$ appears as a fraction, all its results where n and r are non-negative integers are positive integers. Also notice the **symmetry** of the coefficient in Example 3.23. This is a property that you are asked to prove in the exercises:

$$_nC_r = {}_nC_{n-r}$$

Example 3.24

Calculate each binomial coefficient:

$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5}, \binom{6}{6}$

Solution

$\binom{6}{0} = 1 \quad \binom{6}{1} = 6 \quad \binom{6}{2} = 15 \quad \binom{6}{3} = 20 \quad \binom{6}{4} = 15 \quad \binom{6}{5} = 6 \quad \binom{6}{6} = 1$

The values in Example 3.24 are the entries in the 6th row of Pascal's triangle.

We can write Pascal's triangle in the following manner:

$$\begin{matrix} \binom{0}{0} \\ \binom{1}{0}\ \binom{1}{1} \\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\ \vdots\ \ \vdots\ \ \vdots\ \ \vdots \\ \binom{n}{0}\ \binom{n}{1}\ \cdots\ \cdots\ \cdots\ \cdots\ \binom{n}{n} \end{matrix}$$

Example 3.25

Calculate $_nC_{r-1} + {}_nC_r$ (This is called Pascal's rule.)

Solution

$$\begin{aligned} {}_nC_{r-1} + {}_nC_r &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n! \cdot r}{r \cdot (r-1)!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r)! \cdot (n-r+1)} \\ &= \frac{n! \cdot r}{r!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r+1)!} \\ &= \frac{n!\,(r+n-r+1)}{r!(n-r+1)!} \\ &= \frac{(n+1)!}{r!(n+1-r)!} \\ &= {}_{n+1}C_r \end{aligned}$$

You will be able to provide reasons for the steps after you do the exercises.

If we read the result in Example 3.5 carefully, it says that the sum of the terms in the nth row, $(r-1)$th and rth columns, is equal to the entry in the $(n+1)$th row and rth column.

$$\begin{array}{lccc} & (r-1)\text{th column} & & r\text{th column} \\ n\text{th row} & _nC_{r-1} & + & _nC_r \\ & & & \| \\ (n+1)\text{th row} & & & _{n+1}C_r \end{array}$$

That is, the two entries on the left are adjacent entries in the nth row of Pascal's triangle and the entry on the right is the entry in the $(n+1)$th row directly below the rightmost entry. This is precisely the principle behind Pascal's triangle!

Using the binomial theorem

We are now prepared to state the binomial theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

or

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \cdots + {}_nC_ny^n$$

In a compact form, we can use sigma notation to express the theorem as follows:

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i = \sum_{i=0}^{n}{}_nC_ix^{n-i}y^i$$

The proof of the binomial theorem is optional and will require mathematical induction. We will develop the proof in Chapter 5.

Why is the binomial theorem related to the number of combinations of n elements taken r at a time?

Consider evaluating $(x+y)^n$. In doing so, we must multiply $(x+y)$ by itself n times. As we know, one term has to be x^n. How do we get this term? x^n is the result of multiplying x in each of the n factors $(x+y)$ and that can happen in only one way. However, consider the term containing x^r. Having a power of r over the x means that the x in each of r factors must be multiplied, and the rest will be the $(n-r)$ y-terms. This can happen in $_nC_r$ ways. Hence the coefficient of the term x^ry^{n-r} is $_nC_r$

Example 3.26

Use the binomial theorem to expand $(x+y)^7$

Solution

$$\begin{aligned} (x+y)^7 &= \binom{7}{0}x^7 + \binom{7}{1}x^{7-1}y + \binom{7}{2}x^{7-2}y^2 + \binom{7}{3}x^{7-3}y^3 + \binom{7}{4}x^{7-4}y^4 \\ &\quad + \binom{7}{5}x^{7-5}y^5 + \binom{7}{6}x^{7-6}y^6 + \binom{7}{7}y^7 \\ &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7 \end{aligned}$$

Example 3.27

Find the expansion for $(2k-3)^5$

Solution

$$\begin{aligned} (2k-3)^5 &= \binom{5}{0}(2k)^5 + \binom{5}{1}(2k)^4(-3) + \binom{5}{2}(2k)^3(-3)^2 + \binom{5}{3}(2k)^2(-3)^3 \\ &\quad + \binom{5}{4}(2k)(-3)^4 + \binom{5}{5}(-3)^5 \\ &= 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243 \end{aligned}$$

Example 3.28

Find the term containing a^3 in the expansion $(2a - 3b)^9$

Solution

To find the term, we do not need to expand the whole expression.

Since $(x + y)^n = \sum_{i=0}^{n} {}_nC_i\, x^{n-i} y^i$, the term containing a^3 is the term where $n - i = 3$, i.e. when $i = 6$. So the required term is

${}_9C_6 (2a)^{9-6}(-3b)^6 = 84 \cdot 8a^3 \cdot 729b^6 = 489\,888a^3b^6$

Example 3.29

Find the term independent of x in:

(a) $\left(2x^2 - \frac{3}{x}\right)^9$ (b) $\left(4x^3 - \frac{2}{x^2}\right)^5$

Solution

(a) 'Independent of x' means the term with no x variable, i.e. the constant term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^9 = \sum_{k=0}^{9} {}_9C_k (2x^2)^k \left(-\frac{3}{x}\right)^{9-k}$

The constant term contains x^0. Thus it is the term where $(x^2)^k \left(\frac{1}{x}\right)^{9-k} = x^0$,

i.e. $2k = 9 - k \Rightarrow k = 3$, so the term is ${}_9C_3 (2x^2)^3 \left(-\frac{3}{x}\right)^6 = 84 \cdot 8x^6 \cdot \frac{729}{x^6}$

$= 489\,888$

(b) Similarly, $(x^3)^k \left(\frac{1}{x^2}\right)^{5-k} = x^0$, i.e., $3k = 10 - 2k \Rightarrow k = 2$, so the term is

${}_5C_2 (4x^3)^2 \left(-\frac{2}{x^2}\right)^3 = -1280$

```
(9C3)×(2³)×(-3)⁶
                489888
(5C2)×4²×(-2)³
                 -1280
```

Example 3.30

Find the coefficient of b^6 in the expansion of $\left(2b^2 - \frac{1}{b}\right)^{12}$

Solution

The general term is $\binom{12}{i}(2b^2)^{12-i}\left(-\frac{1}{b}\right)^i = \binom{12}{i}(2)^{12-i}(b^2)^{12-i}\left(-\frac{1}{b}\right)^i$

$= \binom{12}{i}(2)^{12-i} b^{24-2i} b^{-i}(-1)^i$

$= \binom{12}{i}(2)^{12-i} b^{24-3i}(-1)^i$

$24 - 3i = 6 \Rightarrow i = 6$. So the coefficient in question is $\binom{12}{6}(2)^6(-1)^6 = 59\,136$

Exercise 3.5

1. Use Pascal's triangle to expand each binomial.
   (a) $(x + 2y)^5$ (b) $(a - b)^4$ (c) $(x - 3)^6$
   (d) $(2 - x^3)^4$ (e) $(x - 3b)^7$ (f) $\left(2n + \frac{1}{n^2}\right)^6$
   (g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$

2. Evaluate each expression.
   (a) $\binom{8}{3}$ (b) $\binom{18}{5} - \binom{18}{13}$ (c) $\binom{7}{4}\binom{7}{3}$
   (d) $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$
   (e) $\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6}$

3. Use the binomial theorem to expand each expression.
   (a) $(x - 2y)^7$ (b) $(2a - b)^6$ (c) $(x - 4)^5$
   (d) $(2 + x^3)^6$ (e) $(3x - b)^7$ (f) $\left(2n - \frac{1}{n^2}\right)^6$
   (g) $\left(\frac{2}{x} - 3\sqrt{x}\right)^4$ (h) $(1 + \sqrt{5})^4 + (1 + \sqrt{5})^4$
   (i) $(\sqrt{3} + 1)^8 - (\sqrt{3} + 1)^8$

4. Consider the expression $\left(x - \frac{2}{x}\right)^{45}$
   (a) Find the first three terms of this expansion.
   (b) Find the constant term if it exists or justify why it doesn't exist.
   (c) Find the last three terms of the expansion.
   (d) Find the term containing x^3 if it exists or justify why it doesn't exist.

5. Prove that ${}_nC_k = {}_nC_{n-k}$ for all $n, k \in \mathbb{N}$ and $n \geqslant k$

6. Prove that for any positive integer n,
   $\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n - 1.$

   For question 6: $2^n = (1 + 1)^n$

7. Consider all $n, k \in \mathbb{N}$ and $n \geqslant k$
   (a) Verify that $k! = k(k - 1)!$
   (b) Verify that $(n - k + 1)! = (n - k + 1)(n - k)!$
   (c) Justify the steps given in the proof of $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ in Example 3.25.

8. Find the value of the expression:
   $\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \ldots + \binom{6}{6}\left(\frac{2}{3}\right)^6$

9. Find the value of the expression:

$$\binom{8}{0}\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right) + \binom{8}{2}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^2 + \cdots + \binom{8}{8}\left(\frac{3}{5}\right)^8$$

10. Find the value of the expression:

$$\binom{n}{0}\left(\frac{1}{7}\right)^n + \binom{n}{1}\left(\frac{1}{7}\right)^{n-1}\left(\frac{6}{7}\right) + \binom{n}{2}\left(\frac{1}{7}\right)^{n-2}\left(\frac{6}{7}\right)^2 + \cdots + \binom{n}{n}\left(\frac{6}{7}\right)^n$$

11. Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{x}\right)^6$

12. Find the term independent of x in the expansion of $\left(3x - \frac{2}{x}\right)^8$

13. Find the term independent of x in the expansion of $\left(2x - \frac{3}{x^3}\right)^8$

14. Find the first three terms in the expansion of $(1 + x)^{10}$ and use them to find an approximation to:

(a) 1.01^{10} (b) 0.99^{10}

15. Show that $\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \binom{n+2}{r+1}$ and interpret your result on the entries in Pascal's triangle.

16. Express each repeating decimal as a fraction:

(a) $0.\overline{7}$ (b) $0.\overline{345}$ (c) $3.2\overline{129}$

17. Find the coefficient of x^6 in the expansion of $(2x - 3)^9$

18. Find the coefficient of x^3b^4 in the expansion of $(ax + b)^7$

19. Find the constant term in the expansion of $\left(\frac{2}{z^2} - z\right)^{15}$

20. Expand $(3n - 2m)^5$

21. Find the coefficient of r^{10} in the expansion of $(4 + 3r^2)^9$

22. In the expansion of $(2 - kx)^5$, the coefficient of x^3 is -1080. Find the constant k.

3.6 Counting principles

Simple counting problems

This section will introduce you to some of the basic principles of counting. In Chapter 5 you will apply some of this in justifying the binomial theorem and in Chapter 11 you will use these principles to tackle many probability problems. We will start with two examples.

Example 3.31

Each integer from 1 to 9 is written on a paper chip. The paper chips are placed in a box. One chip is chosen, the number is recorded, and the chip is put back in the box. This is then repeated for a second chip. The numbers on the chips are added. In how many ways can you get a sum of 8?

Solution

To solve this problem, count the different number of ways that a total of 8 can be obtained:

1st chip	1	2	3	4	5	6	7
2nd chip	7	6	5	4	3	2	1

From this list, it is clear that there are seven different ways of getting a sum of 8.

Example 3.32

Suppose now that the first chip is chosen and the number is recorded, but the chip is not put back in the box, then the second chip is drawn. In how many ways can you get a sum of 8?

Solution

To solve this problem, also count the different number of ways that a total of 8 can be obtained:

1st chip	1	2	3	5	6	7
2nd chip	7	6	5	3	2	1

From this list, it is clear that there six different ways of getting a sum of 8.

The difference between this and Example 3.31 is described by saying that the first random selection is done **with replacement**, while the second is done **without replacement**, which ruled out the drawing of two 4s.

Fundamental principle of counting

Examples 3.31 and 3.32 show you simple counting principles in which you can list every possible way that an event can happen. In many other cases, listing the ways an event can happen may not be feasible. In such cases we need to rely on counting principles. The most important of which is the fundamental principle of counting, also known as the multiplication principle. Consider the following examples.

Example 3.33

For making sandwiches, there are three kinds of bread and four kinds of cheese. Sandwiches can be made with or without pickles. How many different kinds of sandwich can be made?

Solution

There are three kinds of bread, each of which could be combined with one of four kinds of cheese. That would make $3 \times 4 = 12$ sandwiches. But, also, sandwiches can be made with or without pickles, thus 12 with pickles and 12 without. That is, there are $3 \times 4 \times 2 = 24$ possible sandwiches.

Example 3.34

How many 3-digit even numbers are there?

Solution

The first digit cannot be zero, since the number has to be a 3-digit number, so there are 9 possibilities for the hundreds digit. There is no condition on what the tens digit should be, so we have 10 possibilities, and to be an even number, the number must end with 0, 2, 4, 6, or 8. Therefore , there are $9 \times 10 \times 5 = 450$ 3-digit even numbers.

Examples 3.33 and 3.34 are examples of the fundamental principle of counting.

Fundamental Principle of Counting

The fundamental principle of counting states that if there are m ways an event can occur followed by n ways a second event can occur, then there are a total of $m \times n$ ways that the two can occur.

This principle can be extended to more than two events or processes:

If there are k events that can happen in $n_1, n_2, \ldots, n_k$ ways, then the whole sequence can happen in $n_1 \times n_2 \times \ldots \times n_k$ ways.

Example 3.35

A large school issues special coded ID cards that consist of two letters followed by three numerals. For example, AB 737 is such a code. How many different ID cards can be issued if the letters or numbers can be used more than once?

Solution

Since the letters can be used more than once, each letter position can be filled in 26 different ways. That is, the letters can be filled in $26 \times 26 = 676$ ways. Each number position can be filled in 10 different ways; hence the numerals can be filled in $10 \times 10 \times 10 = 1000$ different ways. So, the code can be formed in $676 \times 1000 = 676\,000$ different ways.

Permutations

One major application of the fundamental principle of counting is in determining the number of ways that n objects can be arranged. For example, we have five books we want to put on a shelf: Mathematics (M), Physics (P), English (E), Biology (B), and History (H). In how many ways can we do this?

Consider the positions that we want to place the books in as shown:

_ _ _ _ _
1 2 3 4 5

If we decide to put the Mathematics book in position 1, then there are 4 different ways of putting a book in position 2:

$\underline{M}\,\underline{P}$ _ _ _ (1 2 3 4 5) $\underline{M}\,\underline{E}$ _ _ _ (1 2 3 4 5) $\underline{M}\,\underline{B}$ _ _ _ (1 2 3 4 5) $\underline{M}\,\underline{H}$ _ _ _ (1 2 3 4 5)

Since we can put any of the 5 books in the first position there will be $5 \times 4 = 20$ ways of shelving the first two books. Once we place the books in positions 1 and 2, the third book can be chosen from any one of three books left:

$\underline{M}\,\underline{P}\,\underline{E}$ _ _ (1 2 3 4 5) $\underline{M}\,\underline{P}\,\underline{B}$ _ _ (1 2 3 4 5) $\underline{M}\,\underline{P}\,\underline{H}$ _ _ (1 2 3 4 5)

Once we have selected three books, there are two books for the fourth position and only one way of placing the fifth book. So, the number of ways of arranging all 5 books is

$$5 \times 4 \times 3 \times 2 \times 1 = 120 = 5!$$

Number of permutations of n objects

The books example can be applied to n objects rather than only 5. The number of ways of filling in the first position can be done in n ways.

n ↓ 1, $n-1$ ↓ 2, $n-2$ ↓ 3, $n-3$ ↓ 4, ⋯, 1 ↓ n

Once the first position is filled, the second position can be filled by any of the $n - 1$ objects left, and hence, using the fundamental principle of counting, there will be $n \cdot (n-1)$ different ways for filling the first two positions. Repeating the same procedure until the nth position is filled is therefore:

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

Frequently, we are engaged in arranging a subset of the whole collection of objects rather than the entire collection. For example, suppose we want to shelve three of the books rather than all five of them. The discussion will be similar to the previous situation. However, we have to limit our search to the first three positions only – that is, the number of ways we can shelve three out of the five books is:

$$5 \times 4 \times 3 = 60$$

To change this product into factorial notation, we do the following:

$$5 \times 4 \times 3 = 5 \times 4 \times 3 \times \frac{2!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2!} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

Permutations
An arrangement is called a **permutation**. It is the reorganisation of objects or symbols into distinguishable sequences. When we place things in order, we say we have made an arrangement. When we change the order, we say we have changed the arrangement. Each of the arrangements that can be made by taking some or all of a number of things is known as a permutation.

A permutation of n different objects can be understood as an ordering (arrangement) of the objects such that one object is first, one is second, one is third, and so on.

The number of permutations of n objects taken r at a time is:

$^nP_r = {}_nP_r = P_r^n = P(n, r) = \frac{n!}{(n-r)!}, n \geq r$

This leads us to the result on the left.

To verify the formula, we can proceed in the same manner as with the permutation of n objects.

$$\underset{1}{n} \quad \underset{2}{n-1} \quad \underset{3}{n-2} \quad \underset{4}{n-3} \cdots \underset{r}{n-(r-1)}$$

When we arrive at the rth position, we would have used $r - 1$ objects already, and hence we are left with $n - (r - 1) = n - r + 1$ objects to fill this position. So, the number of ways of arranging n objects taken r at a time is:

$${}_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

Here again, to make the expression more manageable, we can write it in factorial notation:

$${}_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

$$= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \frac{(n-r)!}{(n-r)!}$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Example 3.36

Fifteen drivers are taking part in a car race. In how many different ways can the top six positions be filled?

Solution

Since the drivers are all different, this is a permutation of 15 'objects' taken 6 at a time.

$${}_{15}P_6 = \frac{15!}{(15-6)!} = 3\,603\,600$$

We can also calculate this easily using a GDC.

```
15P6
                3603600
15!÷9!
                3603600
```

Combinations

A **combination** is a selection of some or all of a number of different objects. It is an unordered collection of unique sizes. In a permutation, the order of occurrence of the objects or the arrangement is important, but in a combination, the order of occurrence of the objects is not important. In that sense, a combination of r objects out of n objects is a subset of the set of n objects.

For example, there are 24 permutations of three letters from the letters ABCD. However, there are only 4 combinations. Here is why:

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

For one combination, ABC for example, there are $3! = 6$ permutations. This is true for all combinations. So, the number of permutations is 6 times the number of combinations. That is:

$${}_4P_3 = 3!\,{}_4C_3$$

where ${}_4C_3$ is the number of combinations of the 4 letters taken 3 at a time.

The ISO notation for ${}_nC_r$ is $\binom{n}{r}$. In this book, we will use both notations interchangeably.

According to the previous result, we can write:

$${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{\frac{4!}{(4-3)!}}{3!} = \frac{4!}{3!(4-3)!}$$

The last result can also be generalised to n elements combined r at a time.

Every subset of r objects (combination), gives rise to $r!$ permutations. So, if you have ${}_nC_r$ combinations, these will result in $r!{}_nC_r$ permutations. Therefore:

$${}_nP_r = r!\,{}_nC_r \Leftrightarrow {}_nC_r = \frac{{}_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!r!}$$

Example 3.37

A lottery has 45 numbers. If you buy a ticket, then you choose 6 of these numbers. In how many ways can you select 6 numbers from 45?

Solution

Since 6 numbers will be chosen and order is not an issue here, this is a combination case. The number of possible choices is:

$$\binom{45}{6} = 8\,145\,060$$

This can also be calculated using a GDC:

```
45 nCr 6
               8145060
```

Example 3.38

In some card games, a standard deck of 52 playing cards is used, and a 'hand' is made up of 5 cards.

(a) How many possible hands are there?

(b) How many possible hands are there with 3 diamonds and 2 hearts?

Solution

(a) A player can re-order the cards after receiving them, so the order is not important, and thus this is a combination of 52 cards taken 5 at a time:

$$\binom{52}{5} = 2\,598\,960$$

(b) Since there are 13 diamonds and we want 3 of them, there are $\binom{13}{3} = 286$ ways to get 3 diamonds. Since there are 13 hearts and we want 2 of them, there are $\binom{13}{2} = 78$ ways to get 2 hearts. Since we want them both to occur at the same time, we use the fundamental principle of counting and multiply 286 by 78 to get 22 308 possible hands.

```
52 nCr 5
         2598960
13 nCr 3
             286
13 nCr 2
              78
```

Figure 3.5 GDC screen for solution to Example 3.38 (b)

Example 3.39

A code is made up of 6 different digits. How many possible codes are there?

Solution

Since there are 10 digits and we are choosing 6 of them, and since the order we use these digits makes a difference in the code, this is permutation case. The number of possible codes is:

$$_{10}P_6 = 151\,200$$

Exercise 3.6

1. Evaluate each expression.

(a) $_5P_5$ **(b)** $5!$ **(c)** $_{20}P_1$ **(d)** $_8P_3$

2. Evaluate each expression.

(a) $_5C_5$ **(b)** $_5C_0$ **(c)** $_{10}C_3$ **(d)** $_{10}C_7$

3. Evaluate each expression.

(a) $\binom{7}{3} + \binom{7}{4}$ **(b)** $\binom{8}{4}$ **(c)** $\binom{10}{6} + \binom{10}{7}$ **(d)** $\binom{11}{7}$

4. Evaluate each expression.

(a) $\binom{8}{5} - \binom{8}{3}$ **(b)** $11 \cdot 10!$ **(c)** $\binom{10}{3} - \binom{10}{7}$ **(d)** $\binom{10}{1}$

5. State whether each equation below is true.

(a) $\frac{10!}{5!} = 2!$ **(b)** $(5!)^2 = 25!$ **(c)** $\binom{101}{8} = \binom{101}{93}$

6. You are buying a computer and have the following choices: three types of hard drive, two types of main processor, and four types of graphics card. How many different systems can you choose from?

7. You are going to a restaurant with a set menu. There are three starters, four main meals, two drinks, and three desserts. How many different choices are available for you to choose your meal from?

8. A school needs a teacher for each of three subjects: PE, Mathematics, and English. They have eight applicants for the PE position, three applicants for Mathematics and 13 applicants for English. How many different combinations of choices does the school have?

9. You are given a multiple-choice test where each question has four possible answers. The test is made up of 12 questions and you did not prepare for the test and are thus guessing at random. In how many ways could you answer all the questions on the test?

10. The test in question 9 is divided into two parts, the first six are true or false questions and the last six are multiple choice questions. In how many different ways can you answer all the questions on that test?

11. Passwords on a network are made up of two parts. One part consists of three letters, not necessarily different, the second part consists of five digits, also not necessarily different. How many passwords are possible on this network?

12. In question 11, how many 5-digit numbers can be made if the units digit cannot be 0?

13. Four couples are to be seated in a theatre row. Find how many different ways they can be seated if:

(a) no restrictions are made **(b)** each couple sits together.

14. Five girls and three boys walk single file through a doorway. Find how many different orders they can walk through the doorway if:

(a) there are no constraints **(b)** the girls must go first.

15. Write all the permutations of the letters in JANE.

16. Write all the permutations of the letters in MAGIE taken three at a time.

17. A computer code is made up of three letters followed by four digits.
   (a) How many possible codes are there?
   (b) 97 of the three-letter combinations cannot be used because they are offensive. How many codes are still possible?

18. A local club has 17 members: 10 females and 7 males. They have to elect three officers: president, deputy, and treasurer. Find how many ways to elect the officers if:
   (a) there are no restrictions
   (b) the president is to be a male
   (c) the deputy must be a male, the president can be any gender, but the treasurer must be a female
   (d) the president and deputy are of the same gender.

19. The research and development department for a computer manufacturer has 26 employees: eight mathematicians, 12 computer scientists, and six engineers. They need to select three employees to be leaders of the group. Find how many ways they can do this if:
   (a) the three officers are of the same specialisation
   (b) at least one of them must be an engineer
   (c) two of them must be mathematicians.

20. A combination lock has three numbers, each in the range 1 to 50.
   (a) How many different combinations are possible?
   (b) How many combinations do not have duplicates?
   (c) How many have the first and second numbers matching?
   (d) How many have exactly two of the numbers matching?

21. In how many ways can five couples be seated around a circle so that each couple sits together?

22. (a) How many subsets of {1, 2, 3, …, 9} have two elements?
   (b) How many subsets of {1, 2, 3, …, 9} have an odd number of elements?

23. Nine men and 12 women make up a community archery club. Four members are needed for an upcoming competition.
   (a) How many 4-member teams can they form?
   (b) How many of these 4-member teams have the same number of women and men?
   (c) How many of these 4-member teams have more women than men?

   Tim is the best male archer, and Gwen is the best female archer in the club. Only one of the two can be chosen to be on the team.
   (d) Repeat parts (a)–(c) with these changes.

24. A shipment of 100 hard disks contains four defective disks. A sample of six disks is chosen for inspection.
   (a) How many different possible samples are there?
   (b) How many samples could contain all four defective disks? What percentage of the total number of possible samples is that?
   (c) How many samples could contain at least one defective disk? What percentage of the total number of possible samples is that?

25. There are three political parties represented in a parliament: ten conservatives, eight liberals, and four independents. A committee of six members needs to be set up.
   (a) How many different committees are possible?
   (b) How many committees with equal representation are possible?

26. How many ways are there for nine boys and six girls to stand in a line so that no two girls stand next to each other?

Chapter 3 practice questions

1. In an arithmetic sequence, the first term is 4, the 4th term is 19 and the nth term is 99. Find the common difference and the number of terms, n.

2. How much money should you invest now if you wish to have $3000 in your account after 6 years, if interest is compounded quarterly at an annual rate of 6%?

3. Two students, Nick and Maxine, decide to start preparing for their IB exams 15 weeks ahead of the exams. Nick starts by studying for 12 hours in the first week and plans to increase the amount by 2 hours per week. Maxine starts with 12 hours in the first week and decides to increase her time by 10% every week.
   (a) How many hours will each student study in week 5?
   (b) How many hours in total will each student study for the 15 weeks?
   (c) In which week will Maxine exceed 40 hours per week?
   (d) In which week will Maxine catch up with Nick in the number of hours spent studying per week?

4. Two diet schemes are available for people to lose weight. Plan A promises the patient an initial weight loss of 1000 grams the first month with a steady loss of an additional 80 grams every month after the first, for a maximum duration of 12 months.

   Plan B starts with a weight loss of 1000 grams the first month and an increase in weight loss by 6% more every subsequent month.

   (a) Write down the number of grams lost under Plan B in the second and third months.

   (b) Find the weight lost in the 12th month for each plan.

   (c) Find the total weight loss during a 12 month period under

   (i) Plan A **(ii)** Plan B.

5. You start a savings plan to buy a car, where you invest €500 at the beginning of each year for 10 years. Your bank offers a fixed rate of 6% per year, compounded annually.

   Calculate, giving your answers to the nearest euro(€):

   (a) how much the first €500 is worth at the end of 10 years

   (b) the total value of your investment at the end of the 10 years.

6. The first three terms of an arithmetic sequence are 6, 9.5, and 13.

   (a) What is the 40th term of the sequence?

   (b) What is the sum of the first 103 terms of the sequence?

7. $\{a_n\}$ is defined as follows:

   $$a_n = \sqrt[3]{8 - a_{n-1}^3}$$

   (a) Given that $a_1 = 1$, evaluate a_2, a_3, and a_4. Describe $\{a_n\}$.

   (b) Given that $a_1 = 2$, evaluate a_2, a_3, and a_4. Describe $\{a_n\}$.

8. A marathon runner plans her training program for a 20 km race. On the first day she plans to run 2 km, then she wants to increase her distance by 500 m on each subsequent training day.

   (a) On which day of her training does she first run a distance of 20 km?

   (b) By the time she manages to run the 20 km distance, what is the total distance she would have run for the whole training program?

9. In a certain country, smartphones were first introduced in the year 2010. During the first year, 1600 people bought a smartphone. In 2011, the number of new participants was 2400, and in 2012 the new participants numbered 3600.

   (a) You notice that the trend follows a geometric sequence. Find the common ratio.

   (b) Assuming that the trend continues:

   (i) how many participants will join in 2022?

   (ii) in what year would the number of new participants first exceed 50 000?

   Between 2010 and 2012, the total number of participants reached 7600.

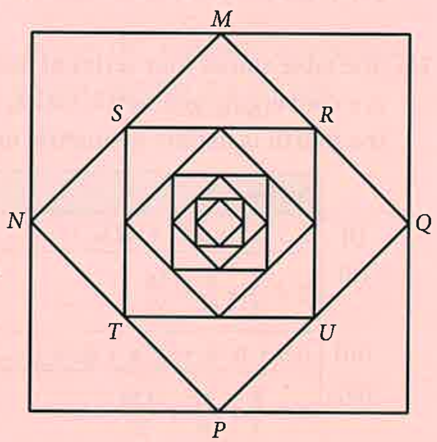
   (c) What is the total number of participants between 2010 and 2022?

   During this period, the total adult population remains approximately 800 000.

   (d) Use this information to suggest a reason why this trend in growth would not continue.

10. In an arithmetic sequence, the first term is 25, the fourth term is 13, and the nth term is $-11\,995$. Find the common difference d and the number of terms n.

11. The midpoints M, N, P, and Q of the sides of a square of side 1 cm are joined to form a new square.



   (a) Show that the side length of the square $MNPQ$ is $\frac{\sqrt{2}}{2}$

   (b) Find the area of square $MNPQ$.

   A new third square $RSTU$ is constructed in the same manner.

   (c) **(i)** Find the area of $RSTU$.

   (ii) Show that the areas of the squares are in a geometric sequence and find the common ratio.

   The procedure continues indefinitely.

   (d) **(i)** Find the area of the 10th square.

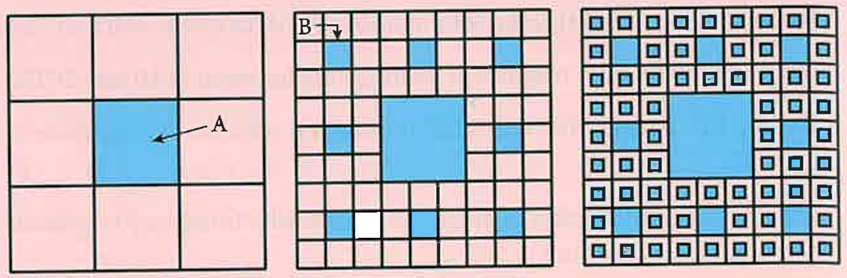
   (ii) Find the sum of the areas of all the squares.

12. Aristede is a dedicated swimmer. He goes swimming once every week. He starts the first week of the year by swimming 200 metres. Each week after that he swims 20 metres more than the previous week. He does this for the whole year (52 weeks).

   (a) How far does he swim in the final week?

   (b) How far does he swim altogether?

13. The diagram shows three iterations of constructing squares in the following manner: A square of side 3 units is drawn, then it is divided into nine smaller squares and the middle square is shaded (below, left). Each of the unshaded squares is in turn divided into nine squares and the process is repeated. The area of the first shaded square is 1 unit.



(a) Find the area of each of the squares A and B.

(b) Find the area of any small square in the third diagram.

(c) Find the area of the shaded regions in the second and third iterations.

(d) If the process was continued indefinitely, find the area left unshaded.

14. The table shows four series of numbers. One is an arithmetic series, one is a converging geometric series, one is a diverging geometric series, and the fourth is neither geometric nor arithmetic.

	Series	Type of series
(i)	$2 + 22 + 222 + 2222 + \dots$	
(ii)	$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \dots$	
(iii)	$0.8 + 0.78 + 0.76 + 0.74 + \dots$	
(iv)	$2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \dots$	

(a) Copy and complete the table by stating the type of each series.

(b) Find the sum of the infinite geometric series.

15. Two IT companies offer apparently similar salary schemes for their new appointees. Kell offers a starting salary of €18,000 per year and an annual increase of €400 each year after the first. YBO offers a starting salary of €17,000 per year and an annual increase of 7% each year after the first.

(a) **(i)** Write down the salary paid in the 2nd and 3rd years for each company.

(ii) Calculate the total amount than an employee working for 10 years will accumulate over 10 years in each company.

(iii) Calculate the salary paid in the tenth year in each company.

(b) Tim works at Kell and Merijayne works at YBO.

(i) When would Merijayne start earning more than Tim?

(ii) What is the minimum number of years that Merijayne requires so that her total earnings exceed Tim's total earnings?

16. A theatre has 24 rows of seats. There are 16 seats in the first row, and each successive row increases by 2 seats.

(a) Calculate the number of seats in the 24th row.

(b) Calculate the number of seats in the whole theatre.

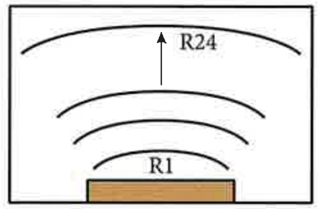


Figure 3.6 Diagram for question 16

17. The amount of €7000 is invested at 5.25% annual compound interest.

(a) Write down an expression for the value of this investment after t full years.

(b) Calculate the minimum number of years required for this amount to become €10,000.

(c) For the same number of years as in part **(b)**, would an investment of the same amount be better if it were invested at a 5% rate compounded quarterly?

18. With S_n denoting the sum of the first n terms of an arithmetic sequence, we are given that $S_1 = 9$ and $S_2 = 20$.

(a) Find the second term.

(b) Calculate the common difference of the sequence.

(c) Find the fourth term.

19. Consider an arithmetic sequence whose second term is 7. The sum of the first four terms of this sequence is 12. Find the first term and the common difference of the sequence.

20. Given that

$$(1 + x)^5(1 + ax)^6 \equiv 1 + bx + 10x^2 + \dots + a^6x^{11},$$

find the values of $a, b \in \mathbb{Z}$.

21. In an arithmetic sequence of positive terms, a_n represents the nth term.

Given that $\frac{a_5}{a_{12}} = \frac{6}{13}$ and $a_1 \times a_3 = 32$, find $\sum_{i=1}^{100} a_i$

22. In an arithmetic sequence, $a_1 = 5$ and $a_2 = 13$.

(a) Write down, in terms of n, an expression for the nth term, a_n.

(b) Find n such that $a_n < 400$.

23. Find the coefficient of x^7 in the expansion of $(2 + 3x)^{10}$, giving your answer as a whole number.

24. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term, u_n.

25. Mr Blue, Mr Black, Mr Green, Mrs White, Mrs Yellow and Mrs Red sit around a circular table for a meeting. Mr Black and Mrs White must not sit together.

Calculate the number of different ways these six people can sit at the table without Mr Black and Mrs White sitting together.

26. Consider the arithmetic sequence 85, 78, 71, …
Find the sum of its positive terms.

27. When we expand $\left(x + \frac{1}{kx^2}\right)^7$, the coefficient of x is $\frac{7}{3}$.
Find all possible values of k.

28. The sum to infinity of a geometric sequence is $\frac{27}{2}$, and the sum of its first three terms is 13. Find the first term.

29. In how many ways can six different books be divided between two students so that each student receives at least one book?

30. Find the sum to infinity of the geometric series $-12 + 8 - \frac{16}{3} \ldots$

31. A geometric sequence is defined by $u_n = 3(4)^{n+1}, n \in \mathbb{Z}^+$, where u_n is the nth term.

(a) Find the common ratio r.

(b) Hence, find S_n, the sum of the first n terms of this sequence.

32. Consider the infinite geometric series:

$$1 + \left(\frac{3x}{5}\right) + \left(\frac{3x}{5}\right)^2 + \left(\frac{3x}{5}\right)^3 + \ldots$$

(a) For what values of x does the series converge?

(b) Find the sum of the series if $x = 1.5$

33. How many four-digit numbers are there that contain at least one digit 3?

34. Consider the arithmetic series $S_n = 2 + 5 + 8 + \ldots$

(a) Find an expression for the partial sum S_n, in terms of n.

(b) For what value of n is $S_n = 1365$?

35. Find the coefficient of x^3 when the binomial $\left(1 - \frac{1}{2}x\right)^8$ is expanded.

36. Find $\sum_{r=1}^{50} \ln(2^r)$, giving the answer in the form $a \ln 2$, where $a \in \mathbb{Q}$

37. Consider the sequence $\{a_n\}$ defined recursively by:

$$a_{n+1} = 3a_n - 2a_{n-1}, n \in \mathbb{Z}^+, \text{ with } a_0 = 1, a_1 = 2$$

(a) Find a_2, a_3, and a_4.

(b) (i) Find the explicit form for a_n in terms of n.

(ii) Verify that your answer to part (i) satisfies the given recursive definition.

38. The sum to infinity of a geometric sequence with all positive terms is 27, and the sum of the first two terms is 15. Find the value of:

(a) the common ratio

(b) the first term.

39. The first four terms of an arithmetic sequence are 2, $a - b$, $2a + b + 7$, and $a - 3b$, where a and b are constants. Find a and b.

40. A school's Mathematics Club has eight members. A team of four will be chosen for an upcoming event. However, the two oldest members cannot both be chosen. Find the number of ways the team may be chosen.

41. Three consecutive terms of an arithmetic sequence are: a, 1, and b. The terms 1, a, and b are consecutive terms of a geometric sequence. If $a \neq b$, find the value of a and of b.

42. The diagram opposite shows a sector AOB of a circle of radius 1 and centre O, where $A\hat{O}B = \theta$.

The lines (AB_1), (A_1B_2), and (A_2B_3) are perpendicular to OB. A_1B_1 and A_2B_2 are arcs of circles with centre O.

Calculate the sum to infinity of the arc lengths:

$AB + A_1B_1 + A_2B_2 + A_3B_3 + \ldots$

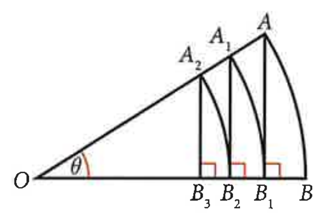


Figure 3.7 Diagram for question 42

43. The sum of the first n terms of a sequence is given by $S_n = 2n^2 - n$

(a) Find the first three terms of the sequence.

(b) Find an expression for the nth term of the sequence, giving your answer in terms of n.

44. (a) Expand $(2 + x)^5$, giving your answer in ascending powers of x.

(b) Hence, find the exact value of 2.01^5

45. You invest \$5000 at an annual compound interest rate of 6.3%.

(a) Write an expression for the value of this investment after t full years.

(b) Find the value of this investment at the end of five years.

(c) After how many full years will the value of the investment exceed \$10,000?

46. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m.

47. The sum of the first 16 terms of an arithmetic sequence $\{u_n\}$ is 12. Find the first term and the common difference if the ninth term is zero.

48. (a) Write down the first four terms of the expansion of $(1 - x)^n$, with $n > 2$, in ascending powers of x.

(b) The absolute values of the coefficients of 2nd, 3rd, and 4th terms of the expansion in **(a)** are consecutive terms in an arithmetic sequence.

(i) Show that $n^3 - 9n^2 + 14n = 0$

(ii) Find the value of n.

49. From a group of five seniors and six juniors, four students are chosen.

(a) Determine how many possible groups can be chosen.

(b) Determine how many groups can be formed consisting of two seniors and two juniors.

(c) Determine how many groups can be formed consisting of at least one junior.

50. (a) Write down the full expansion of $(3 + x)^4$ in ascending powers of x.

(b) Find the exact value of 3.1^4

51. Find the number of ways in which seven different books can be given to three students, if the youngest student is to receive three books and the others receive two books each.

52. (a) Write down how many integers between 10 and 300 are divisible by 7.

(b) Express the sum of these integers in sigma notation.

(c) Find the sum above.

(d) Given an arithmetic sequence with first term 1000 and common difference -7, find the smallest n so that the sum of the first n terms of this sequence is negative.

53. Jim uses a pack of nine cards with integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ written on them. Each card displays one of these integers. Jim is going to select four cards at random from this pack.

(a) Find the number of selections Jim could make if the largest integer drawn among the four cards is either a 5, a 6, or a 7.

(b) Find the number of selections Jim could make if at least two of the four integers drawn are even.

54. Three wives and their husbands are to sit on a bench for a photograph.

(a) Find the number of ways this can be done if the three wives want to sit together.

(b) Find the number of ways this can be done if the three wives will all sit apart.

55. Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term a and common difference d, where $d \neq 0$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

(a) (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant.

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d, and n.

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

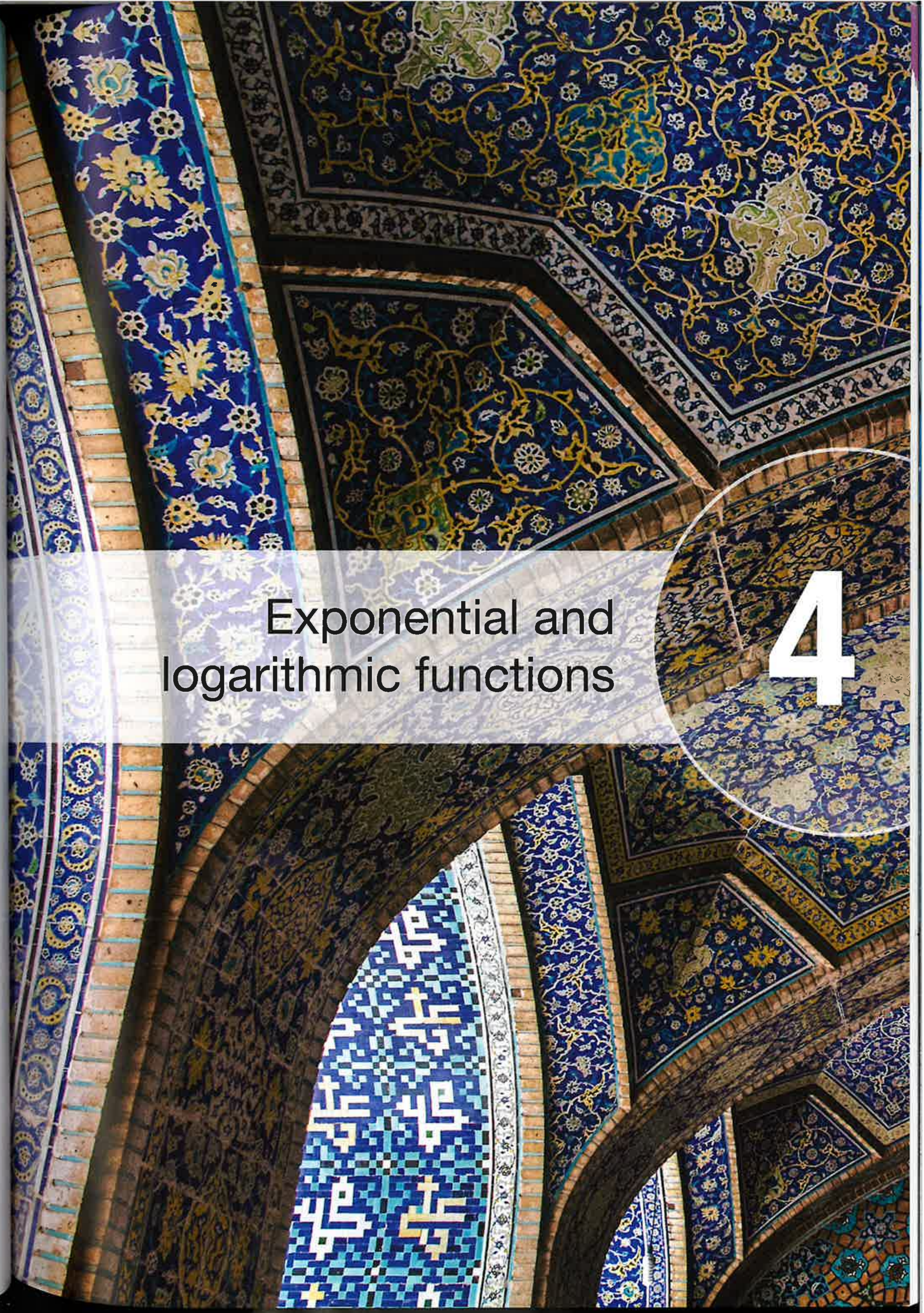
(b) **(i)** Find S_n in terms of a, d, and n.

(ii) Find the values of d for which $\sum_{1}^{\infty} v_i$ exists.

You are now told that $\sum_{1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d.

(iv) Given that $S_\infty = 2^{a+1}$, find the value of d.



4 Exponential and logarithmic functions