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# Mathematics: analysis and approaches

## Higher level

### Paper 3

21 May 2025

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour 15 minutes

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 23]

**This question asks you to use polynomial functions to model some situations in probability.**

Two unbiased tetrahedral (four-sided) dice with faces labelled 1, 2, 3 and 4 are thrown and the scores recorded.

The random variable  $M$  denotes the maximum of these two scores.

The probability distribution of  $M$  is given in the following table.

$m$	1	2	3	4
$P(M = m)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

(a) Find  $E(M)$ . [2]

An alternative way to represent the probability distribution of  $M$  is to use a polynomial

function,  $G$ , where  $G(t) = \sum_{m=1}^4 P(M = m)t^m$ .

Hence, for the distribution of  $M$ ,  $G(t) = \frac{1}{16}t + \frac{3}{16}t^2 + \frac{5}{16}t^3 + \frac{7}{16}t^4$ .

(b) Find  $G(1)$ . [1]

(c) (i) Find  $G'(t)$ . [2]

(ii) Hence, show that  $G'(1) = E(M)$ . [3]

**(This question continues on the following page)**

**(Question 1 continued)**

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable  $X$  denotes the total number of red balls selected.

The probability distribution of  $X$  can be represented by the polynomial function,  $G_X$ , where

$$G_X(t) = \sum_{x=0}^2 P(X=x)t^x.$$

- (d) Show that  $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$ , making it clear how the coefficients of  $G_X(t)$  have been determined. [5]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is  $p$ .

The random variable  $Y$  denotes the total number of tails obtained from tossing both coins.

The probability distribution of  $Y$  can be represented by the polynomial function,  $G_Y$ , where

$$G_Y(t) = \sum_{y=0}^2 P(Y=y)t^y.$$

- (e) Given that the coefficient of  $t^2$  in  $G_Y(t)$  is  $\frac{1}{3}$ , find

(i) the value of  $p$ ; [2]

(ii) an expression for  $G_Y(t)$ . [4]

The random variable  $Z$  denotes the sum of the total number of red balls selected,  $X$ , and the total number of tails obtained from tossing both coins,  $Y$ .

The probability distribution of  $Z$  can be represented by the function,  $G_Z$ , where

$$G_Z(t) = G_X(t)G_Y(t).$$

- (f) For random variable  $Z$ , it can be shown that  $G'_Z(1) = E(Z)$ .

Use this result to find  $E(Z)$ . [4]

2. [Maximum mark: 32]

**Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.**

Consider any function  $f$  that can be differentiated twice.

The curvature,  $k$ , of any function  $f$  is defined by  $k(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}}$ .

Consider the family of linear functions  $g(x) = mx + c$ , where  $x \in \mathbb{R}$  and  $m, c \in \mathbb{R}$ .

(a) Show that  $k(x) = 0$  for this family of linear functions. [2]

Consider the family of quadratic functions  $h(x) = ax^2 + bx + c$  for  $x \in \mathbb{R}$ , where  $a \in \mathbb{R}$ ,  $a \neq 0$  and  $b, c \in \mathbb{R}$ .

For this family of quadratic functions, it is given that

$$k(x) = \frac{2|a|}{\left(1 + (2ax + b)^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = -\frac{12a|a|(2ax + b)}{\left(1 + (2ax + b)^2\right)^{\frac{5}{2}}}.$$

The maximum value of  $k(x)$  is denoted by  $k_{\max}$ .

(b) (i) By solving  $k'(x) = 0$ , find the value of  $x$  where  $k_{\max}$  occurs.  
You are **not** required to justify that this value of  $x$  leads to a maximum. [1]

(ii) Determine an expression for  $k_{\max}$ , in terms of  $a$  only. [2]

(iii) State the value of  $\lim_{x \rightarrow \infty} k(x)$  and explain briefly the significance of this result. [2]

(iv) Consider the quadratic functions

$$p(x) = -2x^2 + 2x - 10 \text{ where } x \in \mathbb{R} \text{ and}$$

$$q(x) = 2x^2 + 5x + 25 \text{ where } x \in \mathbb{R}.$$

State which one of the following statements is true and justify your answer.

A.  $k_{\max}$  of  $p > k_{\max}$  of  $q$

B.  $k_{\max}$  of  $p < k_{\max}$  of  $q$

C.  $k_{\max}$  of  $p = k_{\max}$  of  $q$

[2]

(This question continues on the following page)

**(Question 2 continued)**

Consider the function  $v(x) = \ln x$ , where  $x \in \mathbb{R}$ ,  $x > 0$ .

For  $v$ , it is given that

$$k(x) = \frac{x}{(1+x^2)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}.$$

- (c) (i) Determine the exact value of  $x$  where  $k_{\max}$  occurs.

You are **not** required to justify that this value of  $x$  leads to a maximum. [2]

- (ii) Show that  $k_{\max} = \frac{2\sqrt{3}}{9}$ . [4]

Consider the function  $w(x) = e^x$ , where  $x \in \mathbb{R}$ .

For  $w$ , it is given that

$$k(x) = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{\frac{5}{2}}}.$$

- (d) (i) Show that  $k_{\max} = \frac{2\sqrt{3}}{9}$ . [5]

- (ii) Suggest a reason why  $v$  and  $w$  have the same  $k_{\max}$ . [1]

Consider a family of curves  $y = \sqrt{r^2 - x^2}$ , where  $-r < x < r$ ,  $y > 0$  and  $r$  is a positive constant.

- (e) (i) Show that  $\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$ . [6]

- (ii) Hence, show that the curvature,  $k$ , is constant for this family of curves. [5]