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Mathematics: analysis and approaches

Higher level

Paper 3

21 May 2025

Zone A afternoon | Zone B afternoon | Zone C afternoon

1 hour 15 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

The following question explores features of a family of curves. The family is then linked to a homogeneous differential equation.

Consider the curve given by $y = \frac{x(x^2 - 16)}{x^2 + 16}$.

- (a) (i) Sketch the curve of y for $-10 \leq x \leq 10$. [1]
 - (ii) State the coordinates of the points where the curve crosses the x -axis. [1]
 - (iii) State the coordinates of the local maximum point and the coordinates of the local minimum point. [2]
- (b) State whether the function $f(x) = \frac{x(x^2 - 16)}{x^2 + 16}$ is odd, even or neither. Justify your answer. [2]

Now consider the general curve given by $y = \frac{x(x^2 - A)}{x^2 + A}$, where A is a positive constant and $x \in \mathbb{R}$.

- (c) Given $f(x) = \frac{x(x^2 - A)}{x^2 + A}$, prove that $f'(\sqrt{A})$ is independent of A . [4]
- (d) (i) Show that $x - \frac{2Ax}{x^2 + A} \equiv \frac{x(x^2 - A)}{x^2 + A}$. [2]
- (ii) Hence, determine the equation of the oblique asymptote to the curve. [1]
- (iii) Write down the coordinates of a point on the curve where the oblique asymptote is parallel to the tangent to the curve at that point. [1]

(This question continues on the following page)

(Question 1 continued)

Now consider the differential equation $x^2 \frac{dy}{dx} = x(x+y) - y^2$, where $x \neq 0$, $y \neq \pm x$.

Using the substitution $y = vx$, the differential equation can be written as $x \frac{dv}{dx} = 1 - v^2$.

(e) Using partial fractions, show that $\int \frac{1}{1-v^2} dv = \frac{1}{2} \ln \left| \frac{A(1+v)}{1-v} \right|$, where A is a positive constant. [5]

(f) Hence, show that a solution to the original differential equation may be expressed in the form $x^2 = \left| \frac{A(x+y)}{x-y} \right|$, where A is a positive constant. [5]

Now consider only the case where $\frac{A(x+y)}{x-y} > 0$.

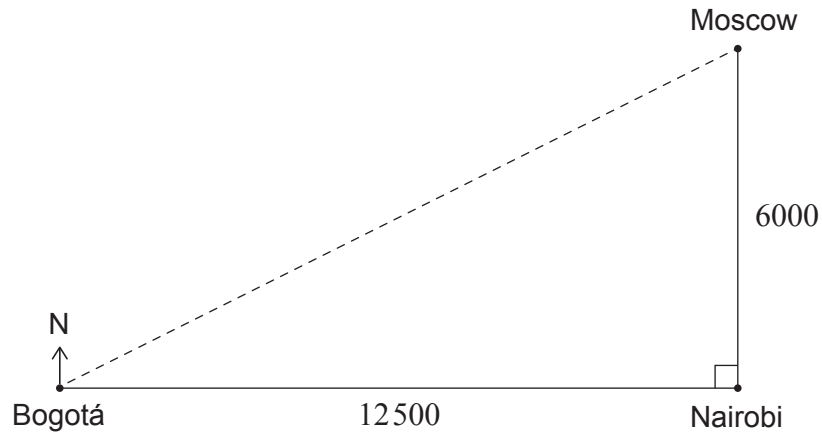
(g) Show that a solution to the original differential equation is $y = \frac{x(x^2 - A)}{x^2 + A}$. [4]

2. [Maximum mark: 27]

The following question compares the distance and direction between cities on a flat surface to the distance and direction between cities on a sphere.

Consider a model where the cities of Bogotá, Moscow, and Nairobi lie on a flat surface. In this model, Nairobi is 6000 km due south of Moscow and Bogotá is 12 500 km due west of Nairobi, as shown in the following diagram.

diagram not to scale



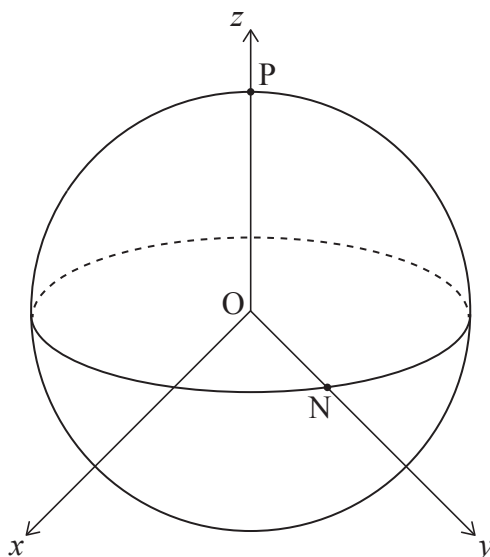
- (a) (i) Find the distance from Bogotá to Moscow. [2]
- (ii) Find the bearing of Moscow from Bogotá. Give your answer in degrees. [3]

(This question continues on the following page)

(Question 2 continued)

In reality, these three cities lie on the curved surface of the Earth which will change the distances and directions found in part (a).

Now consider a curved model using a coordinate system (x, y, z) with its origin, O , at the centre of the Earth. The units of this system are thousands of kilometres and the Earth is modelled as a sphere with radius 6000 km. The North Pole, P , lies on the z -axis, and Nairobi, N , is modelled as being on the equator and lying on the y -axis.



P has position vector $\vec{OP} = \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ and N has position vector $\vec{ON} = \mathbf{n} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$.

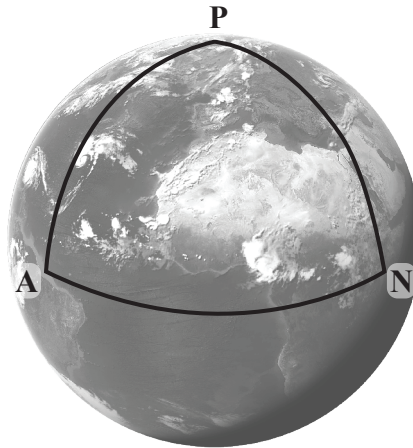
(b) (i) Use the scalar product to find the angle between \mathbf{p} and \mathbf{n} . [2]

(ii) Show that the distance between P and N along the arc from P to N is 3000π km. [2]

(This question continues on the following page)

(Question 2 continued)

Point A , which is also on the equator, has position vector $\mathbf{a} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ as shown in the following diagram.



P , N and A , and the arcs connecting them, form a spherical triangle.

The angle at vertex A is defined as the angle between the vectors $\mathbf{a} \times \mathbf{p}$ and $\mathbf{a} \times \mathbf{n}$.

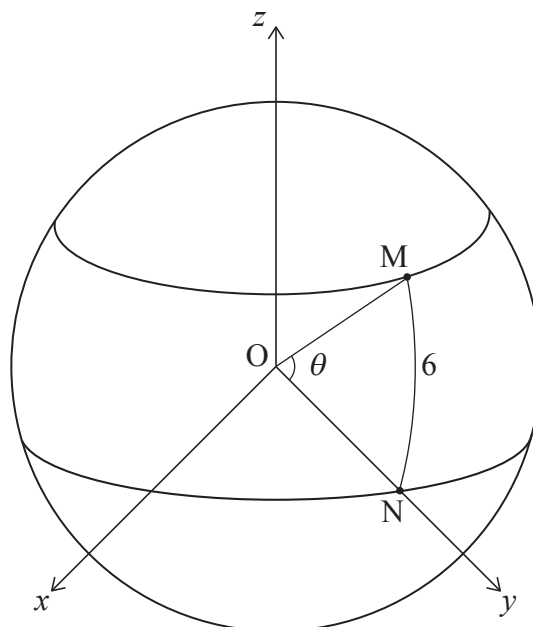
(c) (i) Find the vector $\mathbf{a} \times \mathbf{p}$. [2]

(ii) Show that the angle at vertex A in the spherical triangle is 90° . [3]

(This question continues on the following page)

(Question 2 continued)

Moscow, M, has position vector $\vec{OM} = \mathbf{m} = \begin{pmatrix} 0 \\ 6 \cos \theta \\ 6 \sin \theta \end{pmatrix}$, as shown in the following diagram.



The shortest distance between two points on the sphere lies along an arc of a circle on the sphere with centre O. In this model the shortest distance from Moscow to Nairobi is 6000 km.

- (d) Show that $\theta = 57.3^\circ$, correct to three significant figures. [2]

Bogotá, B, is west of Nairobi and has position vector $\vec{OB} = \mathbf{b} = \begin{pmatrix} 6 \sin 120^\circ \\ 6 \cos 120^\circ \\ 0 \end{pmatrix}$.

- (e) Find the shortest distance from Bogotá to Moscow on the sphere. [5]

The bearing from B to M is defined as the angle at vertex B in the spherical triangle

containing B, M and P. It is given that $\mathbf{b} \times \mathbf{p} = \begin{pmatrix} 36 \cos 120^\circ \\ -36 \sin 120^\circ \\ 0 \end{pmatrix}$.

- (f) Using the method from part (c), find the bearing from Bogotá to Moscow. [6]

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References:

- 2(c)** Google Maps/Google Earth. Data from Data SIO, NOAA, U.S. Navy, NGA, GEBCO Landsat / Copernicus, IBCAO, U.S. Geological Survey, PGC/NASA. Imagery from 14/12/2015. Image available at: https://earth.google.com/web/@12.01529518,-18.56070747,-158.39383184a,23597813.93249989d,30.00008083y,359.99981502h,0t,0r/data=CgRCaggBOgMKATBCAggASg0I_____ARAA.

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