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# Mathematics: analysis and approaches

## Higher level

### Paper 3

21 May 2025

**Zone A** afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

**This question asks you to explore self-composite linear functions, of the form  $f(x) = mx + c$ , for varying values of  $m$ .**

A function composed with itself is called a self-composite function.

For a function  $f$ , the function composition with itself is given by  $(f \circ f)(x) = f(f(x))$ .

Let  $f^n$  denote the  $n$ th composition of  $f$  with itself where  $f^n(x) = \underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(x)$ .

Hence, for example,  $f^2(x) = (f \circ f)(x)$  and  $f^3(x) = (f \circ f \circ f)(x) = f(f(f(x)))$ .

Consider the linear function  $f(x) = mx + c$ , where  $x \in \mathbb{R}$  and  $m, c \in \mathbb{R}$ .

(a) Show that

$$(i) \quad f^2(x) = m^2x + c(1 + m); \quad [3]$$

$$(ii) \quad f^3(x) = m^3x + c(1 + m + m^2). \quad [2]$$

(b) (i) Write down an expression for  $f^4(x)$ . [1]

(ii) Suggest a similar expression for  $f^n(x)$ ,  $n \in \mathbb{Z}^+$ . [2]

(iii) By using your expression from part (b)(ii), or otherwise, find an expression in terms of  $n$  for  $f^n(x)$  when  $m = 1$ . [3]

(c) For  $m \neq 1$ , use mathematical induction to prove that

$$f^n(x) = m^n x + c \left( \frac{1 - m^n}{1 - m} \right) \text{ for } n \in \mathbb{Z}^+. \quad [8]$$

**(This question continues on the following page)**

**(Question 1 continued)**

Consider  $f^n(x) = m^n x + c \left( \frac{1 - m^n}{1 - m} \right)$  where  $-1 < m < 1$ .

- (d) As  $n \rightarrow \infty$ , the family of graphs  $y = f^n(x)$  approaches the graph of a straight line,  $L$ . Determine the equation of  $L$ , giving your answer in terms of  $c$  and  $m$ . [4]

Consider  $f^n(x) = m^n x + c \left( \frac{1 - m^n}{1 - m} \right)$  where  $m = -1$ .

- (e) (i) Show that  $f^n(x) = -x + c$  when  $n$  is odd. [2]  
 (ii) Find an expression for  $f^n(x)$  when  $n$  is even. [2]

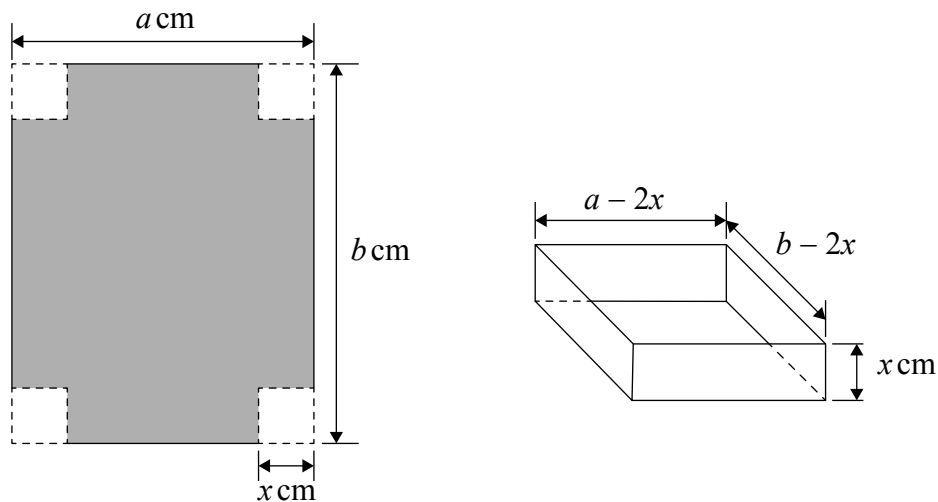
2. [Maximum mark: 28]

**This question asks you to consider open-topped boxes formed by cutting square corners from an  $a \times b$  sheet of cardboard and folding the sides. In particular, you are asked to investigate conditions for which the maximum volume of a box is an integer.**

A rectangular sheet of cardboard measures  $a$  cm by  $b$  cm where  $a < b$ .

A square of side length  $x$  cm is cut from each corner and the resulting sheet of cardboard is then folded to form an open-topped box with a rectangular base.

This is shown in the following diagrams.



The volume,  $V \text{ cm}^3$ , of a box is given by  $V = x(a - 2x)(b - 2x)$  where  $0 \leq x \leq \frac{a}{2}$ .

(a) Write down the value of  $V$  when

(i)  $x = 0$ ; [1]

(ii)  $x = \frac{a}{2}$ . [1]

(b) Show that  $V = 4x^3 - 2(a + b)x^2 + abx$ . [2]

(c) Show that the only solutions of  $\frac{dV}{dx} = 0$  are  $x = \frac{(a + b) \pm \sqrt{a^2 - ab + b^2}}{6}$ . [6]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), the following three results are given.

$$x = \frac{(a+b) + \sqrt{a^2 - ab + b^2}}{6} \text{ lies outside the interval } 0 \leq x \leq \frac{a}{2}.$$

$$x = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6} \text{ lies inside the interval } 0 \leq x \leq \frac{a}{2}.$$

$$\sqrt{a^2 - ab + b^2} > 0.$$

Let  $x_m = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6}.$

(d) Use  $\frac{d^2V}{dx^2}$  to show that there is a local maximum of  $V$  at  $x = x_m$ . [5]

(e) Use your answers to parts (a) and (d) to explain why  $x_m$  will give the maximum volume,  $V_m$ , of the box. [1]

In parts (f) to (h), consider a method for generating sets of positive integer values for  $x_m$  and hence  $V_m$ .

Consider the case where  $a = 2st - s^2$ ,  $b = t^2 - s^2$ , and  $s, t \in \mathbb{Z}^+$ .

(f) Given that  $a < b$ , show that  $t > 2s$ . [2]

It is given that  $\sqrt{a^2 - ab + b^2} = s^2 - st + t^2$ . You are **not** required to show this result.

(g) By substituting for each of  $a$ ,  $b$  and  $\sqrt{a^2 - ab + b^2}$  in terms of  $s$  and  $t$ , show that  $x_m = \frac{s(t-s)}{2}$ . [3]

(h) (i) Given that  $a, b \in \mathbb{Z}^+$ , use  $V_m = x_m(a - 2x_m)(b - 2x_m)$  to explain why  $s$  being even implies that  $V_m$  is an integer. [2]

(ii) State another condition on  $s$  and a corresponding condition on  $t$ , that implies that both  $x_m$  and  $V_m$  are integers. [1]

(iii) Hence, or otherwise, find an integer value of  $x_m$  and the corresponding integer value of  $V_m$ . [4]