# 1 Measuring space: accuracy and geometry

In this chapter you will approximate and calculate measures of angles, distances, areas and volumes in two and three dimensions. You will determine how these tools are used in astronomy to measure the distance to nearby stars, in cartography to measure distances between landmarks, and in navigation to determine the course of ships and aircrafts. In addition, you will explore other applications in surveying, architecture, disaster assessment and biology.

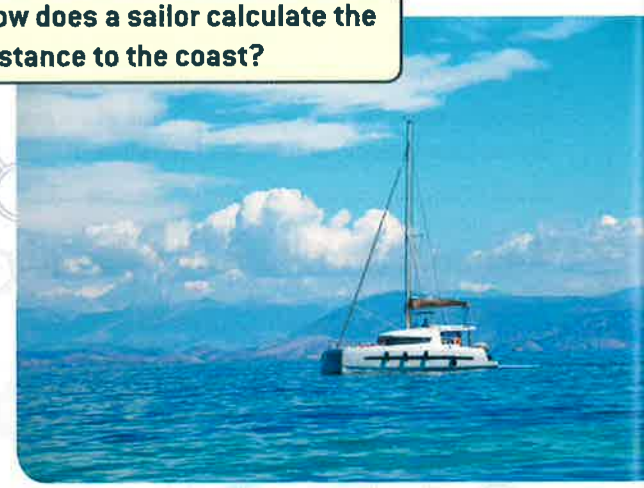
**Concepts**

- Space
- Quantity

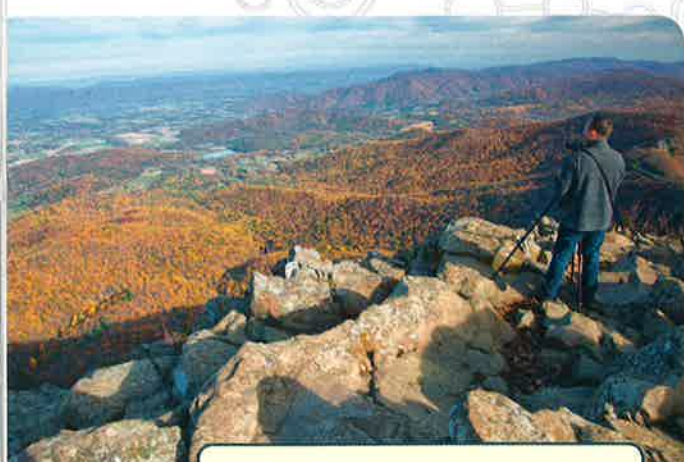
**Microconcepts**

- Rounding
- Percentage error
- Standard form
- Operations with rational exponents
- Right triangles: Sine, cosine, tangent ratios
- Angles of elevation and depression
- Bearings
- Non-right triangles: Sine rule and cosine rule
- Circles: Length of arc, area of sector
- Volume and surface area of 3D figures
- Angle between line and plane

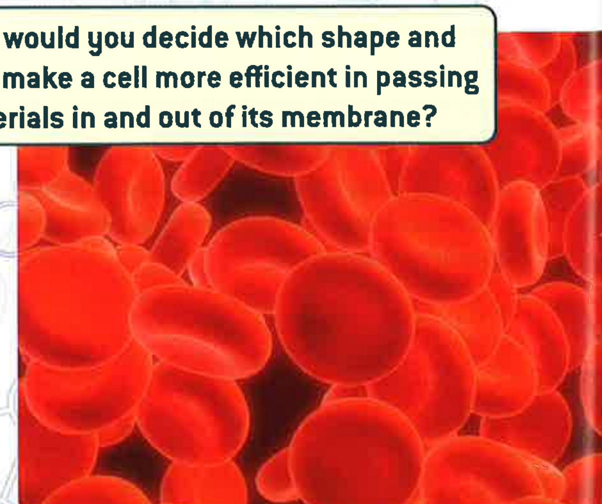
**How can the distance to a nearby star be measured?**



**How does a sailor calculate the distance to the coast?**

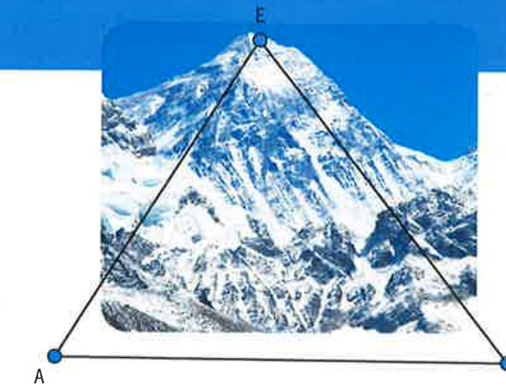


**How could you find the height of a mountain peak?**



**How would you decide which shape and size make a cell more efficient in passing materials in and out of its membrane?**






In 1856 The Great Trigonometric Survey of India measured the height of Mount Everest, known in Napali as Sagarmatha and in Tibetan as Chomolungma. The surveyors measured the distance between two points at sea level and then measured the angles between the top of the mountain and each point.

The summit of Mount Everest is labeled E, the two points A and B are roughly at sea level and are 33 km apart.

If $\hat{E} = 90°$, what would be the maximum possible height of Mount Everest?

**Think** about and then **write down** your own intuitive answer to each question. **Discuss** your answer with a friend then **share** your ideas with your class.

- Why do you think it took so long to determine the elevation of Mount Everest?
- Why do you think surveyors prefer to measure angles, but not lengths of sides?
- What assumptions are made?

## Developing inquiry skills

Write down any similar inquiry questions you might ask if you were asked to find the height of tree, the distance between two towns or the distance between two stars.

What questions might you need to ask in these scenarios which differ from the scenario where you are estimating the height of Mount Everest?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

## Before you start

### You should know how to:

1 Round to significant figures and decimal places.

eg Write the number 80.426579 to:

**a** 2 decimal places

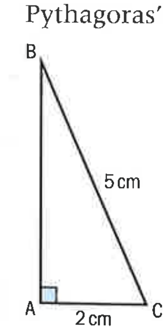
**b** 3 significant figures.

**a** 80.43 **b** 80.4

2 Evaluate integer and rational exponents.

eg $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$, $16^{\frac{1}{2}} = \sqrt{16} = 4$

3 Use properties of triangles, including Pythagoras' theorem.




eg Find the length of AB in this triangle.

$AB^2 = BC^2 - AC^2$

$AB^2 = 5^2 - 2^2$

$AB^2 = 21$

$AB = \sqrt{21} = 4.58$ cm (3 s.f.)

### Skills check

Click here for help with this skills check

1 Write down each number rounded to:

**a** 2 decimal places

**b** 3 significant figures.

**i** 0.6942 **ii** 28.706

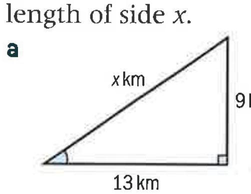
**iii** 77.984561

2 Evaluate

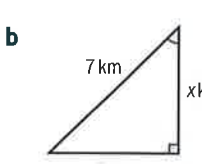
**a** $2^{-3}$ **b** $27^{\frac{1}{3}}$

3 In each of these right triangles, find the length of side $x$.

**a**




**b**



# 1.1 Representing numbers exactly and approximately

In mathematics and in everyday life, we frequently encounter numbers that have been measured or estimated. In this section you will investigate how we can quantify uncertainty in the numbers and calculations we use throughout this course.

### Investigation 1

Margaret Hamilton worked for NASA as the lead developer for Apollo flight software. The photo here shows her in 1969, standing next to the books of navigation software code that she and her team produced for the Apollo mission that first sent humans to the Moon.

1. Estimate the height of the books of code stacked together, as shown in the image. What assumptions are you making?
2. Estimate the number of pages of code for the Apollo mission. How would you go about making this estimation? What assumptions are you making?
3. **Factual** What is an estimate? What is estimation? How would you go about estimating? How can comparing measures help you estimate?
4. **Conceptual** Why are estimations useful?

Recall that we can specify accuracy using **significant figures (digits)**. The digits that can be determined accurately are called **significant figures**. Thus, a scale that could only register up to the hundredths of a gram mass until 99.99 g, could only measure up to 4 figures of accuracy (4 significant figures).

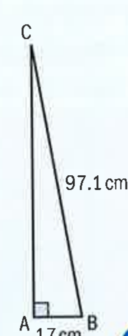
In your IA or on specific exam questions, you may need to choose an appropriate degree of accuracy. When performing operations on measurements, round answers to the same number of significant figures as the least accurate measurement. This is illustrated in the example below.

**EXAM HINT**

Write your final answers as exact values or rounded to at least 3 s.f., unless otherwise instructed. Round only your final answer and not any intermediate calculations.

### Example 1

A component of an aircraft wing is being designed in the shape of a right triangle. One of the legs must measure 17 cm and the hypotenuse must measure 97.1 cm, as shown in the diagram.




**a** Find the height of the triangle, rounding your answer **to the given degree of accuracy**.

**i** To 4 d.p. (decimal places) **ii** To 2 s.f.

**b** Find the area of the material (in cm$^2$) necessary to manufacture the component to an appropriate degree of accuracy.

**c** Show that intermediate rounding to 2 s.f. leads to an inaccurate answer.





| | |
|---|---|
| **a** **i** 95.6003 cm<br>**ii** 96 cm | Use Pythagoras to find the height of the triangle: $\sqrt{97.1^2 - 17^2} = 95.60026151\ldots$<br>If the question did not specify a level of accuracy, we would round to 3 s.f.: 95.6 cm. |
| **b** $A = \frac{1}{2}bh = \frac{1}{2}\times 17 \times 95.60026151\ldots$<br>$= 812.602\ldots$<br>$= 810\text{ cm}^2$ (2 s.f.) | Use the most accurate value (store or copy/paste with technology) when calculating. You could also write down your work as $A = \frac{1}{2}\times 17\times 95.6$. Do not use this to calculate!<br>Round area to 2 s.f. because the least accurate measurement, 17 cm, has 2 s.f. |
| **c** $A = \frac{1}{2}\times 17\times 96 = 816$<br>$= 820\text{ cm}^2$ (2 s.f.)<br>$\neq 810\text{ cm}^2$ | This shows why rounding intermediate answers should be avoided in all calculations. |

### Exercise 1A

1. A restaurant is remodelling and replacing its circular tables with square tables. They want the new tables to have the same area as the old ones. The circumference of the circular tables is measured to be 4.1 m. Find the side length of the new square tables
   - **i** to an appropriate degree of accuracy
   - **ii** to 3 s.f.
2. The heights of 10 koalas, measured to the nearest cm, are: 81, 73, 71, 80, 76, 84, 73, 88, 91, 75.

   Find the mean (average) height of the koalas to an appropriate degree of accuracy.

## Bounds and error

Suppose you find the weight of a bag of coffee as 541.5 g, accurate to the nearest 0.1 g. Then the exact weight $w$ of the bag could be anywhere in the interval $541.45\text{ g} \le w < 541.55\text{ g}$, as all of these values would round to 514.5 g.

If a measurement $M$ is accurate to a particular unit $u$, then its exact value $V$ lies in the interval

$$M - 0.5u \le V < M + 0.5u$$

The endpoints of this interval are called the **lower** and **upper bounds**. Upper and lower bounds provide one way to quantify the **uncertainty** of a measurement when the exact value is unknown.

**TOK**

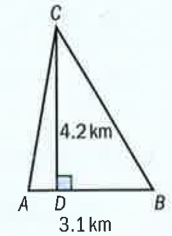
How does the perception of the language being used distort our understanding?

Number and algebra

Geometry and trigonometry

## Example 2

A state park is created on a triangular area between roads. The triangular area is measured to have a base length of 3.1 km and corresponding height of 4.2 km. The measurement tool is accurate to the nearest tenth of a kilometre. Find the upper bound to the area of the park.




| | |
|---|---|
| $3.1 + 0.05 = 3.15\,\text{km}$<br>$4.2 + 0.05 = 4.25\,\text{km}$ | The upper and lower bounds for the base length will be<br>$3.1 - 0.05 \le b < 3.1 + 0.05$<br>$3.05 \le b < 3.15$<br>and for the height<br>$4.2 - 0.05 \le h < 4.2 + 0.05$<br>$4.15 \le h < 4.25$ |
| Area of park $< \frac{1}{2} \times 3.15 \times 4.25$ | Using the area of triangle and upper bounds. |
| Area of park $< 6.69\,\text{km}^2$ (3 s.f.) | Area of a triangle $= \frac{1}{2} \times$ base $\times$ height |

Since measurements are approximate there is always error in the measurement results. A measurement error is the difference between the exact value ($V_E$) and the approximate value ($V_A$), ie:

$$\text{Measurement error} = V_A - V_E$$

### Investigation 2

Tomi and Massimo measured the length of a yardstick and the length of a foot and obtained 92.44 cm for the length of a yard and 31.48 cm for the length of a foot.

1 Given that the exact values of 1 yard is 91.44 cm and 1 foot is 30.48 cm, find the measurement error in the two measurements obtained by Tomi and Massimo.

Tomi thinks that the two measurements were equally inaccurate. Massimo thinks that one of the two measurements is more accurate than the other.

2 Who do you agree with: Tomi or Massimo? Explain why.

Massimo decides to find the magnitude of the error as a percentage of the measured length.

3 Write down the measuring error in the length of one yard as a fraction of the exact length of the yard. Give your answer as percentage.

4 Write down the measuring error in the length of one foot as a fraction of the exact length of one foot. Give your answer as percentage.

5 **Conceptual** In what ways is expressing measuring errors as a percentage of the measured length helpful?

6 **Conceptual** How can measurement errors be compared?

When the exact value of a quantity is known, the error of a measured (approximate) value can be found as a percentage of the exact value:

**Percentage error formula**

Percentage error $= \left|\frac{V_A - V_E}{V_E}\right| \times 100\%$, where $V_A$ is the approximate (or measurement) value and $V_E$ is the exact value.

## Example 3

The fraction $\frac{22}{7}$ is often used as an approximation of $\pi$.

**a** Find the percentage error of this approximation.

**b** Find the least accurate decimal approximation of $\pi$ needed to approximate $\pi$ within 0.001% of the true value.

| | |
|---|---|
| **a** Percentage error $= \left|\frac{\frac{22}{7} - \pi}{\pi}\right| \times 100\%$<br>$= 0.0402\%$ (3 s.f.) | Note the absolute value creates a positive value. |
| **b** $\left|\frac{V_A - \pi}{\pi}\right| \le 0.00001$ | Translating 0.001% to a decimal |
| $-0.00001 \le \frac{V_A - \pi}{\pi} \le 0.00001$ | $\lvert x\rvert \le C$ means $-C \le x \le C$ |
| $\pi - 0.00001\pi \le V_A \le \pi + 0.00001\pi$ | Keep values exact until the last step, then evaluate $\pi$ with technology. |
| $3.14156 \le V_A \le 3.14162$<br>So $V_A = 3.1416$ will approximate $\pi$ to within 0.001% | Write enough digits of the upper and lower bounds to see where they differ. |





**TOK**

In many cases, our measurements and calculations include errors. Very often courts rely on interpretations of forensic data and invite experts to courtrooms to give their opinion. If the percentage error of a certain DNA testing or drug testing to be found as 0.0526, can we be certain the person in question is guilty? What would be considered as an acceptable error rate especially when the stakes are so high?

## Exercise 1B

1 Find the range of possible values for the following measurements, which were rounded to the indicated degrees of accuracy:

- **a** 24 mm (nearest mm)
- **b** 3.2 m (tenth of a metre)
- **c** 1.75 kg (0.01 kg)
- **d** 1400 g (3 s.f.)

2 In 1856, Andrew Waugh announced Mount Everest as 8840 m high, after several years of calculations based on observations made by the Great Trigonometric Survey. More recent surveys confirmed the height at 8848 m.

- **a** Assuming the more recent survey is an exact value, calculate the percentage error made in the earlier survey.
- **b** If the more recent survey was accurate to the nearest metre, find the range of possible values for the exact height of Mount Everest.

3 Two lab groups in a Physics class measure the times for a ball to fall 1 metre and record the times in the following tables.

| Group 1 | |
|---|---|
| Trial | Time (s) |
| 1 | 0.45 |
| 2 | 0.53 |
| 3 | 0.47 |
| 4 | 0.55 |
| 5 | 0.43 |
| 6 | 0.67 |

| Group 2 | |
|---|---|
| Trial | Time (s) |
| 1 | 0.48 |
| 2 | 0.56 |
| 3 | 0.34 |
| 4 | 0.49 |
| 5 | 0.30 |
| 6 | 0.45 |

- **a** Find the average of all the measurements for each group.

Using the laws of Physics the true value for the time of the fall is 0.452 seconds given air resistance can be ignored.

- **b** Calculate the percentage error for each set of data.
- **c** Based on your calculations, comment on the uncertainty of the results of each group.

4 With 72 million bicycles, correct to the nearest million, Japan is at the top of the list of countries with most bicycles per capita. On average, Japanese people travel about 2 km, correct to the nearest km, on their bicycles each day. Calculate the upper bound for total distance travelled by all the bicycles in Japan per year.

5 To determine if a business is making enough profit the following formula is used $P = \frac{S - C}{S}$ where $P$ is relative profit, $S$ is sales income and $C$ is costs. If a company has \$340 000 worth of sales and \$230 000 as costs, each correct to 2 significant figures, calculate the maximum and minimum relative profit to an appropriate degree of accuracy.







6 The temperature today in Chicago is 50 °F. Being used to Celsius, Tommaso wants to convert the °F to °C to know what to wear outside. But instead of using the standard conversion formula $°C = \frac{5}{9} \times (°F - 32)$ he uses his grandmother's rule that is easier, but gives an approximate value: "Subtract 32° from the value in °F and multiply the result by 0.5".

- **a** Calculate the actual and an approximate temperature value in °C using the standard formula and Tommaso's grandmother's rule.
- **b** Calculate the percentage error of the approximate temperature value, in °C.

7 A factory produces circular slabs for use in construction. They guarantee that all slabs produced will have an area within 0.2% of the "target" of 163 m².

- **a** Find the range of values for the radius that will ensure all slabs produced are within this range.
- **b** Determine how accurately the radius must be measured during production to ensure that it will fall within this range.

## Exponents and standard form

Exponents can make representing numbers and performing calculations easier and more exact. In particular, when dealing with very large of very small numbers, such as in astronomy, macroeconomics, or chemistry, **standard form** (or **scientific notation**) can be more efficient.

**International-mindedness**

Where did numbers come from?



Recall that a number can be written in the **standard form** $a \times 10^k$ with **coefficient** $1 \le a < 10$ and **exponent** $k \in \mathbb{Z}$.

### Investigation 3

1 The three countries with the largest populations in 2017 were:

India: 1.34 billion

USA: $3.24 \times 10^8$

China: 1 409 517 397

- **a** Convert all numbers to standard form. Round to 3 s.f. as needed.
- **b** Explain how you can easily order these numbers from smallest to largest when they are converted to standard form.

Gross Domestic Product (GDP) measures the total value of goods and services produced by a country and is one way to measure the wealth of countries. The GDP per capita (per person) of three countries for 2017 is given in the table below.

| Country | GDP per capita (\$/person) |
|---|---|
| India | 1983 |
| USA | 59 501 |
| China | 8643 |

To find the total GDP of each country, multiply GDP per capita by population. First you will investigate how to multiply numbers in standard form by hand.

Continued on next page

2 Complete the examples below using technology:

| | $x$ | $y$ | $xy$ |
|---|---|---|---|
| (a) | $3 \times 10^5$ | $2 \times 10^9$ | |
| (b) | $8 \times 10^1$ | $1 \times 10^4$ | |
| (c) | $2 \times 10^{-3}$ | $4 \times 10^{17}$ | |
| (d) | $5 \times 10^6$ | $3 \times 10^{12}$ | |
| (e) | $4 \times 10^5$ | $9 \times 10^{-7}$ | |

3 How can you find the product $(b \times 10^m)(c \times 10^n)$ in the form $a \times 10^k$ where $1 \le a < 10$ and $k$ is an integer? Make sure your process is consistent with all five examples above.

4 How does your process for multiplication relate to the law of exponents, $x^p \times x^q = x^{p+q}$?

5 Now estimate and calculate the GDP of each country.

a Write each GDP per capita in standard form.

b Estimate the GDP of each country, without use of technology. Round your numbers as needed.

c Use your GDC to calculate the GDP of each country.

d Compare your estimates with the calculations. Was the magnitude (power of 10) of your estimate correct?

6 **Conceptual** How does standard form help with calculations?

**HINT**

Recall that you can enter numbers on your GDC in standard form: $3.4 \times 10^7$ is usually entered as 3.4 E 7. Make sure that you translate this calculator notation back to proper standard form when transferring answers from technology.

**TOK**

What might be the ethical implications of rounding numbers?

Numbers in standard form can be multiplied or divided following rules for exponents:

$(b \times 10^m)(c \times 10^n) = bc \times 10^{m+n}$ and $\dfrac{b \times 10^m}{c \times 10^n} = \dfrac{b}{c} \times 10^{m-n}$

After performing the operation, ensure your answer is given in the form $a \times 10^p$, where $0 \le a < 10$ by adjusting the exponent as needed.

## Example 4

Light travels at a speed of $3 \times 10^8$ m/s. The Earth is approximately 150 million kilometres from the Sun. Estimate the time, in seconds, that light takes to travel from the Sun to the Earth.

| | |
|---|---|
| Since distance = speed × time, | Convert the distance to metres in standard form: |
| | 150 million km = $1.50 \times 10^8$ km <br> $= 1.50 \times 10^{11}$ m |
| $\text{time} = \dfrac{1.50 \times 10^{11}\text{ m}}{3 \times 10^8\text{ m/s}}$ <br> $= 0.5 \times 10^3$ s <br> $= 500$ s | Divide coefficients and subtract exponents. <br> Note that technology could also be used to divide. |





A **negative exponent** represents a reciprocal power:

$x^{-n} = \dfrac{1}{x^n}$

A **rational exponent** represents a power of a root:

$x^{\frac{p}{q}} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p, p, q \in \mathbb{Z}$

In particular, $x^{\frac{1}{2}} = \sqrt{x}$.

The following rules of exponents hold for $a > 0$ and $m, n \in \mathbb{Q}$

$a^m \times a^n = a^{m+n}$

$(a^m)^n = a^{mn}$

$\dfrac{a^m}{a^n} = a^{m-n}$

**HINT**

You should be able to evaluate exponents and roots by hand or with technology, such as:

$\sqrt[3]{50} = 50^{\frac{1}{3}} \approx 3.68$

$2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{16}$

Number and algebra

Geometry and trigonometry

While we will frequently use technology to approximate values for rational exponents in this course, exact values allow us to calculate more precisely and to use rules of exponents.

## Example 5

**a** Find the surface area of a cube with volume 50 cm³

**i** exactly

**ii** approximately.

**b** **i** Find a general formula for the surface area of a cube with volume $V$ cm³.

**ii** Hence, determine the volume needed for a surface area of 1000 cm².

| | |
|---|---|
| **a** $V = 50 = s^3$, so $s = 50^{\frac{1}{3}}$. | Solve for the side length ($s$). |
| $SA = 6s^2 = 6\left(50^{\frac{1}{3}}\right)^2$ | Substitute into the surface area formula. |
| **i** $= 6 \times 50^{\frac{2}{3}}$ cm² | Apply the rule $(a^m)^n = a^{mn}$. |
| **ii** 81.4 cm² | Evaluate with technology. |
| **b** **i** $SA = 6V^{\frac{2}{3}}$ | Generalizing 50 to $V$. |
| **ii** $1000 = 6V^{\frac{2}{3}}$ <br> $\left(V^{\frac{2}{3}}\right)^{\frac{3}{2}} = \left(\dfrac{1000}{6}\right)^{\frac{3}{2}}$ <br> $V = 2150$ cm³ (3 s.f.) | Isolate $V$ by dividing by six and then raising both sides to the reciprocal power. The exponent on $V$ becomes $\dfrac{2}{3} \times \dfrac{3}{2} = 1$ |

## Example 6

A stuffed animal company finds that each store can sell stuffed bears for a price of $p = \frac{240}{\sqrt{x}}$, where $x$ represents the population of the city in which the store operates, in thousands. Research also shows that the weekly quantity $q$ of stuffed bears that will be sold can be found with the formula $q = 0.9x^{\frac{3}{4}}$. The total weekly revenue of a store is the product of its price and quantity sold.

**a** Determine an expression for the price in the form $ax^m$, where $a \in \mathbb{R}$ and $m \in \mathbb{Q}$.

**b** Determine store revenue in the form $ax^m$, where $a \in \mathbb{R}$ and $m \in \mathbb{Q}$.

**c** If the company wants to open a store that will make at least \$1500 per week, determine the smallest population of city they should consider.

| | |
|---|---|
| **a** $p = \frac{240}{\sqrt{x}} = 240x^{-\frac{1}{2}}$ | |
| **b** $r = 240x^{-\frac{1}{2}} \times 0.9x^{\frac{3}{4}} = 216x^{\frac{1}{4}}$ | Multiply price by quantity. Multiply coefficients and add the exponents. |
| **c** $1500 = 216x^{\frac{1}{4}}$ | Isolate $x$. |
| $x = \left(\frac{1500}{216}\right)^4 = 2330$ (3 s.f.) | "Undo" the $\frac{1}{4}$ exponent by raising both sides to the 4th power. |
| The city should have a population of at least 2 330 000. | Interpret the solution; $x$ is measured in thousands. |

### Exercise 1C

1 For each operation, **i** estimate a value for the answer without technology, **ii** find the exact value using technology. Express all answers in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

**a** $(1.08 \times 10^{-3})(9.2 \times 10^{7})$ **b** $\frac{7 \times 10^4}{7.24 \times 10^{-6}}$

2 Calculate each expression using technology to 3 s.f. Express all answers in the form $a \times 10^n$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

**a** $(2.35 \times 10^{-6})(4 \times 10^{1})$

**b** $\frac{7.1 \times 10^6}{8.5 \times 10^2}$

**c** $\frac{4}{3}\pi(5 \times 10^{-7})^3$

**d** $\frac{50}{(8.8 \times 10^{-5})^{-2}}$

**HINT**

Remember to put brackets around numbers in standard form when you perform operations on them.

3 Simplify each expression and write your solution

**i** without negative exponents

**ii** in the form $ab^c$, where $a$, $b$, and $c$ are values or variables.

**a** $\frac{15x^{\frac{1}{2}}}{x}$ **b** $7^{-\frac{1}{3}} \times 7^{\frac{4}{3}}$

**c** $\frac{5 \times 2^{-3t}}{40}$ **d** $\left(\frac{5}{3^x}\right)^2$

4 There are initially 120 bacteria in a Petri dish and the population doubles approximately every hour, which can be represented by the formula $B = 120 \times 2^t$ where $B$ is the number of bacteria and $t$ is the time in hours since the bacteria began growing.

**a** Find $B$ when $t = 1$, $t = \frac{3}{2}$, and $t = 2$.

**b** Comment on what your answers tell you about the growth of the bacteria. What is the meaning of the value obtained when $t = \frac{3}{2}$.

5 The half-life of iodine-131 is approximately 8 days, which means the mass of a sample of iodine decays by half every 8 days. The amount remaining can be calculated using the formula $I = 1600 \times 2^{-\frac{t}{8}}$, where $t$ is the time in days since the beginning of the sample.

**a** Write down this formula without negative exponents.

**b** Find the amount of material remaining after 4 days as an exact and approximate value.

6 The image of a speck of dust in an electron microscope is $1.2 \times 10^2$ mm wide. The image is $5 \times 10^2$ times larger than the actual size. Determine the width of the actual speck of dust.

7 The Earth's mass is $5.97 \times 10^{24}$ kg and Mercury's mass is $3.29 \times 10^{23}$ kg. How many times more massive is the Earth than Mercury?

8 The Earth's surface area is approximately $5.1 \times 10^8$ km$^2$ (2 s.f.) and its population is $7.6 \times 10^9$ (2 s.f.). **Population density** is the number of people per square kilometre. Determine the population density of the Earth assuming all the surface area is habitable by humans.

About 30% of the Earth's surface is land, including Antarctica. Determine the population density of the earth assuming that all the land is habitable, but not the oceans.









# 1.2 Angles and triangles

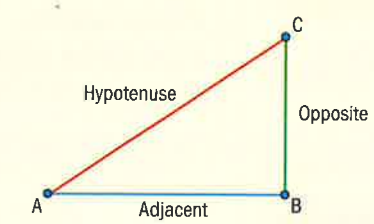
## Trigonometric ratios in a right triangle

The ratios of the sides of a right-angled triangle are called **trigonometric ratios**. The three most common trigonometric ratios are the **sine (sin)**, **cosine (cos)**, and **tangent (tan)**. These are defined for acute angle A in the right-angled triangle below:

$\sin(\hat{A}) = \frac{\text{Opposite}}{\text{Hypotenuse}}$

$\cos(\hat{A}) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

$\tan(\hat{A}) = \frac{\text{Opposite}}{\text{Adjacent}}$




In the definitions above, "opposite" refers to the length of the side opposite angle $\hat{A}$, "adjacent" refers to the length of the side adjacent to angle $\hat{A}$, and "hypotenuse" refers to the length of the hypotenuse (the side opposite the right angle).

**HINT**

Some people use the mnemonic **SOH-CAH-TOA**, pronounced "soh-kuh-toh-uh", to help them remember the definitions of sine, cosine, and tangent.

**International-mindedness**

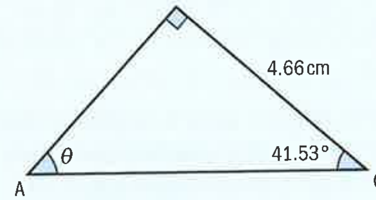
Diagrams of Pythagoras' theorem occur in early Chinese and Indian manuscripts. The earliest references to trigonometry are in Indian mathematics.

**Reflect** What are trigonometric ratios?

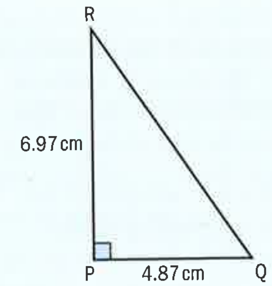
### Example 7

For each triangle, solve for the unknown angles and sides.

**a**




**b**




**a** In ΔABC:

$\theta = 90° - 41.53° = 48.47°$

$\cos(41.53°) = \frac{BC}{AC}$

$AC = \frac{4.66}{\cos(41.53°)} \Rightarrow AC = 6.22 \text{ (3 s.f.)}$

[BC] is adjacent to angle $\hat{C}$, use the cosine ratio to find hypotenuse AC. Ensure technology is set to degree mode. Round answers to 3 s.f.

$AB = \sqrt{AC^2 - BC^2} = 4.13 \text{ cm (3 s.f.)}$

Use Pythagoras' theorem to find AB.

Use the exact value of AC stored in your calculator (not the rounded value) when performing your calculations.

**b** In ΔPQR:

Use Pythagoras' theorem to find RQ.

$RQ = \sqrt{PR^2 + PQ^2} = 8.50 \text{(3 s.f.)}$

$\tan(P\hat{Q}R) = \frac{PR}{PQ} = \frac{6.97}{4.87}$

$P\hat{Q}R = \tan^{-1}\left(\frac{6.97}{4.87}\right) = 55.1° \text{(3 s.f.)}$

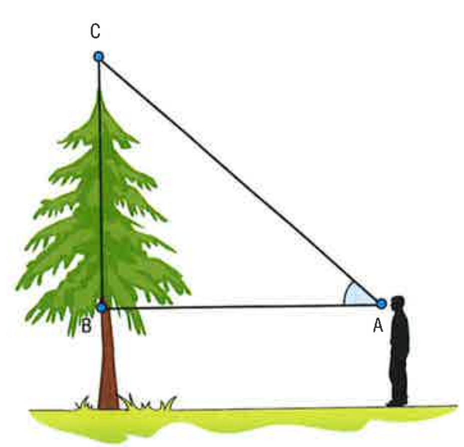
$P\hat{R}Q = 90 - 55.1 = 34.9°$

When solving for an unknown angle, determine a trig ratio with two known sides (use exact lengths when possible). PR is opposite side to $P\hat{Q}R$ and PQ is adjacent, so choose tangent.

Then use an **inverse trig function**, in this case $\tan^{-1}(x)$, to find the angle.

## Angles of elevation and depression

Suppose an observer is standing in front of a tree, with their eyes at point A as shown in the diagram. Angle $B\hat{A}C$ is formed when the observer looks up at the top of the tree. This angle is called an **angle of elevation** above the horizontal line at eye level (AB).



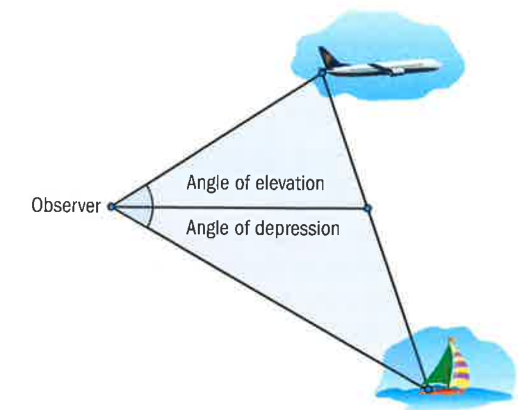





Number and algebra

Geometry and trigonometry

Similarly, when the object of sight falls below the horizontal at the eye level, an **angle of depression** is formed, as shown in the diagram.




### Example 8

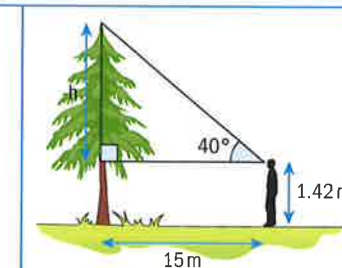
Emma stands 15 m away from a tree. She measures the angle of elevation to the top of the tree as 40° and her height to eye level as 142 cm.

**a** Find the height of the tree.

**b** Frank, whose height is 1.8 m to eye level, is standing on the other side of the tree. His kite is stuck at the very top of the tree. He knows the length of the kite string is 16 m. What is the angle of elevation as Frank looks up at his kite?

**a** $\tan 40° = \frac{h}{15}$

$h = 15\tan 40°$

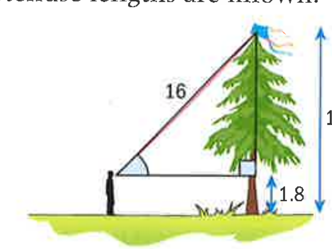



Height of tree $= 15 \tan 40° + 1.42$
$= 14.0 \text{ m (3 s.f.)}$

Add Emma's height to eye level.

**b** $\sin\theta = \frac{15\tan 40° + 1.42 - 1.8}{16} = \frac{12.206...}{16}$

Subtract Frank's height from the height of the tree to find the opposite side length. Use the sine ratio as opposite and hypotenuse lengths are known.



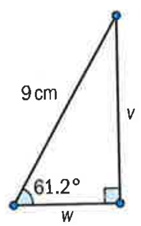

$\theta = \sin^{-1}\left(\frac{12.206}{16}\right) = 49.7° \text{ (3 s.f.)}$

Use inverse sine ($\sin^{-1}$) to find the angle.

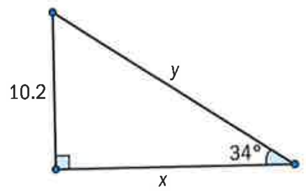
## Exercise 1D

**1** Determine all unknown sides and angles for each of the right triangles below:

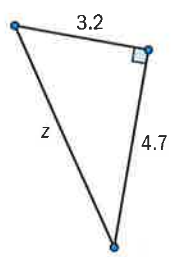
**a**



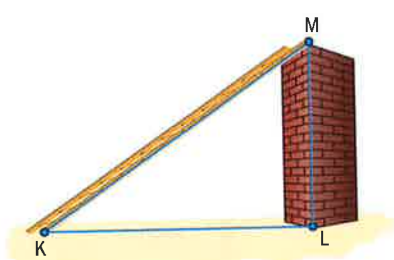

**b**




**c**




**2** A ladder [KM] is 8.5 m long. It currently leans against a vertical wall so that $L\hat{K}M = 30°$.




**i** Find the distance KL.

**ii** Find how far up the wall the ladder reaches.

**iii** The instructions for use of the ladder state that the angle it makes with the ground should not exceed 55°. Find the maximum height that the ladder can reach up the wall.

**3** A hiker, whose eye is 1.6 m above ground level, stands 50 m from the base of a vertical cliff. The angle between the line connecting her eye and the top of the cliff and a horizontal line is 58°.

**a** Draw a diagram representing the situation.

**b** Find the height of the cliff.

**4** The angle of depression from the top of a vertical cliff to a boat in the sea is 17°. The boat is 450 m from the shore.

**a** Draw a diagram.

**b** Find the height of the cliff. Give your answer rounded to the nearest metre.

**5** Your family wants to buy an awning for a window that will be long enough to keep the sun out when it is at its highest point in the sky. The awning is attached to the wall at the top of the window and extends horizontally. The height of the window is 2.80 m. The angle of elevation of the sun at this point is 70°. Find how long the awning should be. Write your answer correct to 2 d.p.

**6** Scientists measure the depths of lunar craters by measuring the shadow length cast by the edge of the crater on photographs. The length of the shadow cast by the edge of the Moltke crater is about 606 m, given to the neasest metre. The sun's angle of elevation "at the time the photograph was taken" is 65°. Find the depth of the crater. State your answer rounded to the nearest metre.

**7** Maatsuyker Island Lighthouse is the last Australian lighthouse still being officially operated by lightkeepers. The lighthouse is 15 m high from its base to the balcony, and located 140 m above sea level.

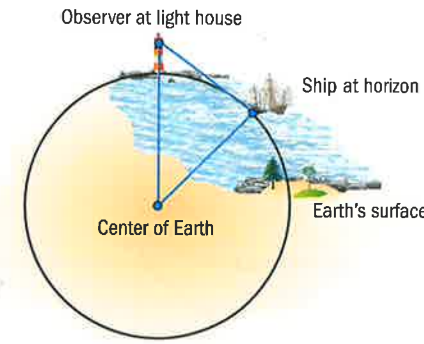
The caretaker is standing at the balcony and notices a ship at the horizon.

Find the straight line distance from the lighthouse balcony to the ship.





You may find it useful to draw a diagram like this




**8** Hans is constructing an accessibility ramp for a library that should reach a height of 27 cm with an angle no greater than 13°.

**a** Find the shortest possible length of ramp to achieve this.

**b** Hans cuts a length of wood to make the ramp. He cuts it to the length calculated in part **a** but can only cut it with an accuracy to the nearest centimetre. If the actual height required by the ramp is 27.43 cm, find the maximum possible percentage error between the desired 13° and the actual angle of the ramp.

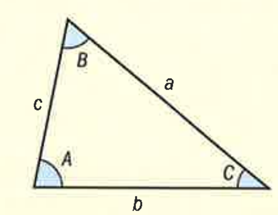
## Non-right triangles and the sine rule

The trigonometric ratios we have used so far require a right triangle. If we have a non-right triangle, can we still find missing sides and angles?

### Investigation 4

**Part 1**

Draw a scalene **obtuse** triangle ΔABC (without a right angle) using dynamic geometry software, and label the vertices.




- Measure all the angles and side lengths of the triangle.
- Find the following **ratios** correct to 3 significant figures.

$\frac{\sin \hat{A}}{a} =$ ______ $\frac{\sin \hat{B}}{b} =$ ______ $\frac{\sin \hat{C}}{c} =$ ______

What do you notice?

**Part 2**

Draw a scalene **acute** triangle ΔDEF (without a right angle) and label the vertices. Measure all the angles and side lengths

Find the following **ratios** correct to 3 significant figures

$\frac{DE}{\sin \hat{F}} =$ ______ $\frac{EF}{\sin \hat{D}} =$ ______ $\frac{DF}{\sin \hat{E}} =$ ______

What do you notice?

**Part 3**

Repeat parts 1 and 2 for a right angled triangle. What do you notice?

**Part 4**

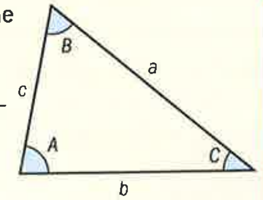
Are there any other types of triangle you could draw? Repeat Parts 1 and 2 for any other types of triangle you draw.

**Conceptual** What can you say about the ratio of the sine of an angle to the length of the side opposite the angle, in any triangle?

When is it most useful to use the sine rule with the angles "on top"? With the side lengths "on top"?

Sine Rule: In any non-right triangle, the ratio of each side to its opposite angle is the same for all three sides.

$$\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c} \quad \text{or} \quad \frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}}$$



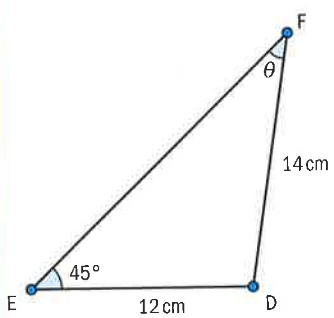

When solving a problem you will use just two of the three sides to set up an equation. When you are solving for a side length, it is easier to use the version with side lengths in the numerator, and similarly for angles.

## Example 9

In a triangle ΔDEF, DE = 12 cm, EF = 14 cm and $D\hat{E}F = 45°$.

Draw a labelled diagram and find the size of the angle $E\hat{F}D$ to the nearest degree.

Let $E\hat{F}D = \theta$




$$\frac{\sin 45°}{14} = \frac{\sin\theta}{12}$$

$$\sin\theta = \frac{12\sin 45°}{14}$$

$$\theta = \sin^{-1}\left(\frac{12\sin 45°}{14}\right)$$

$\theta = 37.30742828°$

$\theta \approx 37°$ (to the nearest degree)

Use the Sine Rule with angles on top to solve for a missing angle more easily.

Solve this equation for $\theta$ using inverse sine.

Keep answers exact until the last step to avoid rounding errors.



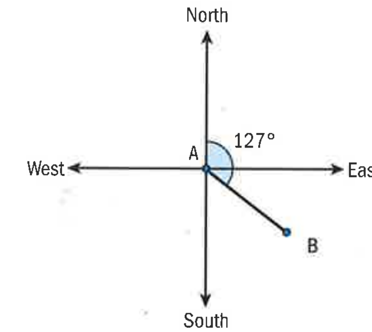


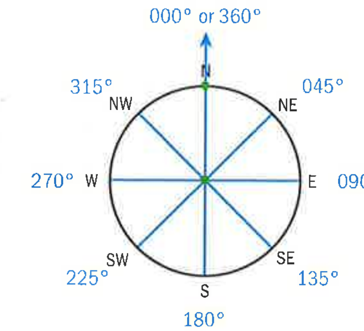




## Bearings

A bearing is an angle measured clockwise from North. The diagram on the left shows a bearing of 127° and the diagram on the right shows the bearings of the major compass directions.




### International-mindedness

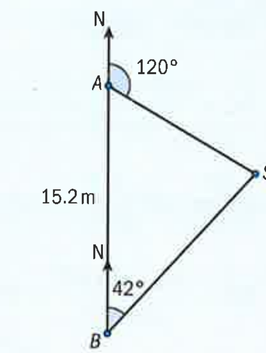
The word "sine" started out as a totally different word and passed through Indian, Arabic and Latin before becoming the word that we use today. Research on the Internet what the original word was.

## Example 10

A ship S is located on a bearing of 120° from port A, and 042° from port B. The port A is directly North of the port B.

The distance between ports A and B is 15.2 miles.

Find the distance from the ship to each port.




$\hat{A} = 180° - 120° = 60°$

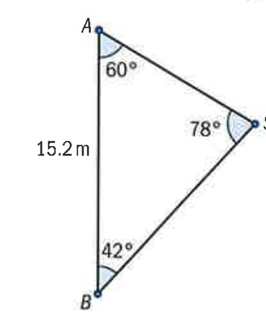
$\hat{S} = 180° - (42° + 60°) = 78°$

$$\frac{AS}{\sin 42°} = \frac{BS}{\sin 60°} = \frac{15.2}{\sin 78°}$$

$$AS = \sin 42° \times \frac{15.2}{\sin 78°} = 10.4 \text{ miles (3 s.f.)}$$

$$BS = \sin 60° \times \frac{15.2}{\sin 78°} = 13.5 \text{ miles (3 s.f.)}$$

Use angle rules to find $\hat{A}$ and $\hat{S}$.




Use the sine rule to find the lengths.

## Triangulation

Surveyors extensively use **triangulation** to indirectly calculate large distances. By measuring the distance between two landmarks and the angles between those landmarks and a third point, the surveyor can calculate the other two distances in the triangle formed by those points. This process can then be repeated to form a chain of triangles and is illustrated in the following example.

### TOK

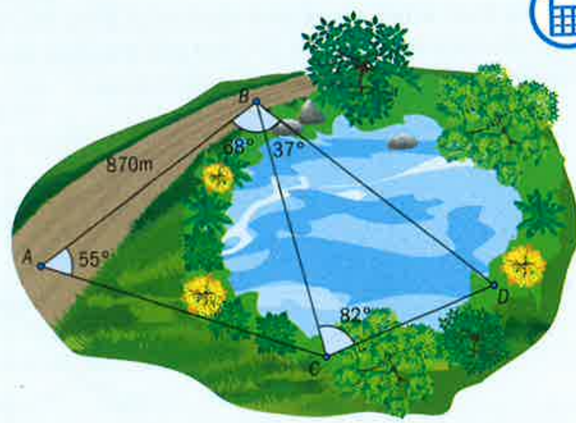
What does it mean to say that mathematics is an axiomatic system?

## Example 11

The diagram shows a lake with three docks at points B, C and D. The distance AB along a highway is known to be 870 m. Surveyors measure the angles as given in the diagram.

**a** Use triangulation to find the distances BC and BD.

**b** Nils, who rows at a speed of 1.5 m/s, starts from dock B. Calculate how much longer will it take him to cross the lake if he rows to the further of the two docks.

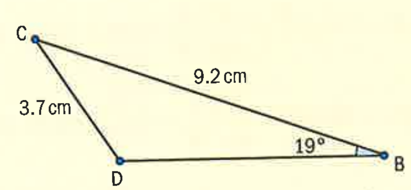



| | |
|---|---|
| **a** $\hat{C} = 180 - 55 - 68 = 57°$ | Find the angle opposite the known side, |
| $\frac{BC}{\sin 55°} = \frac{870}{\sin 57°}$ | $\hat{C} = 180 - \hat{A} - A\hat{B}C$ |
| $BC = \frac{870 \times \sin 55°}{\sin 57°} = 950\text{ m (3 s.f.)}$ | Substitute into the Sine Rule to find BC. |
| $\frac{BC}{\sin 61°} = \frac{BD}{\sin 82°}$ | Repeat the process with angles $B\hat{C}D$ and $\hat{D}$. Remember to use the "exact" value of BC from your calculator. |
| $BD = \frac{BC \times \sin 82°}{\sin 61°} = 962\text{ m (3 s.f.)}$ | |
| **b** $\frac{BD - BC}{1.5} = \frac{112\text{ m}}{1.5} = 74.9\text{ s}$ | Using $\text{time} = \frac{\text{distance}}{\text{speed}}$. |

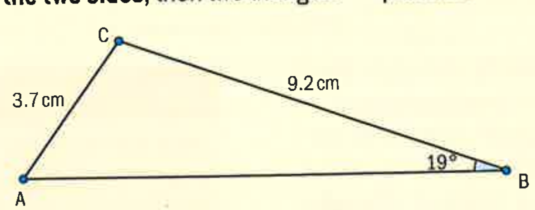
## Investigation 5

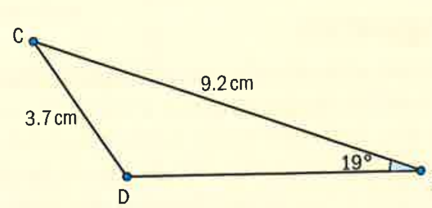
1 Use the Sine Rule to find $\hat{D}$ in the diagram on the right:

2 Explain why the solution you've obtained is not consistent with the diagram.




The issue you have just encountered is known as the **Ambiguous Case** of the Sine Rule. When **two sides and the non-included angle** are known, and the **unknown angle** is **opposite the longer of the two sides**, then two triangles are possible:





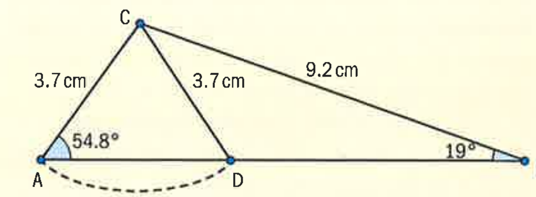







Overlaying these two diagrams shows how the triangles are related:

If you draw a circle with centre C and radius 3.7 cm then the circle will intersect the line at points A and B.




3 Check that $\hat{A}$ matches what you calculated in Question 1.

4 Use the diagram above to explain:

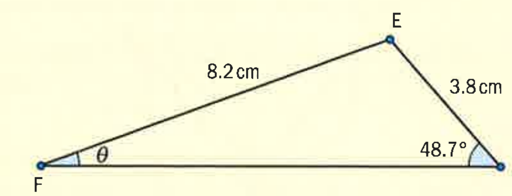
**a** why $\hat{A} = C\hat{D}A$

**b** and hence how angles $\hat{A}$ and $C\hat{D}B$ are related.

5 Use this relationship to find the correct solution to the missing angle in Question 1.

6 If you find one solution for a missing angle in an ambiguous case is 39°, what will be the other solution? What if the angle is $x$°?

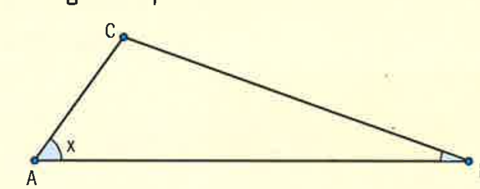
7 In the diagram on the right, the unknown angle is opposite the shorter of the two sides; the angle at G is fixed. Is it possible in this case to draw two different triangles? Draw them or explain why it's not possible.

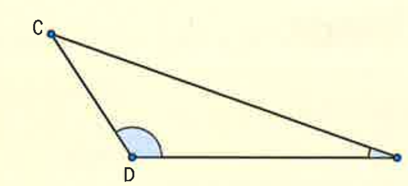



8 **Conceptual** Why does the Sine Rule not always have just one solution?

**Sine Rule – Ambiguous Case**

When **two sides and one angle** are known, and the **unknown angle** is **opposite the longer of the two sides**, then two triangles are possible:




The two possible solutions for the angle opposite the longer side are supplementary (sum to 180°).

Depending on additional information, you may be able to rule out one of the two possible solutions.

You may have noticed in the example above that we encountered obtuse angles. When we defined sine, cosine, and tangent, we did so with right triangles, so all angles were acute. In Chapter 8 you will explore further how we define trig ratios for angles outside this range. For this chapter, it is sufficient to evaluate with technology.

## Exercise 1E

1 For each triangle given below,

i sketch a diagram, labelling known sides and angles

ii state, with a reason, the number of possible triangles that satisfy the given information.

iii For each possible triangle, find all missing lengths and angles.

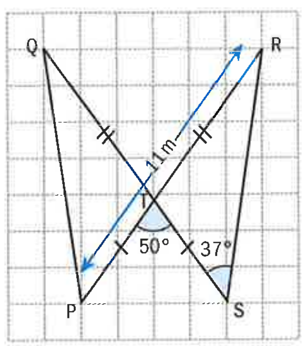
**a** In $\Delta ABC$, AC = 8 cm, $A\hat{C}B = 101°$, and $A\hat{B}C = 32°$.

**b** In $\Delta DEF$, DF = 14.7 cm, EF = 6.2 cm, $\hat{D} = 22°$

**c** In $\Delta GHI$, GH = GI = 209 cm, $\hat{H} = 52°$

2 A surveyor must determine the distance between points A and B that lie on opposite banks of a river. A point C is 450 m from A, on the same side of the river as A. The size of angle $B\hat{A}C$ is 45° and the size of angle $A\hat{C}B$ is 55°. Approximate the distance from A to B to the nearest metre.

3 A portion of a power line support tower is to be constructed as congruent triangles, as shown in the diagram. The crossing beams will each be 11 m long and will intersect at an acute angle of 50°, and $\hat{S} = 37°$. Find the lengths of all sides of the triangle QPT.




4 A ship leaves a port and travels along a bearing of 165° for 270 km. To avoid an incoming storm, it then changes course to a bearing of 203° for another 130 km and anchors. Find its bearing and distance from port.

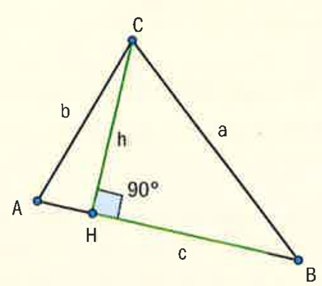
5 Karl and Shayne are located at point A along a beach and wish to swim to an island, represented by point P. Karl swims directly from point A to point P, a distance of 1750 m at a bearing of 297°. Shayne wants to swim no further than 1000 m, so he walks along the shoreline at a bearing of 270 degrees to get closer to the island. Determine the shortest distance he must walk, and the bearing he should follow as he swims, so that his swim from B to P is 1000 m.

## Area of a Triangle

### Investigation 6

Consider a plot of land in the shape of triangle ΔABC with sides of lengths $a$, $b$ and $c$.

The height h is drawn from C towards [AB], so that h is perpendicular to [AB].




1 Write an expression for the area of the triangle in terms of $c$ and $h$.

2 Consider ΔBHC. Write an expression for the height $h$ in terms of $a$ and angle $\hat{B}$.

3 Substitute your expression for h from question **2** into your expression for the area from question **1**.

4 **Factual** What is the formula for the area of any triangle?

5 **Conceptual** How can you find the area of any triangle?

The area of any triangle ABC can be calculated given two sides and the angle between them:

$$\text{area} = \frac{1}{2}ac\sin B$$

**International-mindedness**

In the 17th century, triangulation was used in an argument between England and France over the curvature of the Earth. Research how the curvature of the Earth was first measured in the 18th century.



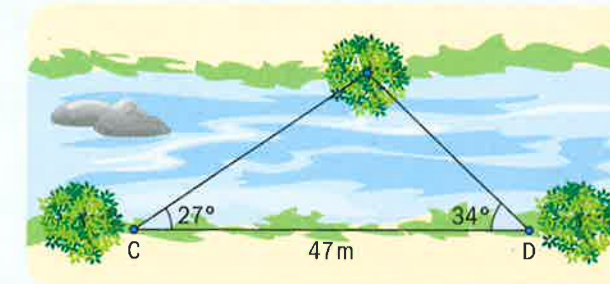






### Example 12

A surveyor triangulating a region uses a tree at the opposite side of the river as reference point. He measures $A\hat{D}C = 34°$. He walks along the bank of the river on a straight line and measures the distance he walked as DC = 47 m, to the nearest metre. From point C, he measures angle $\hat{C} = 27°$.



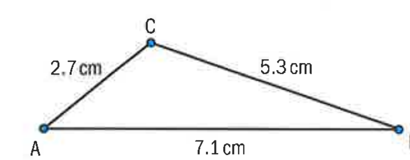

Calculate the area covered.

| | |
|---|---|
| | To find the area we need to find one additional side length; choosing AD. |
| $\hat{A} = 180° - 34° - 27°$<br>$\hat{A} = 119°$ | Determine the angle opposite the known side. |
| $\dfrac{AD}{\sin 27°} = \dfrac{47}{\sin 119°}$<br>$AD = \dfrac{47\sin 27°}{\sin 119°}$.....(1) | When applying the Sine Rule note the ambiguous case does not apply as a length (not an angle) is being found. |
| $\text{Area} = \dfrac{1}{2} \times \dfrac{47\sin 27°}{\sin 119°} \times 47\sin 34°$ | Do not round the answer at this stage. |
| $\text{Area} = 321\ \text{m}^2$ (3 s.f.) | Area $\Delta ACD = \dfrac{1}{2} AD \times DC \times \sin D$; substitute AD from (1). |

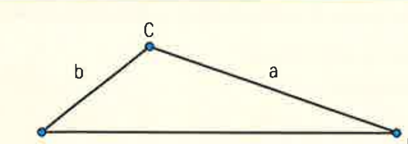
## Cosine rule

Suppose we know all three side lengths $a$, $b$, $c$ of a triangle, but no angles:

Then we do not have enough information to use the Sine Rule, so a different method must be used to find the angles.




The **Cosine Rule** for ΔABC can be used to find an angle given three sides:



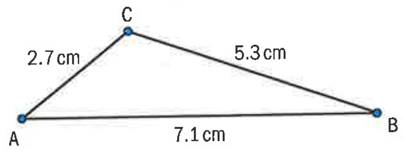

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, where $a$ is the side opposite $\hat{A}$.

It can also be rearranged to solve for a side given two sides and the included angle:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

In the case of the triangle above, you would find angle Â as follows:

$$A = \cos^{-1}\left(\frac{2.7^2 + 7.1^2 - 5.3^2}{2 \cdot 2.7 \cdot 7.1}\right) = 39.4°$$




**HINT**

When entering this into technology, ensure the numerator and denominator are in brackets to correctly represent the fraction:

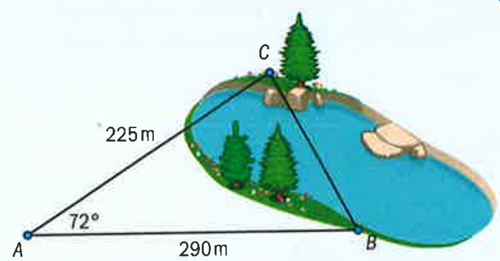
$$\cos^{-1}\left(\frac{(2.7^2 + 7.1^2 - 5.3^2)}{(2 \cdot 2.7 \times 7.1)}\right)$$

### Example 13

A surveyor of a lake measures AC = 225 m, and AB = 290 m, and BÂC = 72° as shown in the diagram.

**a** Find BC.

**b** Find $\hat{C}$.

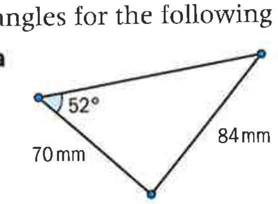



| | |
|---|---|
| **a** $a^2 = b^2 + c^2 - 2bc\cos A$;<br>$CB^2 = 290^2 + 225^2 - 2 \times 225 \times 290 \times \cos 72°$<br>$CB = \sqrt{94398.2...}$<br>CB = 307 m (3 s.f.) | Use the Cosine Rule as two sides and an included angle are given. Substitute the given values for $b$, $c$, and angle $\hat{A}$ in the standard form of the rule. |
| **b** $\frac{\sin C}{290} = \frac{\sin 72°}{BC}$<br>$C = \sin^{-1}\left(\frac{290 \sin 72°}{307}\right)$<br>$= 63.9°$ | Either Sine or Cosine Rule can be used as we now have 3 sides and 1 angle. Since the unknown angle is opposite the shorter side, you do not need to consider the Ambiguous Case. |

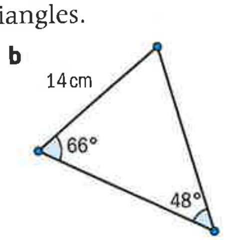
### Exercise 1F

**1** Find the lengths of the missing sides and angles for the following triangles.

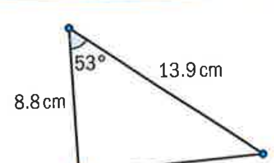
**a**



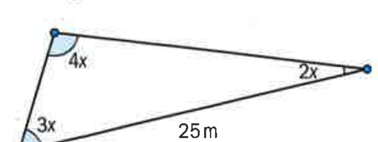

**b**




**c**




**d**








**2** To find the third side of ΔABC with AB = 40 cm, BC = 25 cm and BÂC = 35°, Velina and Kristian offered the following suggestions:

Kristian suggested: "Use the cosine rule as you are given two sides and one angle."

Velina suggested: "Use the sine rule as you are given two sides and an angle opposite to one of them."

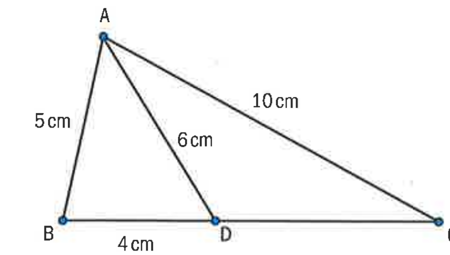
State whose method is correct and justify your statement. Use the correct method to solve the triangle.

**3** A straight air route between two cities A and B is of distance 223 km. Due to bad weather, the pilot had to fly first from city A to city C which was at a distance of 152 km and then turned and flew to city B. The distance between the cities B and C is 285 km. Find the angles between the three cities.

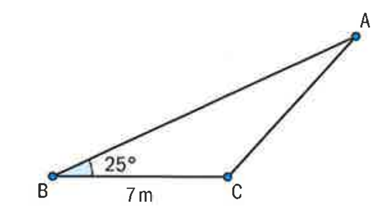
**4** Given point A and collinear points B, D and C as arranged in the diagram shown, find:

**a** the measure of angle B

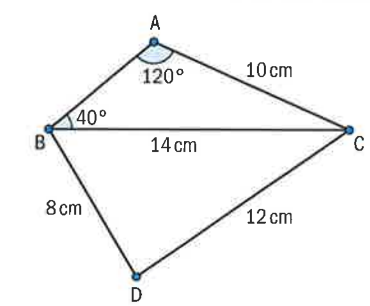
**b** the area of triangle ACD.




**5** The area of ΔABC is 29.6 cm², $\hat{B} = 25°$ and BC = 7 cm. Find the length of side AB.

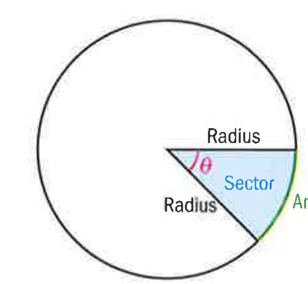



**6** Calculate the area of the quadrilateral ABDC.




## Arc length and area of sector

Sometimes we are interested in only a piece of a circle, either an **arc** of the circumference, or a **sector** of its area, as show in the diagram. The length of an arc will be a fraction of the circumference proportional to its **central angle** $\theta$, and similarly for the area of a sector.




**Length of arc formula**

The length of the arc of a circle with radius $r$ and with central angle $\theta$ (in degrees) is:

$$\frac{\theta}{360°} \times 2\pi r$$

**Area of sector formula**

The area of a sector with central angle $\theta$ is:

$$\frac{\theta}{360°} \times \pi r^2$$

**TOK**

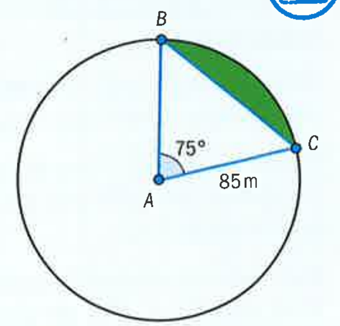
To what extent do instinct and reason create knowledge? Do different geometries (Euclidean and non-Euclidean) refer to or describe different worlds? Is a triangle always made up of straight lines? Is the angle sum of a triangle always 180°?

Number and algebra

Geometry and trigonometry

## Example 14

A city park with a circular perimeter contains several sidewalks represented by the sides of triangle ABC. The length of sidewalk [AC] = 85 m and $B\hat{A}C = 75°$.




**a** Find how much longer the circular arc $\overset{\frown}{BC}$ is than the straight path [BC].

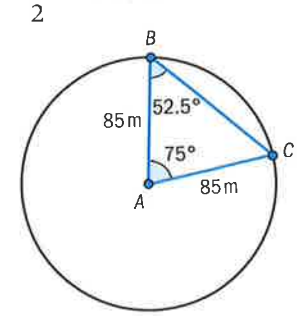
**b** Find the area of the segment that is between the chord [BC] and the arc $\overset{\frown}{BC}$

**a** $\overset{\frown}{BC} = \frac{75°}{360°} \times 2\pi(85) = 111.265$ m

Using length of arc formula.

Since AB = AC = 85 m, ΔABC is isosceles, so

$\hat{B} = \hat{C} = \frac{180 - 75}{2} = 52.5°$.




$\frac{BC}{\sin 75°} = \frac{85}{\sin 52.5°}$

BC = 103.489 m

Now there is sufficient information to use the Sine Rule.

111.265 − 103.489 = 7.78 m longer

Find the difference between the length of the arc and the length of the chord.

**b** $A_{\text{sector}} - A_{\text{triangle}}$

$= \frac{75}{360} \times \pi \times 85^2 - \frac{1}{2}(85)(85)\sin 75°$

$= 1240 \text{ m}^2$

Use formula $A = \frac{1}{2}ab\sin C$ for the area of the triangle.

## Exercise 1G

**1** Determine **i** the length of the each arc **ii** the area of each sector for the radius $r$ and central angle $\alpha$ given below. Give your answer correct to 2 d.p.

- **a** $r = 5$ cm; $\alpha = 70°$
- **b** $r = 4$ cm; $\alpha = 45°$
- **c** $r = 10.5$ cm, $\alpha = 130°$

**2** A clock is circular in shape with diameter 25 cm. Find the length of the arc between the markings 12 and 5 rounded to the nearest tenth of cm.









**3** The London Eye is a giant Ferris wheel in London with a diameter of 120 m. The wheel passenger capsules are attached to the circumference of the wheel, and the wheel rotates at 26 cm per second.

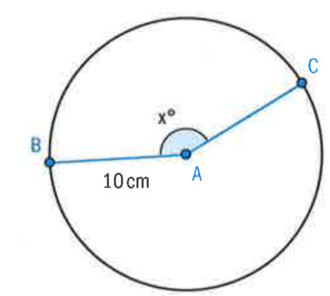
Find:

- **a** the length that a passenger capsule would travel if the wheel makes a rotation of 200°
- **b** the time, in minutes, that it would take for passenger capsule to makes a rotation of 200°
- **c** the time, in minutes, that it would take for passenger capsule to makes a full revolution.

Give your answers to the nearest integer.

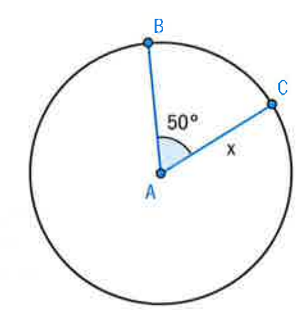
**4** **a** A sector with radius 10 cm and central angle $x°$ has area $48\pi \text{ cm}^2$.

Find the size of the angle $x°$.



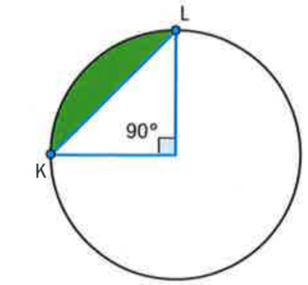

**b** A sector with radius $x$ cm and central angle 50° has area $8\pi \text{ cm}^2$.

Find the length of the radius $x$ cm.

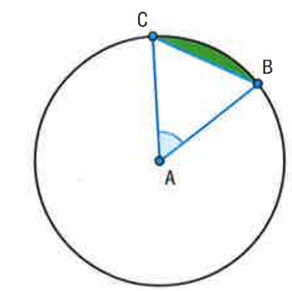



**5** Determine the area of the shaded region of wooded patch in a circular park if the radius of the circle is

**a** 8 cm  **b** 12 cm.



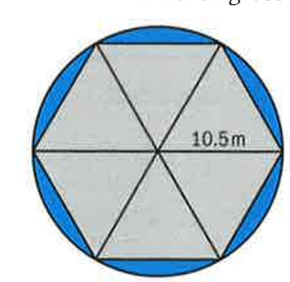

**6** Find the area of the region shaded below if the circle has radius 4 cm and the central angle of sector BAC is 55 degrees.




**7** A landscaper builds a regular hexagonal patio in a circular garden. The area not covered by the patio will be covered in grass.

The radius of the garden is 10.5 m.

Find the area of grass in the garden.



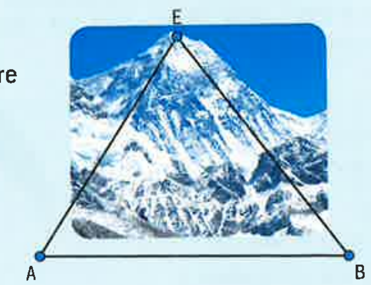
## Developing inquiry skills

Look again at the opening problem. The Great Trigonometric Survey used an instrument called a theodolite to measure the angles from points A and B to the summit E.

$B\hat{A}E = 30.5°$ and $A\hat{B}E = 26.2°$.

Points A and B are 33 km apart.

Find the height of Mount Everest to the nearest metre.

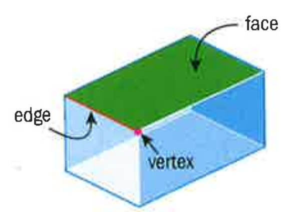



# 1.3 Three dimensional geometry

In this section, you will use the tools of trigonometry to solve problems in three dimensions, including finding lengths, angles, surface areas and volumes.

Shapes with three dimensions (length, width and height) are called **solids**.

A **polyhedron** is a solid composed of polygonal **faces** that connect along line segments called **edges**. Edges meet at a point, called a **vertex** (plural **vertices**).




You are likely to be familiar with the 3D solids we will encounter:

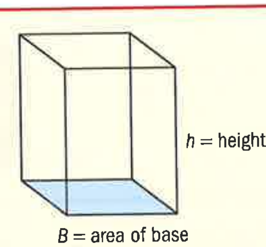
## Volumes of 3D solids

You have already learned how to find the volume of cuboids, prisms and cylinders:

Volume of a prism = area of base × height

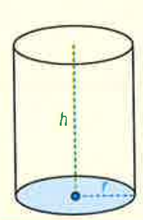
or

$V_{prism} = B \times h$




For a cylinder, $B = \pi r^2$, so

$V_{cylinder} = \pi r^2 h$




**HINT**

The bases are the parallel faces. They are not necessarily the "bottom" of the shape.



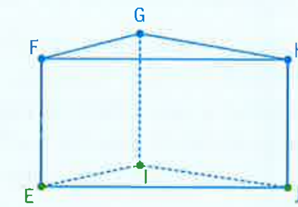


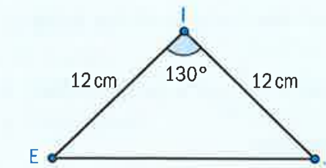




### Example 15

Find the volume of triangular prism whose base is an isosceles triangle with two equal sides of 12 cm. The angle between them is 130°. The height of the prism is 15 cm.



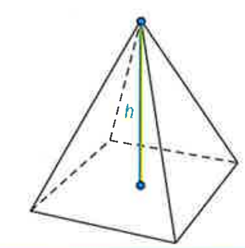
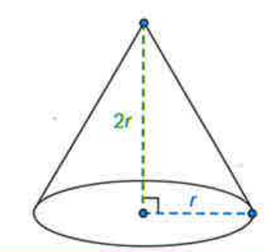
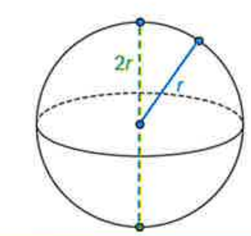




| | |
|---|---|
| $V = \frac{12 \times 12 \times \sin 130°}{2} \times 15$<br>$= 827\ \text{cm}^3$ (3 s.f.) | To find the volume you need to find the area of the triangular base for which you can use the area formula.<br><br>Remember that the volume is measured in cubic units. |

**TOK**

What are the platonic solids and why are they an important part of the language of mathematics?

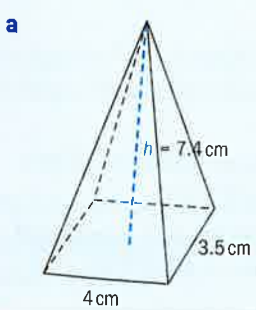
## Volume of pyramids, cones and spheres

| Shape | Pyramid | Cone | Sphere |
|---|---|---|---|
| Volume | $V = \frac{1}{3}(\text{base area} \times \text{height})$  | $V = \frac{1}{3}\pi r^2 h$  | $V = \frac{4}{3}\pi r^3$  |

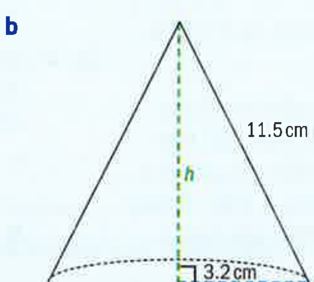
### Example 16

Calculate the volume of each solid.

**a**




**b**




**HINT**

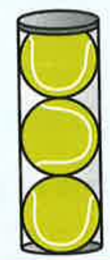
Note that the volume of a pyramid or cone is the one third the volume of a prism or cylinder with the same base area and height.

Continued on next page

| | |
|---|---|
| **a** $V=\frac{1}{3}(4\times3.5\times7.4)$<br>$=34.5\text{ cm}^3$ (3 s.f.) | Use $V=\frac{1}{3}$(base area × height) |
| **b** height $=\sqrt{11.5^2-3.2^2}$<br>$=11.0458...$ cm | Use Pythagoras' theorem to find the height of the cone. |
| $V=\frac{1}{3}\pi\times3.2^2\times11.0458...$<br>$=118\text{ cm}^3$ (3 s.f.) | Use $V=\frac{1}{3}\pi r^2h$ to find the volume. |

## Example 17

A cylindrical can holds three tennis balls. Each ball has a diameter of 6 cm, which is the same diameter as the cylinder, and the cylinder is filled to the top. Calculate the volume of space in the cylinder not taken up by the tennis balls.



| | |
|---|---|
| Volume of cylinder $=\pi\times3^2\times18=508.9\text{ cm}^3$<br>Volume of 3 balls $=3\times\frac{4}{3}\pi\times3^3$<br>$=339.3\text{ cm}^3$<br>Space $=508.9-339.3=169.6\text{ cm}^3$ | The cylinder has radius = 3 cm, height = 18 cm<br>Each ball has radius 3 cm |

### Exercise 1H

**1** Find the volume of each object:

**a** A cone with a base radius of 22 cm and a height that is half of the base's circumference.

**b** A regular hexagonal prism with base side lengths 5 cm and height 23 cm.

**c** A hemisphere whose circular face has a surface area of 412 cm$^2$.

**2** The grain stored inside a cylindrical silo is transferred to a transport container in the shape of a triangular prism. The container's triangular base has side lengths 5.8 m, 5.8 m and 8.1 m and height 7.2 m. The silo has a radius of 5.83 m.

**a** Find the volume of grain that will be removed from the silo if the container is completely filled.

**b** Find the amount by which the height of grain inside the silo will decrease. Give your answers to an appropriate degree of accuracy.

**3 a** Estimate the volume in m$^3$ of each of the following (approximately) spherical objects.

**i** A lithium atom, radius 0.15 nm ($1\text{ nm}=1\times10^{-9}\text{ m}$)

**ii** The Earth, which has equatorial circumference = 40 075 km









**iii** UY Scuti, one of the largest known stars, with diameter equal to 1700 solar radii (radius of Sun = $6.957\times10^5$ km)

**b** Hence, determine which is relatively larger: the Earth compared to an atom, or UY Scuti compared to the Earth.

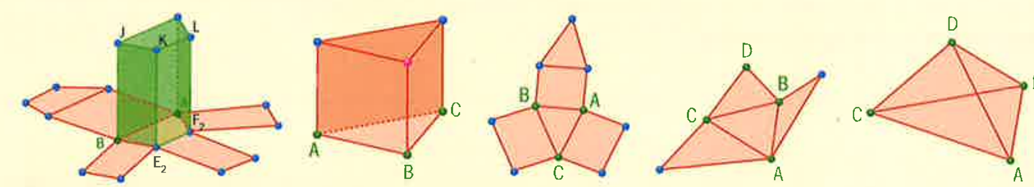
**4 a** Find the volume of material used to construct the Red Pyramid in Cairo, given that it has a height of 105 m, a base length of 220 m, and is estimated to be 96% solid.

**b** Given that the average block of stone used to construct the pyramid had dimensions 130 × 130 × 30 cm and weighed approximately 2250 kg, find the total weight of the pyramid.

**5** A family is replacing the hot water cylindrical tank of the house. They cannot change the height of the boiler but they can double its width. If the previous tank could hold 100 l, predict the volume of the new tank. State the volume of the tank if its width has tripled.

## Surface area

The **surface area** of a solid is the sum of the area of all its faces. Because the faces are two-dimensional, it is often convenient to deconstruct the 3D shape into a 2D net:

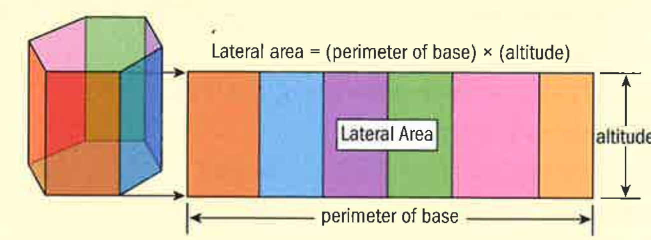


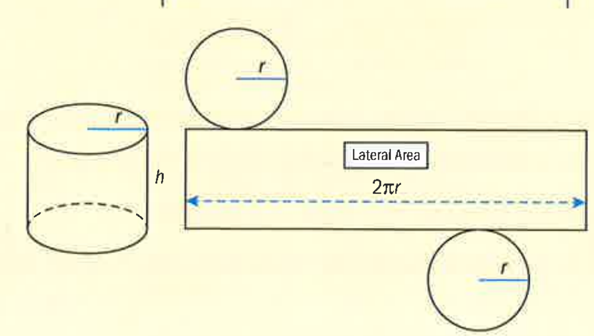

**Surface area of prisms and cylinders**

Recall that for a prism or cylinder, the surface area (SA) is the sum of both bases and the **lateral area** (LA, area of the non-base sides).

$$SA=LA+2B$$

The lateral area of a prism or cylinder is rectangular when drawn as a net, as shown in the diagrams below:




Hence LA $=ph$, where $p$ is the perimeter of the base.

For a cylinder, the surface area formula can be written more specifically as:
$SA=2\pi rh+2\pi r^2$

## Investigation 7

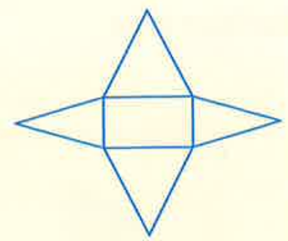
**Surface area of a regular pyramid**

1 Draw a net for each regular pyramid:
 a A square pyramid
 b A triangular-based pyramid
 c A hexagonal pyramid

2 The lateral area is the sum of the sides that are not the base. Describe a process for calculating this area that is consistent with all three pyramids. On your net, label (and draw, if needed) any dimensions necessary for these calculations.

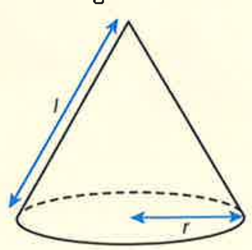
3 Explain how this formula corresponds to the process you described in question 2.

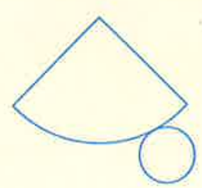
4 If the pyramid is not regular, will this formula still work? Investigate the net of the rectangular pyramid at right to decide.



**Surface area of a cone**

A cone also has a **slant height** $l$ from its vertex to the edge of the base, as shown on the left. If the cone is cut along its slant height and round the edge of the base, the net on the right is created.



5 The lateral area of the cone is a sector of a larger circle.
 a Sketch this larger circle on the net.
 b In terms of the cone's radius $r$ and slant height $l$,
  i What is the radius of the circle, of which the lateral area is a sector?
  ii What is the circumference of this circle?
  iii What is the length of the arc corresponding to this sector?
 c The length of an arc and the area of a sector are proportional:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{length of arc}}{\text{circumference}}$$

 Combining this with what you found in part **b**, determine the area of the sector (lateral area).

6 a **Factual** What are the formulae for finding surface area for these solids?
 b **Conceptual** How are the formulae for surface area derived?
 c **Conceptual** What is the same about finding the surface area of various solids? What is different?



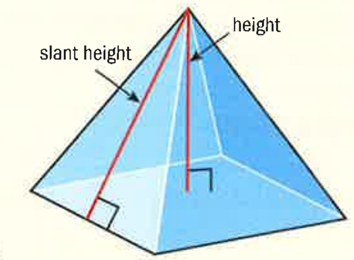


One formula for calculating the **lateral area of a regular pyramid** is:

$$\text{LA} = \frac{1}{2}pl$$

Where $p$ is the perimeter of the base, and $l$ is the **slant height** of the pyramid, or altitude of one of the triangular faces.

The **surface area of a pyramid** is the sum of the lateral area and the area of the base:
$\text{SA} = \text{LA} + \text{B}$

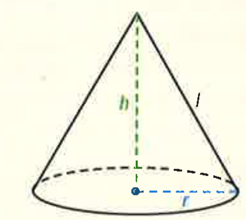



**HINT**

A **right** pyramid is one in which the apex is over the centre of the base, a regular pyramid is one with a regular polygon as a base, and is usually taken to be "right".

Number and algebra

The **lateral area of a cone** is
$\text{LA} = \pi rl$
Where $r$ is the radius of the base and $l$ is the slant height. The **surface area of a cone** is
$\text{SA} = \text{LA} + \text{B} = \pi rl + \pi r^2$

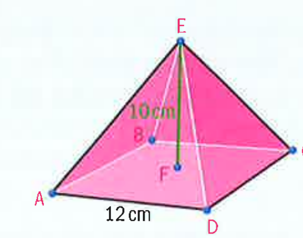


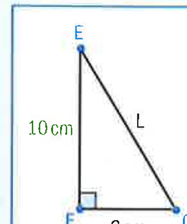
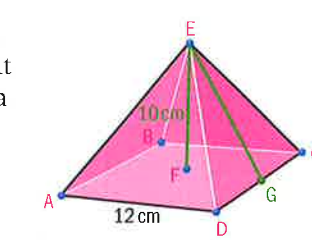

The **surface area of sphere** with radius $r$ is $SA = 4\pi r^2$.

Geometry and trigonometry

### Example 18

Find the surface area of a right square-based pyramid with base side length 12 cm and height 10 cm:




| | |
|---|---|
|  E, 10 cm, L, F, 6 cm, G<br>$l = \sqrt{10^2 + 6^2} = \sqrt{136}$<br>$SA = \frac{1}{2}(4 \times 12)\sqrt{136} + 12^2$<br>$= 424\text{ cm}^2$ | For a regular pyramid,<br>$\text{SA} = \text{LA} + \text{B}$ and $\text{LA} = \frac{1}{2}pl$<br>Since lateral area requires slant height, we draw the slant height EG and recognize that it forms a right triangle EFG with the height.<br>FG = 6 as it is half the side length<br>Using Pythagoras' theorem, solve for the slant height and substitute.  |

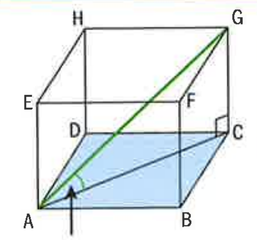
Note that in the example above, we identified a right triangle that was useful in relating different lengths or angles in the 3D diagram.

The triangle EFG also indicates the angle E$\hat{G}$F between the slant height and the base, which is generalized below..

## Angle between a line and a plane

To draw the angle between the line AG and the plane ABCD:

- Draw a line from G that meets the plane ABCD at 90°. Here that line is [GC]
- Then connect A to point C to form a right angled triangle. Angle G$\hat{A}$C is the angle between line [AG] and plane ABCD.




Angle between line [AG] and the plane ABCD

### Example 19

A right cone has a radius 5 cm and a total surface area of 300 cm², rounded to the nearest integer.

Find:

**a** the slant height, $l$, of the cone

**b** the height, $h$, of the cone

**c** the volume of the cone

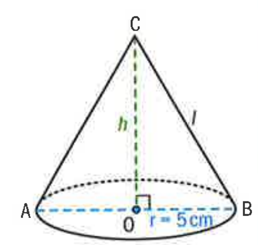
**d** the angle that the slant height makes with the base of the cone.

**a** $300 = \pi \times 5 \times l + \pi \times 5^2$

$l = 14.0985... = 14.1$ cm (3 s.f.)

Draw a diagram and label known information.
Total surface area $= \pi rl + \pi r^2$

Substitute the given values in order to find the slant height, $l$.




**b** $h^2 = l^2 - r^2$

$h = \sqrt{14.0985...^2 - 5^2} = 13.2$ cm (3 s.f.)

Use triangle OBC and Pythagoras' theorem to find the height from the slant height.

**c** $V = \frac{1}{3} \times \pi \times 5^2 \times 13.1821...$

$= 345$ cm³





**d** $\cos C\hat{B}O = \frac{r}{l} = \frac{5}{14.0985...}$

$C\hat{B}O = \cos^{-1}\left(\frac{5}{14.0985...}\right) = 69.2°$ (3 s.f.)

To find the angle between [BC] and the base, draw a segment from C perpendicular to the base – this is [CO]. Connect to point B to create triangle CBO. Angle CBO is the angle between the line and the plane.

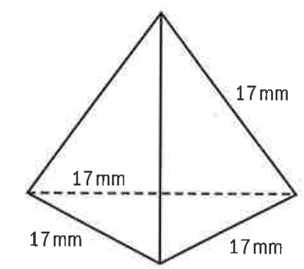
### Exercise 1I

**1** Find the surface area of the following solids.

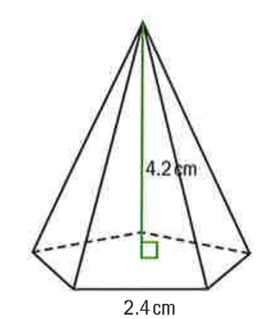
**a** A cylinder with radius 2.5 cm and height 7.3 cm.

**b** A right cone with radius 3.5 cm and height 12 cm.

**c** A regular triangular pyramid with base side length and slant height 17 mm.




**d** A regular pentagonal pyramid with base side length 2.4 cm and vertical height 4.2 cm.




**2** Find the missing dimension for the objects below given that each has a surface area of 525 cm².

**i** The radius of a sphere.

**ii** The height of cylinder, given that its radius is half of its height.

**iii** The height of a cone with base circumference 25 cm.

**3** Cells must exchange different molecules across their boundaries in order to survive. The amount of material that can be taken in or released is limited by the surface area of the cell.

**a** If the ratio of surface area to volume is too low, then the cell is unable to exchange enough material.

**i** Find, in terms of its radius $r$, the ratio of surface area to volume of a spherical cell.

**ii** Find the largest possible radius of the cell, in metres, if the ratio of surface area to volume must be greater than 955 000 (with the radius measured in metres).

**b** Find an expression in terms of radius $r$ for the ratio of surface area to volume for each of the following objects, given that their volume is equal to that of the sphere of radius $r$:

**i** A cube.

**ii** A cylinder with two half-spheres attached at its bases, and height of cylinder equal to five times the radius of its base.

**iii** Hence, if the sphere has unit volume, find how many times more surface area each object will have than the sphere.

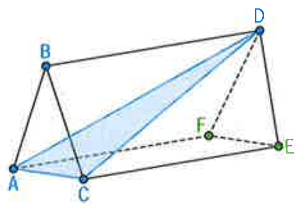
**4** A polyhedron whose faces are congruent regular polygons is known as a **Platonic solid**. A cube is one familiar example; another is a **tetrahedron** – a triangular pyramid with all faces equilateral triangles. A company produces tetrahedral dice for games with a side length of 1.2 cm. They sell approximately 3.5 million dice per year.

**a** Find the total surface area that will be painted on the dice.

**b** The paint is sold in cylindrical cans with a height of 25.4 cm and diameter 6.5 cm. Determine how many cans must be bought if the average thickness of the paint is 25.4 microns (1 micron $= 1 \times 10^{-6}$ m).

**c** Shipment costs \$16 per kg. Determine the total shipping cost of the paint.

**5**




The triangular prism given above has the following side lengths.

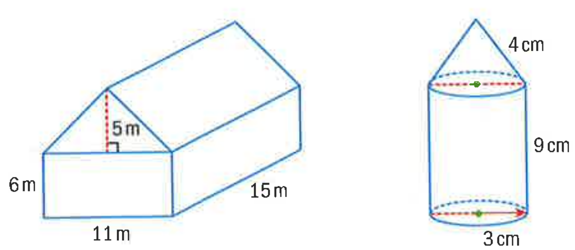
AB = BC = 6 cm

AC = 4 cm

BD = 8 cm

Calculate the area of ΔACD correct to 2 decimal places.

**6** Find the volume and surface area of the following composite solids:



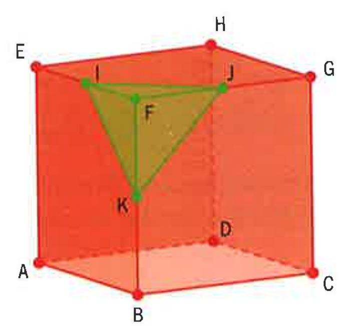

**7**

A silo consists of a cylinder with a hemispherical roof attached to its top base. The radius of the cylinder is 3 m and its height is 12 m.

**a** Find the volume of the silo.

**b** The entire silo is to be painted. Find how much paint is needed if 1 l of paint covers 8.5 m² of surface.

**8** ABCDEFGH is a cube with edge length 6 cm. I, J and K are midpoints of the respective edges.




The corner of the cube is cut off as shown in the diagram. Find the remaining volume and surface area.

## Chapter summary

- If a measurement $M$ is accurate to a particular unit $u$, then its exact value $V$ lies in the interval: $M - 0.5u \le V < M + 0.5u$

  The endpoints of this interval are called the **lower** and **upper bounds**. **Measurement error** $= V_A - V_E$
- **Percentage error formula**

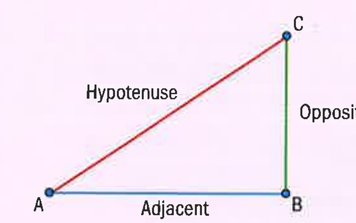
  Percentage error $= \left|\dfrac{V_A - V_E}{V_E}\right| \times 100\%$, where $V_A$ is the approximate (or measured) value and $V_E$ is the exact value.





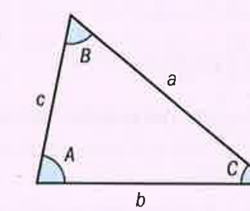
- The following rules of exponents hold for $a > 0$ and $m, n \in \mathbb{Q}$
  - $a^m \times a^n = a^{m+n}$
  - $(a^m)^n = a^{mn}$
  - $\dfrac{a^m}{a^n} = a^{m-n}$
- The general form for **standard form** is $a \times 10^k$ with coefficient $1 \le a < 10$ and exponent $k \in \mathbb{Z}$.
- Numbers in standard form can be multiplied or divided following rules for exponents:

  $(b \times 10^m)(c \times 10^n) = bc \times 10^{m+n}$ and $\dfrac{b \times 10^m}{c \times 10^n} = \dfrac{b}{c} \times 10^{m-n}$
- A **negative exponent** represents a reciprocal power: $x^{-n} = \dfrac{1}{x^n}$
- A **rational exponent** represents a power of a root: $x^{\frac{p}{q}} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p$, $p, q \in \mathbb{Z}$
- In a right triangle ABC:
  - $\sin(A) = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$
  - $\cos(A) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$
  - $\tan(A) = \dfrac{\text{Opposite}}{\text{Adjacent}}$



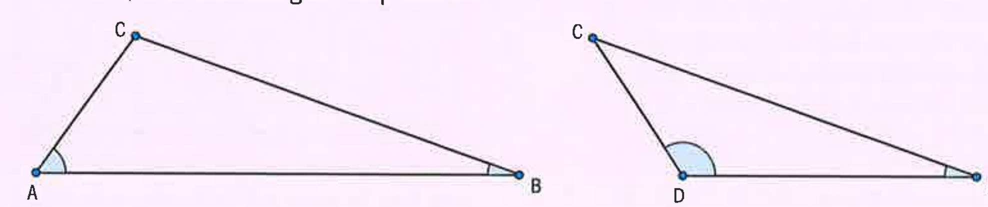

- The **Sine Rule** for triangle ABC can be used to find missing sides and angles in right or non-right triangles:

  $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ or $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$




- **Sine Rule – Ambiguous Case**

  When **two sides and one angle** are known, and the **unknown angle** is **opposite the longer of the two sides**, then two triangles are possible:



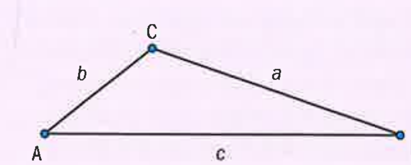

- The area of any triangle ABC is given the formula:

  area $= \dfrac{1}{2} bc \sin A$
- The **Cosine Rule** for ΔABC can be used to find an angle given three sides:

  $\cos \hat{A} = \dfrac{b^2 + c^2 - a^2}{2bc}$, where $a$ is the side opposite $\hat{A}$.

  Or to find a side:

  $a^2 = b^2 + c^2 - 2bc \cos \hat{A}$

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- **Length of Arc Formula**

  The length of the arc of a circle with radius $r$ and with central angle $\theta$ (in degrees) is:

  $\frac{\theta}{360°} \times 2\pi r$

- **Area of Sector Formula**

  The area of a sector with central angle $\theta$ is: $\frac{\theta}{360°} \times \pi r^2$

| Shape | Pyramid | Cone | Sphere | Prism | Cylinder |
|---|---|---|---|---|---|
| Volume | $V = \frac{1}{3}(\text{base area} \times \text{height})$ | $V = \frac{1}{3}\pi r^2 h$ | $V = \frac{4}{3}\pi r^3$ | $V = \text{base area} \times \text{height}$ | $V = \pi r^2 h$ |

- **Surface Area**

  For prisms:

  $SA = LA + 2B$

  For cylinders:

  $SA = 2\pi rh + 2\pi r^2$

- For a pyramid or cone:

  $SA = LA + B$

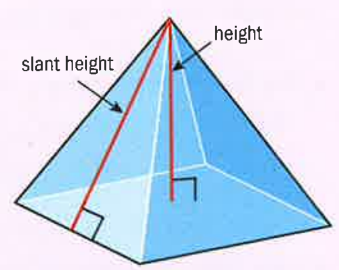
  For a regular pyramid:

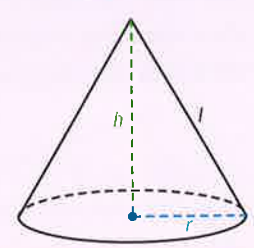
  $LA = \frac{1}{2}pl$, where $p$ = perimeter of base, $l$ = slant height

- For a cone,

  $SA = LA + B = \pi rl + \pi r^2$

- The **surface area of sphere** with radius $r$ is $SA = 4\pi r^2$.




## Developing inquiry skills

Look back at the opening problem. Mount Everest can be modelled as a right cone with base radius equal to approximately 16 km, and the average snow depth is approximately 4 m. Estimate the amount of snow at Mount Everest.








# Chapter review

**1** Calculate each of the following, giving the answers **i** to one decimal place **ii** to three significant figures.

**a** $\frac{5^{\frac{1}{3}} + 8 - \sqrt{6}}{47}$ **b** $(5+8)^{\frac{1}{3}} - \frac{\sqrt{6}}{47}$

**c** $(5+8)^{\frac{1}{3}} - \sqrt{\frac{6}{47}}$ **d** $5^{\frac{1}{3}} + \frac{8 - \sqrt{6}}{47}$

**e** $\frac{\left(5 + 8 - \sqrt{6}^{\frac{1}{3}}\right)}{47}$

**2** **a** An estimate of the number of settings on the WWII Enigma code machine is estimated at $1.5 \times 10^{19}$. Given that the exact value is 15 896 255 521 782 636 000, determine the percentage error of this estimate.

**b** Assuming that the age of the universe is 13.82 billion years, and that each of the Enigma machine settings is written down every second, determine whether the 13.82 billion years would be enough time to write down all the settings.

**3** Cardano published this formula in 1545 to find one solution of the equation $x^3 + px = q$:

$$x = \sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2} - \frac{q}{2}}$$

Use this formula to find a solution of $x^3 + 3x = 5$ and find the percentage error when compared to the answer you get using technology.

**4** In a triangle $\Delta PQR$, PQ = 13.4 cm, QR = 15 cm and angle $\hat{P} = 31°$. Find PR by

**a** one application of the cosine rule

**b** two applications of the sine rule.

**5** In New York City, the heights of the observatories of the One World Trade Centre, the Empire State Building and the Rockefeller Center are 382.2 m, 320 m and 259 m, respectively. The horizontal direct distance between the Empire State Building and the Rockefeller Center is 1310 m and the direct distance between the One World Trade Center and the Empire State Building is 4600 m.

Find the angle of elevation from the observation decks of the Rockefeller Centre to the Empire State and the angle of depression from the One World Trade Center to the Empire State.

**6** Sophie learns that the Cheops pyramid on the Giza plateau in Egypt is constructed on a vast scale. She wants to know more.

**a** Given that the base of the pyramid is approximately square in shape with sides of length 230 m to the nearest metre, calculate the upper and lower bounds of the exact area $A$ of the base.

You are given that a football pitch which meets the requirements for an international football fixture must have width at least 64 metres and no more than 75 m, and that the length must be between 100 m and 110 m inclusive.

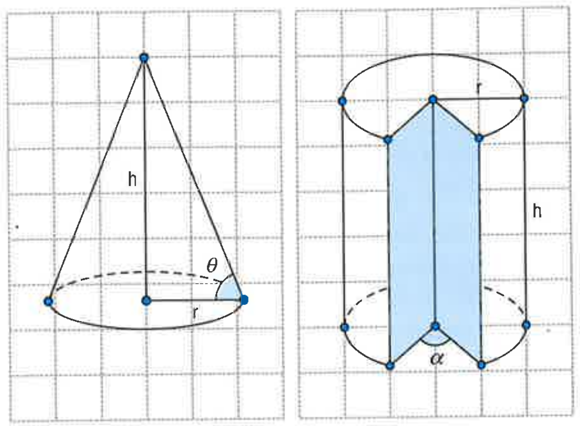
**b** Find the upper and lower bounds for the number of football pitches that fit into the base area of the Cheops pyramid **i** as a ratio $R$ **ii** as a number $P$ of complete pitches.

**7** The speed of light is approximately $3 \times 10^8$ m/s.

Determine the radius of a circular clock if the tip of its second hand is to move at the speed of light. Express your answer in km using standard form correct to 4 significant figures.

**8** A cone has base radius $r$ and height $h$. Let $\theta$ be the angle that the slant height of the cone makes with its base. A cylinder has the same height and radius as the cone.

The cylinder is modified so that the section of the cylinder corresponding to a sector of the base with central angle $\alpha$ is removed:

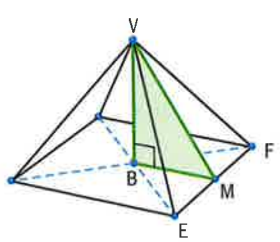



**a** Find the value of $\alpha$ so that the volume of the cone and the volume of the cylinder are equal.

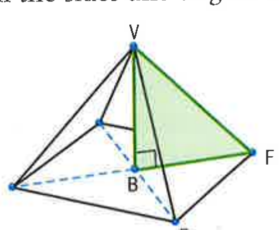
**b** For this value of $\alpha$, find the value of $\theta$ so that the lateral areas of the cone and the modified cylinder are equal.

**9** The diagrams represent the pyramid of Cheops. You are given that the vertical height VB = 146.64954 and the side of the square base EF = 230.35655 m. M is the midpoint of [EF].

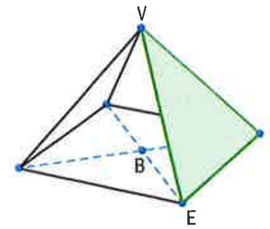
**a** Find all the sides and angles of the $\Delta$VBM.




**b** Find all the sides and angles of $\Delta$VBF.




**c** Find all the sides and angles of $\Delta$VEF and its area.




**d** The Ancient Greek historian Herodotus claimed that the Cheops pyramid was designed so that the square of its height was equal to the area of each of its faces. Find the percentage error in the claim of Herodotus.

**10** A highway is to be built in a desert. The highway forms a straight line in the locality of a town Alphaville which is 5 km from the highway. The highway is planned to pass through Betatown which is situated 13 km from Alphaville. The planners need to build a straight access road from Alphaville to the highway to meet the highway at a junction J. The speed limit on the access road will be 70 km/h and on the highway 110 km/h.

**a** Determine the time taken for the journey from Alphaville to Betatown via J in terms of the distance from J to Betatown.

**b** Determine the time taken for the journey from Alphaville (A) to Betatown (B) via J in terms of the angle between AJ and JB.

## Exam-style questions

**11 P1:** In an electric circuit, the resistance ($R$ ohms) of a component may be calculated using the formula $R = \frac{V}{I}$, where $V$ is the potential difference across the component and $I$ is the current through the component.

The potential difference is measured as 6 V (to the nearest volt), and the current is measured as 0.2 Amps, to the nearest tenth of an Amp.









**a** Calculate lower and upper bounds for the resistance. (3 marks)

**b** The actual value of the resistor is 30 Ω. Calculate the maximum possible percentage error that could be obtained. (2 marks)

**12 P2:** Simplify the following algebraic expressions.

**a** $\sqrt{3x^3 \times 12x^0 \times 4x^5}$ (3 marks)

**b** $\dfrac{(x^{-2})^5}{(x^3)^{-4}}$ (3 marks)

**c** $\left(\dfrac{343x^9}{27y^6}\right)^{-\frac{2}{3}}$ (5 marks)

**13 P1:** The following table gives the masses (in kg) of the various particles which make up an atom.

| Particle | Mass (kg) |
|---|---|
| Neutron | $1.675 \times 10^{-27}$ |
| Proton | $1.673 \times 10^{-27}$ |
| Electron | $9.109 \times 10^{-31}$ |

**a** Find the average (mean) mass of all three particles, giving your answer in standard form to 3 s.f. (2 marks)

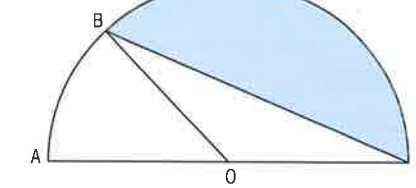
A helium atom consists of two protons, two neutrons, and two electrons.

**b** Determine the ratio of electron mass to neutron mass, giving your answer in the form $1 : x$ where $x$ is accurate to 3 s.f. (2 marks)

**c** Calculate the percentage error in mass when taking the mass of an electron to be $1 \times 10^{-30}$ kg. (2 marks)

**14 P2:** A cylindrical block of metal has base radius 12 cm and height 50 cm. The block is to be melted down and used to produce a number of metal ball bearings, each of radius 2 cm. Calculate the number of ball bearings that can be produced from this block. (6 marks)

**15 P1:** The following diagram shows a semi-circle, centre O. AO = 15 cm and arc AB = 10 cm.




**a** Find the area of the shaded region, giving your answer to 3 significant figures. (4 marks)

**b** Find the perimeter of the shaded region, giving your answer to 3 significant figures. (5 marks)

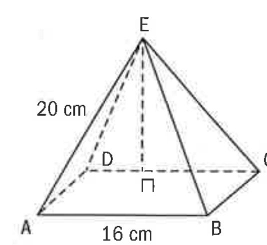
**16 P1:** In a triangle ABC, AB = $x$, AC = $x$ + 1 and B$\hat{A}$C = 60°.

The area of the triangle is $14\sqrt{3}$ cm².

Find the perimeter of the triangle. (8 marks)

**17 P2:** ABCDE is a square-based pyramid.

The vertex E is situated directly above the centre of the face ABCD. AB = 16 cm and AE = 20 cm




**a** Calculate the angle between the line AE and the plane face ABCD. (4 marks)

**b** Calculate the angle between the plane face BCE and the face ABCD. (4 marks)

**c** Calculate the angle between the line AE and the line EC. (2 marks)

**d** Find the volume of the pyramid. (2 marks)

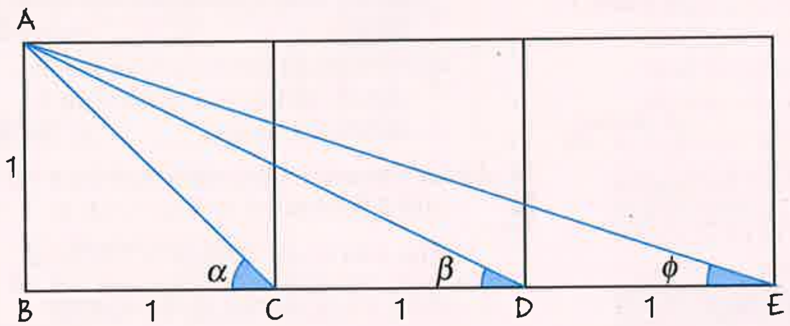
**e** Find the total surface area of the pyramid. (3 marks)

# Three squares

**Approaches to learning:** Research, Critical thinking
**Exploration criteria:** Personal engagement (C), Use of mathematics (E)
**IB topic:** Proof, Geometry, Trigonometry

### The problem

Three identical squares with length of 1 are adjacent to one another. A line is connected from one corner of the first square to the opposite corner of the same square, another to the opposite corner of the second square and another to the opposite corner of the third square:




**Find the sum of the three angles $\alpha$, $\beta$ and $\varphi$.**

## Exploring the problem

Look at the diagram

What do you think the answer may be?

Use a protractor if that helps.

How did you come to this conjecture?

Is it convincing?

This is not an accepted mathematical truth. It is a conjecture, based on observation.

**You now have the conjecture $\alpha + \beta + \varphi = 90°$ to be proved mathematically.**

## Direct proof

What is the value of $\alpha$?

Given that $\alpha + \beta + \varphi = 90°$, what does this tell you about $\alpha$ and $\beta + \varphi$?

What are the lengths of the three hypotenuses of $\Delta ABC$, $\Delta ABD$ and $\Delta ABE$?

Hence explain how you know that $\Delta ACD$ and $\Delta ACE$ are similar.

What can you therefore conclude about $C\hat{A}D$ and $C\hat{E}A$?

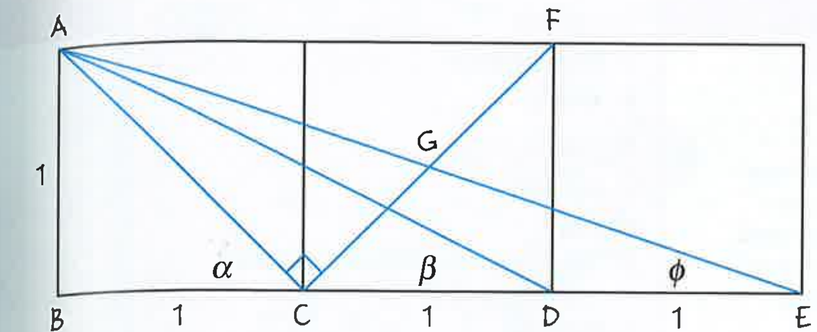
Hence determine why $A\hat{C}B = C\hat{A}D + A\hat{D}C$ and conclude the proof.





## Proof using an auxiliary line

An additional diagonal line, CF, is drawn in the second square and the intersection point between CF and AE is labelled G:




Explain why $B\hat{A}C = \alpha$.

Explain why $E\hat{A}F = \varphi$.

If you show that $G\hat{A}C = \beta$, how will this complete the proof?

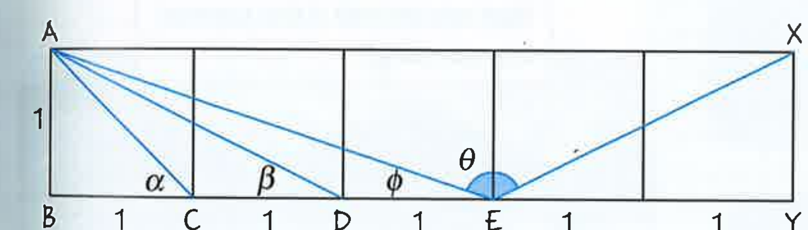
Explain how you know that $\Delta GAC$ and $\Delta ABD$ are similar.

Hence explain how you know that $G\hat{A}C = \angle B\hat{D}A = \beta$.

Hence complete the proof.

## Proof using the cosine rule

The diagram is extended and the additional vertices of the large rectangle are labelled X and Y and the angle is labelled $\theta$:




Explain why $X\hat{E}Y = \beta$.

Calculate the lengths of AE and AY.

Now calculate $A\hat{E}Y$ ($\theta$) using the cosine rule.

Hence explain how you know that $\beta + \varphi = 45°$.

Hence complete the proof.

### Extension

Research other proofs on the Internet.

You could also try to produce a proof yourself.

You do not have to stop working when you have the proof.

What could you do next?

Modelling and investigation activity