3 Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines

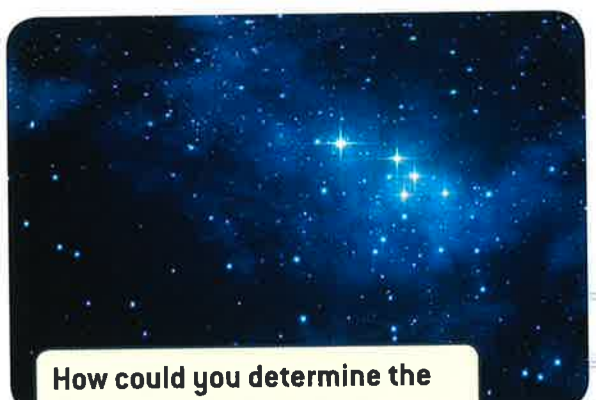
This chapter explores two- and three-dimensional space and the relationship between points and lines in space. In two-dimensional space the world we live can be represented by a map and our position given by our coordinates as defined on the map. We can use these coordinates to see where objects are in relation to each other, and calculate their distance apart. In three dimensional space we need a third coordinate for height. Straight line motion through this space can be defined using a vector equation of the line, and with this equation we can see if objects will collide, or how long they would take to travel a given distance.

Concepts

- Space
- Relationships

Microconcepts

- Gradient
- Equations of straight lines
- Perpendicular lines
- Parallel lines
- Perpendicular bisectors
- Points of intersections of lines
- Coordinates of mid-point in 3D
- Distance in 3D
- Voronoi diagrams
- Area of Voronoi diagrams
- Vectors
- Displacement vectors
- Parallel and perpendicular vectors
- Normalising a vector
- Scalar product
- Application of scalar product to finding the angle between two vectors/lines
- Vector product
- Applications of vector product to areas
- Vector equation of a line
- Motion with constant velocity in 2D and 3D

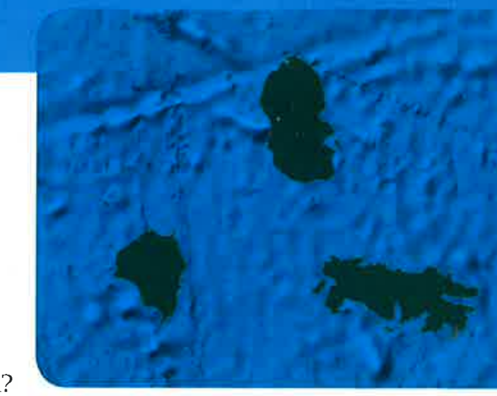


How could you determine the distance between stars?

How could a town with four fire stations be divided into regions so that the nearest fire truck is dispatched to the fire location?

How can you find out whether two aircraft will collide if they maintain their current flight paths?

How can you find the surface area of a crystal when given the coordinates of its vertices.



An island has control of all economic resources within its territorial waters. These resources may include fishing, mining, or offshore oil exploration.

The positions of three small islands are shown.

- How could you model the positions of the three islands?
- How could you use your model to decide how to divide the territorial waters between the islands?
- What information do you need to be able to answer this question?
- What assumptions would you need to make?
- What factors might influence how to divide the waters between the islands?

Developing inquiry skills

Write down any similar inquiry questions you might ask if you were asked to divide the area between different landmarks; for example, deciding which fire station should assume responsibility for different areas of a town or deciding which hospital is closer to your home.

How are these questions different from those used to investigate how territorial waters are being divided between three islands?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Find the distance between two points.
eg Find the distance between A(2, 3) and B(6, 2).

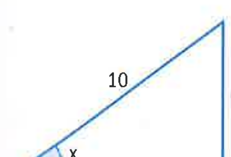
$$d = \sqrt{(6-2)^2 + (2-3)^2} = \sqrt{17}$$

2 Find the midpoint of the line segment joining two points.

eg The midpoint of A(2, 3) and B(6, 2) is

$$\left(\frac{2+6}{2}, \frac{1+2}{2}\right) = (4, 1.5)$$

3 Find angles and sides in right-angled triangles.
eg



$$\sin x = \frac{6}{10}$$

$$x = \arcsin\left(\frac{6}{10}\right)$$

$$= 36.9° \text{ (3 s.f.)}$$

4 Use bearings.

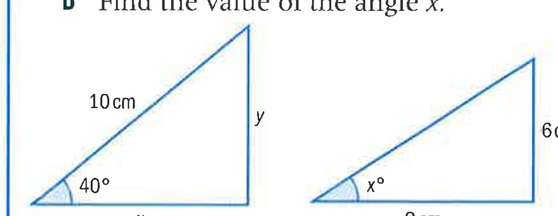
Skills check

Click here for help with this skills check

1 Find the length of the line segment joining A(1, 2) and B(5, 4).

2 Find the midpoint of the line segment joining A(1, 2) and B(5, 4).

3 **a** Find the values of x and y.
b Find the value of the angle x.



4 Town A and B are 50 km apart and the bearing of B from A is 060°
a Find the distance B is
i east of A   ii north of A.
b Find the bearing of A from B.

3.1 Coordinate geometry in 2 and 3 dimensions

Coordinates in 3 dimensions

The position of a point in 3-dimensional space is given by three coordinates along three mutually perpendicular axes.

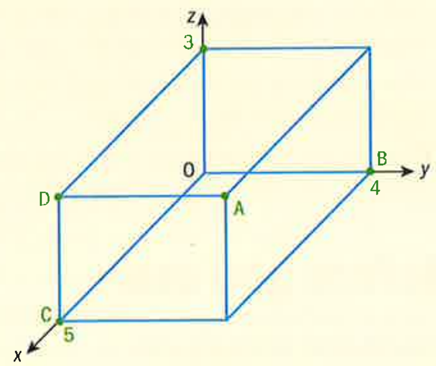
For example, suppose that your classroom is a cuboid with width 7 m, length 8 m and height is 3 m. If one corner of the room is chosen as the origin, then a light in the centre of the ceiling will have coordinates (3.5, 4, 3).

TOK

Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation?

Investigation 1

A cuboid with sides of length 5, 4 and 3 is shown in the diagram. The origin is at the point O and the coordinates of the point A are [5, 4, 3].



1 Write down the coordinates of B, C and D.

If A and B are two points [AB] should be read as "the line segment with A and B as end points".

2 Write down the coordinates of the midpoint of [OA] and [BC].

3 Hence conjecture a formula for finding the mid-point of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2).

4 a Use your formula to find the midpoint of [BD].

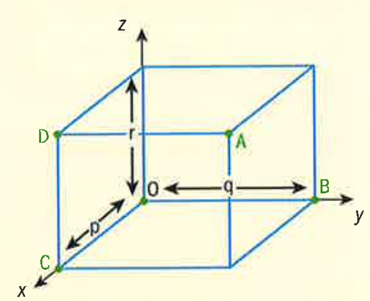
b How can you use a previous answer to verify the formula works?

If A and B are two points AB should be read as "the length of the line segment with A and B as end points" or "the length of [AB]".

5 a Use Pythagoras' theorem to find BC.

b Hence find BD.

A cuboid with sides of length p, q and r is shown in the diagram below. The origin is at the point O and the coordinates of the point A are (p, q, r).



6 Use Pythagoras' theorem to find an expression for the length BC.

7 Hence write down an expression for the length BD in terms of p, q and r.

8 Use your answer to **5** to verify your formula.

9 Hence conjecture a formula for finding the length of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). Justify your answer.

Factual What is the formula for finding the midpoint of the line segment joining two points in 3D?

Factual What is the formula for finding the distance between 2 points in 3D?

Conceptual How are the formulae for distance between two points and midpoint of a line segment in 3 dimensions related to the same formulae in 2 dimensions?

Both the formula for the midpoint of a line segment and the length of a line segment between two points in 2 dimensions can be extended to 3 dimensions.

Given the two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

- the midpoint of [AB] is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
- the length $AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

Exercise 3A

1 A and B are two points with coordinates as given below. Find:

a the distance between A and B

b the midpoint of the line segment joining A and B.

i (2, 1), (1, −4)

ii (2, 4, −3), (0, 3, −2)

2 A and B are two points with coordinates as given below. Find:

a AB

b the midpoint of [AB]

i (21, −13), (−3, 14)

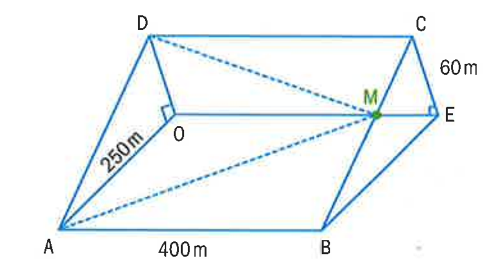
ii (−17, 11, 0), (−2, 8, −12)

3 A tracking station lies at the origin of a coordinate system with the x-axis due east, the y-axis due north and the z-axis vertical with units in kilometers. At a particular time two aircraft have coordinates (20, 25, 11) and (26, 31, 12) relative to the tracking station.

a Find the distance the two aircraft are apart at this time.

b The radar at the tracking station has a range of 40 km. Determine whether it would be able to detect both aircraft at this time.

4 The side of a hill can be regarded as a right angled triangular prism as shown. A path goes in a straight line from the point A to the midpoint of [BC], M, and then to D. Distances are all given in metres.



Take O as the origin of a coordinate system. With [OA] lying along the x-axis and [OE] lying along the y-axis and with the z-axis vertical.

a Write down the coordinates of A, B, C and D.

b Find the coordinates of M.

c Find the total length of the path from A to D.

5 A surveyor records the coordinates of all the vertices of the base of the Great Pyramid at Giza. Unfortunately he loses two of them so he only has values for A and B at diagonally opposite corners.

The origin for the coordinate system is some distance away from the pyramid and is on land 21 m higher than the base of the pyramid.

Functions

Geometry and trigonometry

The coordinates he has are A(97, 77, −21), B(340, −139, −21). The base of the pyramid forms a square.

a Find the area of the base of the pyramid.

The surveyor knows from other sources that the height of the pyramid is 138 m.

b Find the volume of the pyramid.

c Find the coordinates of the vertex of the pyramid.

d Find the shortest distance from one of the corners of the base to the vertex of the pyramid.

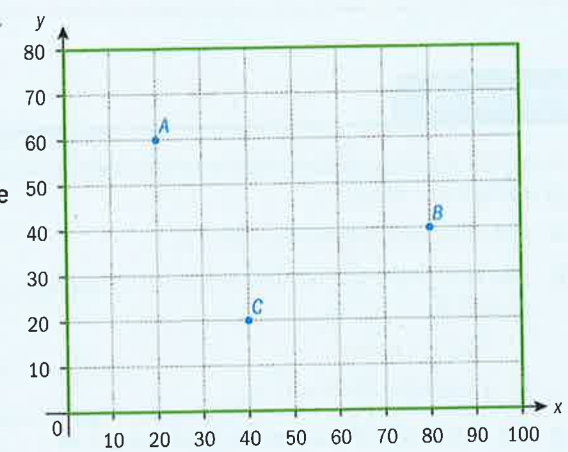
Developing inquiry skills

Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

The positions of the islands can be modelled as shown:

The lines $x = 0$, $x = 100$, $y = 0$ and $y = 80$ mark the boundary of international waters and distances are given in kilometres. The islands are given exclusive fishing rights within these boundaries with the island closest to a point having the rights at that point.

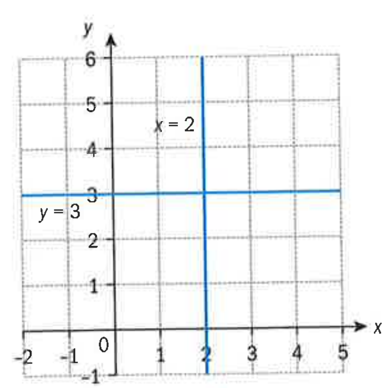
How can you find the distances between each of the islands?



3.2 The equation of a straight line in 2 dimensions

Vertical and horizontal lines

The vertical line on which the value of each x-coordinate is 2 has equation $x = 2$. The horizontal line on which the value of each y-coordinate is 3 has equation $y = 3$.



Reflect What is the form of an equation of a vertical/horizontal line?

The gradient of the line segment joining two points

The gradient of the line segment joining two point can be calculated by finding how far the line segment goes "up" divided by how far it goes across.

The gradient (m) of the line segment joining (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

International-mindedness

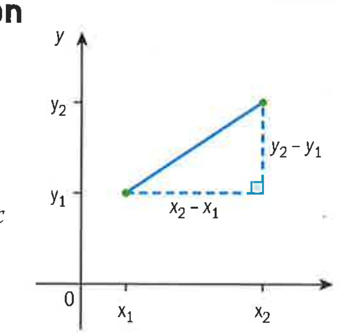
Cartesian coordinates are named after Frenchman René Descartes.

The gradient intercept form for the equation of a straight line

The equation of the line passing through (x_1, y_1) and (x_2, y_2) can be given in **gradient-intercept** form as $y = mx + c$ where c is the value of the y-intercept.

The gradient can be found as above and the value of c can be found by substituting one of the points.

This is the form that is usually used when entering the equation of a straight line into a graphical display calculator.



For two points A and B (AB) should be read as "the line containing the points A and B".

Functions

Geometry and trigonometry

Example 1

For the two points A(2, 2) and B(6, 1)

a Find the gradient m of (AB) (the line passing through A and B).

b Find the equation of (AB) in the form $y = mx + c$.

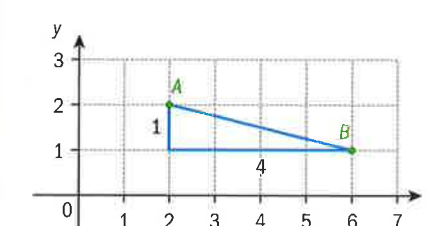
c Sketch the line for $-2 \le x \le 12$.

d Find:

i the value of y when x is 4.7 **ii** the y-intercept.

a $m = \frac{1-2}{6-2} = \frac{-1}{4}$

a It is important you are careful to subtract both coordinates in the same order. The gradient can also be found from a sketch:



From the diagram $m = \frac{-1}{4}$

Because the line goes down as you look from left to right the gradient is negative.

Continued on next page

b $y = -\frac{1}{4}x + c$

Substitute one of the points on the line; for example, (2, 2).

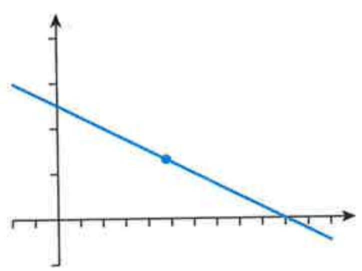
$2 = -\frac{1}{4} \times 2 + c$

$c = 2.5$

Equation is $y = -\frac{1}{4}x + 2.5$

b The same result would have been obtained if (6, 1) had been substituted.

c



c The question will often specify the domain and hence the required range on the *x*-axis. The required range for the *y*-axis can, if necessary, be found from the "table" function on the GDC.

d **i** When $x = 4.7$ $y = 1.325$

ii 2.5

d These are easy to calculate without the graph but make sure you know how to get these values from your GDC.

Reflect What is the formula for calculating the gradient of a line?

Exercise 3B

1 Write down:

  a the equation of the vertical line passing through (2, 7)

  b the equation of the horizontal line passing through (4, 6)

  c the point of intersection of the two lines.

2 Find the equation of the following lines.

  a gradient = 3, *y*-intercept 5

  b gradient = –2, *y*-intercept 0.4

  c gradient = 4.5, passing through (0, 5)

  d gradient = 2, passing through (3, 5).

3 Find the equation of the line passing through the two points given.

  a (3, 5) and (5, –3)

  b (2, –1) and (3, 4)

  c (–3, 2)and (3, 4)

4 A Cartesian grid is superimposed on a map of a city. The *y*-axis lies in the direction of North and the *x*-axis due East. The North-South boundaries of the city lie along the lines $x = 0$, $x = 22$ and the distances are in kilometres.

A straight road passes through the city. The road passes through the points with coordinates (6, 8) and (12, –4) and stays within the city boundaries for $0 \le x \le 22$.

  a Find the equation of the road.

  b Find the coordinates of the road as it crosses:

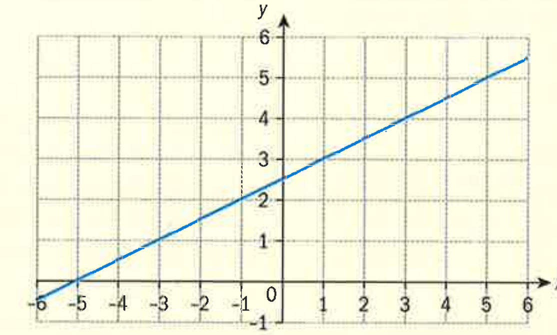
    i the western boundary of the city

    ii the eastern boundary of the city.

  c Find the length of the road within the city boundary.

Investigation 2

Consider the straight line passing through the point (1, 3), shown below



1 Take any point on the line and show that the gradient of the line joining it to (1, 3) is equal to 0.5.

2 Let (x, y) be a point on the line. Explain why $0.5 = \frac{y-3}{x-1}$

3 Generalise the result from question 2 to give an expression for the gradient in terms of *x* and *y* for a straight line with gradient m that passes through a known point (x_1, y_1).

4 Show that the expression derived in question 3 can be written as $y - y_1 = m(x - x_1)$.

$y - y_1 = m(x - x_1)$ is the **point gradient** form of the equation of a line.

5 Use your answer to question 2 to write the equation of the line given in point-gradient form.

6 Write the equation found in questions 5 in the form $y = mx + c$ by expanding the brackets and simplifying.

7 Write the equation derived in question 6 in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$

$ax = by + d = 0$ is the **general** form of the equation of a straight line.

Conceptual What are the three forms of the equation that give the relation between the *x*-coordinates and the *y*-coordinates of all points on a straight line?

Reflect How do you find the *x*- and the *y*-intercepts from the equation of a line? In which form of a line are intercepts easier to find?

The **point-gradient** form for the equation of a line with gradient equal to m, which passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

The **general** form of the equation of a straight line is written as $ax + by + d = 0$ where a, b and d are all integers.

EXAM HINT

Point-gradient form can be easily rearranged to give the equation in gradient-intercept form, but if an exam question does not ask for a particular form it is perfectly acceptable not to rearrange.

The point-gradient form is the easiest to find if given two points or one point and the gradient.

The general form is the easiest for finding intercepts and is often the form used when solving systems of linear equations (simultaneous equations).

Functions

Geometry and trigonometry

Exercise 3C

1 Write down the equation of a line with gradient, m, and which passes through the given point, in each of the following forms:
  i point-gradient form
  ii gradient-intercept form
  iii general form.
  a $m = 2, (3, 9)$
  b $m = \frac{1}{2}, (6, 5)$
  c $m = -\frac{1}{3}, (6, -7)$

2 Find the equations of the lines passing through the points given in
  i point-gradient form
  ii gradient-intercept form
  iii general form.
  a $(2, 5)$ and $(5, 11)$
  b $(0, 4)$ and $(2, 2)$
  c $(2, 6)$ and $(3, 9)$
  d $(-2, -6)$ and $(1, -8)$

3 Find the gradient of each of the lines
  a $2x - 3y - 7 = 0$
  b $4x + 7y - 6 = 0$
  c $ax + by + d = 0$

4 The plan of a triangular garden is drawn on a set of coordinate axes in which one unit represents 1 m. Two of the sides of the garden are formed by the x- and y-axes. The third side passes through the points $(1, 5)$ and $(3, 2)$.
  a Find the equation of the line which includes the third side.
  b Write this equation in general form.

  The owner of the garden wishes to cover it completely in grass.
  c Find the area of turf (grass) he will need to buy.

Intersections of Lines

You need to be able to solve systems of two linear equations "by hand" but also by using applications on your GDC or by plotting their graphs.

If using graphs, the solution to the system will be given by the coordinates of the intersection of the two lines.

For example, if given the starting positions and the directions of two straight railway tracks, it would be possible to find the equations of the lines that the tracks follow and hence find where they will meet.

Example 2

a Find the coordinates of the x- and y-intercepts for the graph of $2x + 3y - 6 = 0$.

b Write the following equation in general form, $y = x - \frac{1}{2}$

c Find the point of intersection of the two lines **i** analytically **ii** using an appropriate application on your technology.

a When $x = 0$, $3y = 6$ so $y = 2$ When $y = 0$, $2x = 6$ so $x = 3$ Coordinates are $(0, 2)$ and $(3, 0)$	**a** x-intercepts occur when $y = 0$ and y-intercepts occur when $x = 0$. The intercepts are the values of x or y so you need to check whether the question is asking for the intercepts or the coordinates of the intercepts.
b $2x - 2y - 1 = 0$	**b** It is usual to avoid beginning the equation with a negative coefficient but $-2x + 2y + 1 = 0$ is an equally valid answer.
c i $2x + 3y = 6$ $2x - 2y = 1$ subtract the two equations to get $5y = 5 \Rightarrow y = 1$ substitute this value into either equation to get $x = 1.5$	**c i** The solution can be found using elimination or substitution, for example by replacing y in $2x + 3y - 6 = 0$ with $y = x - \frac{1}{2}$ to get $2x + 3\left(x - \frac{1}{2}\right) - 6 = 0$ $\Rightarrow 5x - 1.5 - 6 = 0 \Rightarrow x = 1.5$
ii $2x + 3y = 6$ $2x - 2y = 1$ $x = 1.5,\ y = 1$	**ii** When solving a system of equations using your technology, there is no need to write down details of the method. Some calculators have an inbuilt simultaneous equation solver. If not the solution can be found by drawing both lines and finding the point of intersection, $(1.5, 1)$

Reflect How do you find the point of intersection of two non-parallel lines?

Exercise 3D

1 a Find the solutions to
  i $2x + y = 8$, $3x - 2y = 33$
  ii $2x + 10y = 3$, $3x + 15y = 4.5$
  iii $y = 2x + 1$, $4x - y = 5$
  iv $y - 2 = 3(x - 4)$, $y = 2x - 9$

  b Solve
  i $x + 3y = 1$, $5x + 16y = 8$
  ii $3x + 2y = 4$, $5x - 4y = 20.6$

Functions

Geometry and trigonometry

2 Two friends, Alison and Bernard are walking along two different roads. The roads can be represented in the Cartesian plane by the lines with equations

$y = -x + 410$ and $y = \frac{1}{2}x - 100$.

At 2:00 pm Alison is on the first road at the point with coordinates (0, 410) and Bernard is at the point with coordinates (50, −75), where the units are in meters.

a Verify that Bernard is on the road with equation $y = \frac{1}{2}x - 100$ at 2:00 pm.

b Find the coordinates of the point of intersection of the two roads.

At 2:00 pm the two friends begin walking at 4 kmh^{-1} towards the intersection.

c i Show that Bernard arrives at the intersection first.

ii Find the length of time he needs to wait before Alison arrives.

3 Road signs showing the steepness of hills are often given as percentages where the figure is derived using the following formula

$$\frac{\text{vertical height gained or lost}}{\text{horizontal distance covered}} \times 100$$

a A road gains 5 m while covering 20 m horizontally. State the percentage that would be written on the road sign.

There is a triangular hill directly outside my house. On the way up the hill from my house I pass a sign indicating the slope is 10%. On the way down the other side of the hill, I pass one indicating the slope is 15%.

b State which road is steeper.

I decide to take my house to be the origin for a coordinate system and one day I go over the hill to the other side and reach my local shop. My GPS tells me the horizontal distance of the shop from my home is 2.45 km and I am at the same level as my house.

c Assuming the roads up and down the hill are straight lines and lie in the plane of the coordinate system find

i the equation of the road going up from my house

ii the equation of the road going down from the top of the hill to the shop.

d Find the height of the hill

e Find the total distance of my journey from my house to the shop.

4 A straight line makes an angle α with the x-axis where $0 \le \alpha < 90°$.

a Explain why the gradient of the line is equal to $\tan \alpha$.

The air traffic control tower at an airport is taken as the origin of a coordinate system. An aircraft begins its descent 7500 m from the control tower and from a height of 580 m. Let x be the horizontal distance from the control tower and y the height of the aircraft. The angle of descent of the aircraft is 4° to the horizontal and its path will take it directly above the control tower.

The runway begins 700 m from the control tower and in the same direction as the aircraft's approach.

b Find the equation for the line of descent for the aircraft in the form $y = mx + c$.

c Find the height of the aircraft as it passes over the control tower.

d Find the distance of the aircraft from the start of the runway when it lands.

International-mindedness

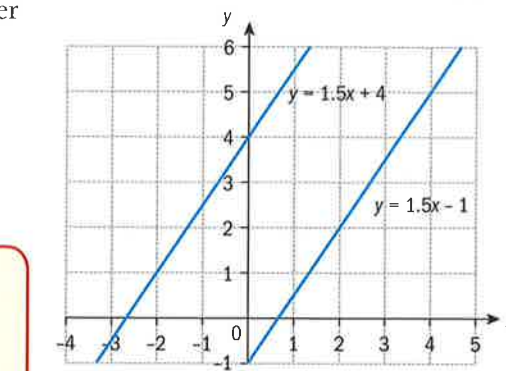
Road steepness is often given as a ratio or percentage. What is it in your country? What others can you find?

Parallel and perpendicular lines

Parallel lines will have the same gradient as each other but a different intercept on the y-axis.

For example $y = 1.5x + 4$ is parallel to $y = 1.5x - 1$.

Lines which are perpendicular intersect at 90°.

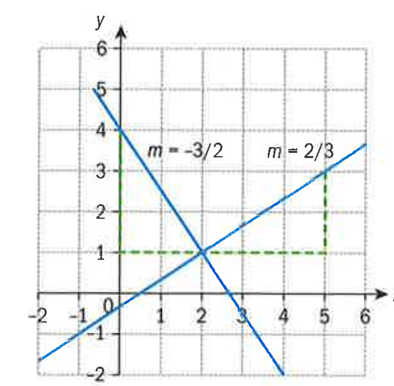


In general $y = mx + c_1$ is parallel to $y = mx + c_2$

If $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular (and neither m_1 or m_2 are equal to 0) then $m_1m_2 = -1$ or

$m_2 = -\frac{1}{m_1}$ (the negative reciprocal of m_1)

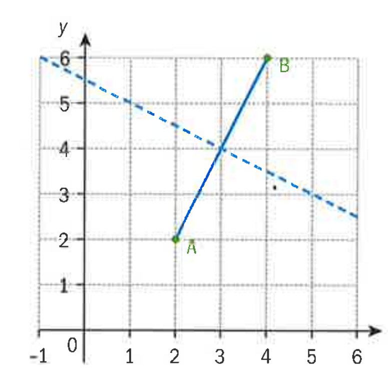
For example, if a line has a gradient of $\frac{2}{3}$ the line perpendicular to it will have gradient $-\frac{1}{\left(\frac{2}{3}\right)} = -\frac{3}{2}$



The perpendicular bisector

The perpendicular bisector of the line segment joining two points is the line that passes through the midpoint of the line segment and is perpendicular to it.

The perpendicular bisector of the line segment joining A(2, 2) and B(4, 6) is shown below.



Example 3

Find the equation of the perpendicular bisector of the line segment joining A(2, 2) and B(4, 6).

Midpoint of [AB] is (3, 4)	Using the formula for midpoint.
Gradient of [AB] is $\frac{4}{2} = \frac{1}{2}$	Using the formula for gradient.
Equation of perpendicular bisector is $y - 4 = -2(x - 3)$	Find the negative reciprocal to get the gradient of the perpendicular line and use point-gradient form to get the equation.
$y = -2x + 10$	This last stage was not necessary as a particular form was not specified but it does give the equation in a simpler form.

Reflect What is the gradient of a line parallel to a line with gradient m?

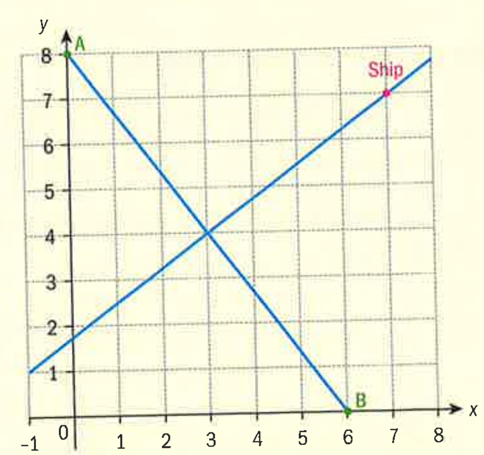
What is the gradient of a line perpendicular to a line with gradient m?

How do you find the equations of perpendicular bisectors given either two points, or the equation of a line segment and its midpoint?

Investigation 3

A ship is sailing along the line forming the perpendicular bisector of the line segment between two lighthouses (A and B), as shown.

The coordinates of A are (0, 8) and the coordinates of B are (6, 0). At 1 pm the ship is at (7, 7). All units are kilometres.



1 Find the equation of the perpendicular bisector of [AB].

2 Verify that the ship is on the perpendicular bisector of [AB].

3 Find the distance of the ship from

a A **b** B.

Comment on your results.

4 From the definition of the perpendicular bisector use the above diagram to prove that the ship will always be equidistant from the two lighthouses so long as it stays on this course.

Conceptual What is the relation between all points on the perpendicular bisector of the line segment joining two points and those two points?

Exercise 3E

1 Write down the gradients of the lines which are perpendicular to the lines given.

a $y = -2x + 7$ **b** $y - 3 = \frac{1}{3}(x - 21)$

c $2x - 4y - 5 = 0$ **d** $6x + 7y + 12 = 0$

2 a Verify that $bx - ay = d_1$ is perpendicular to $ax + by = d_2$.

b Hence find the equation of the line perpendicular to $x + 2y - 10 = 0$ which passes through (2, 5).

c Find the equation of the line perpendicular to $3x - 2y = 7$ which passes through (6, 5).

3 a Explain why the shortest distance from a point A to a line lies on the line through A, perpendicular to the line.

The line l has equation $3x - 4y + 7 = 0$ and the point A has coordinates (5, −7).

b Find the equation of the line perpendicular to l which passes through A.

c Find the point of intersection of the line found in part **b** and l.

d Hence find the shortest distance of A from l.

4 Find the shortest distance from

a (2, 4) to $3x + 5y + 8 = 0$

b (5, −1) to $y = 3x - 2$.

5 Let A and B be two lighthouses with coordinates (12, 18) and (17, 16). A ship S is travelling on the line that keeps it an equal distance from the two lighthouses.

Find the position of S when it is north-east of the lighthouse A.

6 Two towns have coordinates (5, 18) and (17, 24). A rail track is laid along the line with equation $y = x + 10$ and a station is to be built to serve the two towns. Distances are in kilometres.

a Find the position the station should be built to minimize the total distance between the station and the two towns and state this distance.

b It is decided that the station will be built so it is an equal distance from both towns. Find the position the station should be built and state the distance between the two towns.

Developing inquiry skills

Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

1 Find the equations of the perpendicular bisectors between the islands.

2 Find the coordinates of the point where the perpendicular bisectors between A and B and A and C meet.

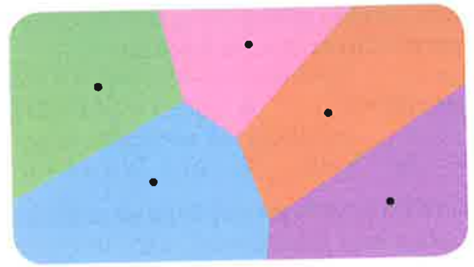
Verify that the perpendicular bisector between B and C also passes through this point. Do you think these lines divide the waters in a fair way? Justify your answer.

TOK

How does the variation in language and symbols affect the knowledge gained and communicated in mathematics?

3.3 Voronoi diagrams

There are five airports in a state, shown as dots on the diagram. The coloured regions on the diagram indicate which of the airports is closest to a given position. This means if there is an emergency and the position of the plane is known the information about the nearest airport is very easily obtained.



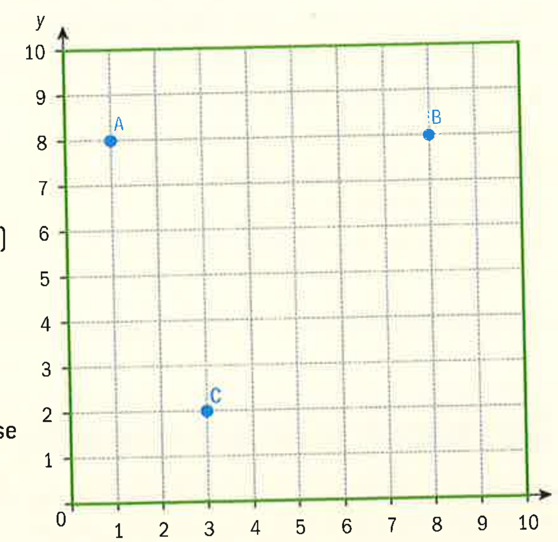
This is an example of a Voronoi diagram, named after Georgy Voronoy (1868–1908). It shows the sets of points that are closer to a chosen point (a **site**) than to any other sites on the plane. The regions formed are known as **cells**.

Reflect What does a Voronoi diagram show?

Investigation 4

Three points A, B and C lie inside a square of side length 10 units whose sides are formed by the lines $x=0$, $x=10$, $y=0$ and $y=10$.

The points A, B and C have coordinates $(1, 8)$, $(8, 8)$ and $(3, 2)$.



1. Construct a copy of the diagram either by using online software (for example Geogebra or Desmos) or by copying onto graph paper.
2. State the equation of the line which contains all points equidistant from A and B and show this on the diagram.
3. Add the lines to the diagram which contain all those points equidistant from
   - **a** A and C
   - **b** B and C.
4. By inspection shade all those points which are closest to each of A, B or C.

Conceptual What can be said about the boundaries of a Voronoi diagram?

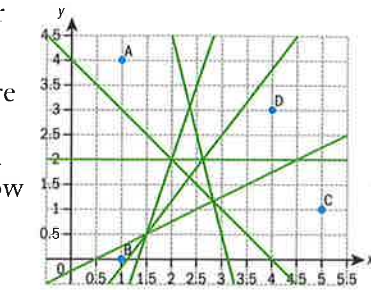
Reflect How do you find the equation of the boundary of a cell on a Voronoi diagram?

The boundaries of the **cells** in a Voronoi diagram are formed by perpendicular bisectors of the line segments joining the sites.

International-mindedness

Georgy Voronoy was a Russian mathematician who studied at St Petersburg University.

The best approach is rarely to draw all the perpendicular bisectors at the same time and then create the diagram.



This diagram shows that even with just 4 points there are difficulties in deciding which of them form boundaries to the cells, and with more points it would become even more difficult. The incremental algorithm described below avoids this problem by adding each of the sites one at a time.

Investigation 5

There are four fire stations (A, B, C and D) in a town. The coordinates of the fire stations are $A(1, 4)$, $B(1, 0)$, $C(5, 1)$ and $D(4, 3)$.

In order to improve response time the town has installed a new centralized fire response system, which allows a dispatcher to send a fire truck from the nearest fire station to the location of fire. How should the town be divided into areas so that there is one fire station in each area and this fire station is the closest one for each house in the section?

The solution will be found through constructing a Voronoi diagram using an **incremental algorithm.**

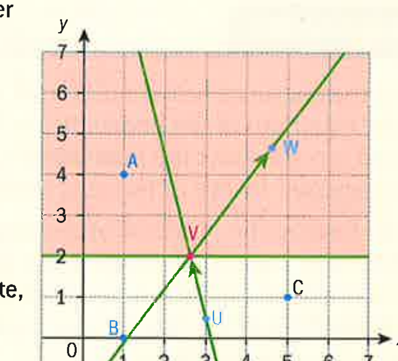
HINT

Many software packages allow you to draw and obtain the equation of perpendicular bisectors directly. If not using software a perpendicular bisector can be constructed using a compass or by finding its equation and drawing it.

Method

1. Plot the fire stations A and B on coordinate axes, either by hand or using a software package. Draw the perpendicular bisector between A and B and gently shade those points nearest to A.
2. Add fire station C to the diagram and find the equations of the perpendicular bisectors of [AC] and [BC] and add these to the diagram. What do you notice?
3. The incremental Algorithm:
   - **i** Begin with the perpendicular bisector which lies between the new site (C) and the site in whose cell this site currently lies (B).
   - **ii** Move along this line until you reach an intersection with another of the perpendicular bisectors between the new site and an existing one (this will also be on a boundary of the previous Voronoi diagram).
   - **iii** Leave the intersection along the other new perpendicular bisector in the direction that lies entirely in the cell surrounding another of the sites (this will be the direction that creates a convex polygon around the new site).

     Hence you should trace out the edge of the new cell in the order U, V and W shown in the diagram.
   - **iv** The algorithm stops either when you return to your starting point (if the cell is bounded) or if there are no more intersections (if the cell is unbounded). In this case you may need to reverse the direction of the algorithm to ensure all sides have been found.

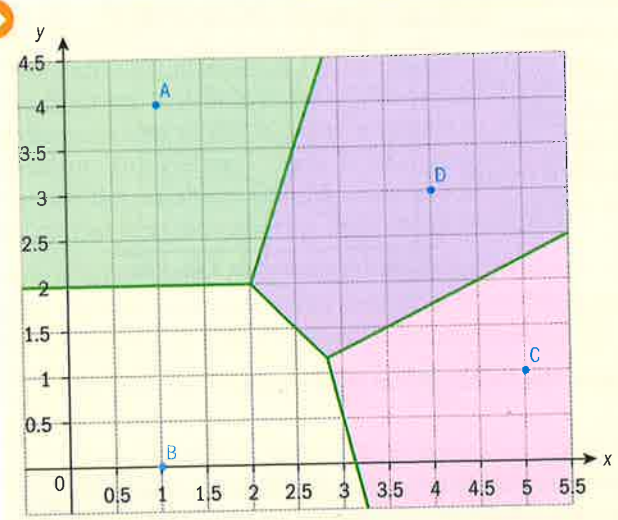


4. Shade the region containing the new site.
5. Having completed the diagram for the first three sites the final site, D, is now added and the process described in stages 2 and 3 is repeated. This time there will be two intersection points with the perpendicular bisectors of [BD] and [AD].

Continued on next page

Functions

Geometry and trigonometry



HINT

Normally a final version of the Voronoi diagram will have the perpendicular bisectors removed so only the edges of the regions remain as in the diagram shown.

A few years later a new fire station is built in the town at the point with coordinates (2, 1).

6 Use the incremental algorithm to construct the new Voronoi diagram showing the areas served by each of the fire stations.

7 Place the fifth fire station on the diagram given just before the start of the investigation which shows the positions of A,B,C and D and their perpendicular bisectors. How easy would it have been to draw a Voronoi diagram without using the incremental algorithm?

8 **Conceptual** Why is the incremental algorithm used in the construction of a Voronoi diagram?

Reflect What is the incremental algorithm?
How many edges meet at a vertex of a Voronoi diagram?

International-mindedness

Voronoi diagrams are used in computer graphics, epidemiology, geophysics, and meteorology.

Nearest neighbour interpolation

If each site is assigned a numerical value (such as the amount of rain that fell on a particular day or a level of pollutant) then the value of all points in each site's cell is assumed to equal that value.

Exercise 3F

In the questions below unless told to calculate the equations of the perpendicular bisectors you can construct the lines using a pair of compasses, by eye or by using software.

1 By finding the perpendicular bisectors between each of the points, use the incremental algorithm to complete the Voronoi diagrams for the given sites.

a (1, 1), (3, 1), (2, 3)

b (1, 1), (5, 1), (5, 5), (3, 5)

2 An Internet weather website uses readings taken at three different stations. A visitor to the website will be told the temperature of the station nearest to their location (nearest neighbour interpolation). The weather stations are at the points with coordinates

A (1, 1) **B** (3, 1) **C** (3, 5).

a Use the incremental algorithm to construct a Voronoi diagram showing the regions that would be assigned to each of the weather stations.

At a particular time a visitor to the website is at point (1, 4) and the readings at the different stations are A: 22°C, B: 24°C and C: 21°C.

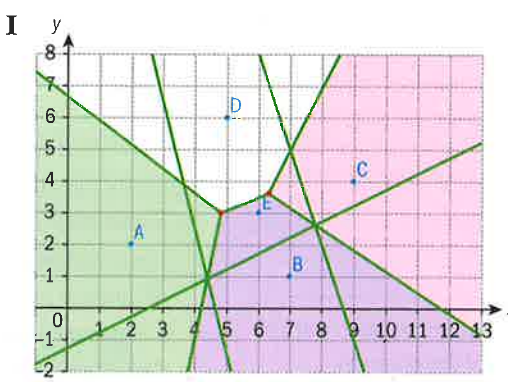
b Write down the temperature that would be given to the visitor to the site.

3 A company that collects meteorological data has many rainfall collection points. In two of their areas (I and II) in which they had four collection points they decide to add a fifth.

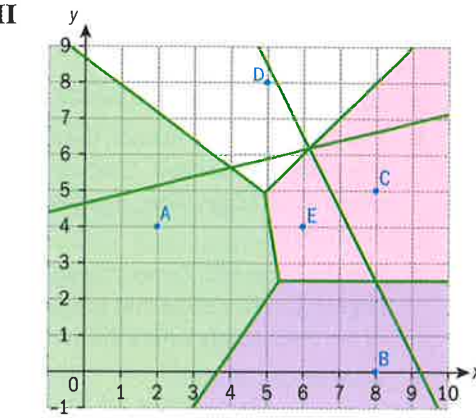
The original points are shown as A to D in the following diagrams and the new point is labelled E.

a The point E and some of the perpendicular bisectors between E and the other points are shown. For each area find the equation of the missing perpendicular bisector(s), add it to the diagram, and indicate the new cells.

I



II



b It is given that on a particular day the rainfall in area I is recorded as

A: 22 mm B: 31 mm C: 24 mm
D: 19 mm E: 21 mm

Use nearest neighbour interpolation to give an estimate for the amount of rain that fell at a point with coordinates (5, 4)

i before point E was added

ii after point E was added.

c It is given that on a particular day the rainfall in area II is recorded as

A: 9 mm B: 11 mm C: 14 mm
D: 8 mm E: 12 mm.

Use nearest neighbour interpolation to give an estimate for the amount of rain that fell at a point with coordinates (4.5, 1)

i before point E was added

ii after point E was added.

The toxic waste dump problem

This problem is to find the point on the Voronoi diagram that is as far as possible from any of the sites. It is called **the toxic waste dump problem** because one application might be to find where waste can be dumped so that it is as far as possible from habitation.

However, it is more frequently used in consideration of where to place businesses or shops.

Another way of thinking about the problem is to say it is at the centre of the largest circle that can be drawn on the diagram that does not contain any of the sites.

Investigation 6

1 **a** Add five points randomly placed on a Geogebra worksheet.

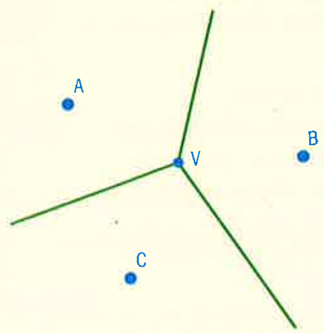
b Use the command, *Voronoi*, to construct a Voronoi diagram with these five points as the sites. Check that you have three distinct points at which three edges meet. If you do not, move one of the original points until you do.

c Use the *Circle with Centre through Point* command to consruct the largest possible circle that contains no sites within it, centred on each of the three points where the vertices intersect.

d What do you notice?

2 **a** Explain using the diagram why there will always be three sites equidistant from a vertex of a Voronoi diagram which has three edges incident to it.

b Explain why the circle with V as a center which passes through A will also pass through B and C.



3 Explain why another site D cannot be inside this circle with centre V.

This circle is known as the largest empty circle because it does not contain sites and to extend it further would mean it would be no longer be empty.

4 **Conceptual** Where on a Voronoi diagram would you look for the solution to the toxic waste dump problem?

International-mindedness

Voronoi diagrams were used in the analysis of the 1854 cholera epidemic in London, in which physician John Snow determined a strong correlation of deaths with proximity to a particular (and infected) water pump on Broad Street.

Within a Voronoi diagram the solution to the toxic waste dump problem will be at an intersection of cell boundaries or on the boundary of the diagram.

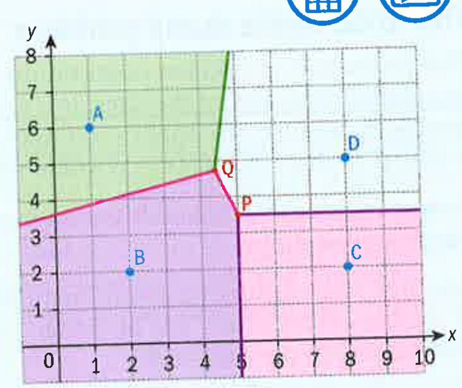
EXAM HINT

In exams the solution will always be one of the internal vertices rather than a boundary edge.

Example 4

A town has four coffee shops, A, B, C and D. An entrepreneur wishes to open a new shop in the town but would like it to be as far as possible from all the other four coffee shops.

Consider the Voronoi diagram showing the positions of the 4 coffee shops on a set of coordinate axes. A(1, 6), B (2, 2), C(8, 2) D(8, 5) where one unit represents 1 km.



a Find the coordinates of the vertices P and Q in the Voronoi diagram.

b Determine the best position for the new shop so as to be as far as possible from any other shop.

a P is the point where $x = 5$ and $y = 3.5$ meet: (5, 3.5) The perpendicular bisector of [AB] is $-x + 4y = 14.5$ The perpendicular bisector of [AD] is $7x - y = 26$ The perpendicular bisector of [BD] is $2x + y = 13.5$ The coordinates of Q are (4.39, 4.72)	Three perpendicular bisectors meet at the vertices, finding the intersection of any two will be sufficient to find the point. Any two of these equations need to be calculated by first finding the midpoint and gradient. The coordinates can then be found algebraically or by using a GDC.
b Centred at P: $PD = \sqrt{(5-3.5)^2 + (8-5)^2} = 3.35$ Centred at Q: $QA = \sqrt{(6-4.72)^2 + (4.39-1)^2} = 3.62$ The new coffee shop should therefore be built as close as possible to the point Q.	The solution will be at whichever of the points P and Q is furthest from the three sites nearest to them. Only one length for each needs to be checked as each of the other two points will be an equal distance from the vertex.

Reflect How do you find the distance from an intersection point to a site if given both coordinates?

How do you decide which intersection point is the solution to the toxic waste dump problem?

Exercise 3G

1 A town is divided into a coordinate system with distances measured in kilometres north and east of a fixed origin. Within this town three schools A, B and C are at the points with coordinates A(1, 3), B(6, 4) and C (6, 1).

It is decided that a new school should be built as close as possible to the point which is furthest from all three existing schools.

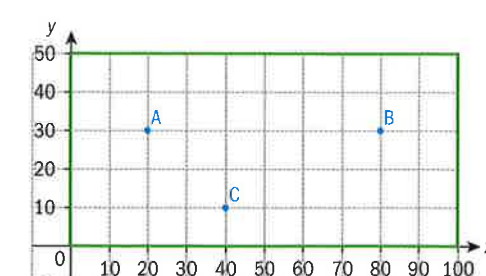
a Explain why this point will be at the intersection of the perpendicular bisectors of [AB], [BC] and [AC].

b Find the equations of the perpendicular bisectors of [AB] and [BC]

c Hence find the coordinates of the point where the new school should be built.

d Determine the distance between the new school and each of the other schools.

2 At a fair there are three hamburger stands, A, B and C. The fairground is in the shape of a rectangle with dimensions 100 m by 50 m. The bottom left-hand side of the field can be regarded as the origin of a coordinate system, with the diagonally opposite corner as (100, 50). The hamburger stands are at the points A(20, 30), B(80, 30) and C (40, 10) as shown on the diagram below.



Functions

Geometry and trigonometry

a Find the equation of the perpendicular bisector of

i A and C **ii** B and C.

People will always go to the hamburger stand that is closest to them.

b Draw the Voronoi diagram that represents this situation.

c Find the proportion of the fair ground in which people would go to

i Stand C **ii** Stand A.

d A fourth hamburger stand is to be added to the fairground at a point as far away as possible from the other three stands.

i State the coordinates of the position at which it should be built.

ii Determine how far it will be from the other hamburger stands.

3 A town can be considered as a rectangle which runs 10 km east to west and 8 km north to south. A coordinate grid is placed on a map of the town with the origin in the south-west corner. There are four schools in the town, A, B, C and D whose coordinates are: A (2, 5), B (3, 3), C (8, 6), D (8, 1).

Children go to the school that is closest to their home when measured by a direct line.

An estate agent wishes to construct a diagram which easily shows in which school's catchment area a house lies.

a Find the perpendicular bisector of [AB].

b Show the positions of A and B and the perpendicular bisector on a diagram of the town.

The perpendicular bisectors of [AC] and [BC] are $y = -6x + 35.5$ and $y = -\frac{5}{3}x + \frac{41}{3}$

c Use the incremental algorithm to construct the Voronoi diagram for the three schools.

d **i** Find the perpendicular bisectors of [BD] and [CD] and show these on the diagram.

ii Explain why there is no need to calculate the perpendicular bisector of [AD].

iii Construct the diagram required by the estate agent.

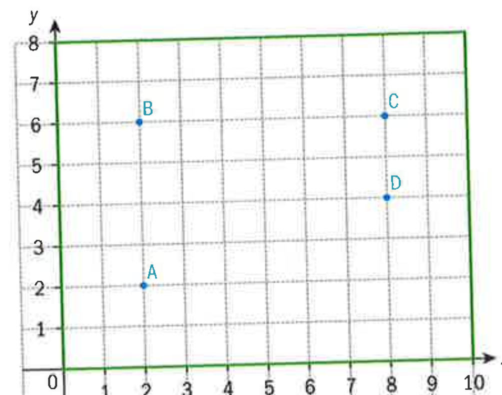
e Find the coordinates of the two vertices where three edges meet.

f Hence find the percentage area of the town covered by school C to 2 significant figures.

g A fifth school is to be built in the town as far as possible from the other schools. Find the coordinates of the point at which it would be built if it was to meet this requirement.

h On the diagram already drawn plot the fifth school at the position found in **g** and **sketch** the new Voronoi diagram. There is no need to find equations for the new perpendicular bisectors.

4 The map of a rectangular province is shown with the positions of the bases for the flying doctors shown on a coordinate grid centred at one of the corners in the province (units are in 100s of miles).



When an emergency occurs the doctor that is based closest to the location of the emergency will fly out to the scene.

a Construct a Voronoi diagram for the four sites.

A neighbouring province asks if the doctor based at point A (2, 6) can also help out in their province. The director replies it would be possible if he is currently covering a smaller area than at least two of the other doctors.

b Find where the perpendicular bisector of AD meets the line $y = 4$.

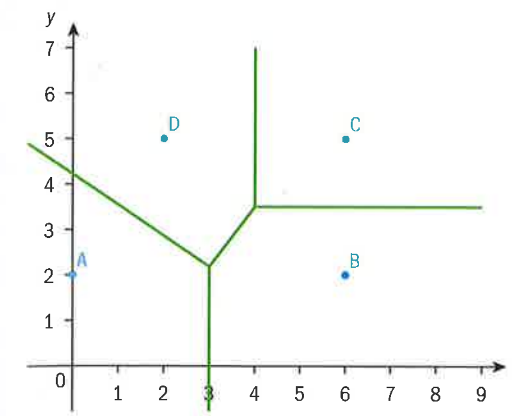
c Find the area of the cells surrounding

i A **ii** B **iii** C **iv** D.

d Hence state whether or not the doctor based at A will be able to support the other province?

5 The diagram below shows the Voronoi diagram for the points A(0,2), B(6,2), C(6, 5), and D(2,5).

The four points represent stations at which environmental readings are taken. The axes are measured in 10 km sections, east and north of a town which is situated at the origin. Hence B is 60 km east and 20 km north of this town.



Each of the towns are connected by straight roads.

An environmental officer decides to drive from A to B, B to C, C to D and back to A.

a **i** Show the total distance driven is 166 km to the nearest km.

ii Find the proportion of the journey during which he is closest to the station at A.

The environmental officer receives a call from the owner of a home at N, whose coordinates are (4,3.5). The owner is concerned at the level of pollution he is experiencing, due to road congestion caused by construction.

The officer decides that he will work out what level of pollution would be expected in this location under normal conditions based on the data he has and will compare it with what the home owner is experiencing.

The pollution readings from each of the stations are (based on Air Quality Health Index):

A 4.5 B 2.1 C 2.6 D 2.8

b The officer decides to use as an expected value the average pollution recorded at 3 of the stations.

i Explain why this is reasonable and which three stations he should use.

ii Find the value he will use.

Readings of the air quality health index at N are taken over a period of 15 days and are shown below

3.0, 2.8, 2.1, 3.4, 3.1, 2.9, 3.0, 3.2, 2.8, 2.7, 2.6, 3.1, 3.0, 3.2, 2.7

c **i** Find the median, lower quartile, upper quartile and inter-quartile range for this data.

ii Show the data on a box-and-whisker plot, indicating the position of any outliers.

d Using the results from parts **b** and **c** comment on the house owner's claim that traffic congestion is causing greater than expected pollution near his home.

Developing inquiry skills

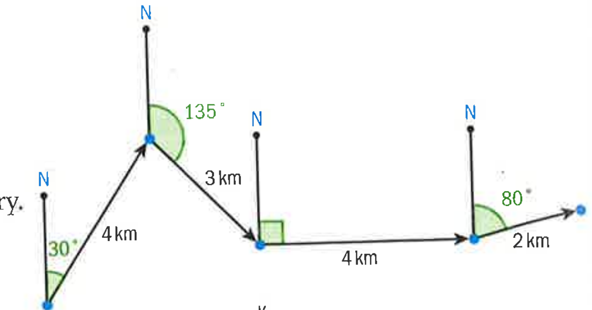
Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

1 Draw the Voronoi diagram showing the regions in which each of the three islands have exclusive fishing rights.

2 Find the area of each of these regions.

3.4 Displacement vectors

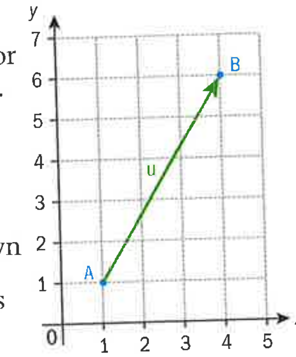
A boat sails 4 km on a bearing of 30°, followed 3 km south-east, then 4 km due east and 2 km on a bearing of 080° as shown. How far is it from its starting point and what bearing would it have to travel on to return directly to the starting point?



Consider how you might solve this using trigonometry.

Vectors provide a straightforward way to answer questions like this as well as many others.

If you move from the point A at (1, 1) to the point B at (4, 6) your movement can be represented by the directed line segment, or **vector**, as shown. A vector has both a magnitude (length) and direction.



Because the vector goes from A to B we can write it as $\overrightarrow{AB}$. Alternatively, we can give it a name such as $\boldsymbol{u}$. In print a vector is named using bold font, when handwritten it is written as $\underline{u}$ or $\vec{u}$.

Vectors are normally described in **component** form. The vector shown can be written as a column vector $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ or, using the **base** vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ as $\overrightarrow{AB} = 3\boldsymbol{i} + 5\boldsymbol{j}$. In each case the first number, or component, indicates movement in the x-direction and the second movement in the y-direction.

The vector $\overrightarrow{BA} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$ because to move from B to A you need to go 3 units to the left and 5 down.

It will always be the case that the vector $\overrightarrow{AB} = -\overrightarrow{BA}$

EXAM HINT

The choice of which notation to use may depend on the context but in an exam, both are equally valid.

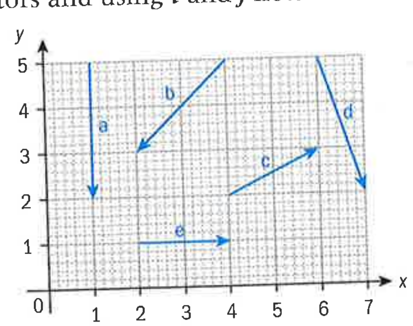
Addition of vectors

Two vectors are added by adding the corresponding components.

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ or $(3\boldsymbol{i} + 5\boldsymbol{j}) + (\boldsymbol{i} + 2\boldsymbol{j}) = (4\boldsymbol{i} + 7\boldsymbol{j})$

Exercise 3H

1 Write the following vectors as column vectors and using $\boldsymbol{i}$ and $\boldsymbol{j}$ notation.



2 Find:

a $\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ **b** $(3\boldsymbol{i} - \boldsymbol{j}) + (4\boldsymbol{i} + 5\boldsymbol{j})$

c $\begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ **d** $(\boldsymbol{i} - 2\boldsymbol{j}) + 4\boldsymbol{i}$

3 Find

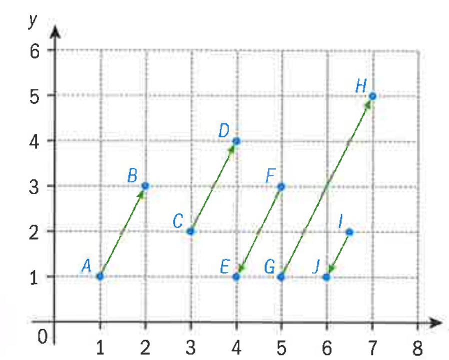
a $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

c Explain how you multiply a vector by a scalar.

d $\boldsymbol{i}$ and $\boldsymbol{j}$ can also be written as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Hence verify $3\boldsymbol{i} + 4\boldsymbol{j} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

4 Let $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as shown.



a Write the following vectors in component form and in terms of the vector $\overrightarrow{AB}$.

i $\overrightarrow{CD}$ **ii** $\overrightarrow{FE}$

iii $\overrightarrow{GH}$ **iv** $\overrightarrow{IJ}$

b Comment on what can be deduced about parallel vectors.

5 State which of the following vectors are parallel to $5\boldsymbol{i} + 2\boldsymbol{j}$.

a $-5\boldsymbol{i} - 2\boldsymbol{j}$ **b** $25\boldsymbol{i} - 10\boldsymbol{j}$

c $-\boldsymbol{i} - 0.4\boldsymbol{j}$ **d** $\begin{pmatrix} 20 \\ 50 \end{pmatrix}$

e $\begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ **f** $2\begin{pmatrix} -10 \\ 4 \end{pmatrix}$

g $2\begin{pmatrix} 3 \\ 3 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

6 a Find p and q if

i $\begin{pmatrix} 4p \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ -2q \end{pmatrix} = \begin{pmatrix} -2p \\ 2 \end{pmatrix}$

ii $\begin{pmatrix} 3p \\ -2q \end{pmatrix} + \begin{pmatrix} 2q \\ p \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

b i Find p if $\begin{pmatrix} p+1 \\ 2p \end{pmatrix}$ is parallel to $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

ii Find q if $\begin{pmatrix} 2q-3 \\ q+6 \end{pmatrix}$ is parallel to $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Reflect How can you demonstrate that two vectors have the same direction?

$\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$, where k is a scalar.

Investigation 7

1 a Plot on graph paper the points P(1, 4) and Q(8, 10) and write down the vector $\overrightarrow{PQ}$.

b Pick another point, A, anywhere on the coordinate grid and add it to your graph.

c Write down the vectors $\overrightarrow{PA}$ and $\overrightarrow{AQ}$ and calculate $\overrightarrow{PA} + \overrightarrow{AQ}$.

Continued on next page

d Pick another point B anywhere on the coordinate grid and add it to your graph.

e Write down the vectors $\overrightarrow{PB}$ and $\overrightarrow{BQ}$ and calculate $\overrightarrow{PB}+\overrightarrow{BQ}$.

f Comment on your results.

2 a Add points C and D anywhere on your graph.

b On your graph draw the vectors $\overrightarrow{PA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CQ}$.

c Find the values of $\overrightarrow{PA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CQ}$ and calculate the sum of these four vectors.

d Conjecture a general property of the addition of vectors.

Conceptual What is the geometric meaning of a vector sum?

From the above we can derive the **triangle law of vector addition.**

$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$

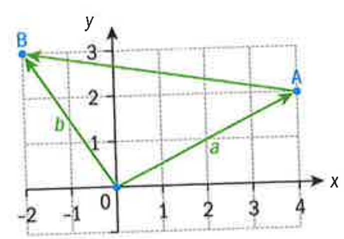
This can be extended to any number of vectors. A consequence of the law is that the sum of two of more displacement vectors is always equal to the final displacement and is independent of the route taken.

Position vectors

A position vector gives a point's displacement from the origin.

The point A with coordinates $(4, 2)$ has position vector $\overrightarrow{OA}=\boldsymbol{a}=\begin{pmatrix}4\\2\end{pmatrix}$

The point B with coordinates $(-2, 3)$ has position vector $\overrightarrow{OB}=\boldsymbol{b}=\begin{pmatrix}-2\\3\end{pmatrix}$



TOK

When is it ethically correct to provide a person's location?

It can be seen from the diagram and using the triangle law of vector addition that $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=-\overrightarrow{OA}+\overrightarrow{OB}$

This is normally written as $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ or $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$.

Example 5

A is the point with coordinates $(6, 1)$ and B is the point with coordinates $(-2, 5)$.

a Find the vector $\overrightarrow{AB}$.

A ship moves from the point A to a point C which has coordinates $(7, 4)$ and then onto point B.

b Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$.

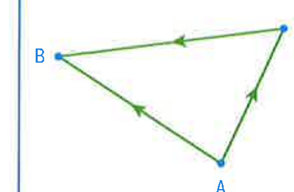
c Write down an equation linking $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{CB}$.

d Verify your answer is true for the values obtained in parts **a** and **b**.

From point B the ship moves on to the point D where $\overrightarrow{BD}=\begin{pmatrix}5\\2\end{pmatrix}$

e Find the vector $\overrightarrow{AD}$.

f Find the coordinates of D.

a $\overrightarrow{AB}=\begin{pmatrix}-2\\5\end{pmatrix}-\begin{pmatrix}6\\1\end{pmatrix}=\begin{pmatrix}-8\\4\end{pmatrix}$	Using $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$
b $\overrightarrow{AC}=\begin{pmatrix}7\\4\end{pmatrix}-\begin{pmatrix}6\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$ $\overrightarrow{CB}=\begin{pmatrix}-2\\5\end{pmatrix}-\begin{pmatrix}7\\4\end{pmatrix}=\begin{pmatrix}-9\\1\end{pmatrix}$	Using $\overrightarrow{AC}=\boldsymbol{c}-\boldsymbol{a}$ and $\overrightarrow{CB}=\boldsymbol{b}-\boldsymbol{c}$
c $\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{CB}$	Care needs to be taken as the triangle law for vector addition requires the vectors to follow on from each other. The endpoint of the first vector must be the first point of the second. The sketch below indicates the order required. 
d $\begin{pmatrix}-8\\4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}-9\\1\end{pmatrix}$	
e $\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BD}$ $\begin{pmatrix}-8\\4\end{pmatrix}+\begin{pmatrix}5\\2\end{pmatrix}=\begin{pmatrix}-3\\6\end{pmatrix}$	
f $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$ $\overrightarrow{OD}=\begin{pmatrix}6\\1\end{pmatrix}+\begin{pmatrix}-3\\6\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix}$ Hence coordinates are $(3, 7)$.	Remember the coordinates of D are obtained from its position vector $\overrightarrow{OD}$. Give the answer as coordinates rather than a position vector if this is what is asked for.

Reflect How do you find the displacement vector between two points A and B?

Functions

Geometry and trigonometry

Exercise 3I

1 The points A, B and C have coordinates (1, 4), (2, 5) and (−1, 6) respectively.

a Find the vectors

i $\overrightarrow{OA}$ ii $\overrightarrow{AB}$ iii $\overrightarrow{AC}$ iv $\overrightarrow{CA}$.

The point D is such that $\overrightarrow{CD}$ is equal to $\begin{pmatrix}4\\1\end{pmatrix}$.

b i Write down an equation that gives $\overrightarrow{BD}$ in terms of $\overrightarrow{BA}$, $\overrightarrow{AC}$ and $\overrightarrow{CD}$.

ii Hence write down an equation that gives $\overrightarrow{BD}$ in terms of $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{CD}$.

iii Hence find $\overrightarrow{BD}$.

iv Find the coordinates of D.

2 a $\overrightarrow{AB} = \begin{pmatrix}1\\4\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}-1\\2\end{pmatrix}$

find i $\overrightarrow{AC}$ and ii $\overrightarrow{CA}$.

b $\overrightarrow{AB} = \begin{pmatrix}2\\4\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}-1\\3\end{pmatrix}$

and $\overrightarrow{AD} = \begin{pmatrix}2\\3\end{pmatrix}$ find $\overrightarrow{DC}$.

3 The points A, B, C and D have coordinates (1, 0), (2, 3), (7, 5) and (6, 2) respectively.

a Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

b State which type of quadrilateral is formed by ABCD. Justify your answer.

c State two other vectors which must be equal.

4 An aircraft flies from an airport at A to one at B and then on to C. The routes taken can be given by the vectors $\overrightarrow{AB} = \begin{pmatrix}75\\90\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}-35\\-100\end{pmatrix}$ km.

a Find the vector $\overrightarrow{AC}$.

b The aircraft then flies directly back to A. Write down the vector that describes this flight.

c Calculate the direct distance from C to A.

5 A surveyor is putting flags out in a large field. His movements between the flags can be described by the vectors $(5\boldsymbol{i} + \boldsymbol{j})$, $(-2\boldsymbol{i} + 4\boldsymbol{j})$, $(4\boldsymbol{i} + 2\boldsymbol{j})$, $(6\boldsymbol{i} + 4\boldsymbol{j})$.

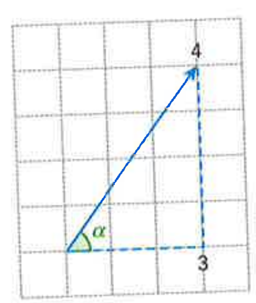
a Find his displacement from his starting position when he puts out the last flag.

b Write down the displacement vector that will take him back to his starting position.

The magnitude and direction of a vector

The magnitude of a vector $\boldsymbol{v}$ is its length. It is written as $|\boldsymbol{v}|$ and can be found using Pythagoras' theorem.

The magnitude of $\begin{pmatrix}3\\4\end{pmatrix} = \begin{vmatrix}3\\4\end{vmatrix} = \sqrt{3^2 + 4^2} = 5$



The direction of a vector is normally given as an angle. Within a Cartesian coordinate system the angle is normally measured anti-clockwise from the positive x-axis.

The direction of the vector $\begin{pmatrix}3\\4\end{pmatrix}$ is the angle α where $\tan\alpha = \frac{4}{3}$ hence $\alpha = 53.1°$.

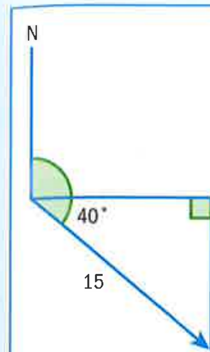
In many contexts it is more natural to describe a vector by giving its magnitude and direction, but because manipulation of vectors is much easier when given in component form it is important to be able to switch between forms.

TOK

How certain is the shared knowledge of mathematics?

Example 6

Write the following displacement as a column vector and in $\boldsymbol{i}, \boldsymbol{j}$ form: 15 m on a bearing of 130°.



Vector is $\begin{pmatrix}15\cos 40°\\-15\sin 40°\end{pmatrix}$

$= \begin{pmatrix}11.5\\-9.64\end{pmatrix}$ m or $11.5\boldsymbol{i} - 9.64\boldsymbol{j}$

In order to find the entries for the column vector first create a right-angled triangle and then use trigonometry.

From the direction of the vector you will be able to see which entries should be positive and which negative.

Reflect How do you find the magnitude and direction of a vector in 2 dimensions?

How do you write a vector given as a magnitude and direction in component form?

International-mindedness

Greek philosopher and mathematician, Aristotle, calculated the combined effect of two or more forces which is called the Parallelogram law.

Investigation 8

1 A boy walks 5 km on a bearing of 045° and then 8 km on a bearing of 120°.

a Show this information on a diagram.

b Use the cosine and sine rules to find his distance and bearing from his starting point at the end of the walk.

c i Write the displacements 5 km on a bearing of 045° and 8 km on a bearing of 120° as column vectors where the first component indicates displacement east and the second displacement north.

ii Use your answer to part **c i** to find how far east and how far north from his starting point the boy is at the end of his walk.

iii Hence give his **resultant** (final) displacement as a column vector.

iv Use your answer to part **c iii** to find his distance and bearing from his starting point at the end of the walk.

Continued on next page

Functions

Geometry and trigonometry

2 A boat sails 60 m on a bearing of 030°, followed by 90 m on a bearing of 160° and 40 m on a bearing of 280° where it touches a buoy.

a By writing each of the displacement vectors in component form find how far the buoy is from the boat's starting position?

b Think about how you would you attempt this question if asked to do it using sine and cosine rules.

Factual How can you represent vectors?

Conceptual Which representation is easier to use to find a resultant?

Reflect How do you find a particle's displacement from its starting point if given its successive individual displacements?

The sum of two or more vectors is called the **resultant** vector.

Exercise 3J

1 For the resultant of each of the vector sums below, find the

i magnitude

ii direction (as an angle anti-clockwise from the direction of $\boldsymbol{i}$).

a $\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}$

b $(5\boldsymbol{i}+2\boldsymbol{j})+(-6\boldsymbol{i}-4\boldsymbol{j})$

c $2\begin{pmatrix}3\\2\end{pmatrix}-3\begin{pmatrix}-4\\-1\end{pmatrix}$

d $5(\boldsymbol{i}+2\boldsymbol{j})+3(\boldsymbol{i}-3\boldsymbol{j})$

2 The magnitude of a vector $\begin{pmatrix}a\\b\end{pmatrix}$ can be written as $\begin{vmatrix}a\\b\end{vmatrix}$.

a Verify that $\begin{vmatrix}48\\20\end{vmatrix}$ is equal to $4\begin{vmatrix}12\\5\end{vmatrix}$.

b By first taking out a factor and without using a GDC, find the magnitude of

i $\begin{pmatrix}18\\24\end{pmatrix}$ ii $\begin{pmatrix}-30\\40\end{pmatrix}$ iii $\begin{pmatrix}28\\-21\end{pmatrix}$

3 A designer needs to construct a line segment of a given length in a given direction. His software requires him to enter the line segment as a single column vector.

Find the column vector he needs to input in the following situations, using the fact that a vector which is in the same direction as a vector $\boldsymbol{u}$ can be written as $k\boldsymbol{u}$. $k > 0$

a A vector that is in the same direction as $\begin{pmatrix}3\\4\end{pmatrix}$ but with a magnitude of 8.

b A single vector which is equivalent to the resultant of a vector in the same direction as $\begin{pmatrix}0\\1\end{pmatrix}$ followed by the vector $\begin{pmatrix}5\\0\end{pmatrix}$ and has magnitude $\sqrt{74}$.

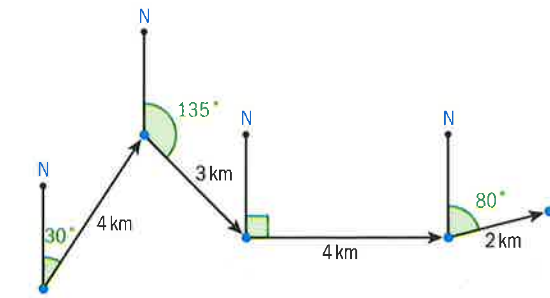
c A vector which is equivalent to the resultant of a vector in the same direction as $\begin{pmatrix}1\\1\end{pmatrix}$, followed by the vector $\begin{pmatrix}1\\-5\end{pmatrix}$ and has magnitude $\sqrt{50}$.

4 A man walking in a large field walks 200 m north-east and 175 m west.

a Write each of the displacements as a column vector.

b Hence find his final distance from his starting point.

5 A boat sails 4 km on a bearing of 030°, followed 3 km south-east, then 4 km due east and 2 km on a bearing of 080° as shown on the right. Determine its final distance from the starting point. Find also the bearing it would have to travel on to return directly to the starting point.



Vectors in 3 dimensions

The work done on vectors so far can be extended to three dimensions.

Example 7

Let A be the point with coordinates (1, 5, 8) and B be the point with coordinates (2, −1, 5) and let O be the origin.

a Show on a sketch the points A, B and O and the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$.

b Write down the position vector of A.

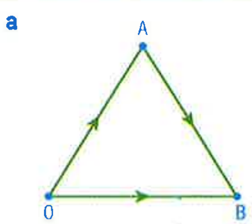
c Find the distance of A from the origin O.

d Find the vector $\overrightarrow{AB}$.

e Find $|\overrightarrow{AB}|$.

f Find the vector that is parallel to $\overrightarrow{AB}$ but with twice the magnitude.

a



It is rarely necessary to draw accurate diagrams. A two-dimensional sketch of a three-dimensional situation is normally sufficient.

b $\overrightarrow{OA}=\begin{pmatrix}1\\5\\8\end{pmatrix}$

This can also be written as $\boldsymbol{i}+5\boldsymbol{j}+8\boldsymbol{k}$ where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the base vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$.

c $|\overrightarrow{OA}|=\sqrt{1^2+5^2+8^2}\approx 9.49$

Pythagoras' theorem can be extended to give the magnitude of vectors in three dimensions.

d $\overrightarrow{AB}=\begin{pmatrix}2\\-1\\5\end{pmatrix}-\begin{pmatrix}1\\5\\8\end{pmatrix}=\begin{pmatrix}1\\-6\\-3\end{pmatrix}$

The triangle law of vector addition also holds and in particular $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$

Continued on next page

Functions

Geometry and trigonometry

e $|\overrightarrow{AB}| = \sqrt{1^2 + 6^2 + 3^2} = \sqrt{46} = 6.78$

f $2\overrightarrow{AB} = \begin{pmatrix} 2 \\ -12 \\ -6 \end{pmatrix}$

Note that this is equivalent to finding the length AB using the formula

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Reflect How do you find the magnitude of a vector in 3 dimensions?

Exercise 3K

1 Find the magnitude of each of the following vectors without using a calculator.

a $8i - 4j + k$ **b** $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

c $3\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$

d $5(2i - 3j + 6k)$

2 A small plane travels along the three vectors $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$ in succession.

State the vector it will have to travel along to return to its starting point

3 **a** If A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and if M is the midpoint of [AB] show that $\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

Three points P, Q and R have coordinates (1, 3, 6), (−1, 0 5) and (2, 4, −1).

b Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

c Hence or otherwise find the vector $\overrightarrow{PR}$.

The quadrilateral PQRS is a parallelogram.

d Find the coordinate of S.

e Find the midpoint of the vector $\overrightarrow{PR}$.

f Find the midpoint of the vector $\overrightarrow{QS}$.

g What do the answers for parts **e** and **f** tell you about the diagonals of a parallelogram?

International-mindedness

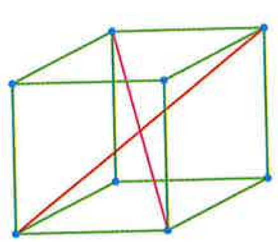
René Descartes used (*x*, *y*, *z*) to represent points in space in the 17th century.

In the 19th century, Arthur Cayley reasoned that we might go further than three values.

3.5 The scalar and vector product

How might you use trigonometry to find the angle between the diagonals of a cube, or the area of a triangular forest if given its coordinates in three dimensions?

This section will demonstrate vector techniques that allow both these questions to be easily solved.



Investigation 9

Let A and B be two points in a plane with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

1 Let A have coordinates (1, 2, 3) and B have coordinates (2, 5, −2).

a Find the vector $\overrightarrow{AB}$.

b Find the value of $|\boldsymbol{a}|^2$, $|\boldsymbol{b}|^2$ and AB^2.

c Hence calculate $\frac{1}{2}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - AB^2\right)$.

The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $\boldsymbol{a} \cdot \boldsymbol{b}$ and is calculated by finding the sum of the product of corresponding components. Hence in 2 dimensions for $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$ and in 3 dimensions for $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

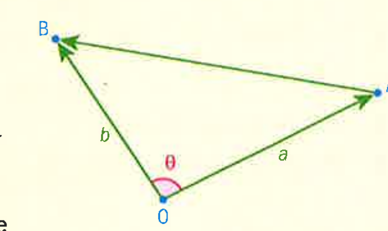
d Evaluate $\boldsymbol{a} \cdot \boldsymbol{b}$ for the values of A and B given above.

e Conjecture an alternative expression for $\boldsymbol{a} \cdot \boldsymbol{b}$.

2 For $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ find expressions for $|\boldsymbol{a}|^2$, $|\boldsymbol{b}|^2$ and AB^2 and hence prove your conjecture in part **1e** for vectors in 2 dimensions.

3 Use $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to show that your conjecture also works in 3 dimensions.

4 Consider the diagram.



a Use the cosine law to prove that $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = \frac{1}{2}\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - AB^2\right)$

b Hence write down an expression which gives $\cos\theta$ in terms of the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ and the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$.

5 Find the scalar product of the two vectors given and hence find the angle between them.

i $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **ii** $\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$

6 **a** Find the scalar product of $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 10 \\ 2 \end{pmatrix}$

b What does this tell you about the angle between the two vectors?

7 **Conceptual** What can the scalar product be used to find?

Reflect How can you calculate the scalar product of two vectors?

The **scalar** or dot product is written as $\boldsymbol{a}\cdot\boldsymbol{b}=a_1b_1+a_2b_2$ for 2 dimensional vectors and $\boldsymbol{a}\cdot\boldsymbol{b}=a_1b_1+a_2b_2+a_3b_3$ for 3 dimensional vectors.

In addition $\boldsymbol{a}\cdot\boldsymbol{b}= =|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

This allows us to find the angle between any two vectors, $\cos\theta=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

If $\boldsymbol{a}\cdot\boldsymbol{b}=0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.

TOK

Why do you think that we use this definition of scalar product? Are different proofs of the same theorem equally valid?

Example 8

Find p if $2p\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$ and $2\boldsymbol{i}+3\boldsymbol{j}+p\boldsymbol{k}$ are perpendicular.

$\boldsymbol{a}\cdot\boldsymbol{b}=4p+3-3p=0$ $p=-3$	When two vectors are perpendicular $\cos\theta=0$ and so $\boldsymbol{a}\cdot\boldsymbol{b}=0$.

Reflect How can you use the scalar product to show two vectors are perpendicular?

Exercise 3L

1 Calculate the angle between the following pairs of vectors.

a $\begin{pmatrix}2\\-1\\4\end{pmatrix}$ and $\begin{pmatrix}3\\1\\2\end{pmatrix}$ b $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\-1\end{pmatrix}$

c $2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ and $3\boldsymbol{i}+2\boldsymbol{j}$

d $2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ and $3\boldsymbol{i}+2\boldsymbol{j}-5\boldsymbol{k}$

e $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\6\end{pmatrix}$ f $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}-4\\0\\-2\end{pmatrix}$

2 A triangle has vertices at the points A(1, 2, 3), B(0, 2, 5) and C(1, 3, −2).

a State which two vectors you could use to find the angle at **i** A **ii** B.

b Find all the angles of the triangle.

c Find the length of the longest side.

3 Find p if the two vectors given are perpendicular.

a $\begin{pmatrix}2\\1\\p\end{pmatrix}$ and $\begin{pmatrix}-3p\\2\\-2\end{pmatrix}$ b $\begin{pmatrix}p\\-2\\4\end{pmatrix}$ and $\begin{pmatrix}p-1\\p\\-1\end{pmatrix}$

4 Triangle ABC has vertices A(1, 3, 2), B(2, 4, −1) and C(2, 3, k).

a Find the possible value of k if the triangle has a right angle at C.

b For these values of k find the CA and CB and hence the area of the triangles.

5 Find the acute angle between the diagonals of a cube.

The vector product

A second way of multiplying vectors is to form the **vector product**. As its name suggests the result of this multiplication will be a third vector.

In three dimensions the vector product can be calculated as follows:

For $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ the vector product $\boldsymbol{a}\times\boldsymbol{b}$ is given by

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}a_2b_3-b_2a_3\\a_3b_1-b_3a_1\\a_1b_2-b_1a_2\end{pmatrix}$$

International-mindedness

Belgian/Dutch mathematician Simon Stevin used vectors in his theoretical work on falling bodies and his treatise "Principles of the art of weighing" in the 16th century.

Investigation 10

This investigation leads to a key property of the vector product.

1 Use the formula given above to find the vector product of $\boldsymbol{a}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$

2 Find the scalar product of your answer to **1** with $\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\1\\-2\end{pmatrix}$

Write down what you notice and what this means about $\boldsymbol{a}\times\boldsymbol{b}$.

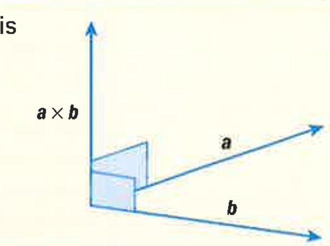
3 Verify your result by repeating **1** and **2** with two vectors of your choosing.

4 Use the formula for $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}a_2b_3-b_2a_3\\a_3b_1-b_3a_1\\a_1b_2-b_1a_2\end{pmatrix}$ to find the result of $(\boldsymbol{a}\times\boldsymbol{b})\bullet\boldsymbol{a}$.

5 **Conceptual** What can you say about the direction of the vector product of two vectors?

The vector product of two non-parallel vectors is perpendicular to both vectors.

Geometrically two vectors that are not in the same line will define a plane and the vector product will be a vector perpendicular to that plane.

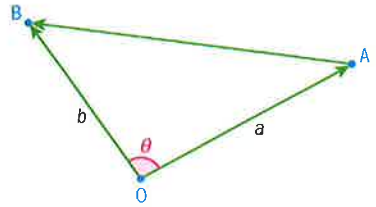


Functions

Geometry and trigonometry

The area of a triangle

Consider the triangle shown.



We know from chapter 1 that the area of a triangle can be found from the formula

$$\text{Area} = \frac{1}{2}ab\sin C$$

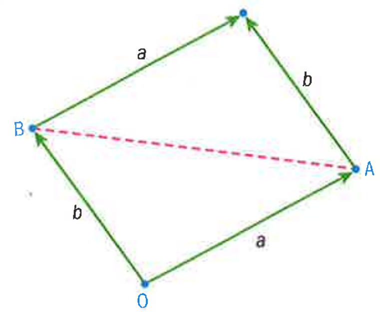
For the triangle above this could be written as $\frac{1}{2}|a||b|\sin\theta$.

International-mindedness

Area can be measured in square units like m^2, but also "packets" like hectares or acres.

Which countries use bigha, mou, feddan rai, tsubo?

To find the area of the triangle a value for θ could be calculated using the scalar product. It is possible though to find the area directly using the **vector product**.



The magnitude of the vector obtained when calculating the vector product is $|a \times b| = |a||b|\sin\theta$

Hence the area of the triangle with vectors a and b as two of its sides will have area $\frac{1}{2}|a \times b|$.

The area of the parallelogram with vectors a and b as two of its sides will have area $|a \times b|$.

The vector product can also be found for two dimensional vectors by taking the third component as equal to 0.

Example 9

a Find the vector product of the vectors $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}4\\1\\-2\end{pmatrix}$. Show it is perpendicular to both of these vectors.

b Find the area of the triangle which has these vectors as two of its sides.

a $\begin{pmatrix}2\\0\\1\end{pmatrix} \times \begin{pmatrix}4\\1\\-2\end{pmatrix} = \begin{pmatrix}-1\\8\\2\end{pmatrix}$

The calculations can be done by excluding the row you are trying to find and with the remaining four numbers finding the product of top left and bottom right and subtracting the product of top right and bottom left.

For the first entry you have $0 \times -2 - 1 \times 1 = -1$

For the second row you repeat this process and make the answer negative or multiply and subtract in the opposite order. $-(2 \times -2 - 4 \times 1) = 8$ or $4 \times 1 - 2 \times -2 = 8$

Final entry is $2 \times 1 - 4 \times 0 = 2$

$\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\8\\2\end{pmatrix} = -2 + 0 + 2 = 0$

$\begin{pmatrix}4\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}-1\\8\\2\end{pmatrix} = -4 + 8 - 4 = 0$

Hence perpendicular.

To show two vectors are perpendicular you need to demonstrate that their scalar product is 0.

b $\text{Area} = \frac{1}{2}|a \times b|$

$= \frac{1}{2}\sqrt{1^2 + 8^2 + 2^2} = 4.15$

Reflect How do you calculate the vector product of two vectors?

How do you find the area of a triangle defined by two vectors?

What does the magnitude of a vector product represent?

Example 10

Find the area of the parallelogram with vertices A (1, 1, 0) B (2, 3, 1), C (4, 2, 4) and D (3, 0, 3).

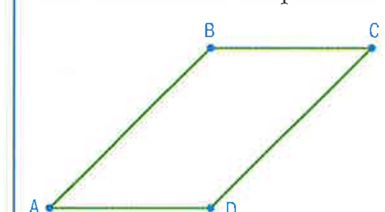
$\overrightarrow{AB} = \begin{pmatrix}1\\2\\1\end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix}5\\-1\\-5\end{pmatrix}$

$\text{Area} = \sqrt{51} = 7.14$

You can take any two of the non-parallel sides to find the area. A sketch makes it clear which ones are possible.



The area of the parallelogram is given by the formula, area $= |a \times b|$.

Reflect How do you find the area of a parallelogram defined by two vectors?

Functions

Geometry and trigonometry

Exercise 3M

1 **a** Find the vector product of the following vectors.

$a = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

b Verify that the vector product is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

2 Find the vector products of the following pairs of vectors.

a $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

c $2i + j - k$ and $3i + 2j$

d $2i - j - 2k$ and $3i + 2j - 5k$

3 Two sides of a triangle ABC are formed by the vectors $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

a Find the vector forming the third side.

b Find the area of the triangle.

4 A triangle has vertices at the points A(1, 2, 3), B(0, 2, 5) and C(1, 3, −2). Find the area of triangle ABC.

5 A parallelogram ABCD has vertices at A(1, 2, −3), B(1, 2, 5) and C(4, 3, 5).

a Find the coordinates of D.

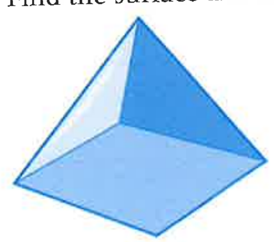
b Find the area of the parallelogram.

c Verify that ABCD is a rectangle.

6 A surveyor is measuring the area of a rectangular roof. His instruments give the coordinates of the corners of the roof as (−2, 4, 2), (−3, 1, 3), (3, −1, 3) and (4, 2, 2) where the coordinates are in metres from a fixed origin. Find the area of the roof.

7 A crystal is in the shape of a square-based pyramid as shown. All the sides are made up of isosceles triangles. The coordinates of the vertex are (1, 3, 2) and the coordinates of two adjacent vertices on the base are (2, 3, −1) and (−1, 3, 1).

Find the surface area of the crystal.



3.6 Vector equations of lines

An air-traffic controller sees that the paths of two planes are about to cross. He knows the regulations state that when the planes cross the vertical height between them must be greater than 300 m and if it is not their distance apart has to be always greater than 10 km. He knows the position and velocities of each aircraft. Does he need to instruct one of them to change their direction?

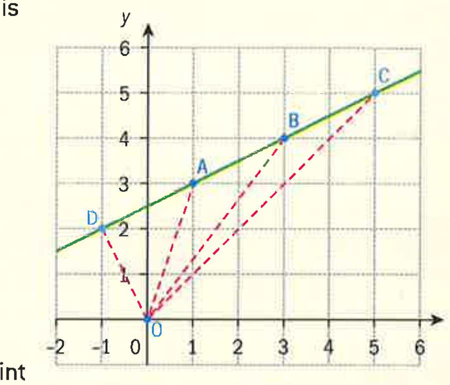
This section will consider what we can deduce about the future paths of objects travelling with constant velocities.

TOK

Do you think that there are times when analytical reasoning is easier to use than sense perception when working in three dimensions?

Investigation 11

Consider the line which passes through a point A(1, 3) and is parallel to the vector $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



1 Explain why the position vector of C can be written as

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2 Write down similar expressions in the form $\boldsymbol{a} + t\boldsymbol{b}$ for the position vectors of A, B and D.

3 What can you deduce about the position vector of any point on the line?

4 Why will this not be true for any point not on the line?

5 Show that A, B, C and D can also be written in the form $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, stating the value of s in each case.

6 **Factual** What is the vector equation of a line in symbols?

7 **Conceptual** How do we express the vector equation of a line in word?

The vector equation of a line is normally written as $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ where $\boldsymbol{r}$ is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in two dimensions, and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in three dimensions, $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{b}$ is a vector parallel to the line – a **direction vector**.

The vector equation of a line is not unique as any point on the line and any vector parallel to the direction of the line could be used.

Every point on the line will have a particular value of the parameter t, where t is a measure of how far the point is away from A as a multiple of the vector $\boldsymbol{b}$.

TOK

Why might it be argued that vector equations are superior to Cartesian equations?

Different forms for the equation of a line

A line passing through A(1, 2, 1) in the direction of the vector $\begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$ can be given in **vector form** as $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$ or

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+t \\ 2+8t \\ 1-2t \end{pmatrix}$$

Functions

Geometry and trigonometry

This equation can also be split up to give the **parametric form of the equation of a line** as three separate equations:

$x = 1 + t$

$y = 2 + 8t$

$z = 1 - 2t$

or even as a general set of coordinates

$(1 + t,\ 2 + 8t,\ 1 - 2t)$

Reflect How do you convert the vector equation of a line to a parametric equation?

Example 11

a Find the equation of the line which passes through the points A(1, 0, 2) and B(2, 3, −1).

b Determine whether or not the point (3, 6, −4) is on the line.

c Find the angle between the line through A and B and the line $\boldsymbol{r} = \begin{pmatrix}3\\6\\-4\end{pmatrix} + t\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

a Direction of line is $\overrightarrow{AB} = \begin{pmatrix}1\\3\\-3\end{pmatrix}$ $\boldsymbol{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\3\\-3\end{pmatrix}$	The vector equation of a line is not unique. Any vector in the same direction as the line can be used and any point. The following would also be correct. $\boldsymbol{r} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + t\begin{pmatrix}-1\\-3\\3\end{pmatrix}$
b $\begin{pmatrix}3\\6\\-4\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\3\\-3\end{pmatrix}$ $3 = 1 + t \Rightarrow t = 2$ $\begin{pmatrix}1\\0\\2\end{pmatrix} + 2\begin{pmatrix}1\\3\\-3\end{pmatrix} = \begin{pmatrix}3\\6\\-4\end{pmatrix}$ Therefore (3, 6, −4) is on the line.	If a point is on the line there must be a value for t that gives all three coordinates. An alternative method is to solve the equation for each coordinate and see if the same value of t is obtained each time.
c $\begin{pmatrix}1\\3\\-3\end{pmatrix} \cdot \begin{pmatrix}-1\\-2\\1\end{pmatrix} = -1 - 6 - 3 = -10$ $\cos\theta = \frac{-10}{\sqrt{19}\sqrt{6}} \Rightarrow \theta = 159°$ or $\theta = 21°$	The lines clearly intersect at (3, 6, −4), so it is meaningful to talk about the angle between the lines. The angle between the lines will be the angle between the direction vectors. The angle between two lines can either be given as an obtuse or an acute angle unless a specific direction is specified.

Reflect How do you find the vector equation of a line if you are given its direction and the coordinates of a point it passes through?

How do you test whether or not a point lies on a given line?

The angle between two intersecting lines is the angle between their **direction vectors.**

Exercise 3N

1 Find a vector equation of the line passing through the following two points

a (2, 4) and (3, 1)

b (2, 1, −1) and (4, 2, 0)

2 a The point $(a, -2, b)$ lies on the line

$$\boldsymbol{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\8\\-2\end{pmatrix}.$$

Find a and b.

b Find the coordinates of the point on the line with x-coordinate equal to 0.

3 a Find the value of s and p if the point $(2, s, p-1)$ lies on the line

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\3\\1\end{pmatrix}$$

b i Verify the point A (1, −3, 8) lies on the line $\boldsymbol{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and state the value of t for this point.

The point B is also on the line and has parameter $t = 5$.

ii Write down the vector $\overrightarrow{AB}$.

4 a Verify that the point (2, 1, 3) lies on both of the lines

$$\boldsymbol{r} = \begin{pmatrix}4\\-5\\1\end{pmatrix} + t\begin{pmatrix}-1\\3\\1\end{pmatrix} \text{ and } \boldsymbol{r} = \begin{pmatrix}4\\2\\0\end{pmatrix} + s\begin{pmatrix}2\\1\\-3\end{pmatrix}$$

b Find the acute angle between the two lines.

5 Two lines l_1 and l_2 have equations

$$\boldsymbol{r} = \begin{pmatrix}3\\5\\2\end{pmatrix} + t\begin{pmatrix}-1\\2\\1\end{pmatrix} \text{ and } \boldsymbol{r} = \begin{pmatrix}3\\5\\2\end{pmatrix} + s\begin{pmatrix}2\\-4\\-2\end{pmatrix}$$

a Explain why l_1 and l_2 are parallel.

b Verify that (1, 9, 4) lies on both lines.

c Explain what this tells you about the two lines.

6 The points A and B have coordinates (1,−5,6) and (5, −3, 11) respectively.

The line l_1 has equation $\boldsymbol{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + s\begin{pmatrix}1\\3\\-2\end{pmatrix}$

a Show A lies on l_1.

b Show $\overrightarrow{AB}$ is perpendicular to l_1.

7 The line l_1 has equation $\boldsymbol{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\3\\2\end{pmatrix}$

a Write down the coordinates of the point P which lies on the line and has parameter t.

b The point A has coordinates (1, 2, −2). Find the vector $\overrightarrow{AP}$.

c Find the value of t for which the vector $\overrightarrow{AP}$ is perpendicular to l_1.

d Find the point on l_1 that is closest to A.

e Hence find the shortest distance from A to the line l_1.

Reflect Which vectors are used to find the angle between two intersecting lines?

How do you test to see if two parallel lines are coincident?

Velocity equations

The **velocity** of an object is a vector quantity which needs both a magnitude and a direction to define it.

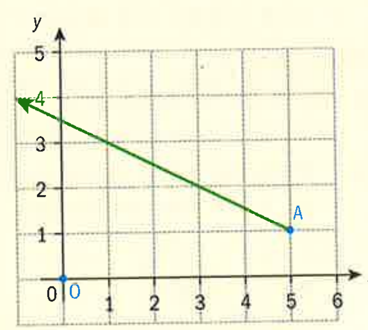
A velocity of $10\,\text{ms}^{-1}$ north is different to a velocity of $10\,\text{ms}^{-1}$ east.

Velocity represents the change in displacement of the object during one unit of time.

An object's **speed** is a measure of **distance** covered in one unit of time, so speed is the magnitude of the velocity vector, $|\boldsymbol{v}|$.

Investigation 12

Consider a boat which is initially at the point A with coordinates (5, 1) and is moving with a velocity of $(-2\boldsymbol{i}+\boldsymbol{j})$ km h^{-1}, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors east and north respectively, measured from a port O.



1 a Write down the position vector of the boat after it has been travelling for one hour.

b Write down the position vector of the boat after it has been travelling for 2 hours.

c Write down the position vector of the boat after it has been travelling for t hours.

2 a What can be said about the x-coordinate of the displacement vector when the boat is directly north of O?

b Find the value of t when the boat is directly north of O.

3 a What can be said about the x- and y-coordinate of the displacement vector when the boat is north-west of O (on a bearing of 315° from O)?

a Find the value of t at which the boat is north-west of O.

4 **Conceptual** How can the motion of a particle travelling with constant velocity be expressed and what do the variables and parameter stand for in this representation?

Reflect What is the displacement equation for an object travelling with velocity $\boldsymbol{v}$ if it is initially at a point with position vector $\boldsymbol{a}$?

HINT

Initially means at the point when $t = 0$.

The position of a particle moving with velocity $\boldsymbol{v}$ and whose position vector at $t = 0$ is $\boldsymbol{r}_0$ can be given by the equation $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}t$.

Example 12

A ship A has position vector $3\boldsymbol{i} + 4\boldsymbol{j}$ at 13:00 hours where $\boldsymbol{i}$ and $\boldsymbol{j}$ represent 1 km east and north of a harbour, respectively. Two hours later the position of A is $2\boldsymbol{i} + 8\boldsymbol{j}$.

a Find the velocity of A.

b Find the speed of A.

c Write down an expression for the position of A t hours after 13:00 as a single vector.

d Find the time at which A is directly north of the harbour.

a $\begin{pmatrix}2\\8\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} + 2\boldsymbol{v}$ $2\boldsymbol{v} = \begin{pmatrix}-1\\4\end{pmatrix}$ $\Rightarrow \boldsymbol{v} = \begin{pmatrix}-0.5\\2\end{pmatrix}$ kmh^{-1}	The velocity can be found by substituting the given values into $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}t$
b Speed $= \sqrt{2^2 + 0.5^2}$ $= 2.06$ kmh^{-1}	Speed is the magnitude of the velocity vector.
c $\boldsymbol{r} = \begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}-0.5\\2\end{pmatrix}t$ $\boldsymbol{r} = \begin{pmatrix}3-0.5t\\4+2t\end{pmatrix}$	Writing an expression for displacement as a single vector can make subsequent calculations easier.
d $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3-0.5t\\4+2t\end{pmatrix}$ $x = 0 \Rightarrow 0.5t = 3 \Rightarrow t = 6$ At $t = 6$ $y = 16$ km Hence A is 16 km north of the harbour.	When the boat is north of the harbour its x-coordinate will be zero.

Unit vectors

Unit vectors are vectors with a length of 1 unit. The unit vector in the same direction as $\boldsymbol{b}$ is written as $\dfrac{\boldsymbol{b}}{|\boldsymbol{b}|}$

If a particle has speed v ms^{-1} in the direction of $\boldsymbol{b}$ its velocity will be $v\dfrac{\boldsymbol{b}}{|\boldsymbol{b}|}$

Changing the magnitude of a vector while keeping its direction constant is referred to as **rescaling** the vector. Rescaling to form a vector with a magnitude of one is referred to as **normalising** the vector.

Example 13

A boat is travelling with a speed of $7.5\,\text{ms}^{-1}$ in a direction parallel to the vector $3\boldsymbol{i} + 4\boldsymbol{j}$.

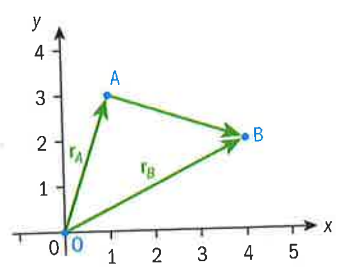
Write the velocity vector of the boat in component form.

Magnitude of $3\boldsymbol{i}+4\boldsymbol{j}$ is 5 Velocity is $\frac{7.5}{5}\begin{pmatrix}3\\4\end{pmatrix} = 1.5\begin{pmatrix}3\\4\end{pmatrix}$ $= \begin{pmatrix}4.5\\6\end{pmatrix}$	The magnitude of the vector is found using Pythagoras. A unit vector in the same direction would therefore be $\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$ and a vector with magnitude 7.5 will be 7.5 times this vector.

Relative position

If A has a position vector $\boldsymbol{r}_A$ and B has position vector $\boldsymbol{r}_B$ then the **relative position** of B from A is the vector $\overrightarrow{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A$

This is useful for finding when one object is in a particular position relative to the other, for example when one is north of the other, and also for finding the distance between the objects at a given time.



Example 14

At time $t = 0$ a model boat A is at (2, 5) and is travelling with a speed of $4\,\text{ms}^{-1}$ in the direction of $\begin{pmatrix}3\\-4\end{pmatrix}$. The x component is the displacement due east from an origin and the y component due north. All distances are in metres.

a Find an expression for the position vector of A ($\boldsymbol{r}_A$) at time t.

A boat B has position vector given by $\boldsymbol{r}_B = \begin{pmatrix}3\\-2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}t$

b Show that the two boats do not collide.

c Find the shortest distance between the two boats and the value of t at which this occurs.

d Find the value of t at which B is due east of A.

a Magnitude of $\begin{pmatrix}3\\-4\end{pmatrix}$ is $\sqrt{3^2+4^2} = 5$ Velocity of A is $\frac{4}{5}\begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}2.4\\-3.2\end{pmatrix}$ Displacement of A is given by $\boldsymbol{r}_A = \begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}2.4\\-3.2\end{pmatrix}t$	**a** Dividing a vector by its magnitude creates a vector in the same direction with a magnitude of 1 (a unit vector). Multiplying this by 4 will give a vector of magnitude (speed) equal to 4.
b Equating x-coordinates $2 + 2.4t = 3 + 2t$ $t = \frac{1}{0.4} = 2.5$	**b** If the boats collide there will be a value of t at which both the x- and y-coordinates are equal.
When $t = 2.5$ the y-coordinate of A is -3.2 and the y-coordinate of B is 0.5. So the two boats do not collide.	An alternative method is to equate the y-coordinates and see if the solution gives the same value of t.
c The distance between the two boats is $\lvert\overrightarrow{AB}\rvert$ $\boldsymbol{r}_A = \begin{pmatrix}2+2.4t\\5-3.2t\end{pmatrix}$, $\boldsymbol{r}_B = \begin{pmatrix}3+2t\\-2+t\end{pmatrix}$	**c** Writing the two displacement vectors as single vectors makes the subtraction easier.
$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix}1-0.4t\\-7+4.2t\end{pmatrix}$ $\lvert\boldsymbol{r}_B - \boldsymbol{r}_A\rvert = \sqrt{(1-0.4t)^2 + (-7+4.2t)^2}$	This is a variation of $\lvert\overrightarrow{AB}\rvert = \boldsymbol{b} - \boldsymbol{a}$ met in section 3.4.
Minimum distance is 0.332 m when $t = 1.67$ s	The minimum of the function is found using a GDC.
d B is due east of A when $-7 + 4.2t = 0$ Hence $t = 0.6$	**d** Using $\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix}1-0.4t\\-7+4.2t\end{pmatrix}$ An alternative approach is just to equate the two y components, so $5 - 3.2t = -2 + t$ without finding the vector for relative position.

Functions

Geometry and trigonometry

Reflect How do you find the relative position of point A from point B if given the position of each?

How might you find the least distance between two objects each moving with constant velocity?

How might you find the time when the bearing of one object to another is in a given direction?

Exercise 30

1 A particle has position vector $\boldsymbol{a}$ at $t = 0$ and position vector $\boldsymbol{b}$ at time t.

Find **i** the velocity **ii** the speed of the particle if

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 4\boldsymbol{i} + 5\boldsymbol{j}$, $t = 2$

b $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $t = 4$

c $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$, $t = 2$

d $\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$, $t = 4$

2 Write the following velocities as column vectors taking the base vectors as due east and north.

a $10\,\text{kmh}^{-1}$ due west

b $7.5\,\text{kmh}^{-1}$ in the direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

c $18\,\text{kmh}^{-1}$ in the direction $\begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}$

d $5\,\text{kmh}^{-1}$ south-west

e $15\,\text{kmh}^{-1}$ on a bearing of 040°

f $12\,\text{kmh}^{-1}$ on a bearing of 120°.

3 A buoy is set as the origin of a coordinate system. At 13.00 a boat is 20 m east and 30 m north of a buoy and has position vector $\begin{pmatrix} 20 \\ 30 \end{pmatrix}$.

a Find the distance of the boat from the buoy at 13:00.

The boat is moving with velocity $\begin{pmatrix} -3 \\ -5 \end{pmatrix}\text{ms}^{-1}$.

b Find the position of the boat t seconds after 13:00.

c Hence find the shortest distance from the boat to the buoy.

4 A particle has position vector $3\boldsymbol{i} + \boldsymbol{j}$ when $t = 0$ and is moving with a velocity of $10\,\text{ms}^{-1}$ in the direction of $3\boldsymbol{i} - 4\boldsymbol{j}$. Find

a the position vector at time t

b its position when $t = 4$

c its distance from the origin at $t = 4$

d the distance travelled in the first four seconds.

5 A particle's displacement from an origin O is given by $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}t$.

a Find the particle's position at $t = 3$.

At $t = 3$ the particle changes its velocity to $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

b Find an equation for the particle's displacement from O at time

i t $(t \geq 3)$

ii t', the time from the change of velocity.

c Find **i** the displacement **ii** the distance of the particle from O when $t = 5$.

6 Let $\boldsymbol{i}$ be the unit vector directed east and $\boldsymbol{j}$ the unit vector directed north and let a harbour H be situated at the origin.

At 10:00 ship A is at the point with position vector $-5\boldsymbol{i} + 10\boldsymbol{j}$ km relative to H and is travelling with velocity $2\boldsymbol{i} + 2\boldsymbol{j}$ km/h. Ship B is at the point $3\boldsymbol{i} + 4\boldsymbol{j}$ km and has velocity $-2\boldsymbol{i} + 5\boldsymbol{j}$ km/h.

a Show the two ships would collide if they maintained these velocities.

In order to avoid a collision A changes its velocity to $(\boldsymbol{i} + 2\boldsymbol{j})$km/h.

b Find the vector $\overrightarrow{AB}$ at time t hours after 10:00.

c Find the distances between the two ships when A is north of B and give the time at which this occurs.

d Find the shortest distance they are apart and the time at which this occurs.

7 A hot air balloon is flying over level ground with a speed of $3.5\,\text{ms}^{-1}$ in the direction $\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ where the first component is due east, the second due north and the third is perpendicular to the ground.

a **i** Find the velocity of the balloon.

ii Write down the speed at which it is ascending.

The balloon passes over a tracking station at time $t = 0$ and its position relative to the tracking station at this time is $\begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix}$

b Find the balloon's displacement relative to the tracking station one minute later.

At this point the balloon begins to descend at $0.6\,\text{ms}^{-1}$ while still travelling on the same bearing.

c Write down the displacement of the balloon from the tracking centre at time t' where t' is the time from the point when the descent begins.

d Find the time it takes for the balloon to reach the ground from the moment it begins its descent.

e Find the distance of the balloon from the tracking station when it reaches the ground.

8 A ship, S, is travelling south with a speed of $9\,\text{kmh}^{-1}$.

a Write down its velocity as a column vector in which the two components are due east and north.

At 10:00, S has displacement $\begin{pmatrix} 20 \\ 15 \end{pmatrix}$ relative to a lighthouse, O. At the same time a speedboat, B, is traveling with velocity $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ and its displacement from O is $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$

b Find the distance between S and B at 10:00.

c Find the relative position of S from B t hours after 10:00.

At the point where S is south-east of B the boat changes direction while maintaining the same speed.

d Find the time at which B changes direction and its displacement from O at which this occurs.

e Write down the displacement of S from O at this point.

f Find the bearing at which B needs to travel in order to intercept S.

Chapter summary

- Given the two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:
  - the midpoint of [AB] is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
  - the length $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- The gradient (m) of the line segment joining (x_1, y_1) and (x_2, y_2) is given by: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Continued on next page

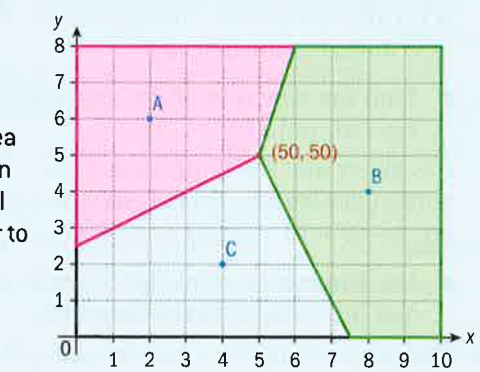
- For two points A and B (AB) should be read as "the line containing the points A and B".
- The **point-gradient** form for the equation of a line with gradient equal to m, which passes through the point (x_1, y_1) is $y - y_1 = m(x - y_1)$.
- The **general** form of the equation of a straight line is written as $ax + by + c = 0$ where a, b and c are all integers.
- $y = mx + c_1$ is parallel to $y = mx + c_2$
- If $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular (and neither m_1 or m_2 are equal to 0) then $m_1m_2 = -1$ or $m_2 = -\frac{1}{m_1}$ (the negative reciprocal of m_1).
- The boundaries of the **cells** in a Voronoi diagram are formed by perpendicular bisectors of the line segments joining the sites.
- **Nearest neighbour interpolation:** Each site is assigned a numerical value and the value of all points in each site's cell is assumed to equal that value.
- Within a Voronoi diagram the solution to the toxic waste dump problem will be at an intersection of cell boundaries or on the boundary of the diagram.
- $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$.
- From the above you can derive the **triangle law of vector addition.**
  $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
- A consequence of the law is that the sum of two of more displacement vectors is always equal to the final displacement and is independent of the route taken.
- The sum of two or more vectors is called **resultant** vector.
- The **scalar** or dot product is written as $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$ for 2 dimensional vectors and $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$ for 3 dimensional vectors.
- In addition $\boldsymbol{a} \cdot \boldsymbol{b} = = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$.
- This allows us to find the angle between any two vectors, $\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- The vector product of two vectors is perpendicular to both vectors.
- Geometrically two vectors that are not in the same line will define a plane and the vector product will be a vector perpendicular to that plane.
- The vector product can also be found for two dimensional vectors by taking the third component as equal to 0.
- The vector equation of a line is normally written as $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ where $\boldsymbol{r}$ is the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in two dimensions, and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in three dimensions where $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{b}$ is a vector parallel to the line – a **direction vector.**
- The vector equation of a line is not unique as any point and any direction vector could be used. Every point on the line will have a particular value of the parameter t, where t is a measure of how far the point is away from A as a multiple of the vector $\boldsymbol{b}$.

- The angle between two intersecting lines is the angle between their **direction vectors.**
- The **velocity** of an object is a vector quantity which needs both a magnitude and a direction to define it.
- An object's **speed** is a measure of **distance** covered in one unit of time, so speed is the magnitude of the velocity vector, $|\boldsymbol{v}|$.
- The position of a particle moving with velocity $\boldsymbol{v}$ and whose displacement at $t = 0$ is $\boldsymbol{r}_0$ can be given by the equation $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}t$.
- **Unit vectors** are vectors with a length of 1 unit. The unit vector in the same direction as $\boldsymbol{b}$ is written as $\frac{\boldsymbol{b}}{|\boldsymbol{b}|}$
- If a particle has speed v ms^{-1} in the direction of $\boldsymbol{b}$ its velocity will be $v\frac{\boldsymbol{b}}{|\boldsymbol{b}|}$
- Changing the magnitude of a vector while keeping its direction constant is referred to as **rescaling** the vector. Rescaling to form a vector with a magnitude of one is referred to as **normalising** the vector.
- If A has a position vector $\boldsymbol{r}_A$ and B has position vector $\boldsymbol{r}_B$ then the relative position of B from A is the vector $\overrightarrow{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A$

Developing inquiry skills

Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

Island A feels it is not getting a fair allocation of the area. An alternative is proposed whereby instead of the previous area it can have exclusive fishing rights for all of the region within 35 km of the centre of the island, including the international waters, except where this would overlap with an area closer to one of the other islands.



Vector methods will be used to find the area of this region.

a Use the diagram to write down the **vector** equations of the three perpendicular bisectors from the opening problem.

Let P and Q be the points on the perpendicular bisectors of [AB] and [AC] which are 35 km from A and on the edges of the Voronoi diagram.

b Find the vectors $\overrightarrow{AP}$ and $\overrightarrow{AQ}$.

c Show the new region in which A has exclusive fishing rights on the diagram.

d Use the scalar product to find $Q\hat{A}P$.

e Find the area of the region in which island A has exclusive fishing rights.

Chapter review

Click here for a mixed review exercise

1 Relative to a radar station on the ground two aircraft, A and B have positions (1.2, 8.5, 3.1) and (−0.2, 9.4, 2.6) respectively, where the units of measurement are kilometres.

a Find the distance the aircraft are from each other.

b Determine which of the aircraft is farthest from the radar station.

2 a Find the equation of the line passing through the two points (1, 5) and (6, 3).

b Find the coordinates of the point where this line meets the line with equation $5x - 2y + 5 = 0$.

3 A, B and C have coordinates (2, 4), (2, 6) and (8, 6) respectively.

a Write down the equations for the perpendicular bisectors of [AB] and [BC].

b Find the equation of the perpendicular bisector of [BC].

c Construct a Voronoi diagram with A, B and C as the three sites.

A further site D with coordinates (4, 3) is now added to the diagram.

d Find the equation of the perpendicular bisector of [AD].

It is given that the equations of the perpendicular bisectors of [BD] and [CD] are $4x - 6y + 15 = 0$ and $4x + 3y - 3 = 0$.

e Use the incremental algorithm to add site D to your Voronoi diagram.

4 The positions of three hamburger outlets in a town can be given on a Cartesian coordinate system as A(2, 4), B(6, 5) and C(3, 2) where the units are in kilometres.

a Find the equations of the perpendicular bisectors of [BC] and [AC].

A rival firm wishes to set up its own outlet in the town but as far as possible from the outlets already in place.

b Find the coordinates of the position where the new outlet should be built to satisfy this requirement.

c Determine its distance from the other three outlets.

5 The displacement of a toy boat at time t minutes is given by the equation

$\boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + t\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ where distances are measured in metres.

a Write down the position of the boat when $t = 0$ and $t = 2$.

b Find the speed of the boat.

c Find an expression for the distance of the boat from the origin at time t.

d Hence find the minimum distance of the boat from the origin and the time at which it occurs.

6 Let $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ p \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ q-1 \\ 1 \end{pmatrix}$

a If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular find a relationship between p and q.

b Find the values of p and q for which $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

c Find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ when $p = 3$ and $q = 2$.

7 A tetrahedron is resting on a flat surface. The coordinates of the vertices are (4.5, 1.5, 0), (2.0, 3.5, 0), (3.5, 3.0, 0) and (2.5, 2.5, 2.0).

Find the volume of the tetrahedron.

8 The line l_1 has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

a The point A(−8, 34, n) lies on l_1 find the value of n

The line $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ u \end{pmatrix} + s\begin{pmatrix} -1 \\ p \\ q \end{pmatrix}$ intersects l_1 at A and is perpendicular to l_1.

b Find the values of p, q and u.

9 A plane A is flying north-east at 750 kmh^{-1} and is climbing at a rate of 2 kmh^{-1}. At 15:00 it is directly over a tracking station and is at a height of 8.2 km.

a Find an expression for the displacement of the plane from the tracking station at time t hours after 15:00.

At 15:30 a second plane B is 13.1 km directly above the tracking station flying on a bearing of 030° at 800 kmh^{-1} and descending at a rate of 1 kmh^{-1}.

b Find an expression for the displacement of B from the tracking station t hours after 15:00.

c Find the distance the two planes are apart when they have the same height.

Exam-style questions

10 P1: The coordinates of point P are (−3, 8) and the coordinates of point Q are (5, 3). Point M is the midpoint of PQ.

a Find the coordinates of M. (2 marks)

L_1 is the line through P and Q.

b Find the gradient of L_1. (2 marks)

The line L_2 is perpendicular to L_1 and passes through M.

c i Write down the gradient of L_2.

ii Find the equation of L_2. Give your answer in the form $y = mx + c$. (3 marks)

11 P2: The line, L, has equation $y = 3x - 5$. For the lines given below, state with reasons if they are parallel to L, perpendicular to L, or neither. (6 marks)

i $y = \frac{1}{3}x - 7$

ii $-6x + 2y + 8 = 0$

iii $y - 5 = 2(x - 7)$

iv $y = -\frac{1}{3}x + 4$

v $x + 3y + 9 = 0$

12 P1: A ski resort is designing two new ski lifts. One lift connects station B (at the base of a mountain) and station P at the top of a ski run. The other lift connects station P with station Q, which is at the top of another ski run.

The three stations are placed on a three-dimensional coordinate system (measured in metres). The coordinates of each station are $B = (0, 0, 0)$, $P = (500, 400, 300)$, $Q = (900, 600, 700)$.

A skier wishes to reach the top of the run located at P from the base of the mountain.

a Determine the distance covered by the skier on the ski lift from the base of the mountain to P. (2 marks)

In order for a skier to reach the top of the ski run at Q they must take the lift from the base of the mountain to P, and then take a separate lift from P to Q.

b Determine the total distance covered by a skier on the ski lifts from the base of the mountain to Q. (3 marks)

13 P1: A triangle ABC is defined by the position vectors $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$.

a Calculate $\overrightarrow{AB} \times \overrightarrow{AC}$. (3 marks)

b Hence find the area of triangle ABC. (2 marks)

14 P1: Lines L_1 and L_2 are given by the equations

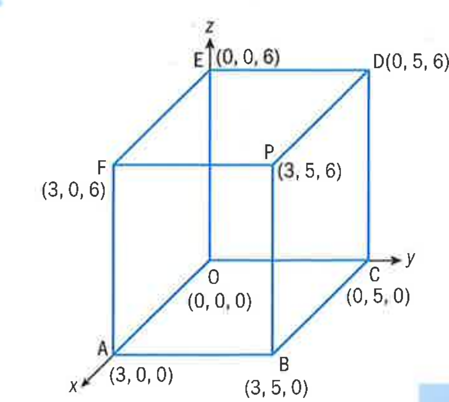
L_1: $ax - 3y = 9$ $\quad$ L_2: $y = \frac{2}{3}x + 4$

The two lines are perpendicular.

a Find the value of a. (3 marks)

b Hence, determine the coordinates of the intersection point of the lines. (2 marks)

15 P2: The cuboid ABCOFPDE has vertices with coordinates shown in the diagram.



a Find the surface area of the cuboid. (2 marks)

b Find the length of the diagonal BE. (2 marks)

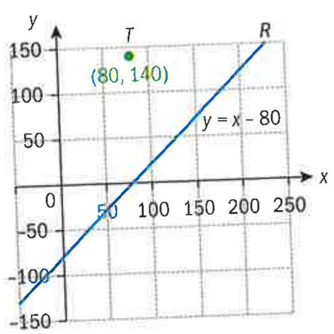
Diagonals AD and BE intersect at the point M.

c **i** Find the coordinates of M.

ii Find angle $A\widehat{M}B$, in degrees, using a vector method. You must show all your working. (7 marks)

16 P1: A new airport, *S*, is to be constructed at some point along a straight road, *R*, such that its distance from a nearby town, *T*, is a minimum.

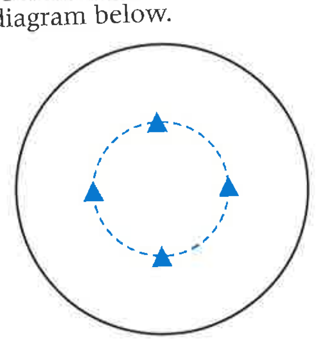
The town, *T*, and the road, *R*, are placed on a plane where town, *T*, has coordinates (80, 140) and the road, *R*, has equation $y = x - 80$. All coordinates are given in kilometres.



a Determine the coordinates of *S*, the new airport. (6 marks)

b Find the distance between *T* and the new airport. (2 marks)

17 P1: Four mathematicians live on the bottom floor of a circular tower of radius 10 m. They sit 5 m from the centre of the circle equally spaced around it as shown in the diagram below.



Each mathematician scatters papers, with equations written on them, on the floor around them but always ensures that his papers are nearer to him than to another mathematician.

a Copy the above diagram and sketch a Voronoi diagram on it, showing where each mathematician's papers can be situated. (2 marks)

b Calculate the area of the floor that each mathematician uses. (2 marks)

Another mathematician joins the group and sits in the centre of the circle. All of the other mathematicians rearrange their papers according to the same rule as before.

c Make another copy of the original diagram and sketch on it a new Voronoi diagram, to represent the new situation. (4 marks)

d Find the area of floor that the 5th mathematician ends up using. (2 marks)

e Calculate the area of floor that each of the original four mathematicians now uses. (2 marks)

f State how many points on the floor there are that are equidistant from any three mathematicians. (1 mark)

18 P2: An aircraft takes off from an airfield. The position of the aircraft at time *t* hours after takeoff is given by the vector

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 50 \\ 60 \\ 1 \end{pmatrix}.$$

Distances are measured in kilometres.

a Find the position vector of the aircraft 4 hours after takeoff. (7 marks)

A second aircraft takes off from a different airfield. The position vector of this aircraft is given by the vector

$$\mathbf{s} = \begin{pmatrix} -90 \\ -100 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 60 \\ 70 \\ 1 \end{pmatrix}.$$

b Determine if the two flight paths intersect and, if so, state the point of intersection. (6 marks)

The two aircraft took off at the same time, so $\lambda = t$.

c State, with a reason, whether the two aircraft actually collide. (2 marks)

d Calculate the distance between the two airfields. (2 marks)

e Calculate the shortest distance that ever exists between the two aircraft and the time when this occurs. (5 marks)

Click here for further exam practice

Real-life Voronoi

Approaches to learning: Thinking skills: Evaluate, Critiquing, Applying

Exploration criteria: Presentation (A), Personal engagement (C), Use of mathematics (E)

IB topic: Voronoi diagrams

Voronoi diagrams

In this chapter you have been introduced to Voronoi diagrams.

Voronoi Diagrams have been used here to answer questions about **distance** and **area**:

1. Distance

   Finding points that are equidistant to several sites or a path that stays far away from sites or where something can be located so that it is as far as possible from existing sites.

   Which questions from the chapter could be listed in this group?

2. Area

   Considering the territories and regions of influence of animals or retail or food places.

   Which questions from the chapter could be listed in this group?

   You can use Voronoi diagrams to answer similar questions that you create for yourself in explorations.

Exploration

As part of your exploration you will need to provide an aim and a context for your choice of topic.

First write down **all** potentially interesting areas for you to consider for an exploration.

You are now going to try to create an exploration that you could use Voronoi diagrams to answer.

Consider the type of questions that are possible and combine this with something(s) from your list of ideas.

Write the aim and context for your choice.

It may be helpful to think about how you would write the problem as a textbook question for someone else to answer.

Think about:

- Can you give a clear rationale for your choice of topic?
- Is the aim clearly stated?
- Do you have clearly articulated parameters for the problem?
- Is the aim manageable and completable within a sensible page limit and timeframe?
- Are you going to be able to find the relevant information?

You could now solve the problem you have created using the incremental algorithm introduced in the chapter.

Which method is best for answering your question given the available information and the form that this information is given in?

Could a computer software program (such as Geogebra) be used?

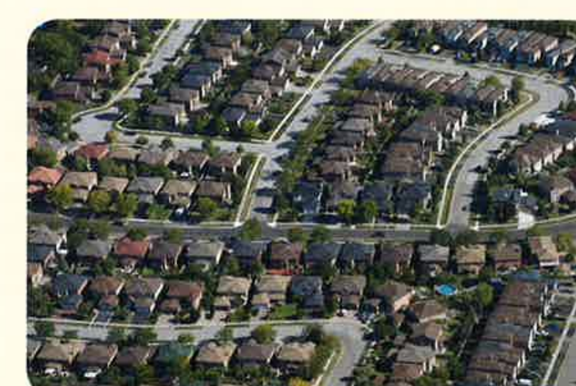
Or could the question be answered by hand using a compass and ruler for construction or by superimposing on a grid and using the equation of a perpendicular bisector?

Extension

Find out about "Nearest neighbour interpolation".

This is an important application of Voronoi diagrams where the aim is to interpolate values of a function at points near to sites, given the value at those sites.

This could be used, for example, to estimate or predict pollution or precipitation levels when you know the pollution or precipitation readings only at particular sites in a region.



Paper 3 question and comments

Paper 3 in the IBDP Mathematics Higher Level course consists of two extended, closed questions, each of which should take approximately 30 minutes to complete.

Usually the questions will provide an opportunity for you to apply the mathematics you have learned to solve a "real-life problem".

Often you will be using mathematics in situations you have not encountered before, and certainly some of the later parts of the questions will have a "problem solving" aspect where you have to choose the best method – out of possibly several different approaches – available to you.

But because paper 3 questions are "closed" problems, there will always be a solution for you to find.

The question will be structured so that the easier parts will normally be at the beginning of the question and, as the question progresses, less and less direction will be provided by the questions.

Paper 3 questions do require a particular set of skills. It is important to make use of the five-minute reading time so you can best assess all that the question is asking of you: You need to be familiar with the overall shape of the question. You need to look at which parts you are confident you can do, and which will require more thought.

In these questions, not being able to do an earlier part should not prevent you answering some of the later parts. Always read through the question carefully and check which parts you can do even if you have not completed all the previous parts. A question part which asks you to "show that…" is often an indicator that the result will be needed in a later part of the question.

The question that follows is typical of the style of a paper 3 question. The notes given are general exam hints, and do not contain any instructions with regard to how to do this particular question.

Give yourself 30 minutes and see how much you can do in that time. If you are not used to extended questions you might struggle to complete it, but with practice you will quickly be able to maximize the marks you can attain.

HINT

Most paper 3 questions will begin with a statement of the task.

1 [Maximum marks: 27]

The aim of the question is to find the shortest distance between Lima in Peru and Tokyo in Japan, to an appropriate degree of accuracy.

A point on the surface of the Earth can be described using two angles. Its longitude measures how far east the point is from the prime meridian, 0°. Its latitude measures how far north the point is from the equator. A negative longitude indicates the point is west of the prime meridian and a negative latitude indicates it is south of the equator.

Let the Earth be modelled as a sphere of radius 6370 km. Let the centre of the Earth be the origin (O) of a coordinate system in which the z-axis passes through the two poles, and the x- and y-axis lie in the plane containing the equator with the x-axis passing through the point on the equator with longitude 0°.

HINT

Because this information is given in a "stem" rather than within a question part, the information applies to the whole question.

a The diagram shows the equator set in the given coordinate system. You are looking at the x- and y-axis from above.

HINT

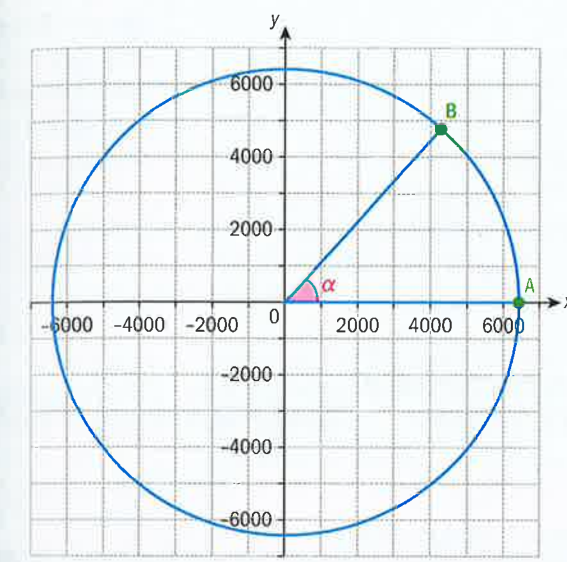
This is setting up the information for part **a**. Because it is given within the part you will not need to use this information for the later parts.

HINT

This part is not directly related to solving the main problem but it develops some of the insights needed as well as providing a straight forward introduction to the question.

Two towns A and B lie on the equator. A has coordinates $(6370, 0, 0)$. The longitude of town B is given by the angle α.

α is positive for counter clockwise rotations and negative for clockwise rotations.



i When $\alpha = 50°$ show that the coordinates of B, to three significant figures, are $(4090, 4880, 0)$.

HINT

The command term is "show that". It is likely this answer will be required in a later part.

Because it is "show that", you must make sure you show all your working.

HINT

All IB exam papers will have the instruction on the cover sheet to give answers "exactly or to 3 significant figures". Often the best policy is to write down a long answer and then round it to 3 sf for your final answer. If using an intermediate answer in a later calculation make sure you use the full result from your GDC rather than a rounded answer.

ii Find the shortest distance around the surface of the Earth from A to B.

iii Find the shortest straight-line distance between A and B.

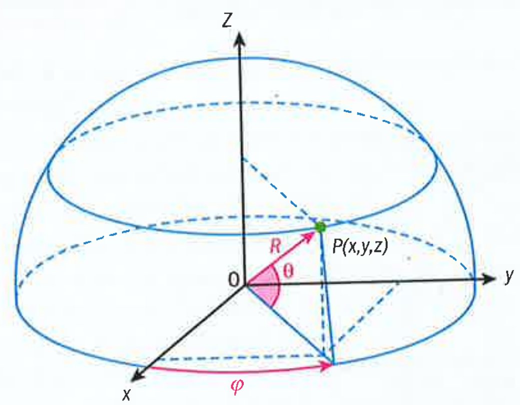
iv Find the percentage error in using the straight-line distance as an approximation for the distance along the surface of the Earth. (9 marks)

HINT

Notice that you could do part **ii** even if you could not do part **i**, and you could do part **iii** even if you could not do either of parts **i** or **ii**.

Part **iv** is dependent on the answers to parts **ii** and **iii**. If you could not answer parts **ii** or **iii**, but you can do percentage error, you should write down two sensible values for answers to parts **ii** and **iii** and you could get full marks for part **iv**.

Consider the point P, which lies on the surface of the sphere at the point with latitude θ and longitude ϕ, as shown in the diagram below. Let the radius of the Earth be R.



b i Write down the z-coordinate of P in terms of R and θ.

ii Find the coordinates of P in terms of R, θ and ϕ. (4 marks)

HINT

The command term "write down" means you should not need to do any working out.

HINT

This part is more abstract than part **a**, and so is likely to be more difficult. If you cannot answer it, check ahead to see where you can "get back into" the question.

In this case, if you cannot answer part **b**, you can can start again with part **d**, as it requires you to use values that are given in the question.

In this model, Lima in Peru lies at the point with latitude −12.05 and longitude −77.04.

c Show that the coordinates of Lima are $(1397, -6071, -1329)$. (2 marks)

HINT

Once again, the command "show that" indicates that you might need this result later. Take note of it, even if you cannot answer part **b** (and so cannot do part **c** either).

In this same model, Tokyo lies at the point with latitude 35.69 and longitude 139.70. This point has coordinates $(-3945, 3346, 3717)$.

Let Lima be at the point L and Tokyo be at the point T.

d i Write down $|\vec{OL}|$

HINT

The command term is "write down" so do not be tempted to try and do lots of working out. Pause and consider whether or not there is a quick route to the answer.

Also there has been a switch from cartesian coordinates to vector notation here. This could be a hint for the best way to approach later parts of the question.

HINT

As the question moves on, there is more of a problem solving element to it.

There is more than one way to solve part **d (ii)**, and some ways are quicker than others.

ii Find the angle $L\hat{O}T$.

iii Hence find the shortest distance around the surface of the earth between Lima and Tokyo. (8 marks)

HINT

"Hence" tells you that you have to use the earlier part. If you could not find the angle but know how to find the shortest distance, write down a sensible value for part **ii** and you may get follow through marks.

In reality, the Earth is not a perfect sphere and the distance from the centre of the Earth to the surface is between 6353 km and 6384 km.

HINT

Often when looking at real-life situations, mathematical models are necessarily simplifications and so consideration of limitations of the model, or error bounds to your solutions, become important.

e i Find upper and lower bounds for the distance between Lima and Tokyo along the shortest route, assuming that both are the same distance from the centre of the Earth.

HINT

The command term is "find", so there is some work to be done, but part **e** only has 4 marks to it and requires you to give two bounds and an approximation in part **ii**. This appears to be quite a lot of work for relatively few marks, so there must be an easy solution. Try to think of a simple approach to this problem before working through the whole process for a second and then a third time.

ii Hence give the distance between the two cities to an appropriate degree of accuracy. (4 marks)