4 Modelling constant rates of change: linear functions and regressions

All of these questions are about situations that can be represented by a mathematical model. These models will allow you to study relationships between the variables and to make predictions.

A rental car costs a fixed amount plus $t per kilometre. How can I find the maximum number of kilometres that can be paid with $A?

It's a hot summer's day – how long will it take to fill your swimming pool?

In general, is there a relationship between the number of children in a family and the family's income?

How much do you have to invest in your bank account to grow your savings so you can buy that yacht? And how much does the interest rate matter?

Concepts

- Change
- Modelling

Microconcepts

- Function
- Domain
- Range
- Graph of a function
- Dependent and independent variables
- Linear models and their parameters
- Gradient
- Intercept
- Inverse function
- One-to-one function
- Composite function
- Identity function
- Arithmetic sequences and series
- Common difference
- General term
- Simple interest
- Direct variation
- Piecewise linear function
- Least squares regression equation
- Sum of squares residual
- Linear correlation
- Prediction

Stefan drives taxis part-time, usually driving between 10 and 40 km per day. Currently, on each day that he drives he earns a fixed amount of $20 plus $2.50 for every kilometre that the taxi's meter is on. Stefan wants to increase his annual income so that it is equal to the annual average salary of $44 000. He is thinking of increasing the income per kilometre by $0.30 per month every month for the next year.

What will his daily income be if he drives 10, 20, 30 or 40 km?

How will his income change if he increases the income per kilometre by $0.30 each month?

How can you model the daily income for different values of income per kilometre?

Will he reach the annual average salary in one year's time? If not, how long will it take?

What assumptions do you need to make to answer these questions?

Developing inquiry skills

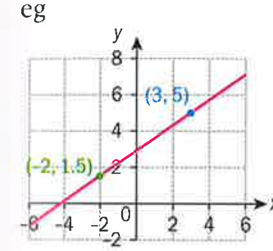
Write down any similar inquiry questions you could ask and investigate for another business or charging structure. What information would you need to find? Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Find the equation of a line given a graph or information about the line.

eg



Calculate the gradient: $m = \frac{5-1.5}{3-2} = 0.7$

Substitute in point-gradient form:

$y - 5 = 0.7(x - 3)$

Convert to gradient-intercept form:

$y - 5 = 0.7x - 2.1$

$y = 0.7x + 2.9$

2 Given an equation, substitute a value for one variable and solve for the other.

eg $y = 7 - 5x$

a Find y when $x = 3$.

$y = 7 - 5(3) = -8$

b Find x when $y = 3$.

$3 = 7 - 5x$

$-4 = -5x$

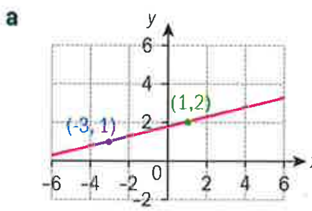
$x = \frac{4}{5}$

Skills check

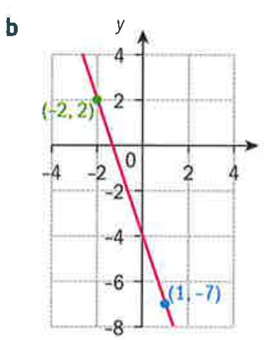
Click here for help with this skills check

1 Find the equation of each line:

a



b



2 Given that $y = 1.5x - 4$,

a find y when $x = 18$

b find x when $y = 2$.

4.1 Functions

Functions help us to model situations with predictable relationships between two variables. In the following investigation you will investigate what a function is and the different ways that you can represent it.

International-mindedness

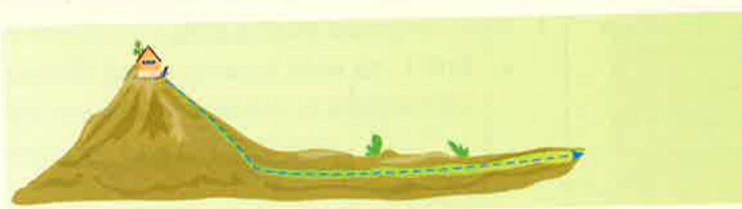
One of the first mathematicians to study the concept of functions was Frenchman Nicole Oresme in the 14th century.

Investigation 1

Amir loves running. Here are the **contexts** of six of his recent runs, 1–6, and some graphs, A–G, that might describe the runs.

Run 1: Amir runs on flat ground for 10 minutes.

Run 2: He runs downhill.



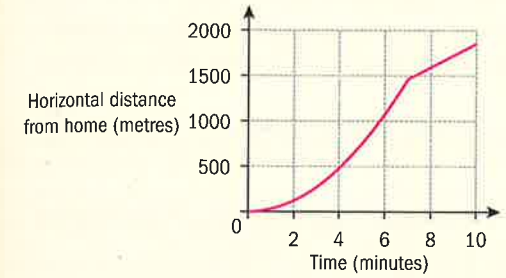
Run 3: He runs 10 km.

Run 4: He runs up a mountain and back.

Run 5: He stops to chat with a friend midway through his run.

Run 6: He encounters a rough trail that slows him down on his way home.

A



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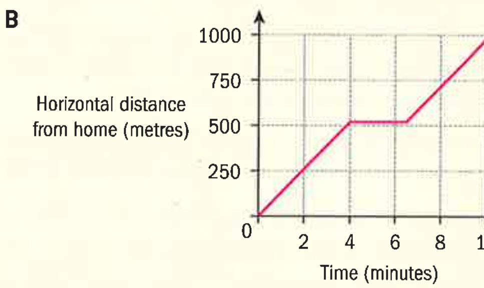

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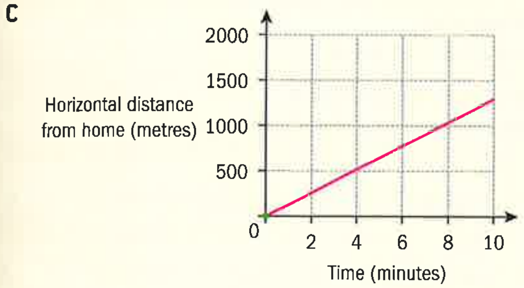

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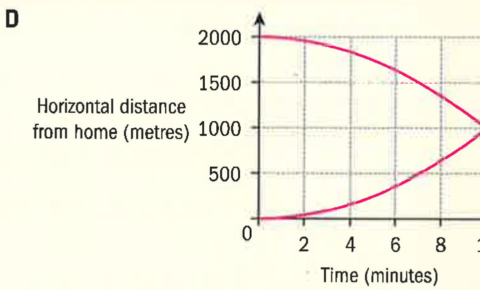

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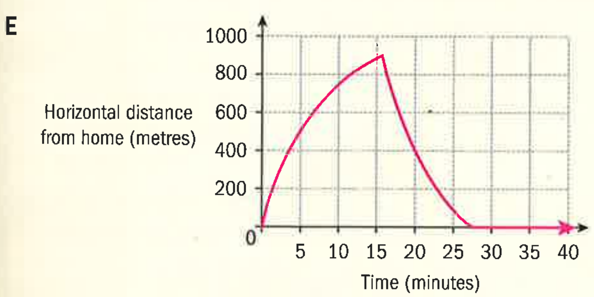

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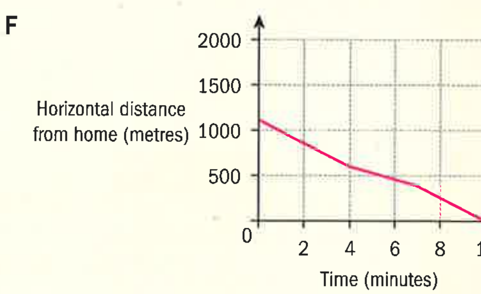
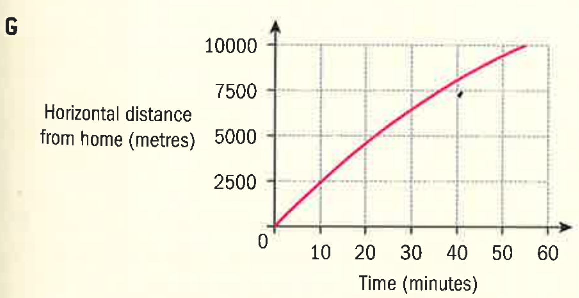


G



1 Match each run to a graph, giving two specific reasons why you know they match.

2 One graph does not match any of the runs. Explain why it is not possible for this graph to represent a person running over time.

Functions

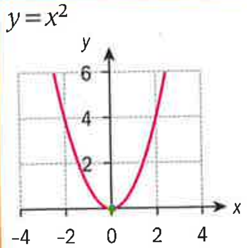
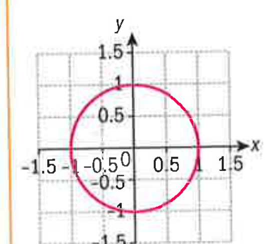
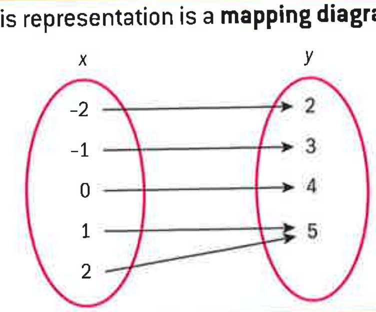
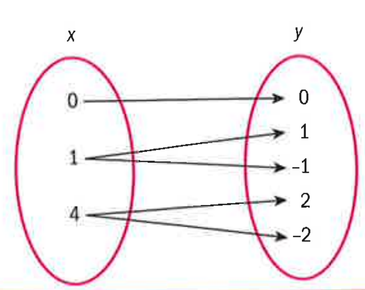
A **relation** is a set of ordered pairs (x, y) that specifies corresponding values of the **independent** (x) and **dependent** (y) variables. A graph is one way to represent a relation between two variables.

Investigation 2

The graphs in Investigation 1 corresponding to the different contexts are examples of representations of **functions**. They show Amir's distance from home as a function of time. The graph that did not match a context is an example of a **non-function**.

Here are some more examples of functions and non-functions and different ways that you can represent them, as equations, tables or mapping diagrams.

1 For each pair of examples, describe how the relationship between the **inputs** (x-values) and **outputs** (y-values) differs between the function and the non-function.

<table>
<tr><th></th><th>Relations that are functions</th><th>Relations that are not functions</th></tr>
<tr><td>a</td><td>x = length of foot in cm
y = shoe size</td><td>x = shoe size
y = length of foot in cm</td></tr>
<tr><td>b</td><td>x = a student in your class
y = their birthday</td><td>x = a student in your class
y = the name of one of their siblings</td></tr>
<tr><td>c</td><td>$y = x^2$
</td><td>$x^2 + y^2 = 1$
</td></tr>
<tr><td>d</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>−7</td><td>−22</td></tr><tr><td>−2</td><td>15</td></tr><tr><td>0</td><td>8</td></tr><tr><td>7</td><td>15</td></tr></table></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>−2</td><td>3</td></tr><tr><td>0</td><td>0</td></tr><tr><td>7</td><td>−14</td></tr><tr><td>7</td><td>11</td></tr></table></td></tr>
<tr><td>e</td><td>This representation is a mapping diagram:
</td><td></td></tr>
</table>

2 Based on your descriptions, what generally is the difference between a relation that is a function and a relation that isn't? Make sure that your generalisation is consistent with all of the examples.

3 **Factual** What types of relations are functions?

4 **Factual** What are the different ways that a function can be represented?

5 **Conceptual** How can you identify that a relation is a function from a graph, table, mapping diagram or context?

A **function** is a relation between two sets in which every element of the first set (input, independent variable) is mapped onto **one and only one** element of the second set (output, dependent variable).

That is, y is a function of x if, for each x-value, there is exactly one corresponding value of y.

Example 1

For each relation, determine, with a reason, whether y is a function of x.

a In an annual survey of panda populations, x = time (in years since 1977), y = number of pandas.

b The relation $x \rightarrow 2x^2 - 1$ with the set of inputs $A = \{-3, -2, 0, 2\}$. Note that the **mapping** notation $x \rightarrow 2x^2 - 1$ represents the same relation as $y = 2x^2 - 1$.

c The relation $x = y^2$.

a Yes, because each year will have exactly one measurement of the panda population.

b Yes, because each input has exactly one corresponding output:

x	$y = 2x^2 - 1$
−3	17
−2	7
0	−1
2	7

Substitute each x-value into the formula to find the corresponding y-value. For example, when $x = -3$:

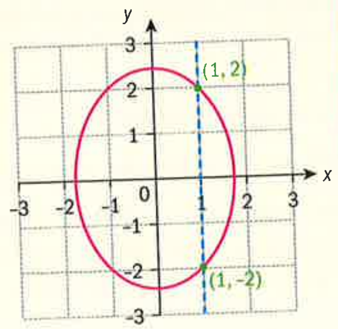
$$y = 2(-3)^2 - 1$$
$$= 2(9) - 1$$
$$= 18 - 1$$
$$= 17$$

Note that two inputs can have the same output.

c No, because an x-value can have two corresponding y-values. For example, $x = 9$ corresponds to $y = 3$ and $y = -3$.

One method for determining whether a graph represents a function is the **vertical line test**: if you can draw a vertical line that intersects the graph more than once, then the graph fails the test and represents a non-function. Otherwise, the graph is a function.

For example, the relation in the graph is not a function, because the vertical line $x = 1$ intersects the graph at (1, 2) and (1, −2).



Reflect Why is the vertical line test consistent with the definition of function?

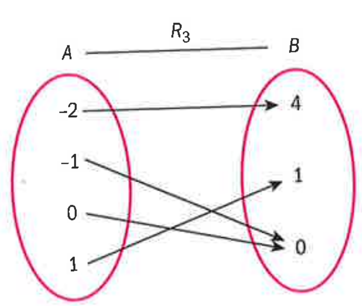
Exercise 4A

1 Determine, with reasons, whether y is a function of x in each context.

- **a** In a group of 500 people, $x = \{1, 2, 3, 4, \ldots, 10, 11, 12\}$ and y = a person born in month x.
- **b** In the same group of people, $x = \{1, 2, 3, 4, \ldots, 10, 11, 12\}$ and y = the number of people in the group born in month x.

2 For each mapping diagram or graph, decide with a reason whether it represents a function.

a



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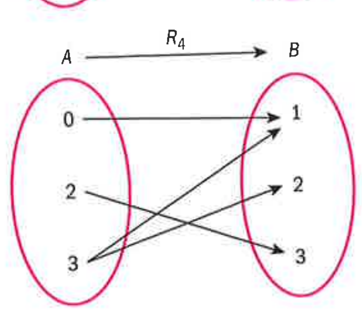

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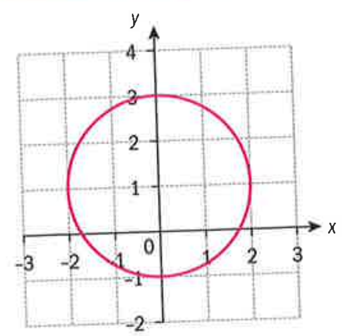

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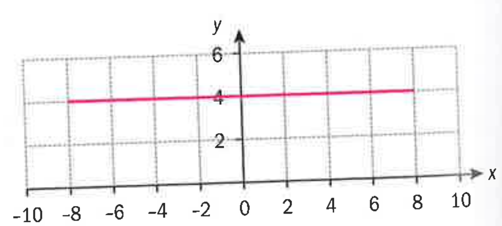

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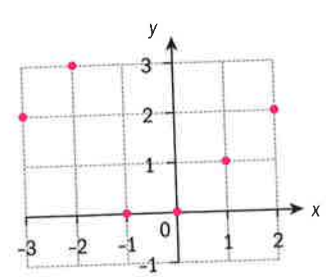

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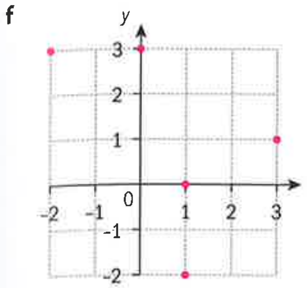
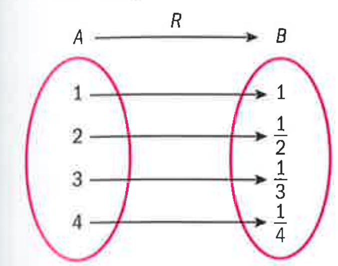


3 a Write down the mapping R in the form $x \to \ldots$



- **b** State whether or not R is a function.
- **c** Write down the output of $x = 2$ under R.
- **d** The set A is extended to $\{1, 2, 3, 4, a\}$, and a is mapped onto 3. State the value of a.

4 Consider the mapping $y = x^3$.

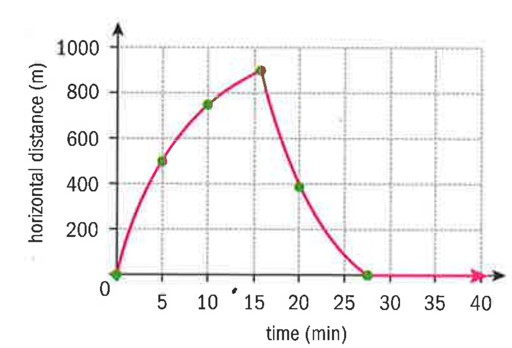
- **a** State the output when the input is $x = -1$.
- **b** State the input when $y = -64$.
- **c** Let the set of inputs be $A = \{-1, 0, 1, 2, 3, 4\}$. Find B, the set of outputs.
- **d** State, with a reason, whether or not the mapping $y = x^3$ from A to B is a function.

Function notation

You can use **function notation** to concisely communicate a function's output for a given input by writing $y = f(x)$ or $f(x) = y$. You also say that y is the **image** of x under the function f.

For example, here is Run 4 from Investigation 1 in both graph and table form.

x	y
0	0
5	520
10	790
15	945
20	390
25	45



From the table you can see that the time $x = 5$ minutes corresponds to a distance of $y = 520$ meters; You could write $f(5) = 520$. This is read "f of 5 is 520" – it does not mean "f times 5".

In this context, it means that after 5 minutes, Amir is 520 meters from home. How can you see this from the graph?

Investigation 3

In this investigation you'll explore what information you can gather from different function representations, including function notation.

Explore function notation by using the graph and table we've just looked at to answer these questions as accurately as possible.

1 What y-value solves $y = f(10)$? What does this mean in context?

2 What two x-values solve $f(x) = 600$? What does this mean in context?

3 What is the furthest distance from home that Amir reaches, and when does he reach it? Express your answer in function notation in the form $f(x) = y$.

4 When does Amir reach home? Express your answer in function notation.

5 **Conceptual** What useful information does function notation communicate?

Example 2

Raquel invests $1200 in a savings account whose value increases over time. The future value, V, of the account is a function of the time t (in years) invested, represented by the equation $V(t) = 1200 \times (1.03)^t$ for $0 \le t \le 50$.

a Find

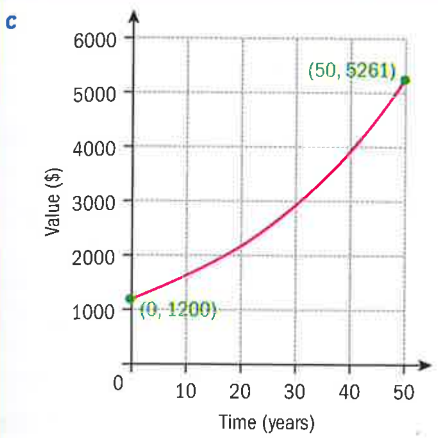
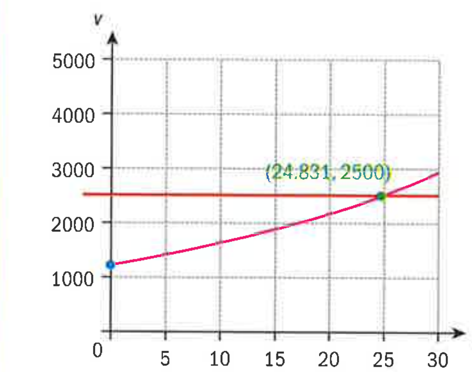
i $V(0)$ **ii** $V(50)$

Interpret each of these in context.

b If Raquel keeps her money invested for 50 years, determine how much she will earn on her initial $1200.

c Sketch a graph of the function V for $0 \le t \le 50$.

d If Raquel invests her money in 2015, determine the year when the value of her account will reach $2500.

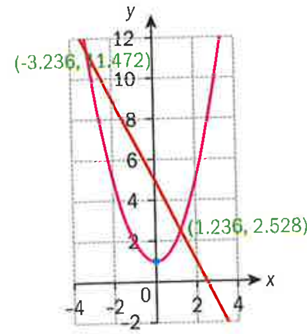
a i $V(0) = \$1200$ This shows that Raquel invested $1200 initially.	Substitute the given inputs ($t = 0$ and $t = 50$) into the function. $V(0) = 1200 \times (1.03)^0 = 1200$
ii $V(50) = \$5260$ (3 s.f.) This shows that Raquel will have $5260 in the account after 50 years.	$V(50) = 1200 \times (1.03)^{50} = 5260.687\ldots$ Alternatively, graph $y = 1200 \times (1.03)^x$ and use the TRACE or TABLE function of your GDC to find the y-values corresponding to $x = 0$ and $x = 50$.
b $4060	Find $V(50) - V(0)$.
c 	Graph the function $y = 1200 \times (1.03)^x$ with technology. Choose an appropriate graphing window: since the given set of input values is $0 \le t \le 50$, set the x-axis to display from 0 to 50. From part **a**, the output grows to approximately $5300, so set the y-axis to display from 0 to 6000.
d $V(t) = 2500$	As you want to find the time when the value is $2500, you are solving $V(t) = 2500$. Graph $y = 2500$ on the same screen as $y = 1200 \times (1.03)^x$. Use technology to find the intersection of these two graphs.
$t = 24.8$	
Raquel's account will reach a value of $2500 during the year 2039.	Since $t = 0$ corresponds to the year 2015, Raquel's account value will reach $2500 during the 24th year after 2015, or 2039.

The graphical method used to solve $V(t) = 2500$ in Example 2 can be generalized to solve any equation of a single variable.

To solve the equation $f(x) = g(x)$, graph both $f(x)$ and $g(x)$ and find their point(s) of intersection. $f(x)$ and $g(x)$ are equal at such points, so the x-value(s) of the coordinate(s) solve the equation.

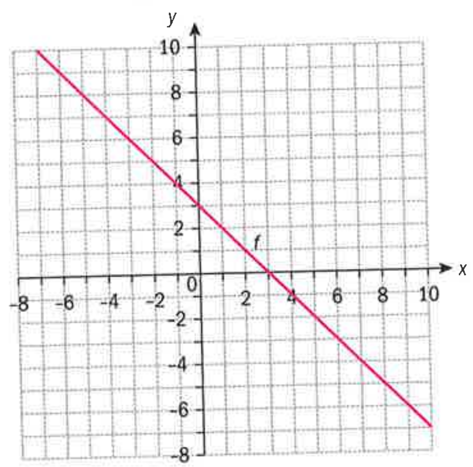
For example, to solve $x^2 + 1 = 5 - 2x$, graph $y = x^2 + 1$ and $y = 5 - 2x$ and find their intersection point(s).

The solutions to the equation $x^2 + 1 = 5 - 2x$ are $x = -3.24$ and $x = 1.24$ (3 sf).



Exercise 4B

1 The function f is shown in the graph.



a Write down the values of

i $f(-2)$

ii $f(6)$.

b Find the value(s) of x that solve each of these equations:

i $f(x) = 3$

ii $f(x) = 0$.

c Find the set of values of x for which $f(x) < 0$.

2 Consider the function f defined by $f(x) = 10 - 4x$.

a Calculate

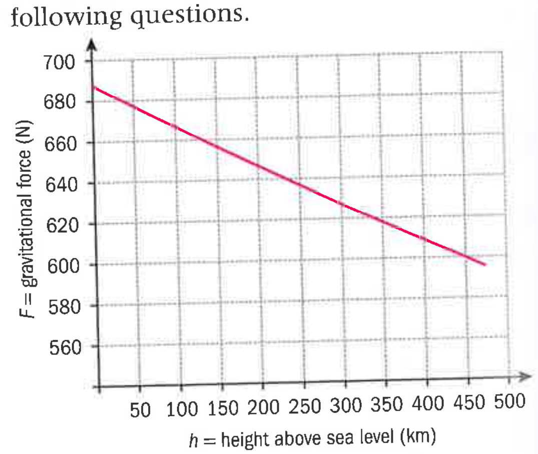
i $f(2)$

ii $f\left(-\frac{1}{2}\right)$

b Show that $f(2.5) = 0$.

c There is a value of x for which $f(x) = -6$. Find this value of x.

3 Gravity decreases the further you are from the centre of the Earth. The force of gravity F, measured in Newtons (N), on Jaime, who weighs 70 kg at home, is a function of his height above sea level h, measured in kilometres. The function $F(h)$ is shown in the graph. Hence estimate answers to the following questions.



a i Find $F(0)$. Interpret its meaning in context.

ii If Jaime is currently an astronaut on the International Space Station, 410 km above sea level, find the gravitational force on him.

iii State your answer to part **ii** as a percentage of the force at sea level. State how much lighter Jaime is on the space station.

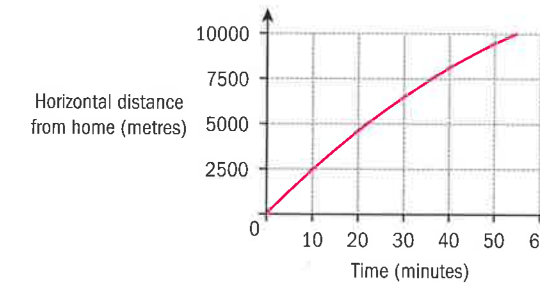
b Solve $F(h) = 625$ and interpret its meaning in context.

c Find the height at which the force of gravity is 5% less than at sea level. Write down your answer in function notation.

Domain and range

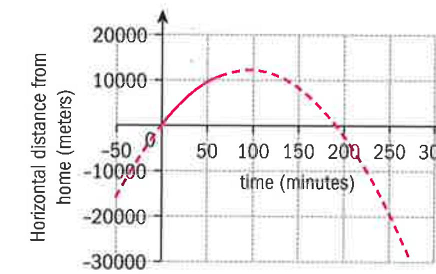
The **domain** of a function is the set of all possible input values x.

Recall Run 3 from Investigation 1, in which Amir ran a 10 km from his house.



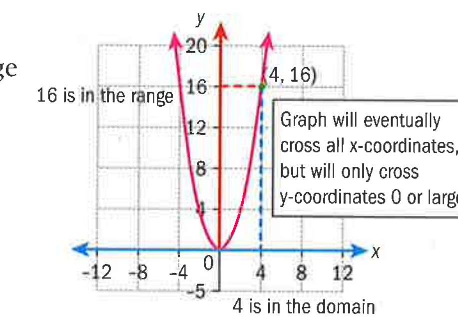
This can modeled by the equation $g(x) = -1.336x^2 + 255x$. If you graph this function with technology you will see that the graph extends beyond the portion shown in the investigation.

For the function $g(x)$, any value can be input as x, so you say that its **domain** is the set of all real numbers, $x \in \mathbb{R}$. (Recall that $x \in S$ means that x is a member of the set S.) $\mathbb{R}$ is the default domain if one is not specified.



The **range** of a function $f(x)$ is the set of y-values (or outputs, or images) corresponding to all the inputs in the domain.

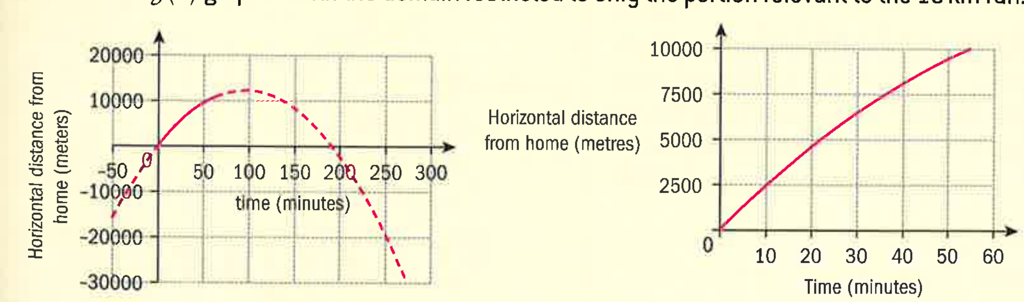
For example, $f(x) = x^2$ has domain $x \in \mathbb{R}$ and range $f(x) \geq 0$ or $y \geq 0$, because squaring all numbers results in a non-negative number. You can see this on the graph of $f(x)$.



Investigation 4

The domain of a function is sometimes **restricted** to a smaller set that makes sense for the real-world context that the function represents. This is the **reasonable domain**.

Consider again the context of Amir's 10 km run. The first graph shows $g(x)$ graphed with domain $x \in \mathbb{R}$ and the second shows $g(x)$ graphed with the domain restricted to only the portion relevant to the 10 km run.



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1 **Factual** How can you see from the graph of $g(x)$ that the domain is $x \in \mathbb{R}$?

2 Estimate a **reasonable domain** for $g(x)$ in the context of the 10 km run.

3 Use the graph of $g(x)$ with domain $x \in \mathbb{R}$ to describe its approximate range.

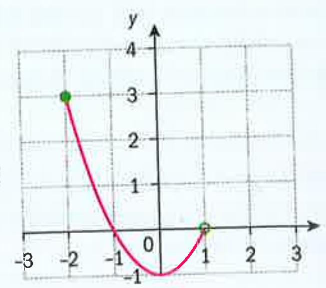
4 The **reasonable range** is the range associated to the reasonable domain; that is, the set of outputs corresponding to the set of inputs in the reasonable domain. What is the **reasonable range** of $g(x)$ in the context of the 10 km run?

5 **Conceptual** How does the real-world context of a function influence its reasonable domain and range?

Example 3

For each of the following functions, find its domain and range.

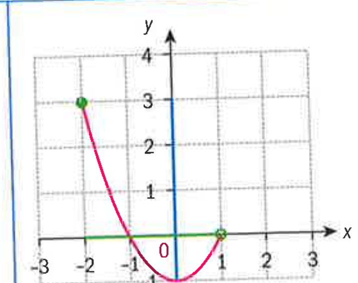
a The function $f(x)$ is defined by the graph. Note that, like a number line diagram, if a graph has a solid dot • then this means that the point is included in the graph. If there is a hollow dot ○ then this means that the point is not included in the graph.



b The function $f(x) = 7 - \frac{1}{3}x$, $-2 \le x < 4$.

a Domain: $-2 \le x < 1$

Range: $-1 \le y \le 3$



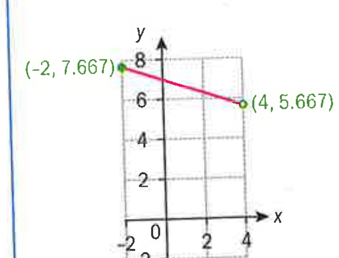
Project the graph onto the x-axis to see that the domain (the green line) is $-2 \le x < 1$. Observe that $x = 1$ is not included in the domain.

Project the graph onto the y-axis to see that the range (the blue line) is $-1 \le y \le 3$.

b Domain: $-2 \le x < 4$

Range: $5.67 < y \le 7.67$

Graph $f(x) = 7 - \frac{1}{3}x$ on the given domain:



The endpoints of the domain will in this case also be the endpoints of the range. Find these coordinates with technology or by substitution:

$f(-2) = 7.67$, $f(4) = 5.67$

Note that the graph contains the upper endpoint but not the lower one.

Exercise 4C

1 A function $T(d)$ gives the average daily temperature T, in °C, of a certain city during last January, where $d = 1$ corresponds to January 1st.

a Explain the meaning of $T(2) = 25$.

b State the domain of the function T.

c Write down the greatest possible number of values in the range of T.

d	2	8	15	18	22	29
T	25	22	26	20	21	28

2 Consider the function $f(x) = -2x + 3$, $-1 \le x \le 3$

a Find

i $f(-1)$ **ii** $f(3)$

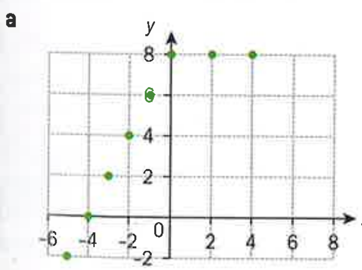
b Find the value of x such that $f(x) = 2$.

c Sketch the graph of this function. Label clearly the end points of the graph.

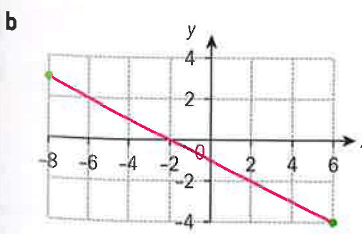
d Hence, find the range of this function.

3 Find the domain and range of each of these functions whose graphs are as follows.

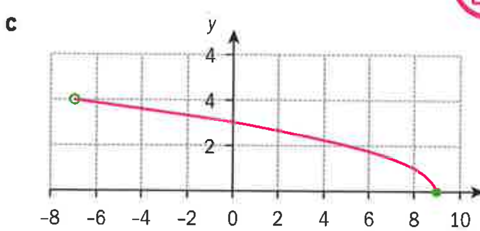
a



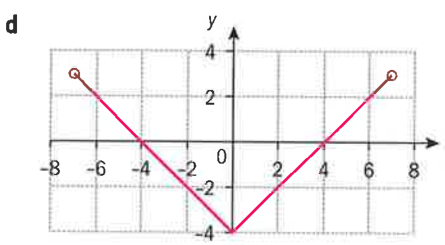
b



c



d



4 Robbin is a mountain guide planning a two-week camping trip that will serve between three and eight clients. She knows that the total weight per person, w kg, of food that must be carried for p people (not including Robbin) can be calculated by the function $w(p) = 9 + \frac{15}{p+1}$.

a Write down the reasonable domain for $w(p)$.

b Find and interpret $w(3)$ and $w(8)$.

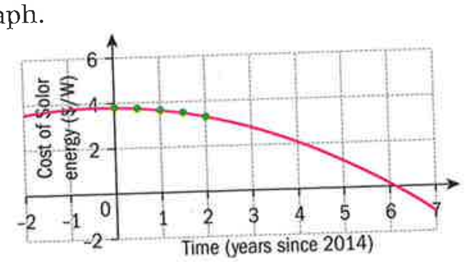
c Find the associated reasonable range for $w(p)$.

d The company Robbin works for advertises that clients can expect to carry 11–12 kg of food each. Find the number of people Robbin can take on the trip in order to meet this requirement.

5 The cost of solar-powered energy has become cheaper over time as technology has improved and more solar power is installed. The table shows the cost of solar energy (measured in dollars per watt of energy) at several recent dates.

Year	Jan 2014	Jun 2014	Jan 2015	Jun 2015	Jan 2016
Cost of solar energy ($ per W)	3.82	3.75	3.65	3.53	3.31

Based on this data, researchers found that the cost of solar energy, S, as a function of time, t (in years since 2014), can be represented as $S(t) = -0.09t^2 - 0.0651t + 3.81$. They expect their model to be valid until 2019. The equation and the points from the table are shown in the graph.



a State the reasonable domain for $S(t)$.

b Estimate the reasonable range for $S(t)$.

c Estimate what the cost of solar was in June of 2016.

d Find the year in which the researchers predict the cost of solar energy will fall below $1.50 per watt.

e Explain why the function may not lead to reasonable predictions for years beyond 2019.

6 A company produces water bottles at a cost of $C(q) = 27\,000 - \dfrac{60\,000\,000}{q+5000}$ euros, where q represents the number of bottles produced, $0 \le q \le 30\,000$. Assuming all bottles are sold, its revenue, or earnings, can be modelled by $R(q) = \dfrac{1.75}{10\,000}q(30\,000 - q)$. In the following questions, round all monetary values to the nearest euro.

a Find the revenue earned when 2500 water bottles are sold. Express your answer in function notation.

b Calculate how much more revenue the company will earn selling 5000 water bottles than selling 2500.

c Solve $C(q) = 20\,000$ and interpret your answer in context.

d Sketch both functions on the same graph using technology. Find the **break-even point(s)**: the number of bottles the company must sell so that its costs and revenues are equal.

e The company makes a profit if its revenue is greater than its cost. Determine the values of q for which the company will make a profit.

7 The value of Renee's car, in UK£, is changing as a function of time. This relationship can be expressed by the equation $V(t) = \dfrac{8500}{t+1} + 5(t-15)^2$, where t is the time in years since she purchased the car in 2008, $0 \le t \le 60$.

a Using the equation or a graphical representation of the function, find the values of a and b in the table. If more than one value is possible, list all possible values.

t	$V(t)$
0	a
1	b
3	2845
c	4000

b Find the year in which the car's value will drop below UK£1000.

c If Renee keeps her car long enough, it will become an antique and will begin to increase in value. Find the year in which her car's value will be as high as when she bought it.

4.2 Linear models

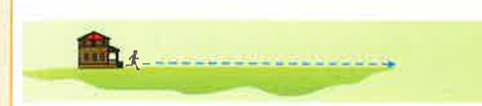
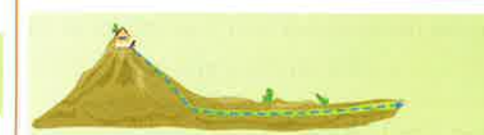
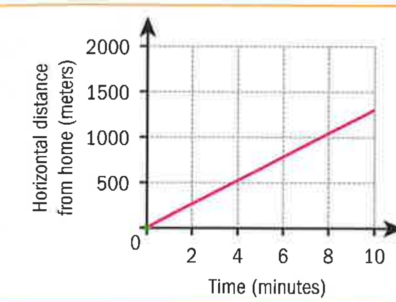
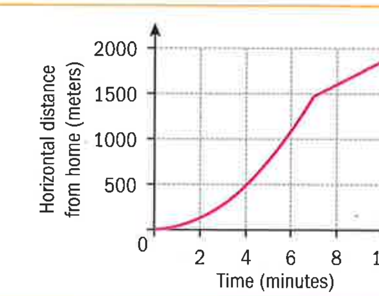
In this section you will explore several specific types of linear functions and the real-world situations that they model.

TOK

What is the relationship between real-world problems and mathematical models?

Investigation 5

Use the graphs of Amir's Runs 1 and 2 from Investigation 1 to answer the questions as precisely as possible.

Run 1	Run 2
	
	

1 What is Amir's total distance travelled and total time taken in each run?

2 Average speed can be calculated by

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}.$$

What is Amir's average speed in each run?

3 Is Amir travelling at this average speed for the whole of Run 1? For Run 2? How can you tell?

4 The **rate of change** between two variables is the amount of change in the dependent variable per unit change in the independent variable. For Run 1, how far does Amir run in one minute? Use this to calculate the rate of change of his distance per unit time.

5 What does the rate of change tell you about Amir's run?

6 Find the equation of the line that represents Run 1 in the form $y = mx + c$.

7 How do the gradient and y-intercept parameters in your equation correspond to the context?

8 **Factual** What do the letters m and c stand for in $y = mx + c$.

9 **Conceptual** What does the constant rate of change represent in a linear function and what does the y intercept represent in terms of the dependent and independent variables?

Functions

A **linear function** $f(x) = mx + c$, where m and c are constants, represents a context with a **constant rate of change**. The constants in an equation are called **parameters**. A linear function has the parameters gradient (m) and y-intercept (c).

Example 4

If you have travelled between lower and higher altitudes, you may have noticed that the air pressure changes. Air pressure at sea level (0 km) is defined as 1 atmosphere (atm). At an altitude of 5000 feet, or 1.524 km, above sea level, air pressure is 83.7% of the pressure at sea level, or 0.837 atm. Assume that the relationship between air pressure and altitude is linear.

a Find an equation to express air pressure P (in atm) as a function of altitude a (in km).

b Interpret the gradient and y-intercept of $P(a)$ in context.

c If $(k, 0.5)$ is a point on the graph of $P(a)$, find the value of k and interpret its meaning in context.

a $P(a) = 1 - 0.107a$

Use the given pairs of independent and dependent variables to calculate the gradient of the linear function: (0, 1) and (1.524, 0.837).

The y-intercept in this case is given as one of the coordinates.

So, $c = 1$ and $m = \frac{0.837 - 1}{1.524 - 0} = -0.107$.

b As the altitude increases, the air pressure reduces at a rate of 0.107 atm per kilometre.

The atmospheric pressure at ground level (0 km) is 1 atm.

The gradient is the rate of change between the dependent variable (pressure in atm) and independent variable (altitude in km).

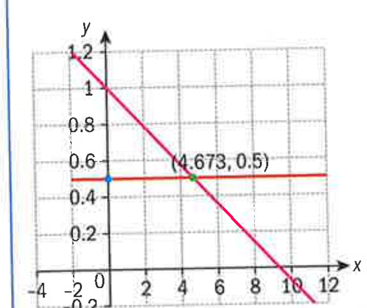
The y-intercept occurs at $a = 0$.

c $k = 4.67$

The air pressure will be 50% of the pressure at sea level at an altitude of 4.67 km.

Method 1:

Use technology to find the intersection of the function with the line $y = 0.5$:



Method 2:

Substitute $P(a) = 0.5$ in the equation and solve for a:

$0.5 = -0.107a + 1$

$a = \frac{-0.5}{-0.107} = 4.67$

Recall that there are three common ways to represent a linear equation:

- Gradient-intercept form: $y = mx + c$
- Point-gradient form: $y - y_1 = m(x - x_1)$, where (x_1, y_1) is any point on the graph.
- Standard or general form: $ax + by + d = 0$, where a, b and d are constants.

The gradient-intercept form translates directly to the linear function $f(x) = mx + c$, but the other two forms can be useful starting points depending on the information given in the problem.

International-mindedness

The word "modeling" is derived from the Latin word "modellus", which means a human way of dealing with reality

Functions

Example 5

A water tank drains at a constant rate. It contains 930 litres of water 3.5 minutes after it starts to drain. It takes 50 minutes for the tank to empty. Let W be the amount of water in the tank (in litres) t minutes after it started to drain.

a Find a gradient–intercept model for $W(t)$, the amount of water in the tank with respect to time.

b Write down the amount of water in the tank when it starts to drain.

c Write down the rate at which the water tank is emptying.

d Use your model to find the amount of water after 30 minutes.

a $W(t) = -20t + 1000$

If the tank drains at a constant rate then it is a linear function:

$W(t) = mt + c$

From the information given you can create the ordered pairs (3.5, 930) and (50,0).

Method 1:

As you have two points, use these to calculate the gradient:

$m = \frac{0 - 930}{50 - 3.5} = -20$

Substitute the gradient and one point into the point-gradient form:

$y - 0 = -20(x - 50)$

Convert to gradient-intercept form.

Method 2:

Substituting each point into the equation

$W(t) = mt + c$ gives: $3.5m + c = 930$
$50m + c = 0$

Use technology to solve these simultaneous equations.

$m = -20$, $c = 1000$

Continued on next page

b 1000 litres	When it starts to drain, $t = 0$. This is the y-intercept of the function.
c The tank drains at 20 litres per minute.	The rate at which the tank drains (the rate of change) is given by the gradient, m.
d 400 litres	Substitute $t = 30$ into the formula: $W(30) = -20 \times 30 + 1000 = 400$

Exercise 4D

1 State which of the following are linear functions. If the function is linear, state its gradient.

a $f(x) = 3$ **b** $g(x) = 5 - 2x$

c $h(x) = \frac{2}{x} + 3$ **d** $t(x) = 5(x - 2)$

2 For each context

i identify the independent and dependent variables and their units

ii determine whether a linear function can be used to model the relationship between the two variables, and if so, state the rate of change.

a Parking at an airport is charged at \$18 for the first hour parked and \$12.50 for each subsequent hour.

b The population of fish in a lake declines by 7% each year.

c Italy charges a value-added tax (VAT) of 22% on all purchases. The amount of tax you pay depends on the amount of your purchase (in euros).

d Sal's Ski Resort notices the following trend in sales of daily passes:

Daily high temperature, °C	−4	−1	3	7
Number of daily passes sold	430	406	374	342

3 The table shows values for a function $y = f(x)$.

x	1	3	5	8
y	3	7	11	15

Determine whether $y = f(x)$ is a linear function or not.

4 An online shipping company charges a fixed cost plus US\$100 per kilogram of shipment. The total price of shipping is given by the function $P(x) = 100x + 30$, where x is the weight of the shipment.

a Write down the fixed cost.

b Find the price when the shipment weighs 1.5 kg.

Daniel pays US\$310 for a shipment.

c Find the weight of this shipment.

5 Ewout is 13 km from home and is walking at a velocity of 65 metres per minute. Velocity is the rate of change of distance over time.

a Find an equation to represent Ewout's distance from home, d (in kilometres), as a function of time, t (in minutes).

b Determine the distance from home Ewout will be after one hour.

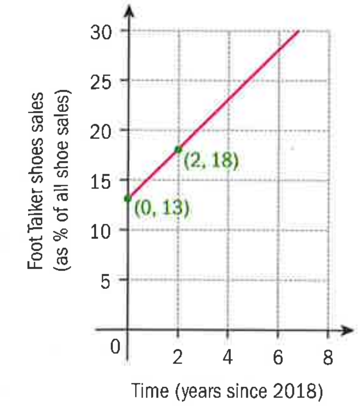
c Find the x-intercept of the graph of $d(t)$ and interpret its meaning in the context of the problem.

6 In 2018, Sneakies was the largest shoe company in the industry: their sales represented 64% of all shoe sales by all companies. However, analysts projected that in 2019 they would sell only 62.65% of all shoes, and that without any changes, their percentage share of shoe sales would decline at the same steady rate for the foreseeable future.

a Find an equation in gradient-intercept form for the function $S(t)$ that represents Sneakies' shoe sales (as a percentage of all shoe sales) t years after 2018.

b Find the year when Sneakies is predicted to no longer sell the majority (over 50%) of shoes.

The graph represents the projected sales of rival company Foot Talker.

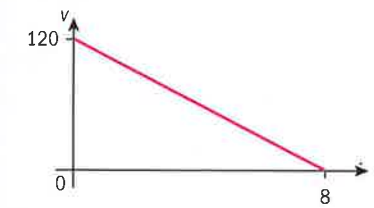


c Show that Foot Talker is growing faster than Sneakies is declining, in terms of their percentage of total shoe sales.

d Find an equation for $F(t)$, the function representing Foot Talker's sales over time.

e Find the year when Foot Talker will sell more shoes than Sneakies.

7 The straight line graph shows the velocity v of a moving body, in metres per second, at time t seconds.



a Write down the initial velocity of the moving body.

b Determine the number of seconds it will take for the body to stop moving.

c Find the rate of change of the velocity with respect to time.

d Find a model for v.

8 A spring is stretched by suspending different weights from it. The length of the spring, L, in centimetres can be modelled by a linear function $L = mW + c$, where W is the weight suspended in grams.

When the weight suspended is 50 g, the length of the spring is 20 cm.

a Write down an equation that shows this information.

When the weight suspended is 80 g, the length of the spring is 35 cm.

b Write down an equation that shows this second piece of information.

c Find the values of m and c.

d Hence, find the length of the spring when a weight of 90 g is suspended.

Functions

Direct variation

Investigation 6

This investigation will help you to determine what direct variation is.

1 For each of the following functions:

i find an equation that expresses the dependent variable in terms of the independent variable

ii sketch its graph on an appropriate domain.

a The circumference C of a circle in terms of its radius r.

b The currency conversion between US\$ d and euros e. One euro is worth US\$1.32.

Continued on next page

TOK

Around the world you will often encounter different words for the same item, like gradient and slope, trapezium and trapezoid or root and surd.

Sometimes more than one type of symbol might have the same meaning, such as interval and set notation.

To what extent does the language we use shape the way we think?

c The price P you pay in Happy Shop in terms of the regular price R if there is a sign that says "Today you pay 15% less for every single item".

d The volume of a certain type of wood in terms of the weight:

Volume V (dm^3)	1	2	3
Weight W (kg)	0.9	1.8	2.7

2 Describe any similarities in the equations of the four functions.

3 Describe any similarities in the graphs of the four functions.

4 **Conceptual** How can you identify a direct variation relationship from a graph, equation or context?

A particular type of linear function known as **direct variation** is when the dependent variable is **directly proportional** to the independent variable. This can be represented as $f(x) = mx$, where $m = \frac{y}{x}$ is the **constant of proportionality**.

Exercise 4E

1 You are given that the conversion rate from Chinese yuan to Australian dollars is CNY1 = AUD0.21.

a Find an equation that will convert AUD to CNY.

b Interpret the gradient of your equation in context.

c Grace has AUD75 and wants to buy a souvenir on her trip to China worth 599 yuan. Determine how much more in AUD (to the nearest dollar) she will need to withdraw from her bank to make the purchase (assuming that her bank uses this conversion rate and charges no extra fees).

2 Hooke's law, $F = kx$, states that the force F (in Newtons or N) required to stretch a spring is directly proportional to the displacement, which is the length x (in metres) that the spring is stretched beyond its natural length. Andrew adds a mass corresponding to a force of 15 N to the bottom of a hanging spring and finds that his spring stretches 0.64 m.

a Find an equation to express the force as a function of the displacement.

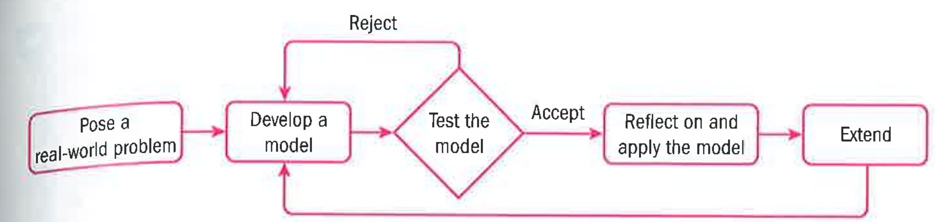
b Use your function to predict the length of the spring when a force of 80 N is applied.

If instead the spring is compressed (shortened), then both a negative force and negative displacement occur.

c Find the force needed to compress the spring by 1.5 m from its natural length.

The modelling process

When you describe a real-world situation with a linear function, you say that the function is a **model** of the situation. An appropriate model represents essential features of the situation in a way that can be used to analyse the situation and make predictions and decisions about it. The **modelling process** consists of some or all of the steps illustrated in the diagram and in the following example.



Example 6

Lucy is researching shipping companies for her business to use. She ships between 200 and 500 kg of products each week. Ted's Transport charges a rate of \$15.99 per kg, plus a flat fee. She knows her friend used Ted's Transport and paid about \$2800 to ship 170 kg of belongings.

a State a real-world problem:

List some relevant questions you might ask here.

b Develop a model:

i State the independent and dependent variables in this situation.

ii Explain why a linear model is appropriate for this situation.

iii Find an equation for your linear model in gradient-intercept form.

c Test the model:

The following week, Lucy makes her first shipment of 310 kg and pays \$5031.90. Comment on whether this is consistent with your model. If not, find an alternative model, giving reasons for how you choose to revise it.

d Apply your model:

The following week, Lucy budgets \$4500 for shipping. State the maximum weight she can ship, to the nearest kilogram.

e Reflect on the model:

i Find $C(0)$, and comment on its meaning in the context of the problem.

ii State a reasonable domain and associated range if Lucy uses it to predict weekly shopping costs.

a One possible question could be:

How can you calculate the cost of a shipment from its weight?

b **i** The dependent variable cost (C in \$) is a function of the independent variable weight (w in kg).

ii A linear model is appropriate because a constant rate of change between the two variables is specified: \$15.99 per kg.

iii $C(w) = 15.99w + 81.70$

Use the coordinate pair (170, 2800) and the known gradient, 15.99, to create an equation in point-gradient form:

$C - 2800 = 15.99(w - 170)$

Convert to gradient-intercept:

$C = 15.99w - 15.99 \times 170 + 2800$

$= 15.99w + 81.70$

Continued on next page

Functions

c $C(310) = 5038.60$

The model's prediction is close, but too high by \$6.70.

Because the gradient must be 15.99, you adjust the y-intercept by subtracting 6.70:

$C(w) = 15.99w + 75$

d 276 kg

Since $C \leq 4500$, you solve

$4500 \leq 15.99w + 75$

$w \leq 276.7$

e **i** $C(0) = 75$.

This shows that the flat fee is \$75.

ii Domain: $200 \leq w \leq 500$

Range: $3272 \leq C \leq 8070$

The problem specifies that she will ship between 200 and 500 kg.

Because the model is linear, the lowest and highest costs will occur at the endpoints: $C(200) = 3273$, $C(500) = 8070$.

Reflect What does it mean to make a mathematical prediction?

How do you use the modelling process to represent and make predictions about real-world contexts?

International-mindedness

The development of functions bridged many countries including France (René Descartes), Germany (Gottfried Wilhelm Leibniz), and Switzerland (Leonhard Euler).

The following example illustrates why the standard form of the equation of a straight line can be a useful representation.

Example 7

Siria reads her English textbook at a pace of 2 minutes per page and her Biology textbook at 3 minutes per page. She has two hours available to read.

a Write an equation that shows the relationship between the number of pages of English (x) and of Biology (y) that Siria can read in this time. Define all the variables.

b **i** Find the x- and y-intercepts of the graph of your equation.

ii Use these to sketch a graph of the equation.

iii Interpret each intercept in the context of the problem.

c Siria ends up reading 45 pages in total. Determine how many pages of English and of Biology she read.

a x = pages of English read

y = pages of Biology read

$2x + 3y = 120$

Siria will take $2x$ minutes to read x pages of English and $3y$ minutes to read y pages of Biology. These two times will add up to 2 hours (120 minutes).

b **i** y-intercept: $(0, 40)$

x-intercept: $(60, 0)$

To find the y-intercept, substitute $x = 0$ and solve for y. Similarly, for the x-intercept substitute $y = 0$ and solve for x.

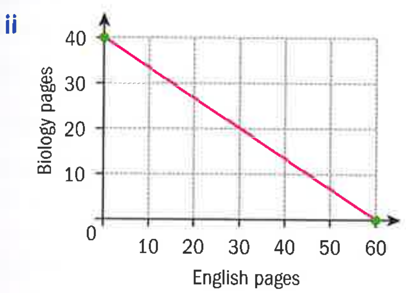
Note: The equation can also be converted to gradient-intercept form:

$2x + 3y = 120$

$3y = 120 - 2x$

$y = 40 - \frac{2}{3}x$

ii



Label bvoth axes.

Plot both intercepts and connect them with a line.

iii If Siria reads only English then she can read 60 pages in two hours. If she reads only Biology, she can read 40 pages.

c Siria read 15 pages of English and 30 pages of Biology.

Because the total number of pages must be 45, you have a second equation:

$x + y = 45$

Solving with technology,

$$\begin{cases} 2x + 3y = 120 \\ x + y = 45 \end{cases}$$

has the solution $(15, 30)$

Exercise 4F

1 UK£1 is worth approximately US\$1.33.

a Find a formula for $u(x)$, where u is the amount in US\$ corresponding to UK£x.

b Find the amount in UK£ equivalent to US\$100.

c Your bank charges you a fixed fee to exchange currency. You exchange UK£500 and receive US\$661.72.

i Find the amount of the fixed fee in US\$.

ii Write down the amount in US\$ you receive from the bank, B, as a function of the amount in UK£, x.

iii Determine the amount you must exchange to have at least US\$1000

2 Economists often use a **demand function** to express the relationship between the price of something and the number of people willing to buy it at that price. Seong Woo performs market research and determines that the number of people, N, who are willing to buy a concert ticket for price p (in euros) can be modelled by the equation $N(p) = 5000 - 36p$.

a State the meaning of the gradient parameter in this context. Explain the meaning of the y-intercept.

b Find the number of people willing to buy a concert ticket if it costs 75 euros.

c Solve $N(p) = 0$ and interpret the meaning of your answer.

d State the reasonable domain and associated range of this model.

e A **supply function** shows the relationship between the price of something and the quantity of it that a company is willing to sell. If the company putting on the concert has a supply function of $S(p) = 28p - 504$, find the price at which the supply and demand for tickets will be equal (This price is called the **equilibrium price**.)

3 The air pressure P (in atm) and the boiling point of water B (in degrees Celsius) can be found as functions of altitude (in kilometres above sea level) using the following functions:

$P(a) = -0.107a + 1$

$B(a) = -11.7a + 100$

As the boiling point of water decreases, food takes longer to cook. The United States Department of Agriculture recommends altering cooking procedures for raw meat at heights above 3000 feet, or 918 m.

a **i** Find the percentage of the sea-level atmospheric pressure at this altitude.

ii Find water's boiling point at this altitude.

A recipe book lists its cooking times for sea level and suggests adding 2 minutes of cooking time for every 5 °C that the boiling point lowers.

b Explain why a linear model is an appropriate choice to model the relationship between boiling point and cooking time.

c Find an equation to express the cooking time T (in minutes) as a function of boiling point B (in Celsius) for a recipe that takes 15 minutes to cook at sea level.

d Calculate how much longer a 15-minute recipe will take to cook in the world's highest city, La Rinconada, Peru, located at an altitude of 5130 m above sea level.

e State the reasonable domain and associated range of the model $T(B)$.

4 Alfie is buying some meat for pizza toppings, Pepperoni costs €3.50 per 100 g and Parma ham costs €6.50 per 100 g. He wants to spend €25 in total.

a Write an equation that represents the relationship between the amount of pepperoni and the amount of ham Alfie can order. Define all the variables.

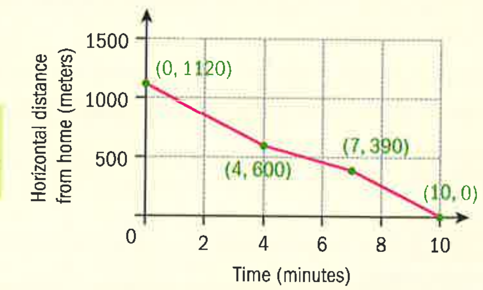
b Find the intercepts of the graph that represents your equation and interpret their meaning in context.

c Alfie decided that he would like to order 200 g more pepperoni than ham. Determine the amount of each topping he should buy.

Piecewise functions

Investigation 7

Amir's Run 6 from Investigation 1 is shown, along with a more detailed graph.



1 What does the graph tell you about the rate of change of each section of Amir's run? Explain how this relates to the context.

2 Why can this particular context not be modelled accurately by a single linear function?

3 Describe how you could model sections of Amir's run accurately with linear functions.

4 Find a linear equation in point-gradient form to model each section separately, filling in the table.

	Starting point	Ending point	Gradient	Point-gradient equation
Section 1				
Section 2				
Section 3				

5 Convert each equation to gradient-intercept form. In the "domain" column, write the domain of x-values to which the equation applies.

	Equation	Domain
Section 1		
Section 2		
Section 3		

6 How fast is Amir running on each piece of his run, and for how long does he run at that speed?

The set of equations you have created can be used to form a piecewise function:

$$f(x) = \begin{cases} -130x + 1120 & 0 \le x < 4 \\ -70x + 880 & 4 \le x < 7 \\ -130x + 1300 & 7 \le x \le 10 \end{cases}$$

To find the value of a piecewise function for a given value of x, use the formula corresponding to the domain in which x lies. For example, if $x = 5$, since $4 \le 5 < 7$ you use the second piece and find that $f(5) = -70(5) + 880 = 530$.

7 **a** If $x = 7$, would you use the second or third piece to evaluate $f(7)$? Justify your answer.

b Based on this, why is it important that the pieces of the function do not overlap on their domains?

Continued on next page

Functions

8 Use your piecewise model to find each of the following and interpret it in the context of the problem:

a $f(3)$ b $f(8)$ c x if $f(x) = 500$

A piecewise function is **continuous** if one piece connects to the next with no breaks or jumps.

9 Check that your piecewise function is continuous with the following steps:

a Verify using the equations that the endpoint of the first section matches the beginning point of the second section.

b Repeat for the endpoint of the second section and the beginning point of the third section.

c Why can there not be any other breaks or jumps elsewhere in the function?

10 **Factual** What is a piecewise function and how do you evaluate a piecewise function for a specific value?

11 **Factual** How do you check that a piecewise function is continuous?

12 **Conceptual** When is a piecewise linear function a useful model?

Example 8

Consider the piecewise function

$$h(t) = \begin{cases} \frac{1}{30}t & 0 \le t < 315 \\ 15 - \frac{1}{70}t & t \ge 315 \end{cases}$$

a Sketch the graph of the function.

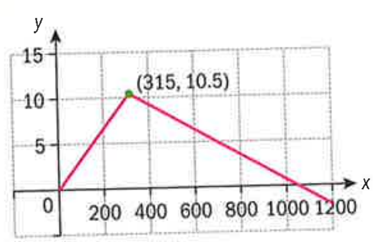
Suppose that $h(t)$ is modelling the height h (in centimetres) of water in a bathtub as a function of time t (in seconds).

b Give a possible explanation for what happens at $t = 315$.

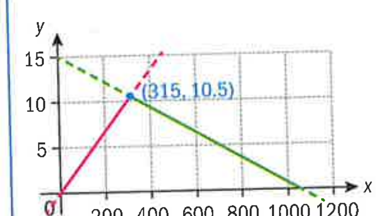
c Find the number of minutes until the bathtub is empty.

d Hence, write down a practical domain for $h(t)$.

a



Graph both lines completely using technology or by hand. Then remove the portions of the lines that do not fall within the domain of each piece.



b The bathtub begins to drain at this time, as the height of the water changes from increasing to decreasing.

Use technology or substitution to find the point where the pieces connect, at $t = 315$.

c $t = 1050$ seconds or 17.5 minutes

The bathtub will empty when $h = 0$. Find the t-intercept with technology or by substitution.

d $0 \le t \le 17.5$

Exercise 4G

1 Consider $f(x) = \begin{cases} 2x - 1 & -5 \le x < 2 \\ 4 - \frac{1}{2}x & 2 \le x < 7 \end{cases}$

a Sketch a graph of $f(x)$.

b Find

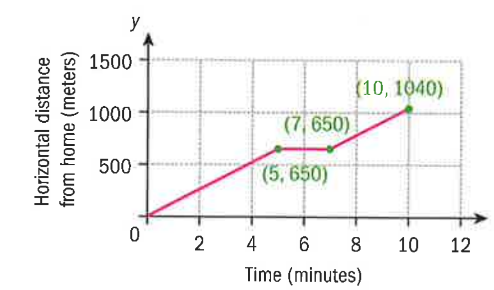
i $f(5.7)$ ii $f(-3.2)$

c Find the value(s) of x for which $f(x) = 2$.

d Verify that $f(x)$ is continuous.

e State the domain and range of $f(x)$.

2 a Construct a piecewise function for Amir's Run 5 from Investigation 1.



b Verify from the formulas that the function is continuous.

c Determine how long it took Amir to run from 300 m to 800 m away from home.

3 A cell phone company offers a plan with a flat rate of \$35 per month for up to one gigabyte (GB) of data. Any data used beyond 1 GB costs 6 cents per megabyte (MB), and 1000 MB = 1 GB.

a Find a piecewise linear model for the monthly cost C (in \$) of the phone as a function of the amount of data d (in GB) used.

b Determine the monthly cost if

i 500 MB is used ii 2 GB is used.

c The cell phone company offers an alternative plan that costs \$59/month for unlimited data. Determine when this plan is the better deal.

d It is January 3rd, and Yaqeen has used 172 MB of data. She is currently on the original \$35/month plan but is considering switching. Assume that her data usage will continue at the same constant rate.

i Estimate her data usage for the month of January.

ii Determine which plan is cheaper for Yaqeen, and how much cheaper it will be than the other plan.

Functions

Developing inquiry skills

Looking back at the opening problem, will the taxi driver reach his desired annual salary if he keeps his profit per kilometer at \$2.50?

How can you model the taxi driver's daily profit for different values of profit per kilometer? What about monthly profit? State any assumptions that you make in your model.

TOK

"The object of mathematical rigour is to sanction and legitimize the conquests of intuition"

—Jacques Hadamard

Do you think that studying the graph of a function contains the same level of mathematical rigour as studying the function algebraically?

4.3 Inverse functions

Investigation 8

Jamie moves from the United States to Thailand. She would like to convert Celsius temperatures back to the Fahrenheit scale as it is more familiar to her. Jamie knows that water freezes at 0 degrees Celsius or 32 degrees Fahrenheit, and that it boils at 100 degrees Celsius or 212 degrees Fahrenheit. She also knows that both scales have a constant rate of change.

1 Use this information to plot two points representing equivalent temperatures on a graph with temperatures in Celsius as the independent variable and temperatures in Fahrenheit as the dependent variable.

2 Use these two points to create a linear model $F(x) = mx + c$ that converts a Celsius temperature of x degrees to its equivalent temperature in Fahrenheit.

3 Normal human body temperature is 98.6°F, or 37°C. Test the accuracy of your model by checking that it predicts this temperature correctly.

After ten years, Jamie moves back to the US from Thailand. Now she's more familiar with the Celsius scale and would like a way to convert Fahrenheit temperatures to Celsius.

4 Explain why the independent and dependent variables will be reversed.

5 Plot the two points representing the boiling and freezing points of water on a graph with axes corresponding to your independent and dependent variables.

6 As you did before, create a linear model $C(x)$ in gradient-intercept form that converts a Fahrenheit temperature of x degrees to its equivalent temperature C in Celsius. Then test the accuracy of your model.

7 Absolute zero, the coldest possible temperature, is −459.67°F. Find this temperature in Celsius.

8 a Summarize the points that you've found on the graphs of your two functions in the table.

	$(x, C(x))$	$(x, F(x))$
Freezing		
Boiling		
Body temperature		
Absolute zero		

b Generalise your results. What point on the graph of $F(x)$ corresponds to the point (a, b) on the graph of $C(x)$? Explain why this makes sense in the context of temperature conversion.

c Considering that absolute zero is the coldest possible temperature and that there is no upper limit on temperatures, find the practical domain and associated range of each model.

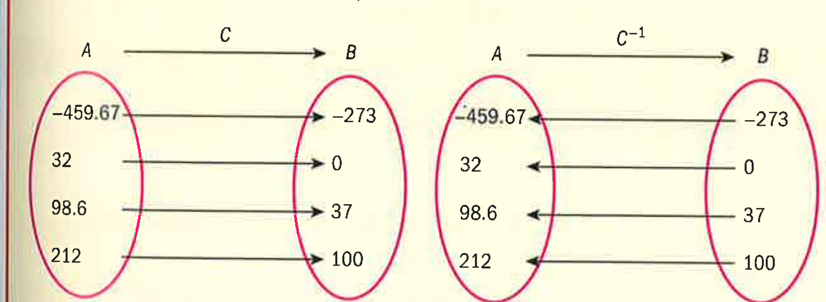
	$C(x)$	$F(x)$
Domain		
Range		

d What is the relationship between the domain and range of C and F? Explain why this makes sense in context.

9 **Conceptual** How are the inputs and outputs of inverse functions related? How are the practical domain and range of a linear model related to the domain and range of its inverse?

In Investigation 8 you created an example of two **inverse functions.** You say that $C(x)$ and $F(x)$ are inverses because if $C(a) = b$ then $F(b) = a$ for any a in the domain of $C(x)$.

- You use the notation $f^{-1}(x)$ to denote the inverse function of $f(x)$. If $f(a) = b$, then $f^{-1}(b) = a$. Informally, you think of the inverse function $f^{-1}(x)$ as "reversing" or "undoing" the function $f(x)$.
- If $C(x)$ is a function that maps the domain A onto the range B, written $C: A \to B$, then $C^{-1}: B \to A$, as illustrated.



For the functions in Investigation 8, you could also write the Fahrenheit function, $F(x)$, as the inverse of the Celsius function: $F(x) = C^{-1}(x)$.

Example 9

The function $f(x) = 7 - \frac{1}{2}x$ is defined on the domain $-5 \le x \le 5$.

a Find $f^{-1}(6)$.

b Determine whether $f^{-1}(11)$ exists. If it does, find it.

c Find the range of $f(x)$.

d State the domain and range of $f^{-1}(x)$.

e Solve $f^{-1}(x) = 0$.

Continued on next page

Functions

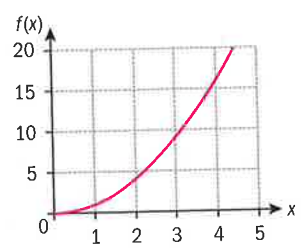
a $f^{-1}(6)=2$	Since $f(a)=b$ means $f^{-1}(b)=a$, $f^{-1}(6)=a$ means $f(a)=6$. Solve $6=7-\frac{1}{2}x$ algebraically or with technology.
b $f^{-1}(11)$ is not defined.	$11=7-\frac{1}{2}x$ means that $x=-8$, which is outside the domain of f.
c Range: $4.5\le y\le 9.5$	You know that $f(x)$ is linear, so you can find its values at the endpoints of the domain to find the range. $f(-5)=9.5$, $f(5)=4.5$
d Domain: $4.5\le x\le 9.5$ Range: $-5\le y\le 5$	The domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.
e $x=7$	$f^{-1}(x)=0$ means $f(0)=x$.

Exercise 4H

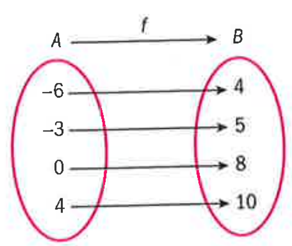
1 For each function, find (or estimate) $f^{-1}(4)$. Justify your answer.

a $f(x)=3x+5$ **b** $x\to\frac{1}{2}x$

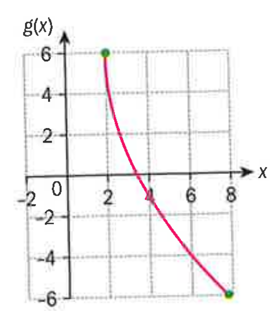
c



d



2 The graph of $g(x)$ is shown.



a Find the domain and range of the inverse, $g^{-1}(x)$.

b Estimate the solution to $g^{-1}(x)=4$.

3 The circumference of a circle is a function of its radius: $C(r)=2\pi r$. Find $C^{-1}(8)$ and interpret its meaning in context.

4 For the function $f(x)=2x+c$, find the value of c such that $f^{-1}(21)=-5$.

Equations and graphs of inverse functions

Because the inverse function $f^{-1}(x)$ reverses the input and output of $f(x)$, you can find its equation by switching the variables x and y in the equation for $f(x)$.

For example, to find the inverse of the °C to °F function $F(x)=\frac{9}{5}x+32$:

Write $F(x)$ as y: $y=\frac{9}{5}x+32$

Swap x and y: $x=\frac{9}{5}y+32$

Solve for y: $x-32=\frac{9}{5}y$

$y=\frac{5}{9}(x-32)$

$y=\frac{5}{9}x-17.8$ (3 s.f.)

So $F^{-1}(x)=\frac{5}{9}x-17.8$.

Investigation 9

1 Verify that the equation for F^{-1} found above agrees with your equation for $F^{-1}(x)$ from Investigation 8.

2 Use graphing technology to plot both $F(x)$ and $F^{-1}(x)$ on the same coordinate plane. Remember to use the domain and range of each function to estimate a reasonable graphing window the equation for F^{-1} found above.

3 Find the inverse of each of the following functions using the algebraic process demonstrated above. Then add both the function and its inverse to your graphing screen. Graph each function and its inverse in the same colour, if possible.

a $f(x)=3x$ **b** $g(x)=5x-100$ **c** $h(x)=-\frac{1}{2}x-25$

4 a Use your graphs and/or equations to complete the table.

Function	y-intercept	x-intercept	Point of intersection with inverse
$F(x)$			
$F^{-1}(x)$			
$f(x)$			
$f^{-1}(x)$			
$g(x)$			
$g^{-1}(x)$			
$h(x)$			
$h^{-1}(x)$			

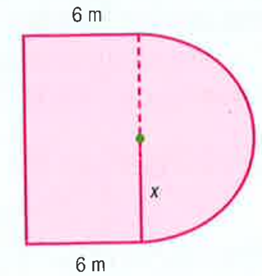
b Describe any patterns you notice in the table.

5 Graph the **identity line** $y=x$ on your coordinate plane. What relationship do you notice between the graph of a function, the graph of its inverse, and the line $y=x$? Can you explain how this relates to the fact that $f(a)=b$ means $f^{-1}(b)=a$?

6 Conceptual How can the relationship between inverse functions be seen in their graphs and their equations?

Example 10

Laura is planning to construct a pool consisting of a rectangle joined to a semicircle of radius x, as shown in the diagram. The sides of the rectangle perpendicular to this side will be of length 6 m. The pool must fit within an 18 m by 18 m square.



a Find a model for the perimeter P (in meters) of this pool as a function of its radius x.

b Find the reasonable domain and associated range of this model.

c Find an equation for the inverse function $P^{-1}(x)$ in gradient–intercept form.

d State the independent and dependent variables of the function $P^{-1}(x)$.

e Edging for the pool comes in three lengths: 15, 30 or 45 metres. Determine which length will give Laura a rectangle that is closest to a square.

a $P(x) = \pi x + 2x + 12$	The rectangle has side lengths 6 and $2x$. The semicircle has circumference $\frac{2\pi x}{2} = \pi x$.
b Domain: $0 < x \leq 9$ Range: $12 < P \leq 58.3$	x represents the radius, which must be positive. As the pool must fit within an 18×18 square, $x + 6 \leq 18$ and $2x \leq 18$, so $x \leq 9$. $P(0) = 12$ and $P(9) = 58.3$.
c $P^{-1}(x) = 0.194x - 2.33$	$x = \pi y + 2y + 12$ Swap x and y. $x - 12 = (\pi + 2)y$ Factorise. $\frac{x-12}{\pi+2} = y$ Divide by the y coefficient to isolate y and convert numbers to decimal approximations.
d Independent variable x is the perimeter. Dependent variable y is the radius associated with that perimeter.	
e Laura should choose the 30 m edging because it results in the rectangle whose dimensions are most similar.	Find the radius for each perimeter by evaluating P^{-1}, then double the result to find one side of the rectangle. The length of the other side is 6 m: $P^{-1}(15) = 0.583$ m: 6 m × 1.166 m $P^{-1}(30) = 3.50$ m: 6 m × 7 m $P^{-1}(45) = 6.62$ m: 6 m × 12.8 m

Exercise 4I

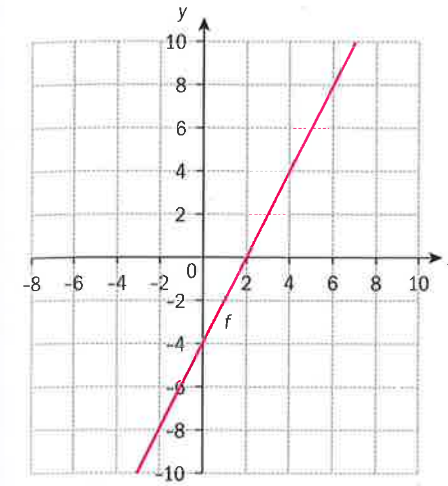
1 For each of the following functions:

i Sketch a graph of the inverse function.

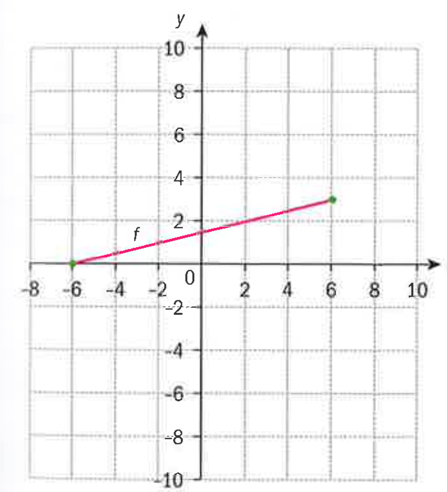
ii State the domain and range of $f^{-1}(x)$.

iii Estimate the solution of $f(x) = f^{-1}(x)$.

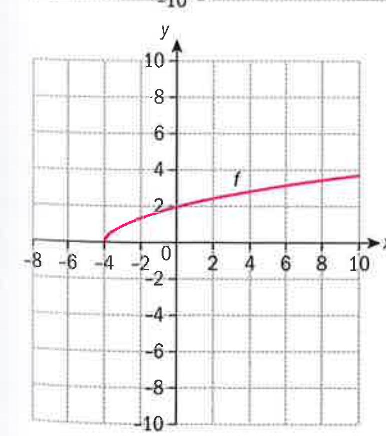
a



b



c



2 Consider $f(x) = -2.5x + 5$ for $0 \leq x \leq 3$.

a Draw the graph of f on a pair of axes. Use the same scale on both axes.

b Point A lies on the graph of f^{-1} and has coordinates $(b, 3)$. Find the value of b.

c Determine the missing entries in the table.

Function	Domain	Range
f	$0 \leq x \leq 3$	
f^{-1}		

d Draw the graph of f^{-1} on the same set of axes used in part **a**.

e Find the coordinates of the point that lies on the graph of f, the graph of f^{-1} and the graph of $y = x$.

3 Dieneke travels from Amsterdam to Budapest, a distance of 1400 km. On the first day she covers 630 km. She notes that her average speed is 90 km hr^{-1} on day 1. Assume that she continues at the same average speed on the second day without any breaks.

a Supposing that she begins driving at 8 am on day 2. Find an equation that expresses her total distance travelled from Amsterdam, d (in kilometres), as a function of time, x (in hours since 8 am of day 2).

b Find an equation for the inverse function $d^{-1}(x)$.

c Use the inverse function to predict the time (to the nearest minute) at which Dieneke will arrive at the following points:

i halfway between Amsterdam and Budapest

ii in Prague, 880 km from Amsterdam

iii in Budapest.

d Comment on which of the three places Dieneke should plan to stop for a midday meal.

Composition and the formal definition of inverse

You think of an inverse function as reversing or undoing a function. To formalise this in a definition, you need a way to show that you are performing more than one function in a row.

For example, at a restaurant in Rome, you notice that a €2 per table charge has been added to your bill for x euros. You also plan to add a 10% tip. If you calculate the tip and then add the table charge, you would perform the calculations

$x \to 1.1x \to 1.1x + 2$

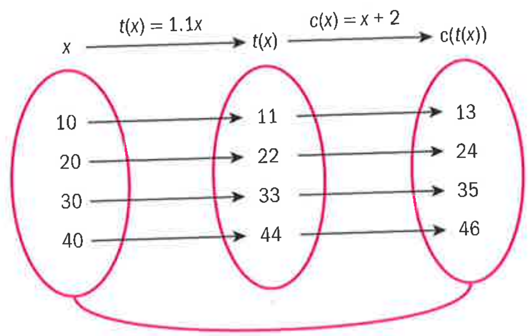
If you add the table charge and then the tip, your process would be

$x \to x + 2 \to 1.1(x + 2)$

Let $c(x) = x + 2$ represent the table charge function and $t(x) = 1.1x$ represent the tip function. Then the first case above can be represented as

$x \to t(x) \to c(t(x))$

The mapping diagram shows the final charge that would result for several different bills.



TOK

Does a graph without labels have meaning?

You call the inputting of one function into another a **composition of functions**, and denote it as $c \circ t(x)$ or $c(t(x))$.

Note that a composition is also a function; in the example above, it is given by the formula $c(t(x)) = 1.1x + 2$.

Example 11

If $f(x) = 5 - \frac{1}{6}x$ and $g(x) = \frac{3}{2}x + 12$, find a simplified expression for

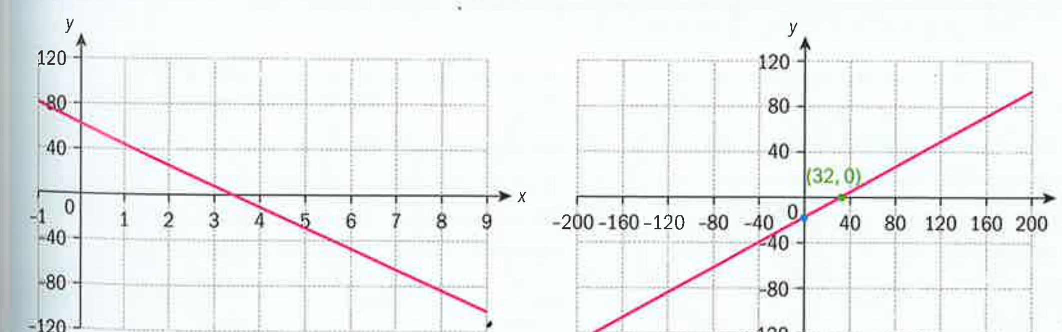
a $f \circ g(8)$

b $f \circ g(x)$

a $f \circ g(8) = f(g(8))$ $= f\left(\frac{3}{2}(8) + 12\right)$ $= f(24)$ $= 5 - \frac{1}{6}(24)$ $= 1$	Note that in the composition $f(g(x))$, the x-value is input first into g. The output of g (in this case, 24) is the input of f.
b $f \circ g(x) = f\left(\frac{3}{2}x + 12\right)$ $= 5 - \frac{1}{6}\left(\frac{3}{2}x + 12\right)$ $= 5 - \frac{1}{4}x - 2$ $= -\frac{1}{4}x + 3$	Here you replace the x in $f(x)$ with the expression for $g(x)$.

Example 12

In the Earth's atmosphere, as altitude increases the average temperature decreases, as shown by the graph of the function $T(x)$ on the left. On the right, the function $C(x)$ converting Fahrenheit to Celsius temperatures is graphed.



a Use the graphs to estimate $C(T(2))$ and interpret your answer in context.

b Predict the temperature in Celsius at the top of Mount Everest (8848 m).

c Solve $C(T(x)) = 0$ and interpret your answer in context.

Continued on next page

Functions

a $C(T(2)) \approx C(20) \approx -5$

This means that at an altitude of 2 km above sea level, the average temperature is −5°C.

From the graphs, $T(2) \approx 20$ and $C(20) \approx -5$.

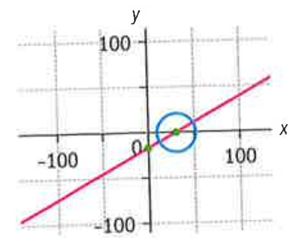
b $C(T(8.848)) \approx C(-100) \approx -75$°C

c $C(T(x)) = 0$ when $T(x) = 32$.

This happens when $x \approx 1.5$.

At an altitude of 1.5 km above sea level, the average temperature is 0°C.

Work backwards: first find the zeroes or x-intercepts of $C(x)$ from the graph. There is one at (32, 0):



So the function $T(x)$ must give an output of 32. Estimate the intersection of $y = 32$ and $T(x)$.

Investigation 10

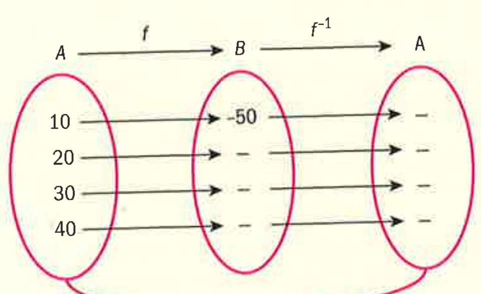
In this investigation you will determine how to use compositions to prove that two functions are inverses.

In Investigation 9 you found that the inverse of the function $f(x) = 5x - 100$ is $f^{-1}(x) = \frac{1}{5}x + 20$.

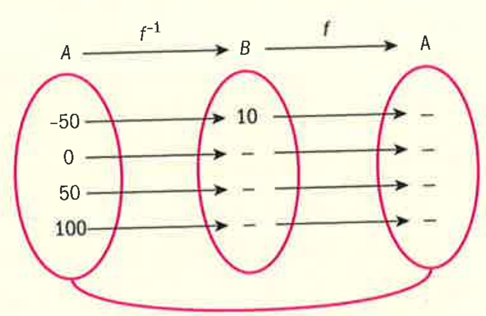
1 Find the following values, and comment on how your results relate to the concept of an inverse function.

a $f(10)$ **b** $f^{-1}(-50)$

2 Complete the following mapping diagrams for the compositions $f^{-1} \circ f(x)$ and $f \circ f^{-1}(x)$.



$f^{-1} \circ f(x)$



$f \circ f^{-1}(x)$

b What do you notice about the compositions $f^{-1} \circ f(x)$ and $f \circ f^{-1}(x)$?

3 Based on this, what is true about the composition of inverses? Why does this make sense?

4 Test your conjecture on the function $f(x) = 3x + 2$ by finding its inverse function and composing the function with its inverse.

5 **Conceptual** How do you prove that two functions are inverses?

You call the function $i(x) = x$ that "does nothing" to x the **identity function**. The composition of a function and its inverse, in either order, will result in the identity: $f \circ f^{-1}(x) = f^{-1}(x) \circ f(x) = x$.

Exercise 4J

1 If $f(x) = 8x - 25$, $g(x) = 4 - x$ and $h(x) = \frac{3}{2}x$, find:

a $f \circ g(1)$

b $h \circ f(4)$

c $g \circ f(x)$

d $h \circ h \circ h(x)$

2 Returning to the example of the bill at the restaurant, recall that $c(x) = x + 2$ represents the table charge function and $t(x) = 1.1x$ represents the tip function.

a Show that $c(t(x)) \neq t(c(x))$ by finding a simplified function for each.

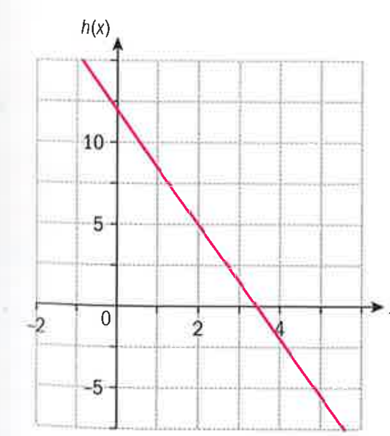
b Determine which order of composition will provide the waiter with the larger tip, and state, in terms of x, how much larger.

3 Find two functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$ if:

a $h(x) = \frac{7}{3}x - 5$

b $h(x) = 4(x - 2)$

c $h(x)$ is shown in the graph.



4 The number of window panes a company can produce is a function of time t, measured in hours of operation per day, and is given by $w(t) = 7t - 3$. The company's profit \$$P$ is a function of the number of windows sold, given by $P(w) = 50w - 2000$.

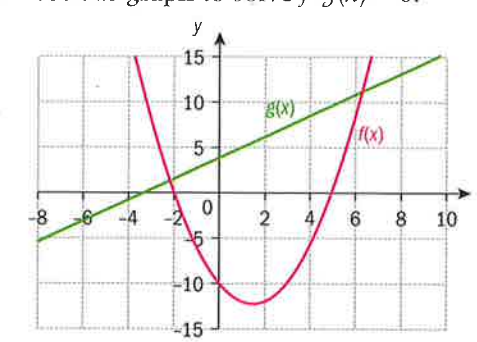
a Find the company's profit (or loss) if it operates for 4 hours a day.

b Find the company's profit as a function of time, expressed in the form $P(t) = mt + c$ where $m, c \in \mathbb{R}$.

c Determine the number of hours (to the nearest hour) that the factory must operate in order to earn a (positive) profit.

5 Show that the temperature conversion functions $C(x) = \frac{5}{9}x - 17.8$ and $F(x) = \frac{9}{5}x + 32$ are approximate inverses of each other.

6 Use the graph to solve $f \circ g(x) = 0$.



4.4 Arithmetic sequences and series

Investigation 11

Pablo starts his first full-time job at the age of 24. In his first year he earns $3250 per month after taxes, or $39 000 per year. He is given a salary schedule that shows how his salary will increase over time:

Years in job	1	2	3	4	5
Annual salary	39 000	39 900	40 800	41 700	42 600

The salaries can also be written as an ordered list of numbers or **sequence:**

39 000, 39 900, 40 800, 41 700, 42 600

Each number is a **term**. To identify a specific term in the sequence, for example, the 4th, you write u_4.

So, in the sequence above,

$u_1 = 39\,000$ (also called the **first term**)
$u_2 = 39\,900$ (the second term)
$u_3 = 40\,800$

and so on.

In general, u_n is the nth term. The entire sequence is denoted by $\{u_n\}$.

1 Complete the table by calculating the difference between every two terms. The first is done as an example. What do you notice?

Years in job (term number)	1	2	3	4	5
Annual salary	39 000	39 900	40 800	41 700	42 600
Difference between terms	–	39 900 − 39 000 = 900			

2 a Plot the first five terms of the sequence from the table above and verify they form a straight line. A sequence that creates a linear graph is called **arithmetic**. What is the gradient of this line?

How can it be predicted from the terms of the sequence?

b What is the y-intercept of the line associated with the sequence?

How can it be predicted from the terms of the sequence?

3 **Factual** How do you determine whether a sequence is arithmetic without graphing it?

4 **Factual** If a sequence is arithmetic, how are the parameters of the corresponding linear function related to the first term and difference between terms of the sequence?

5 Use the pattern in the sequence $\{u_n\}$ to predict Pablo's annual salary if he stays in the same job and retires at the age of 65. You may find it helpful to complete the table:

Age	24	25	26	27	...	64	65
Number of term (n)	1	2	3	4	...		
Number of differences added (d = $900)	0	1	2		...		
Term (u_n)	39 000	39 000 + 900 = 39 900	39 000 + 2 × 900 = 40 800		...		

6 **Conceptual** How can you describe the nth term of an arithmetic sequence?

A sequence is **arithmetic** if the difference between consecutive terms is constant. This difference is called the **common difference**.

To calculate the n **th term** of an arithmetic sequence $\{u_n\}$ with first term a_1 and common difference d:

$u_n = u_1 + (n-1)d$

Verify that this formula is consistent with the method you described in Investigation 11.

Example 13

For each of the following arithmetic sequences:

i State its first term and common difference.
ii Find the 10th term of the sequence.
iii Determine, giving your reasons, whether 49 is an element of the sequence.

a $u_n = 3n + 1, n \in \mathbb{Z}^+$. Remember that $\mathbb{Z}^+$ is the set of positive integers: {1, 2, 3, ...}.
b 206, 199, 192, ...

a i Arithmetic. $u_1 = 4, d = 3$	The formula for calculating u_n is linear, so the sequence is arithmetic. The gradient or common difference is 3 and $u_1 = 3(1) + 1 = 4$.
ii $u_{10} = 31$	Using the formula $u_n = u_1 + (n-1)d$: $u_{10} = 4 + (10-1) \times 3 = 31$
iii Yes, 49 is the 16th term.	You wish to determine whether any whole-number value of n results in $u_n = 49$. Use algebra or technology to solve the following: $49 = 4 + (n-1) \times 3$ This gives $n = 16$.
b i Arithmetic. $u_1 = 206, d = -7$	The sequence decreases by 7.
ii $u_{10} = 143$	As d is negative, you subtract $7(n-1)$: $u_{10} = 206 - 7(10-1) = 143$
iii No, 49 lies between u_{23} and u_{24}.	You solve as in part **b**: $49 = 206 - 7(n-1)$ Solving yields $n = 23.4$ (3 s.f.), a non-integer.

Example 14

A piledriver is a machine used in construction to drive support poles into the ground by repeatedly striking them. Acme construction company uses a piledriver that drives support poles 0.12 m deeper into the ground with each strike. The current support pole has already been driven 13.6 m into the ground.

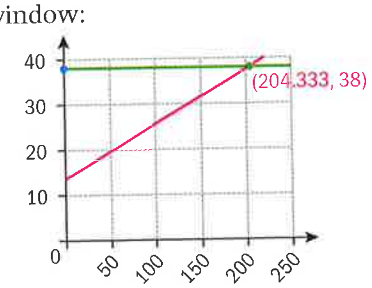
a If the sequence $\{u_n\}$ represents the depth of the support pole after n strikes, find the first three terms of the sequence.

b Write down an expression for the nth term of the sequence.

c The support poles must be driven to a depth of at least 38 m below ground. Determine

i the number of strikes needed to reach this depth

ii the exact depth it will then have reached.

a $u_1 = 13.6$, $u_2 = 13.72$, $u_3 = 13.84$	The pole is at an initial depth of 13.6 m, so $u_1 = 13.6$. The piledriver adds an additional depth of 0.12 m per strike, so $d = 0.12$.
b $u_n = 13.6 + 0.12(n - 1)$	Using $a_n = a_1 + (n - 1)d$.
c i 205 strikes	u_n represents the depth after n strikes, so you need to solve $u_n \geq 38$. Solve directly or using graphing technology, choosing an appropriate window:  This gives $n = 204.33$. As n must be a whole number and the depth must be 38 m or more, you choose the next largest whole number, so $n = 205$.
ii $u_{205} = 38.08$ m	Using the formula, $u_{205} = 13.6 + 0.12(205 - 1)$.

Exercise 4K

1 For each of the following sequences:

i Predict the next three terms of the sequence.

ii Determine whether the sequence is arithmetic, stating the common difference if it is.

a 7.3, 15.9, 24.5, 33.1, …

b 5, −17, −39, −61, …

c 61, 70, 88, 95, …

d 10, 5, 2.5, …

2 For each of the sequences below, use the formula to write down the first three terms if $n \in \mathbb{Z}^+$. Then determine whether the sequence is arithmetic; state the first term and the common difference if it is.

a $a_n = 2.5n + 7$

b $b_n = 5000 - 2350n$

c $c_n = n(2n + 1)$

3 The first terms of an arithmetic sequence are 5, 9, 13, 17, …

a Write an expression for the general (nth) term of this sequence.

b Determine whether 116 is a term of this sequence.

4 When a company opened up it had 85 employees. It decided to increase the number of employees by 10 at the beginning of each year.

a Find the number of employees during the second year and during the third year.

b Determine the number of employees this company will have during the 10th year.

c Determine after how many years the company will have 285 employees.

5 A sequoia tree that was 2.6 m tall when it was planted in 1998 grows at a rate of 1.22 meters per year.

a Write down a formula to represent the height of the tree in years, with a_1 representing the height in 1998.

b Find the height of the tree in 2025.

c The tallest living sequoia tree, the General Sherman tree in the US Sequoia National Park, has a height of 84 m. Determine in what year the tree planted in 1998 would reach that height if it continues with the same rate of growth.

Sometimes you will need to find missing parameters (first term, common difference or number of terms) by interpreting information about your sequence.

Example 15

Julie swims each day in a 25 m pool. Today, she notes that she swims her first warm-up lap in 1 minute and 6 seconds. She then swims the remainder of her laps at a constant speed. After her fifth lap, she checks the clock and sees that 4 minutes 18 seconds have passed. Her entire swim takes 19 minutes 30 seconds.

a If the sequence $\{u_n\}$ represents Julie's total swim time after each lap in minutes, write down u_1 and u_5.

b Find the common difference, d, of this sequence. Explain its meaning in the context of the problem.

c Determine the number of laps that Julie swims.

a $u_1 = 1.1$, $u_5 = 4.3$	You convert minutes and seconds to minutes: 6 seconds $= \frac{6}{60}$ minutes $= 0.1$ minutes.
b $d = 0.8$ Julie takes 0.8 minutes, or 48 seconds, to swim one lap.	Four differences are added between the first and fifth terms, so $\frac{(4.3 - 1.1)}{4} = 0.8$. This can also be solved by substituting into the equation $u_5 = u_1 + 4d$: $4.3 = 1.1 + 4d$, $d = \frac{4.3 - 1.1}{4}$

Continued on next page

c Julie swam 24 laps.	The last (nth) term of the sequence is 19.5: $1.1, u_2, u_3, u_4, 4.3, \ldots, 19.5$ Substitute into the nth-term equation: $1.1 + (n - 1) \times 0.8 = 19.5$ Solving using algebra or technology gives $n = 24$.

Exercise 4L

1 The first term of an arithmetic sequence is −10 and the seventh term is 1.

- **a** Find the value of the common difference.
- **b** Find the 15th term of this sequence.

2 Consider the arithmetic sequence $0, u_2, u_3, u_4, 10, \ldots$

- **a** Find the common difference.
- **b** Find u_3.

3 You enter your pet frog in a 1000 cm jumping contest. He hops the full race and beyond! His distance from the finish line is represented by the sequence $975, 950, \ldots, -225$. Here, u_n represents the distance from the finish line after n hops.

- **a** Write down the common difference and interpret its meaning in context.
- **b** Find the 10th term. Interpret its meaning in context.
- **c** Determine the number of hops it takes your frog to finish the race.
- **d** Determine the number of terms in the sequence. Interpret this number in the context of the problem.

4 In an arithmetic sequence, $u_3 = 12$ and $u_{10} = 43.5$. The common difference is d.

- **a** Write down two equations in u_1 and d to show this information.
- **b** Find the value of u_1 and of d.
- **c** Find the 100th term.

5 The first row of a theatre has 22 seats. The tenth row has 49. The last row has 106. The number of seats per row follows an arithmetic progression.

- **a** Find an expression for the number of seats in the nth row.
- **b** Determine the number of rows in the theatre.

6 A trebuchet is a medieval weapon that hurls a heavy object through the air, similar to a catapult. The distance that the object flies depends linearly on its weight. Tyler builds a trebuchet and collects a set of objects each weighing an an integer value of kilograms. He finds that a 1 kg object travels 12 m. He also notes that an object weighing 9 kg travels twice as far as one weighing 3 kg. Let $\{u_n\}$ represents the distance travelled (in metres) by an object with a weight of n kg.

- **a** Write down u_1.
- **b** Write down expressions for u_3 and u_9 in terms of u_1 and d.
- **c** Write an equation relating u_3 and u_9.
- **d** Solve this equation to find d.
- **e** If Tyler wants to hurl an object 100 m, calculate how heavy this object should be.

Simple interest

The **principal** or **capital** is the money that you initially put in a savings institution or a bank. The **account balance** is the total amount of money either saved or owed at given point in time.

Interest is a percentage of the principal or account balance paid to you (for a savings account) or paid by you (for a loan). **Simple interest** is calculated as a fixed percentage of the principal and hence is a type of arithmetic sequence.

TOK

Do all societies view investment and interest in the same way?

What is your stance?

The total amount of simple interest I earned on a principal P over n years at an interest rate of r% per year can be calculated using the formula

$I = P \times r \times n$

The total savings account balance is given by

$A = P + (P \times r \times n)$

Example 16

An amount of $5000 is invested at a simple interest rate of 3% per annum (p.a., meaning each year) for a period of 8 years.

- **a** Calculate the interest received after 8 years.
- **b** Find the account balance after the 8 years.
- **c** Find the time it will take for the account balance to double.

a $1200	Substituting in the formula $I = P \times r \times n$: $I = 5000 \times 0.03 \times 8 = 1200$
b $6200	The account balance is found by adding the interest to the capital: $5000 + 1200 = 6200$
c 34 years	The account balance must be double its initial principal: $2 \times 5000 = 10000$. Substituting and solving with technology: $10\,000 = 5000 + 5000 \times 0.03 \times n$ $n = 33.3$ (3 s.f.) As the interest is calculated at the end of the year, the principal will not exceed double until the 34th year.

Example 17

Gabe pays a total of $1500 in simple interest on a car loan over 5 years with a 2.3% interest rate p.a. Find, to the nearest dollar, the original value of the car.

$P = \$13\,043$	Using $I = P \times r \times n$: $1500 = P \times 0.023 \times 5$ $P = 13\,043$ Note that for a loan, simple interest is paid to the bank. It is not added to the account balance as savings interest is.

Exercise 4M

1 Calculate the total simple interest and final account balance of a savings account in which $9000 is invested at a rate of 5.9% p.a. for 3 years.

2 Find the amount of money borrowed if after 7 years the interest charged is UK£9000 at a rate of 7.5% per annum.

3 Find the annual interest rate if €1840 in interest is earned after 5 years on a principal of €8000.

4 Stephen deposits $8600 in a bank account that pays simple interest at a rate of 6.5% per annum. Determine the year in which Stephen's money doubles.

Arithmetic series

In Investigation 11 you modelled Pablo's annual salary as an arithmetic sequence with first term $39 000, common difference d = $900, and 42 terms:

$a_n = 39\,000 + 900(n - 1)$

What if Pablo would like to know how much money he will earn in total over the course of his career? You can represent this total as a **series**, or the sum of a sequence:

$S_{42} = 39\,000 + 39\,900 + 40\,800 + \ldots + 75\,900$

Here, S_{42} stands for the sum of the first 42 terms of the sequence.

You can also represent this series with the **sigma notation** introduced in Chapter 2:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{42} (39\,000 + 900(i - 1))$$

n = number of terms (and ending value of i)

Index variable and starting value

Term i (general formula for a term)

How do you evaluate this sum? You will investigate a simpler series to look for a pattern that you can generalise.

Investigation 12

According to mathematical legend, Karl Gauss was asked as a boy to add the numbers 1 to 100 in order to keep him busy during class. He found the sum almost instantly. So will you.

1 The sum of the numbers 1 to 100 is written below. Instead of adding them in order, you will add them up in the indicated pairs. What is the sum of each pair?

2 How many pairs would you have to total, if you continued this process?

3 Use your answers above to determine the sum of the series.

4 Try the same method on this smaller series, and confirm that your method gives you the same sum as if you add it in order:

5 + 8 + 11 + 14 + 17 + 20

5 **Factual** Generalise your method to a formula for finding the sum of any arithmetic series:

If $S_n = u_1 + u_2 + u_3 + \ldots + u_n$, then $S_n = \sum_{i=1}^{n} u_i = ?$

6 Use your formula to find Pablo's lifetime earnings.

7 **Factual** When is it more useful to represent a context as an arithmetic series than as a linear function?

8 **Conceptual** How is an arithmetic series represented and calculated?

The sum of n terms of an arithmetic series with first term u_1 and common difference d is given by

$$S_n = \sum_{i=1}^{n} u_i = \frac{n}{2}(u_1 + u_n)$$

or

$$(2u_1 + (n-1)d) \times \frac{n}{2}$$

Example 18

a Find the sum of the arithmetic series $-10 + (-6) + (-2) + \ldots + 90$.

b Write down this series in sigma notation.

c Find the least number of terms from this series needed to obtain a sum greater than 100.

Continued on next page

Functions

a $S_{26} = 1040$

This is an arithmetic series with common difference 4.

To find n, solve $u_n = 90$:

$90 = -10 + (n - 1) \times 4$

Solving with technology or algebraically, $n = 26$.

Then:

$$S_{26} = \frac{26}{2}(-10 + 90) = 1040$$

b $\sum_{i=1}^{26}(-10 + (i-1) \times 4)$

Since $u_1 = -10$ and $d = 4$, you can write the general term u_i. From part **a** the number of terms $n = 26$, so you represent the series in the sigma notation:

$$\sum_{i=1}^{n} u_i$$

c 11 terms

To find n so that the sum is greater than 100, you set up the inequality:

$$S_n = \frac{n}{2}(2 \times (-10) + (n-1) \times 4) > 100$$

$$= \frac{n}{2}(-20 + (n-1) \times 4)$$

Solve with technology by graphing or using a table.

n	Sum
9	54
10	80
11	110

The sum is smaller than 100 when $n = 10$, and larger when $n = 11$.

Example 19

A skydiver jumping from a plane 4200 m above the ground falls 9.5 metres in the first second. In each succeeding second she falls 7.8 metres more than in the previous one. This continues until she opens her parachute.

a Find the distance that she falls during the 20th second.

b Find the total distance she has fallen after 20 seconds.

c The skydiver should open her parachute at or before 600 m above ground level. Determine the latest time, to the nearest second, that she can open her parachute.

A second skydiver leaving the same plane also falls 9.5 metres in the first second, but falls d metres more each second than the previous one. After 5 seconds she has fallen approximately 120 metres.

d Find d to the nearest tenth of a metre.

e Explain, with a reason, which skydiver is falling faster.

a $u_{20} = 9.5 + (20 - 1) \times 7.8 = 157.7\text{ m}$

The distance the skydiver has travelled after each second is an arithmetic sequence with first term 9.5 and common difference 7.8.

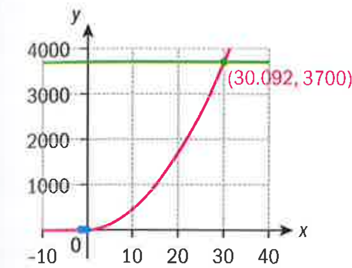
b $S_{20} = (9.5 + 157.7) \times \frac{20}{2} = 1672\text{ m}$

Because the sequence represents the additional distance she falls with each second, her total distance will be the sum of these second-by-second distances.

c $3700 = (2 \times 9.5 + (n-1) \times 7.8) \times \frac{n}{2}$

At a height of 600 m above ground level the skydiver will have fallen a total of $4200 - 600 = 3700\text{ m}$

The sum must therefore be less than 3700. You use this to set up an inequality and solve with technology:



$n \le 30.092$

The latest the skydiver can safely open her parachute is after 30 seconds.

At 30 seconds she will be above 600 m; at 31 seconds she will be below.

d $d = 7.3\text{ m}$

For the second skydiver the first term is still 9.5 but the common difference is unknown.

You know that $S_5 = 120$.

Substitute the known information into the sum formula and use technology to solve for the unknown, d:

$$120 = (2 \times 9.5 + (5-1)d) \times \frac{5}{2}$$

$d = 7.25 = 7.3\text{ m}$ to the nearest tenth of a metre.

e The first skydiver is faster, as her distances are increasing by 7.8 m each second rather than 7.3 m.

Exercise 4N

1 Find the sum of the first 20 terms of the arithmetic series $6 + 3 + 0 - 3 - 6 - \ldots$

2 Find the sum of the first 30 multiples of 8.

3 Consider the series $52 + 62 + 72 + \ldots + 462$.

- **a** Find the number of terms.
- **b** Find the sum of the terms.

4 Janet begins farming on 1 acre of land. Each year, as her business grows, she buys more land to farm. She buys an additional 5 acres in the second year and 9 acres in the third year.

- **a** Assuming that the amount of land she buys each year continues in the same pattern, write down a sum that represents the total land she owns after ten years.
- **b** Write down this sum in sigma notation.
- **c** Calculate S_{10} and find S_n in terms of n.
- **d** Find the value of n for which $S_n = 2000$, and interpret the meaning of n in context.

5 An arithmetic series has $S_1 = 4$ and $d = -3$.

- **a** Write down an expression for S_n in terms of n.
- **b** Find S_{10}.
- **c** Find the smallest n for which $S_n < -250$.

6 Montserrat is training for her first race. In her first training week she runs 3 km, and in the second training week she runs 3.5 km. Every week she runs 0.5 km more than the previous one.

- **a** Determine the number of kilometres Montserrat runs in her 10th training week?
- **b** Calculate the total number of kilometres that Montserrat will have run by her 15th training week.

7 In Exercise 4L question **5**, you found the number of rows of seats in a theatre that has 22 seats in the first row, 49 in the tenth row, and 106 in the last row. Recall that the number of seats in each row increases by a constant amount.

- **a** Find the total number of seats in the theatre.
- **b** An architect is designing another theatre for a university that wants to have 25 rows, a total of at least 6000 seats, and a first row of 16 seats. The number of seats per row will increase by a constant amount. Determine the least possible value for the increase and the total number of seats it will provide.

Approximately arithmetic sequences

Sometimes arithmetic sequences can be used to model a context even if it is not perfectly arithmetic.

TOK

Do you think that mathematics is just the manipulation of symbols under a set of rules?

An **approximately arithmetic sequence** has approximately equal differences between terms and its graph is nearly linear.

When a sequence of real-world data is approximately arithmetic, find a common difference for your model by averaging the common differences of the data.

Exercise 4O

1 Scientists use tree rings to determine the age of trees. Each ring corresponds to one year of the tree's growth (see the picture).

Abby examines a tree stump and measures the radius of each of the first five rings. She records the results in a table.

Ring	1	2	3	4	5
Radius (mm)	27	31	34	38	40

She notes that the radii of the rings continue to increase approximately linearly. The radius of the entire tree stump is 1.8 m. Approximately how old was the tree?

2 Dionissi is an economist who has been tasked with modelling the trade deficit between his country and Australia. (A trade deficit means that his country buys more products from Australia than Australia buys from his country. A trade surplus is the opposite. The relation is measured as a percentage of the country's gross domestic product, or GDP.) Dionissi finds that the deficit has been growing in the following pattern over ten recent years:

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Surplus/deficit (% of GDP)	2.28	1.95	1.57	1.21	0.82	0.45	0.12	−0.24	−0.59	−0.96

Using this data, construct an approximate arithmetic sequence and use it to predict the trade deficit in 2025

Developing inquiry skills

Looking back at the opening problem, how will the taxi driver's income change if he increases the income per kilometre by $0.30 after one month?

How can you model the daily income for different values of income per kilometre?

How have you used the modelling process discussed in this section to investigate this problem?

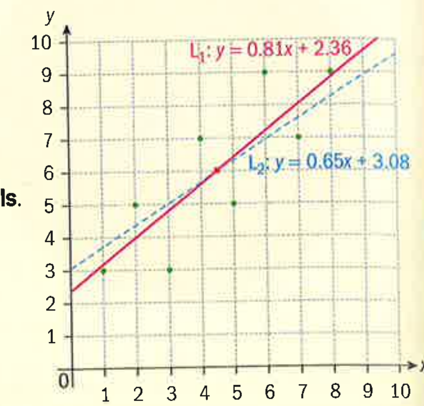
Functions

4.5 Linear regression

In the previous section you developed a method to model approximately arithmetic sequences. In this section, we'll investigate how you can model approximately linear situations with linear models. In this investigation you will improve on the method developed in Chapter 2 for creating a line of best fit.

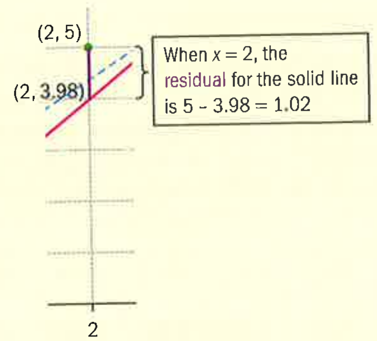
Investigation 13

1 The scatter graph shows a set of eight data points and also two lines of best fit, L_1 ($f_1(x) = 0.81x + 2.36$) and L_2 ($f_2(x) = 0.65x + 3.08$). One of these lines models this set of points better than the other. Which line do you think best fits this data? Why?



One way to measure the fit of a line to a data set is to calculate **residuals**.

A residual is the difference between the actual y-value and the predicted y-value. Therefore, residuals are the errors made when using best-fit lines to make predictions.



2 a What does a positive residual tell you about the predicted y-value compared with the actual y-value?

b Will the line that better fits the data have residuals with generally smaller or larger values? Why?

3 Calculate the residuals for each line by completing the table. (You can leave the square of residuals columns blank for now.) The first point has been done as an example. You may wish to use spreadsheet technology to make your calculations more efficient.

Point	x	y	Predicted y using L_1	Residual using L_1	Square of residual using L_1	Predicted y using L_2	Residual using L_2	Square of residual using L_2
(1, 3)	1	3	$0.81\times1+2.36=3.17$	$3-3.17=-0.17$	0.0289	$0.65\times1+3.08=3.73$	$3-3.73=-0.73$	0.5329
(2, 5)	2							
(3, 3)	3							
(4, 7)	4							
(5, 5)	5							
(6, 9)	6							
(7, 7)	7							
(8, 9)	8							
					SS_{res} =			SS_{res} =

4 To get a total of the residuals for a line, you could simply add them all. Try adding all the residuals of L_1. What happens? Is this a good measure of the overall "error" of the line?

So that positive and negative residuals do not cancel each other, you will square each residual before adding them together. You call this the sum of square residuals, or SS_{res}.

5 Find the squares of the residuals for each line, then add them to find the sum of square residuals.

Point	Residual using L_1	Square of residual using L_1	Residual using L_2	Square of residual using L_2
(1, 3)	−0.17	0.0289	−0.73	0.5329
(2, 5)				
(3, 3)				
(4, 7)				
(5, 5)				
(6, 9)				
(7, 7)				
(8, 9)				
		SS_{res} =		SS_{res} =

6 Based on the values of SS_{res}, which line has the better fit? Does this support your conjecture from step **1**?

You only chose two lines to compare, but there are many other possible choices.

7 To see where the least squares regression line gets its name from, open the least squares regression app on your GDC.

a What happens to the size of the squares as you make the line a worse fit for the data? A better fit?

b How are the areas of the squares connected to the sum of square residuals calculation?

c What parameters can you change about the line?

8 **Conceptual** How do we use sum of square residuals to define the parameters in the least squares regression line and why does this lead to a line of best fit?

- The residual for a point (x_i, y_i) in a data set modelled by the linear function $f(x)$ is given by residual of $x_i = y_i - f(x_i)$.
- For a set of n data points $\{(x_i, y_i)\}$ and approximating linear function $f(x)$, $SS_{res} = \sum_{i=1}^{n}(y_i - f(x_i))^2$.
- The line that minimizes the sum of square residuals, that is, the line that has the smallest possible SS_{res}, is called the **least squares regression line** or **linear regression equation.** If the vertical (y) residuals are minimised the regression line is said to be "y on x". This line is used for predicting y-values from given x-values.

TOK

When students see a familiar equation with a transformation, they will often get a "gut feeling" about what the function looks like.

Respond to this question:

"Is intuition helpful or harmful in mathematics?"

Continued on next page

Functions

- In examinations and most applications, it is sufficient (and more efficient) to calculate the linear regression equation $f(x) = mx + c$ using technology. However, it is possible to calculate the parameters using for a given data set $\{(x_i, y_i)\}$ using the formulas:

$$m = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - x)^2}, \quad c = \bar{y} - m\bar{x}$$

Recall that $\bar{x}$ represents the mean or average of the data set $\{x_i\}$.

You can now calculate the linear regression equation and use it to predict y-values from known x-values, as the following example shows.

Example 20

At a coach station, the maximum temperature in °C (x) and the number of bottles of water sold (y) were recorded over 10 consecutive days. The collected data are summarized in the table.

Day	1	2	3	4	5	6	7	8	9	10
x	20	19	21	21.3	20.7	20.5	21	19.3	18.5	18
y	140	130	140	145	143	145	145	125	120	123

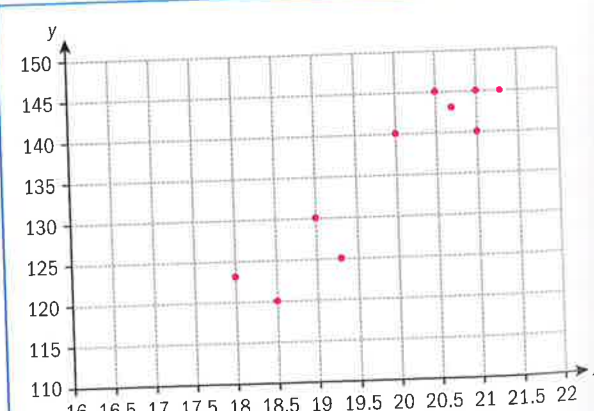
a Use a graph of the data to justify why a linear regression is appropriate.

b Find the regression line of y on x.

c Interpret the gradient and y-intercept of the regression equation in context.

d Use the regression equation to predict the number of bottles that will be sold at a temperature of 19.5°C.

a Because the data is approximately linear, linear regression is appropriate.



b $y = 8.05x - 24.7$

The GDC shows the general form of the equation as $y = ax + b$ with $a = 8.05$ (3 s.f.) and $b = -24.7$ (3 s.f.).

c The gradient of 8.05 indicates that an increase of 1°C corresponds to an increase of about 8 bottles sold.

The y-intercept is outside the range of the data set and is negative, so it does not have meaning in the context.

Substitute $x = 19.5$ in the regression equation, and solve either with technology or algebraically:

$y = 8.05(19.5) - 24.7 = 132.2$

d 132

Exercise 4P

1 The travel time in minutes (x) and the price in euros (y) of ten different train journeys between various places in Spain are shown in the table.

x	128	150	102	140	140	98	75	130	80	132
y	25.95	40	24.85	31.8	30.2	28.95	21.85	34.5	23.25	26

a Plot the data points on a scatter diagram. Use your diagram to justify why a linear regression is appropriate.

b Write down the equation of the regression line of y on x.

c Predict the price of a train journey of 2 hours.

d Comment on whether the regression equation be more reliable in predicting the price for a journey of 10 minutes or 100 minutes. Justify your answer.

2 The heights in metres (x) and weights in kilograms (y) of ten male gorillas are shown in the table.

x	1.9	1.83	1.81	1.79	1.74	1.91	1.93	1.86	1.81	1.95
y	275	267	260	257	258	272	273	268	261	273

a Plot the data points on a scatter diagram. Use your diagram to justify why a linear regression is appropriate.

b Write down the equation of the least squares regression line for this data.

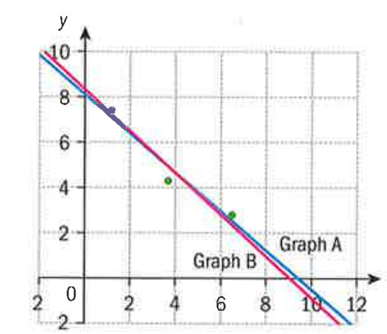
c Predict the weight of a gorilla that is 1.8 m tall.

d Interpret the meaning of the gradient in context.

3 Two potential lines of fit for the data set shown in the table are $f_1(x) = -0.861x + 8.11$ and $f_2(x) = -0.913x + 8.30$. One of these is the linear regression equation.

x	1.2	6.5	3.7
y	7.4	2.8	4.3

a Match each equation to its graph, with reasons.



b Calculate the sum of square residuals for each equation and hence determine which is the linear regression equation.

Validity of predictions

In previous investigations, and in Chapter 2, you found that you cannot always make valid predictions from a linear regression. In the next investigations, you will explore the conditions under which more valid predictions can be made.

International-mindedness

In 1956, Australian statistician, Oliver Lancaster made the first convincing case for a link between exposure to sunlight and skin cancer using statistical tools including correlation and regression.

Functions

Investigation 14

In Chapter 2 you learned how to calculate Pearson's correlation coefficient to measure the strength and direction of a linear correlation. The four sets of data in the table are known as Anscombe's Quartet after their inventor, statistician Francis Anscombe.

I		II		III		IV	
x	y	x	y	x	y	x	y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

1 Find the correlation coefficient of each data set. What does this suggest about the strength of the correlation?

2 Graph each data set using technology.

3 What additional information do the graphs give you that the correlation coefficient did not? What does this tell you about relying solely on the correlation coefficient to measure the appropriateness of a linear regression?

4 For each data set, decide whether it is appropriate to model it with a linear regression, giving reasons.

5 **Conceptual** How does the graph help us to decide whether a linear regression and Pearson's correlation coefficient are appropriate to model a data set?

International-mindedness

Karl Pearson (1857–1936) was an English lawyer and mathematician. His contributions to statistics include the product-moment correlation coefficient and the chi-squared test.

He founded the world's first university statistics department at the University College of London in 1911

Investigation 15

A random sample of 12 students is taken to see if there is any linear relationship between height (in cm) and shoe size or height and age (in years). The data is shown in the table.

Height (cm)	160	187	175	180	186	170	185	172	174	180	165	170
Shoe size	37	42	39	38	40	38	41	39	38	40	37	39
Age (years)	13.7	16.2	16.5	15.8	15.5	14.3	15.2	13.8	14.3	14.7	14.4	15.3

1 Plot shoe size and age against height on two scatter diagrams. Is a linear regression appropriate for each?

2 Calculate Pearson's correlation coefficient for both data sets. Which correlation is stronger?

3 a Find the regression equation for each pair of variables.

b For a student who is 163 cm tall, use your regression equations to predict

i the shoe size of that student ii the age of the student.

c Which prediction do you think is more accurate, and why?

4 **Conceptual** How does the strength of Pearson's correlation coefficient of a data set impact the accuracy of predictions made from its linear regression?

We'll focus now only on the shoe size regression.

5 So far you have only predicted values of y. Can you also predict values of x? You will explore this in what follows.

a Use the regression equation of y on x (or S on h) to predict the height of a student with shoe size 38.

b Now create the regression of x on y (or h on S) by switching the independent and dependent variables and re-running the regression. Compare its correlation coefficient to that of the y on x regression.

c Use the regression of x on y to predict the height of a student with shoe size 38. Compare with your prediction from part **a**. What do you notice?

d Can you think of a reason why predictions of shoe size from height would be more accurate than predictions of height from shoe size?

6 Why can the regression of y on x not always be used to reliably predict x from a given y?

7 **Conceptual** Summarise: When is it appropriate to use a linear regression for prediction?

Predictions from linear regression are more accurate when the correlation coefficient is stronger. At least a moderate correlation and linear relationship should be established before making predictions from a linear regression.

TOK

What is the difference between correlation and causation?

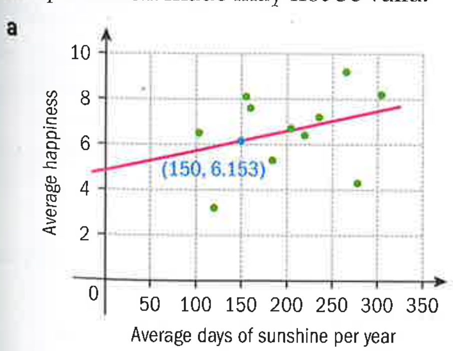
To what extent do these different processes affect the validity of the knowledge obtained?

Functions

Exercise 4Q

1 Kiernan wonders whether there is a correlation between how sunny a city is and how happy its inhabitants are. Suppose that each of the graphs below represent the data he collects. For each data set, state one or more reasons why the prediction made may not be valid.

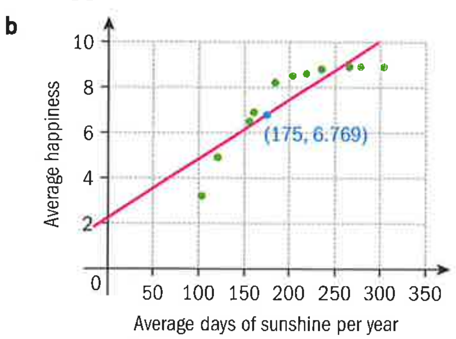
a



$r = 0.319$

Prediction: A city with 150 days of sunshine per year will have an average happiness of 6.15.

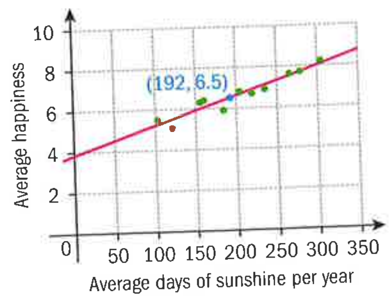
b



$r = 0.880$

Prediction: A city with 175 days of sunshine per year will have an average happiness of 6.80.

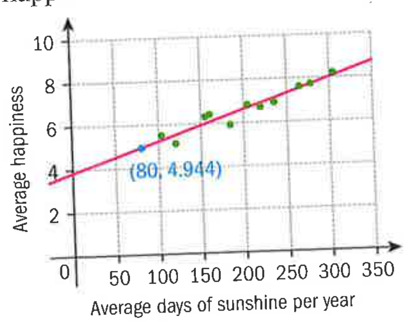
c



$r = 0.955$

Prediction: A city with an average happiness of 6.5 will have 192 sunny days.

d



$r = 0.955$

Prediction: A city with 80 days of sunshine per year will have an average happiness of 4.94.

2 The rate at which crickets chirp can be used to predict the temperature. In the table, data has been collected on the number of chirps in a 15-second period and the surrounding temperature in degrees Fahrenheit.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature (°F)	80.5	70.5	57	66	68	72	52	73.5	53

a Determine whether the data follow a linear trend.

b Find the correlation coefficient and interpret its meaning.

c Based on the above, explain whether it is appropriate to model and make predictions about this data with a linear regression.

d **i** Find the regression equation of the temperature T on the number of chirps c.

ii If the temperature is 75°F, predict the number of chirps or explain why it is not valid to do so.

iii You count 40 chirps a minute. Predict the temperature and explain whether your prediction is valid.

e The *Old Farmer's Almanac* suggests the following rule for predicting temperature from chirps: count the number of chirps in 14 seconds, then add 40. Explain how consistent your regression equation is with this rule.

3 Lee is interested in the cryptocurrency Bitcoin. He knows that computer graphics cards can be used to "mine" Bitcoins and wonders whether there is a relationship between the value of a Bitcoin and the current price of a graphics card. He collects the data shown in the table on different days between August and December 2017.

Bitcoin value ($)	4204	3686	4394	5697	7118	7279	10 859	17 601	13 412	16 178	13 585	10 035
Graphics card price ($)	690	660	680	650	550	650	750	1500	1000	1300	1250	1200

a Determine whether the data follows a linear trend.

b Find the correlation coefficient and interpret its meaning.

c Based on the above, explain whether it is appropriate to model and make predictions about this data with a linear regression.

d **i** Find the regression equation of graphics card price G on Bitcoin value b.

ii On December 8th the value of a Bitcoin was $12 320. Predict the price of a graphics card on this day.

e Interpret the gradient of the regression equation in the context of Bitcoin value and graphics card price.

Chapter summary

- A **function** is a relation between two sets in which every element of the first set is mapped onto **one and only one** element of the second set.
- The **domain** of a function is the set of all input values. The **reasonable domain** of a function modelling a real-world context is the set of input values that are relevant in that context.
- A **linear function** $f(x) = mx + c$, where m and c are constants, represents a context with a **constant rate of change**. It has a **gradient parameter** (m) and a **y-intercept parameter** (c).
- A linear model with formula $y = mx$ relates two variables, x and y, that are in **direct variation** (proportion).
- A **piecewise function** is a function that is defined by a different formula on each piece of is domain.
- You use the notation $f^{-1}(x)$ to denote the **inverse function** of $f(x)$. If $f(a) = b$, then $f^{-1}(b) = a$.
- The graphs of inverse functions are symmetric about the **identity line** $y = x$. The equation of an inverse function can be found by exchanging x and y in the equation for the function.
- You call the inputting of one function into another a **composition of functions**, and denote it $f \circ g(x)$ or $f(g(x))$. Here the function g is performed first and its result input into f.
- By definition, two functions are inverses if their composition is the **identity function** $i(x) = x$: $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$.
- A sequence in which the **difference** between consecutive terms remains constant is called an **arithmetic sequence**. This constant value is called **common difference** of the sequence.
- The general term (or ***n*th term**) of an arithmetic sequence with first term u_1 and common difference d is $u_n = u_1 + (n-1)d$, where $n \in \mathbb{Z}^+$.
- The total amount of **simple interest** I earned on a principal P over n years at an interest rate of $r\%$ per year can be calculated using the formula $I = P \times r \times n$. The total savings account balance A is given by $A = P + P \times r \times n$.
- The sum, S_n, of the first n terms of an arithmetic sequence $u_1, u_2, u_3, \ldots$ can be calculated using the formula $S_n = \frac{n}{2}(2u_1 + (n-1)d)$. The sum of a sequence is a **series**.
- The **sum of square residuals** for a set of n data points $\{(x_i, y_i)\}$ and approximating linear function $f(x)$ is given by $SS_{res} = \sum_{i=1}^{n} (y_i - f(x_i))^2$.
- The linear function $f(x) = mx + c$ that minimises SS_{res} for a given data set is the **least squares regression line** or the **linear regression equation**. It is calculated using technology in examinations.
- Predictions from a linear regression are only appropriate when the data displays a clear linear trend in its scatter plot, the correlation coefficient r is at least moderate, and you predict values of the dependent variable. Predictions are more valid when using **interpolation** to predict within the data set, rather than **extrapolation** that predicts for values outside the data set.

Developing inquiry skills

Has what you have learned in this chapter helped you to answer the questions from the beginning of the chapter about the taxi driver's income?

How could you apply your knowledge to investigate other business charging structures? What information would you need to find?

Thinking about the inquiry questions from the beginning of this chapter:

- Discuss if what you have learned in this chapter has helped you to think about an answer to these questions.
- Consider whether there are any that you are interested in and would like to explore further, perhaps for your internal assessment topic.

Chapter review

Click here for a mixed review exercise

1 Takumi is supervising the cleanup of an oil spill. The surface area A of ocean remaining to be cleaned (in m^2) is a function of the time t worked (in hours), modelled by the equation $A(t) = 1350 - 2.7t$.

a Find the time it will take to clean the spill.

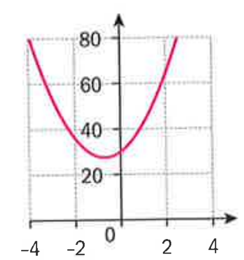
b Find a reasonable domain and range for this model.

c Sketch a graph of $A(t)$ for this domain.

d Find $A(5)$ and interpret its meaning in context.

2 For each of the following relations, determine whether it is a function, and if so, whether it is a one-to-one function.

a



b $3x - 5y = 30$

c x = volume of a cylinder, y = radius of the cylinder

3 Damir is researching a potential link between family size and life expectancy in Asian and Middle Eastern countries. He collects the data shown in the table.

Country	Life expectancy (years)	Babies per woman	Country	Life expectancy (years)	Babies per woman
Japan	84.2	1.48	Jordan	76.7	3.24
China	76.9	1.64	Tajikistan	72.2	3.27
Vietnam	74.9	1.95	Pakistan	68	3.35
Myanmar	70.3	2.17	Papua New Guinea	61.1	3.56
Indonesia	72	2.31	Yemen	67.1	3.79
Saudi Arabia	77.6	2.45	Australia	82.9	1.83
Kazakhstan	72	2.57	Turkmenistan	70.5	2.79
Philippines	70.5	2.86	Nepal	71.5	2.05

Source: Gapminder (http://bit.ly/2Kax5Z6)

a If Damir wants to predict a country's life expectancy given its average babies per woman, define appropriate variables and find an appropriate regression equation to do so.

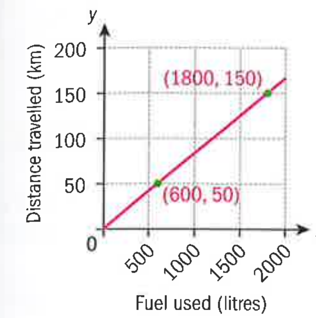
b Write down two reasons why it is valid to make predictions from this regression line.

c Predict the life expectancy of Malaysia, which has an average of 2 babies per woman. Explain whether your prediction is valid.

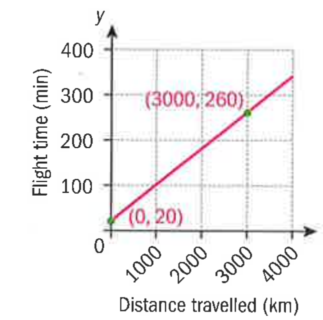
d Interpret the gradient of the regression line in context.

4 The two graphs show the relationship between the distance an aircraft travels (in kilometres), the time it takes to fly that distance (in hours) and the fuel used (in litres).

Graph 1



Graph 2



a Write down which graph represents direct variation, with a reason.

b Find and interpret the gradient of each graph in context.

c Find an equation for the travel time T of the aircraft as a function of litres used, x, and use it to predict how long a flight will be that uses 150 000 litres of fuel.

5 Tiffany is working part-time while she attends college. She is saving money each month for a car costing \$6500, as well as \$3000 in her savings account for unexpected expenses. As of this month, she has \$400 saved and is saving \$150 per month.

a At this rate, determine how many more months it will take Tiffany to save for her goal.

Tiffany decides that she will increase her savings by a fixed amount of $\$k$ each month (starting from \$150) so that she can reach her savings goal within one year.

b Find the smallest value of k (rounded to the nearest dollar) so that Tiffany can achieve her goal.

c If Tiffany continues to increase her monthly savings at this rate and spends none of it, determine after how many more months she will have at least \$20 000 saved.

6 Lydia lives in Amsterdam and is budgeting for three upcoming weekend trips. For each, she can either fly or drive. The car rental company is offering a special deal of €49 for a weekend plus €0.06 per kilometre driven.

a Write a linear function of the rental car's cost €C as a function of distance travelled, d (km).

b Lydia prefers flying to driving, but she will only pay up to €50 more to fly than to drive. Given the information in the table, determine

i whether she should fly or drive for each trip, with reasons

ii how much she should budget for all three trips.

Destination	One-way distance from Amsterdam (km)	Cost of round-trip flight (€)
Brussels	210	130
Hamburg	470	150
Paris	515	240

Functions

7 Nadia remembers that the balance of her savings account, which pays simple interest, was 8% greater 5 years after she first invested it, and that she had €3200 more in 2018 than in 2013. Determine Nadia's initial investment and interest rate.

8 Several countries use a progressive tax system in which additional income is taxed at a higher marginal rate. The table shows Canada's marginal tax rates in 2018.

Income	Marginal tax rate
First CA$46 605	15.0%
Over CA$46 605 up to CA$93 208	20.5%
Over CA$93 208 up to CA$144 489	26.0%
Over CA$144 489 up to CA$205 842	29.0%
Over CA$205 842	33.0%

Source: https://www.taxtips.ca/taxrates/canada.htm

For example, a person earning CA$50 000 will be taxed at 15% on the first CA$46 605 and 20.5% on the remaining CA$3395.

a Ian made CA$122 000 this year. Find how much he will pay in taxes.

b Construct a piecewise function and corresponding reasonable domain to model the tax owed, T, as a function of total income earned, x.

c Ian wins a major prize for excellence in neuroscience and has two options: one payment of CA$40 000, or two equal payments of CA$20 000 over two years.

i Determine the amount of tax he will pay over these two years in each scenario, assuming his other earnings remain the same.

ii Based on this, state the better option if Ian wishes to minimise his tax payment.

9 **a** Find the inverse of the following piecewise function. State its domain and range.

$$f(x)=\begin{cases}5-\frac{1}{3}x & -6\le x\le 3\\ -2x+10 & x>3\end{cases}$$

b Show that the function

$$f(x)=\begin{cases}5-\frac{1}{3}x & -6\le x\le 3\\ 2x-2 & x>3\end{cases}$$

is not invertible.

Exam-style questions

10 **P1:** Paired, bivariate data (x, y) that is strongly correlated has a y on x line of best fit given by $y = mx + c$.

The data represents students' test scores in Geography, x, and test scores in Environmental Systems, y.

When $x = 70$ an estimate for y is 100. When $x = 100$ an estimate for y is 140.

a Find the value of

i m **ii** c (3 marks)

b State whether the correlation is positive or negative. (1 mark)

c Given that the value of $\bar{x}$ is 90, find the value of $\bar{y}$. (3 marks)

d When $x = 60$ find an estimate for the value of y. (2 marks)

11 **P1:** The price of renting a car (£C) from "Cars-R-Us" for d days is given by the formula $C = 30 + 12.5d$.

The price of renting a car (£C) from "Car-nage" for d days is given by the formula $C = 70 + 8.35d$.

Abel wishes to rent a car for the duration of his holiday.

He decides to rent from "Car-nage" as it will be cheaper for him.

a What is the minimum length of Abel's holiday? (3 marks)

b If Abel's holiday is between 14 and 21 days, find an inequality which shows the range in which Abel's car hire bill will lie. (4 marks)

12 **P2:** Paired bivariate data (x, y) is given in the table below.

The data represents the heights (x metres) and lengths (y metres) of a rare type of animal found on a small island.

x	2.4	3.6	2.8	1.8	2.0	2.2	3.0	3.4
y	3.0	4.0	3.0	1.7	2.0	2.3	3.1	2.7

a **i** Calculate the Pearson product moment correlation coefficient for this data.

ii In two words, describe the linear correlation that is exhibited by this data.

iii Calculate the y on x line of best fit. (6 marks)

Another four examples of this rare animal are found on a nearby smaller island.

This extra data is given in the table below.

x	2.3	2.7	3.0	3.5
y	4.1	1.5	4.2	1.5

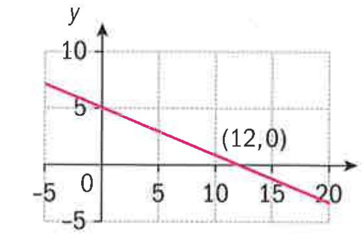
b **i** Calculate the Pearson product moment correlation coefficient for the combined data of all 12 animals.

ii In two words, describe the linear correlation that is exhibited by the combined data.

iii Suggest a reason why it may not be valid to calculate the y on x line of best fit for the combined data. (5 marks)

13 **P1:** The following diagram is a graph of the function $f(x) = a + bx$ $(x \in \mathbb{R})$.

a Determine the value of a and the value of b. (2 marks)



b Find an expression for $f^{-1}(x)$. (4 marks)

c Solve the equation $f(x) = f^{-1}(x)$, giving your answer in an exact form. (3 marks)

d Explain why, for any function $h(x)$, the equation $h(x) = h^{-1}(x)$ will have the same solution(s) as the equation $h(x) = x$. (1 mark)

14 **P1:** The first four terms of an arithmetic sequence are given by 7, $3a + b$, $5a - 6b$, $2a + 9b + 4$.

a Find the value of a and the value of b. (5 marks)

b Find the least number of terms required so that the sum of the series exceeds 1000. (5 marks)

15 **P2:** Functions f and g are given such that $f(x) = x - 24$ $(x \in \mathbb{R})$ and $gf(x) = 2x^2 + 18$ $(x \in \mathbb{R})$.

a State the range of $f(x)$. (1 mark)

b State the range of $gf(x)$. (1 mark)

c Solve the equation $fgf(x) = 0$. (3 marks)

d Determine the function $g(x)$. (6 marks)

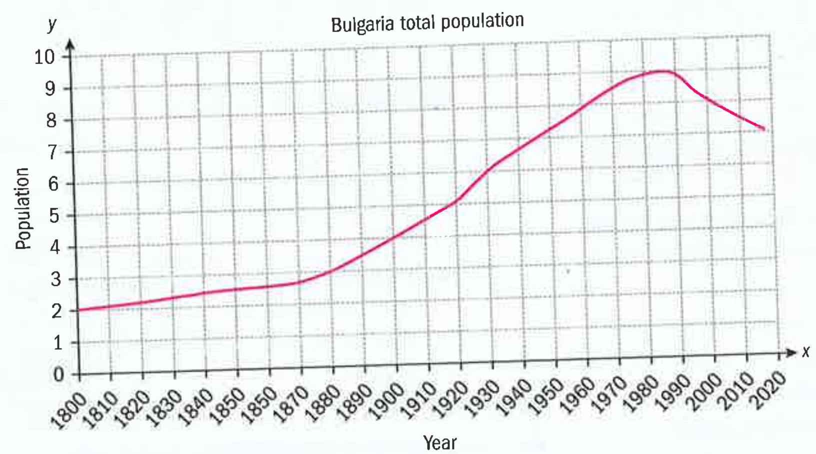
Graphs of functions: describing the "what" and researching the "why"

Approaches to learning: Thinking skills, Communicating, Research

Exploration criteria: Presentation (A), Mathematical communication (B), Personal engagement (C)

IB topic: Graphs, Functions, Domain

Bulgaria population data



This graph includes two essential elements:

- A title.
- x- and y-axes labels with units.

Without any research, write a paragraph to describe this graph.

Do not just describe the graph, but also explain why it might have this shape. Include any interesting points and regions on the graph "where things happen".

Using sources

You can use your general knowledge, printed sources and Internet sources to research data. Different sources can often give different explanations, and not all sources are valid, useful or accurate.

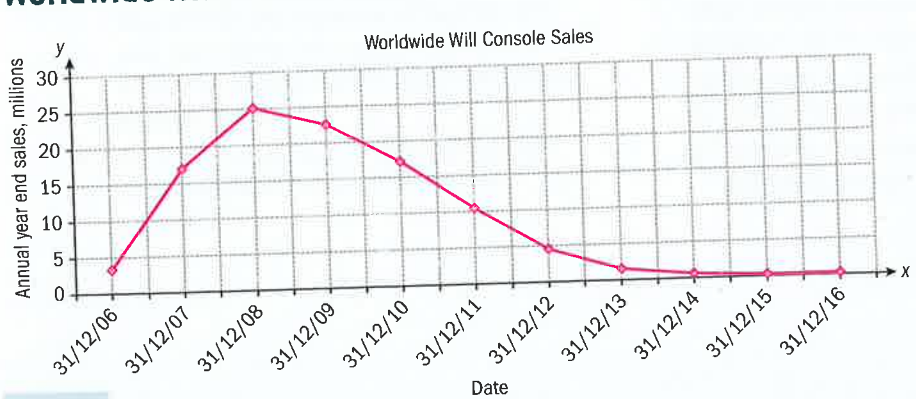
How do you know if a source is reliable?

Use Internet research to find out more precisely what happened at the key dates shown on the Bulgaria population graph.

Keep a record of any sources that you use.

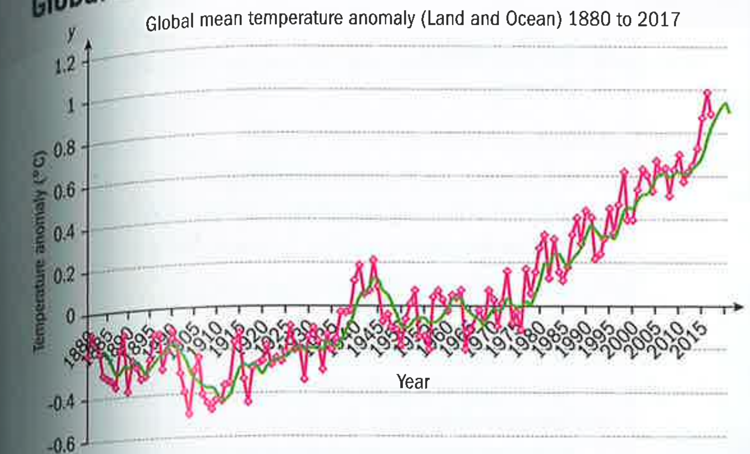
Could you use the graph to predict what might happen to the population of Bulgaria in the future? Explain your answer.

Worldwide Wii console sales



Initially **without research** write a paragraph about this graph, then research the reasons for the shape of this graph.

Global mean temperature anomaly



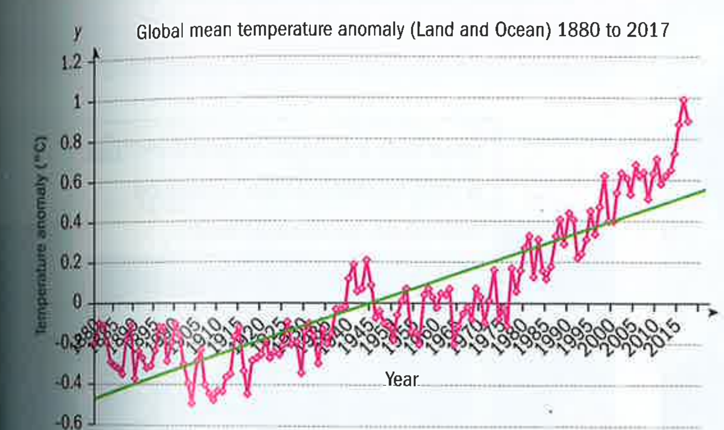
What is the domain?

When is the graph rising (increasing)?

When is the graph falling (decreasing)?

What is the shape of the graph?

Describe and explain.



Research what is meant by Global Mean Temperature Anomaly.

This data is based on deviations from the base average for 1951 to 1980.

On the graph in red the 5-year moving average trend line is included. What is a 5-year moving average?

On the second graph is a linear trend line. What is a linear trend line?

What are the advantages and disadvantages of each of the representations of the data shown?

Describe the data, note any interesting points or trends and to try to explain and investigate why the trends may be as they are.

TOK

This is a potentially controversial topic with many opinions and theories. How can you protect against your own biases?

Extension

Find and research a graph from the news or an academic journal from one of your subjects or another source.

Describe and explain the trends and the reason for the shape of the graph.

Now display or print out the graph.

Write a series of questions for other students to answer that encourage them to describe and explain the graph.