6 Modelling relationships with functions: power and polynomial functions

Concepts

- Modelling
- Relationships

Microconcepts

- Quadratic, cubic, inverse proportional and power functions
- One-to-one and many-to-one functions
- Axis intercepts, zeros, roots, vertex and symmetry of graphs
- Increasing and decreasing functions and concavity
- Vertical and horizontal asymptotes
- Composite functions
- Inverse functions including domain restriction
- Transformations of graphs
- Modelling and regression

If for example you are trying to use mathematics to model the path of a javelin, the shape of bridge or the maximum volume of a container, you will need to study equations of curves. This chapter looks at ways of modelling real-life scenarios with curves, and fitting equations to curves in order to predict the height of the curve (which would tell you, for example, the height of the javelin below) and the distance spanned by a curve (telling you the distance the javelin travels).

How can you predict where a javelin will land? How can you find out when its speed is fastest?

What is the maximum volume of a box made from a piece of card with squares cut from each corner?

How long does it take an object to fall, given that the distance varies directly with the square of the time taken?

How can you find the price of a car, given that the price varies inversely with the age of the car?

Oliver is practising his basketball skills.

- What shape is the path of the ball? Sketch a path for the ball from Oliver's hands to the basketball hoop.
- Sketch a path for the ball from Oliver's hands to the hoop when he is
  - standing further away from the hoop
  - standing closer to the hoop
  - standing on a platform
  - sitting on the floor.
- What do you notice about the shape of the ball's path when Oliver is in different positions? What changes and what is the same?

How can you model a path of a basketball from Oliver's hands to the hoop from any point on the court?

How can the model help you to predict whether the ball will go into the hoop or not?

- What information do you need to know to build this model?
- What assumptions would you need to make in your model?

Developing inquiry skills

Write down any similar inquiry questions you might ask to model the path of something in a different sport, for example determining where an archer's arrow would land, deciding whether a tennis ball would land within the baseline, or considering whether a high-jumper would clear the bar successfully.

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:	Skills check
1 Substitute coordinates into an equation. eg Substitute (1, 3) into $y = 2x^2 + 4x + c$, in order to find the value of parameter c. $3 = 2 \times 1^2 + 4 \times 1 + c \Rightarrow c = -3$	**1** If $y = 2x^2 - 3x + c$, find the value of c at the point $(2, -1)$.
2 Solve quadratic equations using the quadratic formula. eg $2x^2 + 5x - 3 = 0$ $x_{1,2} = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times (-3)}}{2 \times 2}$ $\Rightarrow x_1 = \frac{1}{2}, x_2 = -3$	**2** Solve the following quadratic equations using the quadratic formula: **a** $x^2 - x - 6 = 0$ **b** $3x^2 + x - 2 = 0$
3 Solve quadratic equations by factorising. eg $x^2 - 6x + 5 = 0$ $(x - 5)(x - 6) = 0$ $x = 5$ and $x = 6$	**3** Solve the following quadratic equations by factorising: **a** $x^2 + 2x - 8 = 0$ **b** $x^2 - 6x + 8 = 0$

Click here for help with this skills check

6.1 Quadratic models

A dolphin jumps above the surface of the ocean. The path of the jump can be modelled by the function $f(x) = -0.09375x^2 + 1.875x - 3.375$, in which:

- x represents the horizontal distance, in metres, which the dolphin has travelled from the point where it left the water
- $f(x)$ represents the vertical height, in metres, of the dolphin above the surface of the water.

How can you find out how far the dolphin jumped and how high it jumped?

When a dolphin jumps out of the water, what does the path of its jump look like?

To investigate the dolphin's jump, you could use your GDC to plot a graph of the equation that is used to model the path of the jump. By finding the coordinates of certain points on the graph, you could tell how far and how high the dolphin jumped.

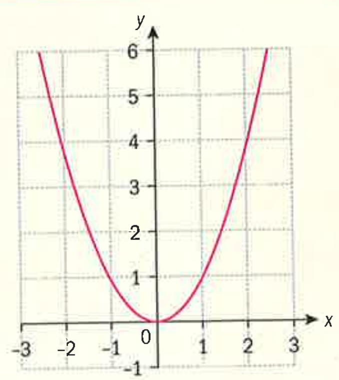
This section shows you how to do this.

Quadratic functions are polynomial functions where the highest power of the independent variable (x) is 2.

For example, $f(x) = ax^2 + bx + c$, $a \neq 0$ and $a, b, c \in \mathbb{R}$ is a quadratic function.

The fundamental (simplest) quadratic function is $f(x) = x^2$.

The shape of the graph of a quadratic function is called a **parabola**.



Reflect State what type of function $f(x) = ax^2 + bx + c$ would be if $a = 0$.

How are quadratic graphs different from linear graphs?

How are they the same?

The maximum or minimum turning point on the graph of a quadratic function is called the **vertex**.

The graph of a quadratic function is symmetric about a vertical line called the **axis of symmetry**, going through its **vertex**.

International-mindedness

The Babylonians (2000–1600 BC) used quadratics to find the areas of fields for agriculture and taxation.

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Frenchman Nicole Oresme was one of the first mathematicians to consider the concept of functions in the 14th century. The term "function" was introduced by the German mathematician, Gottfried Wilhelm Leibniz in the 17th century, and the notation was coined by a Swiss mathematician, Leonhard Euler, in the 18th century.

"Quadratic" comes from the Latin word *quadrare*, meaning "to square".

"Parabola" comes from the Greek word *παραβάλλω*, meaning "to place next to something".

Reflect Describe how to find the domain of a function from its graph.

Describe how to find the range of a function from its graph.

The y-intercept is the value of $f(0)$.

Transferring a graph from screen to paper

When you "**sketch**" a graph, you do not need to be as accurate as when you have to "**draw**" a graph on graph paper, but your sketch should:

- show the general shape of the graph accurately
- label the coordinates of any axes intercepts
- label the coordinates of any vertices.

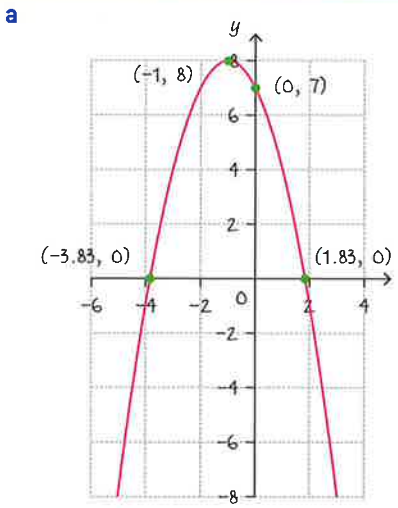
You can sketch (or draw) the graph of a quadratic function by hand with the help of your GDC. In many cases the **domain** of the function that you need to sketch or draw will be explicitly given. In such cases you should only sketch (or draw) the part of the function that is in the given domain.

Example 1

a Sketch the graph of the function $f(x) = 7 - 2x - x^2$, for $-5 \le x \le 3$, and hence determine the range.

b State the range if the domain were unrestricted.

a



The range of the function is $-8 \le y \le 8$.

Use your GDC to sketch a graph within a given domain.

The end points, the y-intercept, the x-intercepts and the vertex (maximum) can all be obtained from the graph on your GDC. Make sure you clearly show these points on your diagram.

Continued on next page

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We have seen the involvement of several nationalities in the development of quadratics.

To what extent do you believe that mathematics is a product of human social collaboration?

EXAM HINT

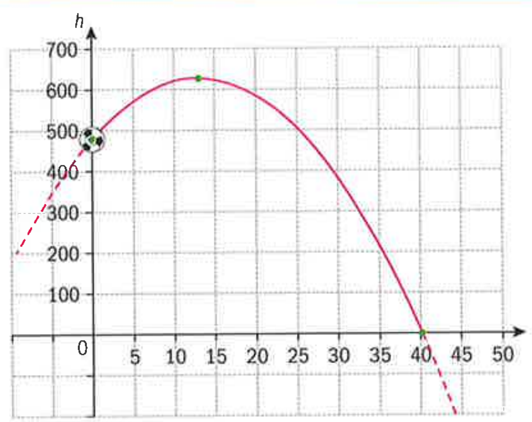
Draw and label your axes. You must draw the shape of the graph correctly and label the points where the graph cuts the axes, as well as the vertex. Since you are asked for a sketch, you do not need to put a full scale on the axes, but you should have at least one number on each axis. In an exam the points that need to be shown on the sketch will be given in the question. These are likely to include the points where the graph cuts the axes as well as any maximum or minimum points.

b The range of the function is $y \leq 8$.	You can evaluate the function f at any real value of x, so the domain is the set of all real numbers. There is still a maximum value of the function, at $(-1, 8)$.

In a model of a real-life situation you might need to consider the domain and the range without them being explicitly given.

Example 2

Sketch the graph of the function $h(t) = -0.846t^2 + 22.2t + 478$, representing the height h, in metres above the ground, of a projectile at time t minutes after it was launched. Find the maximum height that the projectile will reach.

 The maximum height above the ground of the projectile is $h = 624$.	You first need to decide on a reasonable domain and range in the context of the question. Here the input variable is time, so the domain cannot contain negative values, meaning that $t \geq 0$. Similarly, since the output variable is the height in metres above the ground, the range cannot contain negative values. The starting point should be at $(0, 478)$ and the ending point at approximately $(40.3, 0)$. The vertex should be placed at approximately $(13.1, 624)$. This will further restrict the domain. The resulting graph should only appear within the first quadrant.

HINT

When the **domain** of the function is **not explicitly given**, you might need to do some working in order to choose an appropriate view window in your GDC. The most important point of a parabola that should be in your view window is the vertex. This means that finding the vertex analytically or by using the table function in the GDC might prove to be very helpful.

Exercise 6A

1 Find the equation of the axis of symmetry of the following functions.

- **a** $f(x) = x^2 + x + 1$
- **b** $f(x) = -2x^2 + 4x - 3$
- **c** $f(x) = \frac{1}{2}x^2 + 4x + 5$
- **d** $f(x) = x^2 - 6x$
- **e** $f(x) = -5x^2 + 1000x - 49945$

2 Sketch the graph of the following functions, within the specified domains and hence determine their range. On your graph show clearly any intersections with the coordinate axes, the axis of symmetry and the vertex and state the range.

- **a** $f(x) = -x^2$, for $-3 \leq x \leq 4$
- **b** $f(x) = -2x^2 + 4x - 3$, for $0 \leq x \leq 2$
- **c** $f(x) = \frac{1}{2}x^2 - 4x + 5$, for $0 \leq x \leq 4$
- **d** $f(x) = \frac{1}{2}x^2 - 4x + 5$, for $0 \leq x \leq 8$
- **e** $f(x) = -5x^2 + 1000x - 49945$, for $95 \leq x \leq 105$
- **f** $P(q) = 75 - 0.000186q^2$, where P is the price at which q units of a certain good can be sold and where $q \leq 500$

3 Consider the curve defined by the equation $y = 0.4x^2 - 2x - 8$.

- **a** Find the coordinates where the curve crosses the x-axis.
- **b** Find the coordinates of the intercept with the y-axis.
- **c** Find the equation of the axis of symmetry of the curve.
- **d** Find the point of intersection of this curve with the curve given by the equation $y = -5x^3$.

4 Zander is playing a game of baseball. He hits the ball and the height of the ball is modelled by the formula $y = -0.018x^2 + 0.54x + 1.0$, where y is the height of the ball, in metres, and $x > 0$ is the horizontal distance in metres.

- **a** Find the maximum height that the ball reaches.
- **b** Find the positive value for x when the graph crosses the x-axis and explain what this value represents in context.
- **c** Find the y-intercept and explain what this represents in context.

5 Omar is on the school shot-put team. The path of the shot-put is modelled by a quadratic function with equation $y = 1.5 + 0.75x - 0.05x^2$, where y is the height of the shot-put in metres and $x > 0$ is the horizontal distance travelled in metres.

- **a** Find the maximum height that the shot-put reaches.
- **b** Write down the equation of the axis of symmetry.
- **c** Find the positive value for x when the graph crosses the x-axis and explain what this value represents in context.
- **d** Find the y-intercept and explain what this value represents in context.

6 Ziyue is the goalkeeper in his football team. He takes a free kick from the goal and the vertical path of the ball is modelled by the function $f(x) = -0.06x^2 + 1.2x$, where $f(x)$ is the height of the ball, in metres, and x is the horizontal distance travelled by the ball, in metres.

- **a** Find the maximum height that the ball reaches.
- **b** Write down the equation of the axis of symmetry.
- **c** Find the x-intercepts and explain what these values represent.

7 The vertical cross-section of a ramp in a skateboard park is modelled by the function $f(x) = 10.67 - 1.6x + 0.0417x^2$, where x is the horizontal distance in metres from the start and $f(x)$ is the vertical distance above the ground in metres. The ramp has been constructed in such a way that part of its base lies below the level of the ground around the ramp.

- **a** Find the minimum depth of the run.
- **b** Find the x-intercepts and explain what these values represent.

8 Jin throws a stone into the air. The height of the stone above the ground can be modelled by the function $f(t) = 1 + 7.25t - 1.875t^2$, where t is the time, in seconds, that has passed since the stone was thrown, and $f(t)$ is the height of the stone, in metres.

- **a** Find the maximum height that the stone reaches.
- **b** Determine how long it takes for the stone to land on the ground.

9 A bullet is fired from the top of a cliff. The path of the bullet may be modelled by the equation $y = -0.0147x^2 + 2x + 96$ $(x \geq 0)$, where x is the horizontal distance from the foot of the cliff and y is the vertical distance above the sea which meets the foot of the cliff.

- **a** State the height of the cliff.
- **b** Find the maximum height reached by the bullet.
- **c** Find the distance the bullet is from the foot of the cliff, when it hits the sea.

10 Consider an arithmetic sequence with first three terms 13, 9 and 5.

- **a** Show that the sum of the first n terms of the arithmetic sequence is given by the quadratic expression $S_n = 15n - 2n^2$, $n \in \mathbb{Z}^+$.
- **b** Use the table function on the GDC to plot the values of S_n for $1 \leq n \leq 8$.
- **c** Hence, find the maximum possible value of the sequence.
- **d** Determine the greatest value of n for which the sum is positive.

11 The equation of motion when throwing something vertically upwards is given by the formula $h(t) = h_0 + v_0 \times t - \frac{1}{2} g \times t^2$, where h_0 is the initial height above the ground, v_0 is the initial upward speed and g $(= 9.81\,\text{m}\,\text{s}^{-2})$ is the acceleration due to gravity. Consider someone standing on the roof of a building 21 m tall, who throws a ball vertically upwards with an initial speed of $15\,\text{m}\,\text{s}^{-1}$.

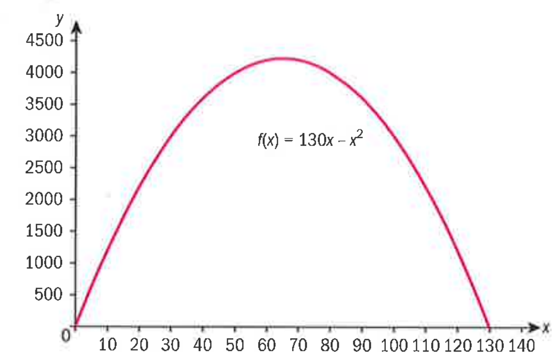
- **a** Sketch the graph of the height of the ball against time.
- **b** Determine the maximum height that the ball will reach.
- **c** Determine how long the ball will need in order to fall to the ground.

Problems involving quadratics

Example 3

A rectangular mirror has perimeter 260 cm.

- **a** If the length of the mirror is x cm, find the height of the mirror in terms of x.
- **b** Find an equation for the area of the mirror, A cm^2, in terms of x.
- **c** Use your GDC to plot a graph of your equation for the area of the mirror, showing area A on the y-axis and length x on the x-axis. Choose a suitable domain and range.
- **d** Find the coordinates of the points where the graph intersects the x-axis.
- **e** State what these two values of x represent.
- **f** Find the equation of the graph's line of symmetry.
- **g** State what the equation of the line of symmetry represents in this context.

a Let the height of the rectangle be y cm. $2x + 2y = 260$ $x + y = 130$ $y = 130 - x$	Label the height y cm and form an equation in x and y for the perimeter of the rectangle.
b Area = length × height $A = xy$ $A = x(130 - x)$	Substitute $y = 130 - x$, which you found in part **a**.
c 	Use your GDC to locate the turning point. A reasonable domain would be from 0 to 150, and range from 0 to 4500.
d The x-intercepts are (0, 0) and (130, 0).	Use your GDC to find the zeros of the function.
e The x-coordinates 0 and 130 are the upper and lower limits between which the value of x must lie.	
f The line of symmetry is $x = 65$	You can find the line of symmetry by finding the coordinates of the vertex on your GDC or by finding the midpoint of the two x-intercepts.
g This is the value of x which gives the largest area of the mirror.	The axis of symmetry passes through the vertex of the graph, which is a maximum in this case.

Exercise 6B

1 A rectangular picture frame has perimeter 100 cm.

- **a** The width of the frame is x cm. Find an expression, in terms of x, for the height of the frame.
- **b** Find an expression for the area, A cm^2, in terms of x.
- **c** Sketch the graph of A against x.
- **d** Find the x-intercepts.
- **e** Find the y-intercept.
- **f** Find the equation of the axis of symmetry.
- **g** Find the coordinates of the vertex.
- **h** Write down the maximum area of the picture frame, and the dimensions of the picture frame that give this maximum area.

Functions

2 A rectangular picture frame has perimeter 70 cm.

a The width of the frame is x cm. Find an expression, in terms of x, for the height of the frame.

b Find an expression for the area of the frame, A cm^2, in terms of x.

c Sketch the graph which shows how A varies with x. Use a suitable domain and range.

d Find the x-intercepts of the graph in part **c**.

e State what these two values of x represent.

f Use the x-intercepts to find the equation of the graph's line of symmetry, and state what this tells you in the context of the problem.

3 A company produces and sells books.

The weekly cost, in euros, for producing x books is €$\left(0.1x^2 + 400\right)$.

The weekly income from selling x books is €$\left(-0.12x^2 + 30x\right)$.

a Show that the weekly profit, $P(x)$, can be written as $P(x) = -0.22x^2 + 30x - 400$.

b Sketch the graph of the profit function showing the coordinates of the vertex and axes intercepts.

c State what the x-intercepts represent in the context of the problem.

d Find the equation of the axis of symmetry of the graph, and state what this tells you in the context of the problem.

4 The first four terms of an arithmetic sequence are:

6 10 14 18

a Show that the sum to n terms can be written as $2n^2 + 4n$.

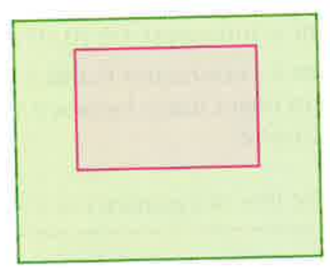
b If the sum to n terms is 880, write a quadratic equation to represent this information.

c Find the set of values of n for which the sum to n terms is greater than 880.

5 A dolphin is jumping out of the water and its motion follows the shape of a parabola. If the dolphin reaches the maximum height at 1.3 seconds after it started jumping out of the water, determine when it will fall back in the water.

6 Marina bought 12 metres of fencing to construct a small playground for her daughter in her rectangular 10 m by 8 m backyard. She wants to decide how to place the fencing in order to construct the largest rectangular playground. Three sides of her backyard are enclosed by walls of adjacent houses and one side is connected to her house through a sliding door.

a The first design that she wanted to analyse was that of a rectangular playground in the middle of the backyard.



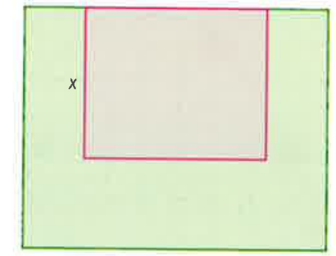
i If the length of one side of the playground is x, find the length of the other side in terms of x.

ii Find an expression for the area of the playground in terms of x.

iii Using an appropriate domain for the area function, sketch its graph.

iv Hence determine the maximum possible area that the playground can have.

b The second design that she wanted to analyse was that of a rectangular playground with one side being the adjacent wall of her backyard.



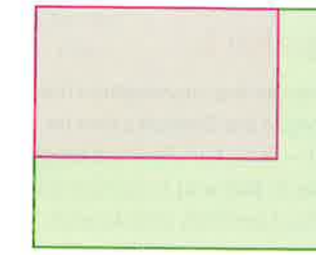
i If the length of one side of the playground is x as shown in the diagram, find the length of the other side in terms of x.

ii Find an expression for the area of the playground in terms of x.

iii Using an appropriate domain for the area function, sketch its graph.

iv Hence determine the maximum possible area that the playground can have.

c A third design that she wanted to analyse was that of a rectangular playground with two sides being on the walls of one of the corners of her backyard.



i If the length of one side of the playground is x, find the length of the other side in terms of x.

ii Find an expression for the area of the playground in terms of x.

iii Using an appropriate domain for the area function, sketch its graph.

iv Hence determine the maximum possible area that the playground can have.

v State the dimensions that produce the maximum area from all the possible designs.

Restricted domains and the inverse of a function

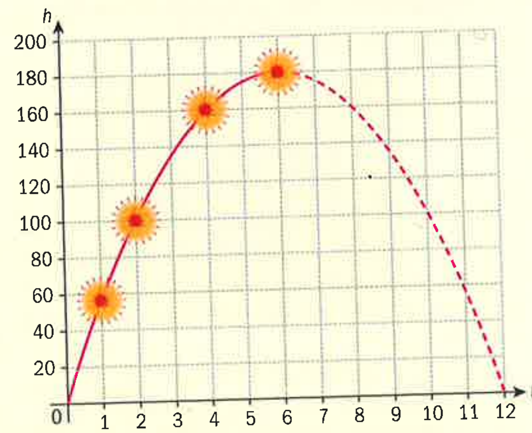
From chapter 4 you will recall the following facts about the inverse function f^{-1}:

- $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$.
- The graph of $y = f^{-1}(x)$ can be obtained by reflecting the graph of $y = f(x)$ in the line $y = x$.
- To find the inverse function write as $y = f(x)$, interchange x and y and rearrange to make y the subject of the expression.

This section will explore a particular requirement for inverse functions to exist.

Investigation 1

A pyrotechnician has developed a model function for the trajectory of the fireworks that he launches: $h(t) = 180 - 5(t-6)^2$. This is a function of height against time. In this way he can predict the height h of the fireworks, t seconds after launch.



He is planning a show with a large number of fireworks. To create the most amazing spectacle, he wants the fireworks to burst at different heights so that they do not overlap, as shown on the graph. Knowing the desired height of burst of each firework, he needs to work out the corresponding time. What he has realized is that, instead of knowing the input of the function, he now knows the output and is looking for the input, which leads to a time-consuming solution of quadratic equations each time.

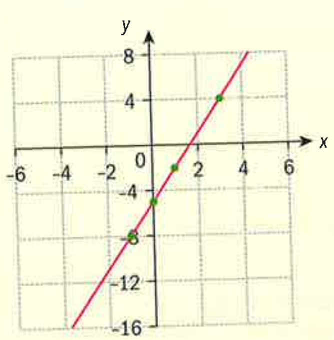
So, he has decided to reverse the variables of the function and make the height h the input and the time t the output. To do this he needs to make t the subject of the equation.

After reaching the maximum height, the fireworks will start falling back to the ground. This means that they will be at a certain desired height at two different times. It is required, though, that the fireworks burst while they are ascending.

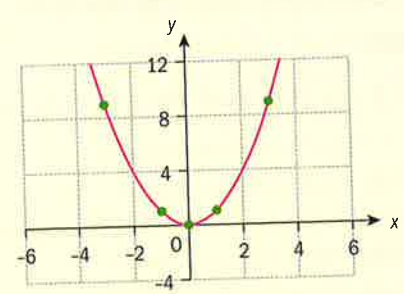
1. How should the domain of the model function be restricted so that each height corresponds to a single time?
2. Rearrange the model function $h(t) = 180 - 5\ (t-6)^2$ so that it becomes a function of time t with respect to height h.
3. What do you notice? How can the restricted domain from question **1** help?

The function $h(t)$ is an example of a **many-to-one** function: two (or more) different inputs map to the same output. When each output corresponds to exactly one input, we say a function is **one-to-one**. Here is an example of each function type:

One-to-one: $y = 3x - 5$



Many-to-one: $y = x^2$



4. You can create the inverse relation of any function by reflecting it across the line $y = x$, as you did for inverse functions.
   - **a** Sketch the graph of the inverse relation of each of the example functions above.
   - **b** Use your graphs to explain why the one-to-one function's inverse relation is a function, but the many-to-one's inverse relation is **not** a function.
5. **Conceptual** Why is a function not always invertible (invertible means that its inverse function exists), and when is it possible for a function to be invertible?

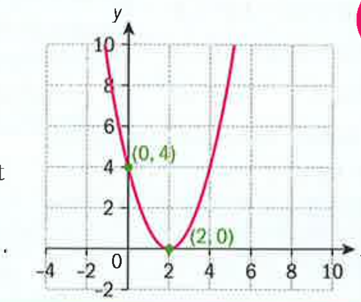
An easy way to test whether or not a function has an inverse is to use the horizontal line test. This test says, if any horizontal line cuts the curve more than once then an inverse function does not exist.

If a function is many-to-one, we can sometimes **restrict its domain** so that it is one-to-one.

Example 4

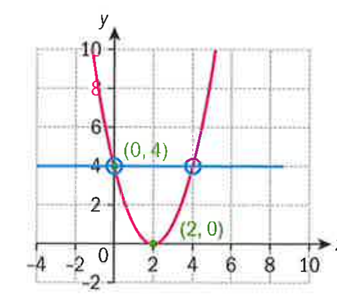
Consider the function $f(x) = (x - 2)^2$.

- **a** State reasons why f^{-1} does not exist if the domain of f is $x \in \mathbb{R}$.
- **b** The domain is now restricted to $x \geq k$. Find the smallest possible value of k such that the function is invertible.
- **c** Restricting $f(x)$ to this domain, sketch the graph of $f^{-1}(x)$.



a f^{-1} does not exist because f is many-to-one. For example, $f(0) = 4$ and $f(4) = 4$.

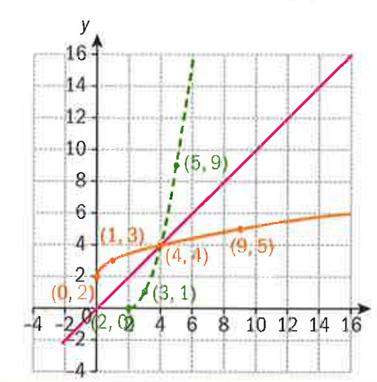
Note that a many-to-one function will not pass the **horizontal line test**: A horizontal line can be drawn that intersects the function more than once.



b $\{x | x \geq 2\}$

The function is symmetric around $x = 2$, so there are two x-values corresponding to each positive y-value. Eliminating the left "half" of the function will remove one of these x-values.

c The graph of f^{-1} is in orange.



Sketch the line $y = x$ to see where the function will be reflected.

Use the table or point-plotting feature of your graphing technology to find several coordinates on the graph of $f(x)$. Invert these to find the corresponding coordinates of $f^{-1}(x)$.

Exercise 6C

1 Restrict the domain of the following quadratic functions so that they can have an inverse and then find their inverse, stating its range.

a $f(x) = x^3 + 4$

b $f(x) = (x-1)^3 - 2$

c $f(x) = 2(x+3)^3$

d $f(x) = 1 - 3(x-2)^3$

2 The model function linking the price €p of a can of soda to its demanded quantity Q, is given by the model function $Q(p) = 175 - 3.5p$.

The cost of producing one can is €5 and the company has a fixed cost of €875 in making these cans.

a Find the modelling function for the revenue of the company.

b Find the modelling function for the costs of the company.

c Hence find the modelling function for the profit of the company.

3 Consider the function $f(x) = (x+5)^2 - 2$.

a Use technology to sketch the graph.

b Explain why f^{-1} does not exist if the domain of f is $x \in \mathbb{R}$.

c Find the largest positive domain such that the function is invertible.

d Sketch the graph of $f^{-1}(x)$ on this restricted domain.

4 Construct a piecewise linear function that meets the following requirements:

a The function has three pieces, two with different positive gradients and one with a negative gradient.

b The function is continuous.

c The function is invertible when the domain is restricted to $x \geq 3$, but is not invertible on the domain $x \geq k$ for k smaller than 3.

TOK

How can you deal with the ethical dilemma of using mathematics to cause harm, such as plotting the course of a missile?

Developing inquiry skills

Look back at the opening problem for this chapter. Oliver was trying to throw a basketball through a hoop.

What type of function could you use to model the path of the basketball?

6.2 Problems involving quadratics

The equation of a linear function has the general form $f(x) = mx + c$, with two parameters (m and c) defining a unique line. To determine the value of these two parameters, you need the coordinates of two points, since there is a single line going through two given points. This will create two linear equations with two unknowns, which you will be able to solve through your GDC.

The equation of a quadratic function has the general form $f(x) = ax^2 + bx + c$, with three parameters (a, b and c) defining a unique parabola. To determine the value of these three parameters, you now need the coordinates of three points, since there is a single parabola going through three given points. This will create three linear equations with three unknowns, which you will be able to solve through your GDC.

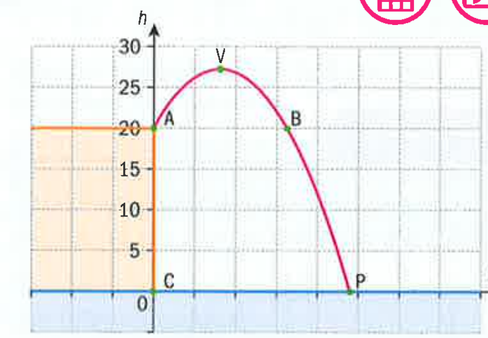
Example 5

A student wants to model the path of a rock with respect to time.

He is standing at the top of a vertical cliff at a height of 20 m and throws a rock upwards as shown in the diagram.

He starts counting time at the instant he throws the rock, at point A.

He measures that the time at which the rock is again at eye level, at point B, is $t = 3.26$ s.



He finally measures that the time at which the rock falls into the sea, at point P, is $t = 4.8$ s.

The simplest function that will model the height of the rock with respect to time is a quadratic of the form $h(t) = at^2 + bt + c$.

Determine this function.

There are two methods you can use to do this.	
Method 1: Using **Quadratic regression** on your GDC.	
You need to identify three clear points on the curve.	You can take any three points on the graph as long as they are exact.
Known points are A(0, 20), B(3.26, 20) and P(4.8, 0).	Use the information from the question. At A, $t = 0$ and $h = 20$. At B the height is the same as at A, 20. At P the rock is at ground level, where $h = 0$.
	In your GDC, put the t values in List 1 and the h values in List 2.
The parabola is given by the equation $h(t) = -2.71t^2 + 8.82t + 20$	Then go to Statistics Quadratic regression. Here you see the values of the parameters a, b and c.

Continued on next page

Method 2: Using **Simultaneous equation solver** on your GDC.

Curve passes through A(0, 20), B(3.26, 20) and P(4.8, 0).

Substituting these points into the given function:

Using point A: $a(0)^2 + b(0) + c = 20 \Rightarrow c = 20$

Using point B: $a(3.26)^2 + b(3.26) + 20 = 20$
$\Rightarrow a(3.26)^2 + b(3.26) = 0$

Using point P: $a(4.8)^2 + b(4.8) + 20 = 0$
$\Rightarrow a(4.8)^2 + b(4.8) = -20$

$h(t) = -2.71t^2 + 8.82t + 20$

First, find the three points on the curve as in Method 1.

For each of the three points, substitute the coordinates of t and h into the general equation of a parabola, $h = at^2 + bt + c$.

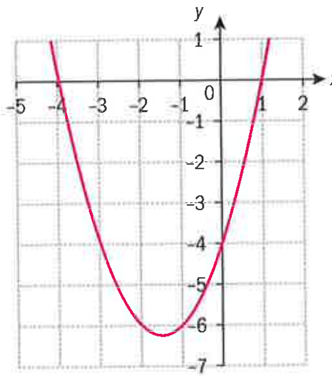
This gives you three simultaneous equations in a, b and c.

Solve these equations on your GDC to give the solution.

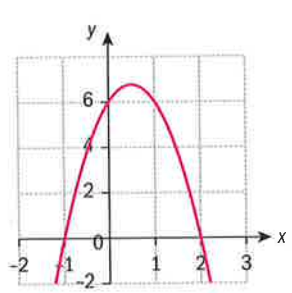
Exercise 6D

Find the quadratic functions represented by the following graphs.

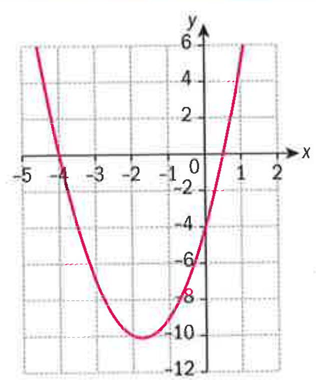
1



2



3



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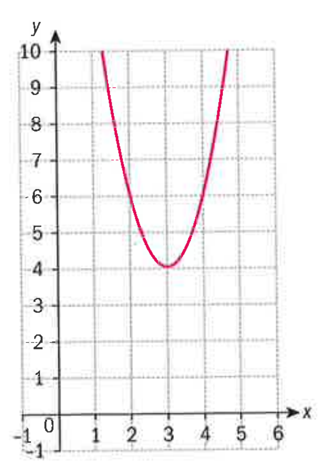
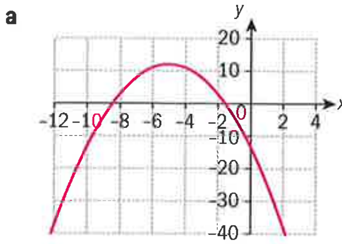
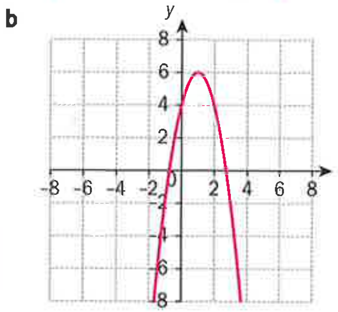


Find approximate quadratic functions represented by the following graphs.

5 a



b



6 The graph of the quadratic function $f(x) = ax^2 + bx + c$ passess through point P(3, −4).

a Show that $9a + 3b + c = -4$.

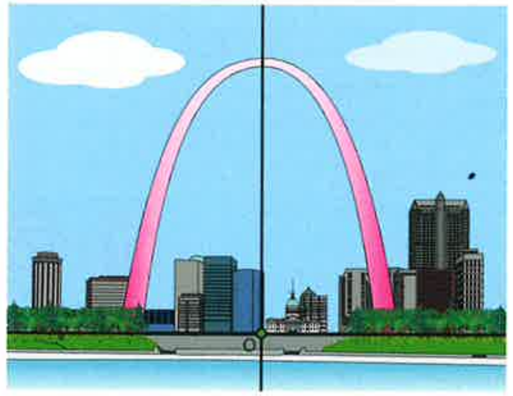
The graph also passess through points Q(−2, 9) and R(5, −30).

b Find two more equations for a, b and c.

c Solve the system of equations to determine the values of a, b and c.

d Sketch the graph of this function clearly showing points P, Q and R.

7 A mathematics student wants to create a model function for the Gateway Arch in St. Louis in the US state of Missouri. The arch is both 192 m wide and high. Find a suitable modelling function.



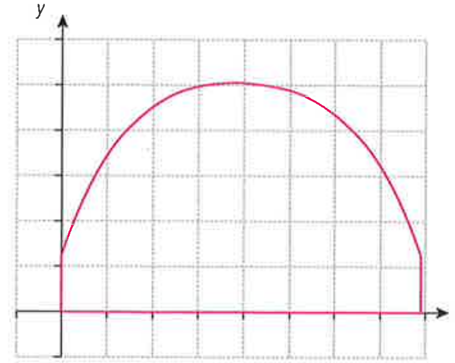
8 In a football game, if a player is fouled outside the penalty area, they are awarded a free kick.

In a free kick, the player sets the ball at the point where they were fouled, and the opponents can make a "wall" at a distance of 9.1 metres. The goal at which the player taking the free kick is aiming, has a height of 2.4 metres. Assuming that the player was fouled about 20 metres from the goal and that the tallest football players who will make the wall have a height of about 2 metres, we are faced with the problem of creating the trajectory to describe the optimum path of the ball towards the net.

Assume the path of the ball can be modelled as a parabola.

Find the equation of the path of the ball which passes just over the head of the tallest players in the wall and then passes just under the crossbar of the goal, as shown in the diagram.

9 The arched truss for an outdoor concert has the following shape:



The function that models this arch is known to be $f(x) = -0.12x^2 + 1.92x + 3$, where x is the horizontal distance from the bottom left corner and y is the height. The curved part of the arch rests on 3m vertical sides.

Because of the wind expected during the concert, the organizers decided to bring the arched part of the stage down by 1 metre while keeping the vertical section at 3 m. Find the new function that will model the arched part of the stage.

Functions

Quadratic regression

In order to choose a quadratic model to describe a certain set of data, the data need to have some specific features. These include:

- the data having a single maximum or a single minimum point (corresponding to the vertex of a parabola)
- the data being symmetric about a certain vertical line (corresponding to the axis of symmetry of a parabola)
- the data having a variable rate of change, creating a curve (corresponding to the shape of a parabola).

The more data you have, the more certain you are about the choice of curve to model them. As you saw before, you need at least three points in order to create a quadratic model.

- If you have exactly three points, there is a single parabola going through them, but not enough evidence that it is the correct curve to use. In such a case, you need to know in advance that you are indeed looking for a parabola in order to go on and find one.

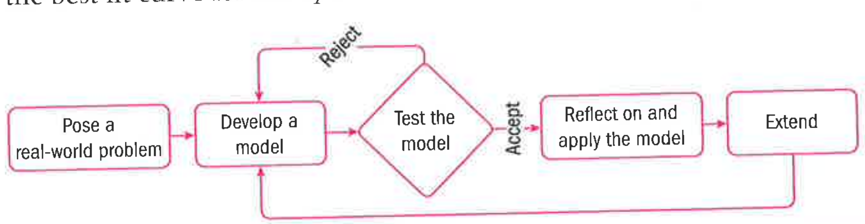
  If you have more than three points from a data collection process, it is almost certain that they will not lie exactly on the graph of a certain quadratic function. This means that you need to determine the parabola of best fit through these given points. Your GDC can easily and quickly perform this operation through quadratic regression.

  The difference from the previous example is that now you need to check whether the parabola fits the data well or not. The measure for this is the **coefficient of determination**, which your GDC calculates. It takes values between 0 and 1 where 1 would indicate that all the points in the data set lie on the curve. The coefficient of determination can be used for any curve. If used to test for a linear function its value will be the same as the square of the PMCC which you met in Chapters 2 and 4.

International-mindedness

The Sulba Sutras (around 500 BCE) and the Bakhshali manuscripts (around 300 AD), both from India, contained an algebraic formula for solving quadratic equations.

You always need to keep in mind the following diagram when finding the best fit curve for messy data.



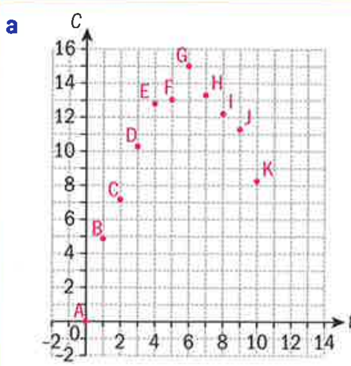
Example 6

A patient takes a specific drug in the form of a pill. Data is collected for the amount of the drug found in the bloodstream of the patient as soon as he has taken the medication. Time (t) is measured in hours and the concentration of the medication (C) is measured in milligrams of the medication found per litre of blood.

t	0	1	2	3	4	5	6	7	8	9	10
$C(t)$	0	4.87	7.17	10.27	12.81	13.05	15.03	13.3	12.22	11.29	8.26

a Plot a scatter plot of the given data.
b State a suitable type of function that would model this set of data points.
c Use your GDC to determine the model function for this set of data.
d Comment on the choice of model by determining the coefficient of determination.
e Sketch the model function over the scatter plot and comment on the closeness of fit to the original data.
f Hence, determine the time at which the medication is at its maximum effect.
g Using the model function, determine the time at which the medication will have been fully absorbed by the patient.
h Use the model to determine the concentration of the medication after 24 hours?

a

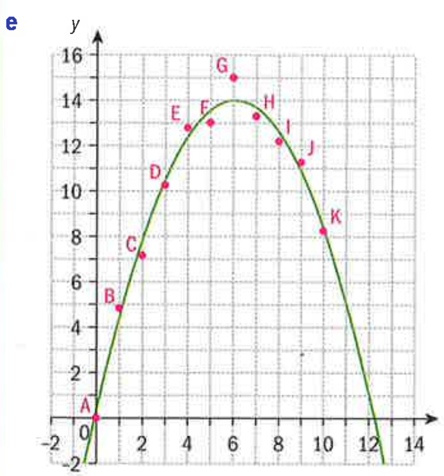


b There is strong evidence that the data depicts quadratic behaviour. There is a maximum point, there seems to be some kind of symmetry and the data follows a parabolic shape.

c $C(t) = -0.3741375t^2 + 4.56328438t + 0.12111888$

d The coefficient of determination is $R^2 = 0.98617438$, which shows very strong quadratic association.

e



Sketching the graph over the data points you get the graph shown.

It is now evident that the regression curve fits the data very closely.

Continued on next page

Functions

f The maximum amount of medication occurs around 6.1 hours after it was taken by the patient.	Using the graph of the model function on your GDC.
g It will take about 12.2 hours for the whole of the medication to be fully absorbed and not be present any more in the bloodstream.	
h No, as the value would be negative. $C(24) < 0$	After the time that the concentration of the medication becomes zero, the model cannot be used anymore as the value would become negative.

Exercise 6E

1 A company's weekly profit, in euros, in relation to the number of units sold each week, is given in the table below.

Number of units sold	200	300	400	500	600	700	800	900	1000	1100
Profit (euros)	4900	8000	10 000	12 000	12 600	13 200	13 000	10 000	8000	6000

a Plot these points on your GDC or other graphing software.
Put the number of units sold on the x-axis and the profit on the y-axis.

b Using your GDC or technology, find the best fit quadratic function through these points.

c State whether the function you found in part **b** is a good fit for this data. Justify your answer.

d Explain whether you could use this function to predict the value of the profit at a particular time during the year.

2 A company's weekly profit, in GBP in relation to the number of units sold each week, is given in the table below.

Number of units sold	0	5	10	15	20	25	30	35	40	45
Profit (GBP)	−350	−100	50	200	300	400	450	350	320	150

a Plot these points on your GDC or other graphing software.
Put the number of units sold on the x-axis and the profit on the y-axis.

b Using your GDC or technology, find the best fit quadratic function through these points.

c State whether the function you found in part **b** is a good fit for this data. Justify your answer.

d The company uses this function to predict what the profit would be in week 52. Explain whether this prediction would be reliable.

3 Mees measures the outside temperature, in °C, every 2 hours over a 24-hour period. His measurements are given in the table below.

Time (hours)	4:00	6:00	8:00	10:00	12:00	14:00	16:00	18:00	20:00	22:00	24:00	2:00
Temperature (°C)	4	5	10	15	18	21	19	14	11	8	7	4

a Use your GDC to plot these points.

b Find the best fit quadratic function through these points.

c Hence, find an estimate of the outside temperature at 17.00. State whether this is likely to be a reasonable estimate, and justify your answer.

4 Cindy tries to make a model of a suspension bridge. To do so, she estimates the distance of various points on the bridge from where she is standing across the harbour, and the height of each point of the suspension cable above the ground. Her data is shown in the table below.

Distance from Cindy (m)	0	100	150	200	300	400	450	500	550	600	700	750
Height of bridge (m)	250	220	180	175	100	50	168	172	150	200	205	210

a Plot these points on your GDC or other graphing software. Put the distance on the x-axis and the height on the y-axis.

b Using your GDC or technology, find the best fit quadratic function through these points.

c Deduce whether Cindy's data will allow her to model the bridge. Justify your answer.

d Using your answer to part **c**, explain whether your best fit quadratic function could be used to find the height of the bridge at 410 metres from Cindy.

5 A shot-put athlete is training for a major competition. He wants to improve his technique. For this he recorded on video a great number of his tries at putting the shot at various release angles. He then analysed the video in order to create pairs of data between the angle of release and the distance at which the shot landed. The data are the following:

Angle	0	7.4	16.6	28.6	36.7	41.8	47.0	51.0	55.0	57.3	63.6
Distance	12.4	13.6	14.9	15.9	15.9	15.6	14.9	14.2	13.4	12.7	10.8
Angle	66.5	68.2	71.0	72.2	75.1	77.9	80.8	82.5	85.9	87.1	88.0
Distance	9.9	9.2	8.2	7.6	6.6	5.3	4.2	3.5	1.8	1.2	0.0

a Use quadratic regression to find the equation of the parabola that best describes the above set of data.

b Determine the optimal angle at which the athlete should put the shot in order to achieve the best results.

c Comment on the appropriateness of using the model function.

Transformations of graphs

Investigation 2

Consider the function $f(x) = x^2$.

1 Let $g(x) = f(x) + 3 = x^2 + 3$.

a Fill in the following table of values.

x	−3	−2	−1	0	1	2	3
$f(x)$							
$g(x)$							

b Plot both graphs on your GDC.

c How does adding a positive number affect the graph? How does it affect the coordinates?

d How would subtracting a positive number from a function affect its graph?

Continued on next page

Functions

2 Let $h(x) = 2 \times f(x) = 2x^2$.

a Fill in the following table of values.

x	−3	−2	−1	0	1	2	3
$f(x)$							
$h(x)$							

b Plot both graphs on your GDC.

c How does multiplying a function by a number greater than 1 affect the graph? How does it affect the coordinates?

d Which points remain invariant under this transformation?

e How would multiplying a function by a number between 0 and 1 affect its graph?

3 Let $k(x) = -f(x) = -x^2$.

a Fill in the following table of values.

x	−3	−2	−1	0	1	2	3
$f(x)$							
$k(x)$							

b Plot both graphs on your GDC.

c How does changing the sign of a function affect the graph? How does it affect the coordinates?

d Which points remain invariant under this transformation?

4 Let $p(x) = f(x+1) = (x+1)^2$.

a Fill in the following table of values.

x	−4	−3	−2	0	1	2	3
$f(x)$							
$p(x)$							

b Plot both graphs on your GDC.

c How does adding a positive number to the independent variable x of a function affect the graph? How does it affect the coordinates?

d How would subtracting a positive number from the independent variable affect its graph?

5 Let $q(x) = p(2x) = (2x+1)^2$.

a Fill in the following tables of values.

x	−4	−3	−2	−1.5	−1	−0.5	0	0.5	1	2
$p(x)$										
$q(x)$										

b Plot both graphs on your GDC.

c How does multiplying the independent variable x of a function affect the graph? How does it affect the coordinates?

d Which points remain invariant under this transformation?

e How would dividing the independent variable by a number affect its graph?

6 Let $r(x) = p(-x) = (-x+1)^2$.

a Fill in the following table of values.

x	−4	−3	−2	−1	0	1	2	3	4
$p(x)$									
$r(x)$									

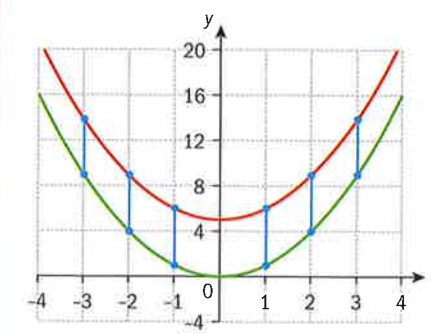
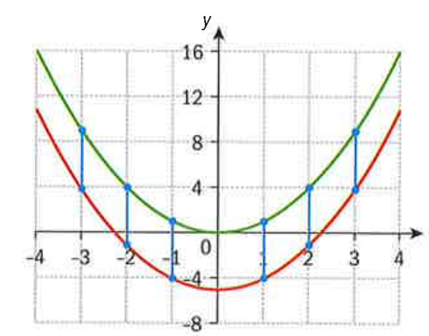
b Plot both graphs on your GDC.

c How does changing the sign of the independent variable of x a function affect the graph? How does it affect the coordinates?

d Which points remain invariant under this transformation?

7 **Conceptual** What is the effect of changing the parameters of the quadratic function $y = a(bx+c)^2 + d$ on its graph?

You can express transformations of functions in three different ways: using mathematical notation, using a geometric description or using a graphical representation.

Mathematical notation	Geometric description	Graphical representation
$y = f(x) + k$	Vertical translation k units up, with vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$	
$y = f(x) - k$	Vertical translation k units down, with vector $\begin{pmatrix} 0 \\ -k \end{pmatrix}$	

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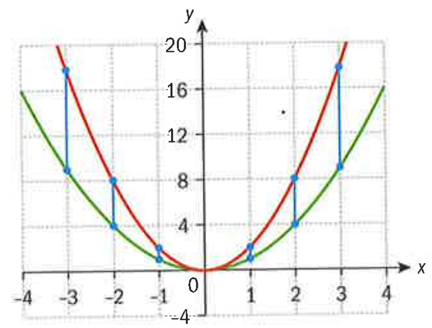
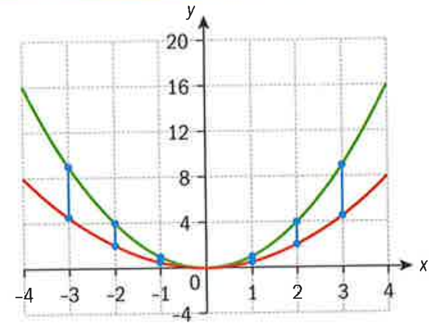
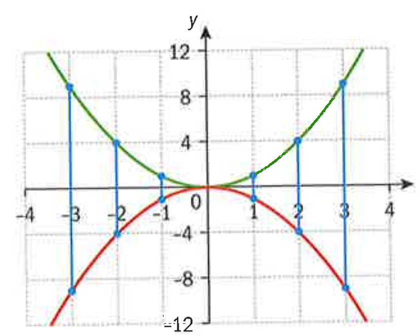
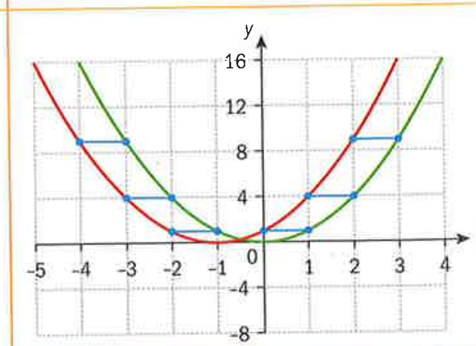
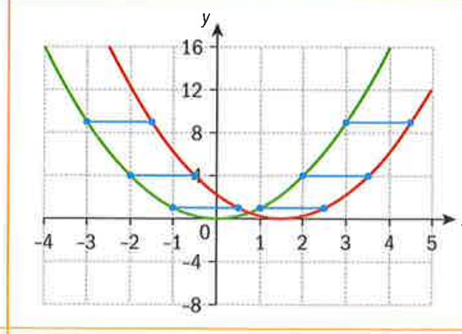
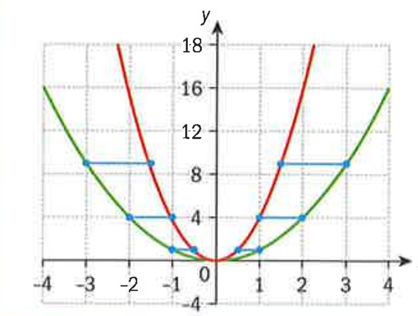
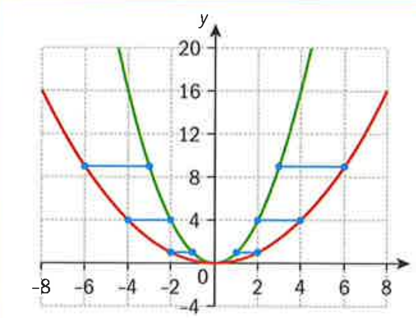
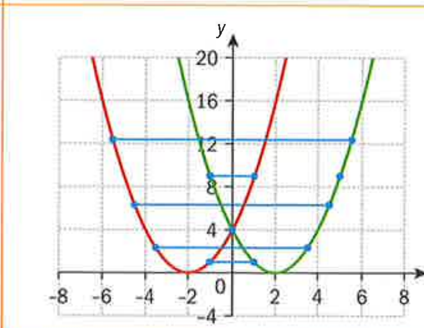
Mathematical notation	Geometric description	Graphical representation
$y=k\times f(x)$	Vertical stretch with scale factor k	
$y=\frac{f(x)}{k}=\frac{1}{k}\times f(x)$	Vertical stretch with scale factor $\frac{1}{k}$	
$y=-f(x)$	Vertical reflection in the x-axis	
$y=f(x+k)$	Horizontal translation k units left, with vector $\begin{pmatrix}-k\\0\end{pmatrix}$	

Mathematical notation	Geometric description	Graphical representation
$y=f(x-k)$	Horizontal translation k units right, with vector $\begin{pmatrix}k\\0\end{pmatrix}$	
$y=f(kx)$	Horizontal stretch with scale factor $\frac{1}{k}$	
$y=f\left(\frac{x}{k}\right)=f\left(\frac{1}{k}\times x\right)$	Horizontal stretch with scale factor k	
$y=f(-x)$	Horizontal reflection in the y-axis	

You can also combine transformations within the same function. Changing the order of transformations can change the result.

Example 7

Write down a full geometric description of a sequence of transformations required to transform the function $f(x) = x^2$ to the function $g(x) = 5 - 2(x + 3)^2$.

	You need to think about the transformations in sequential steps. It is better to work with the algebraic notation first and then give the geometric description.
Horizontal translation 3 units to the left.	$y = x^2 \rightarrow y = (x + 3)^2$
Vertical stretch with scale factor 2.	$y = (x + 3)^2 \rightarrow y = 2(x + 3)^2$
Vertical reflection in the x-axis.	$y = 2(x + 3)^2 \rightarrow y = -2(x + 3)^2$
Vertical translation 5 units up.	$y = -2(x + 3)^2 \rightarrow y = 5 - 2(x + 3)^2$

HINT

When giving a geometric description of a transformation you should include the variable it affects using the terms *vertical* or *horizontal* for transformations on the y- or x-axis respectively, the type of transformation using the terms *translation*, *stretch* or *reflection*, and the "value" of the specific transformations, which will be units and direction for translations, scale factor for stretches, and the mirror line for reflections.

Example 8

The graph of $g(x) = 5 - 2(x + 3)^2$ is transformed into the graph of $f(x)$ by the following sequence of transformations:

- a vertical stretch with scale factor 2 followed by
- a horizontal translation of 4 units to the left followed by
- a vertical translation of 3 units down.

Find an expression for $f(x)$.

Let $p(x)$ represent a vertical stretch with scale factor 2. $(p \circ g)(x) = 2g(x) = 2(5 - 2(x + 3)^2)$ $= 10 - 4(x + 3)^2$ Let $q(x)$ represent a horizontal translation of 4 units to the left.	If an order is specified, you need to use the transformations in the order they are given. When working with a composition of functions, you should work "from the inside out". You can either perform a transformation to the whole of the function or you can perform a transformation only on the input variable.

TOK

How accurate is a visual representation of a mathematical concept?

$$(q \circ p \circ g)(x) = q(10 - 4(x + 3)^2)$$
$$= 10 - 4(x + 3 + 4)^2$$
$$= 10 - 4(x + 7)^2$$

Let $r(x)$ represent a vertical translation of 3 units down.

$$(r \circ q \circ p \circ g)(x) = r(10 - 4(x + 7)^2)$$
$$= 10 - 4(x + 7)^2 - 3$$
$$= 7 - 4(x + 7)^2$$

Exercise 6F

1 Determine a full geometric description of the following transformations:

a from $f(x) = x^2$ to $g(x) = \left(\frac{x}{2}\right)^2 - 3$

b from $f(x) = 2(x - 3)^2 + 4$ to $g(x) = x^2$

c from $f(x) = x^2$ to $g(x) = 4(x - 1)^2 + 2$

d from $f(x) = 2(x - 3)^2 + 4$ to $g(x) = 2 - (2 - x)^2$.

2 Determine the function resulting from the following transformations starting from $f(x) = x^2$:

a translation 3 units to the left, followed by a reflection about the x-axis, followed by a vertical stretch with scale factor 2

b translation 2 units to the left, followed by a horizontal stretch with scale factor 2

c horizontal stretch with scale factor 2, followed by translation 2 units to the left

d translation 1 unit up followed by a translation 2 units left followed by a reflection in the y-axis

3 Consider the curve $y = 2(x - 1)^2 + 5$.

a State the vertex of the graph.

b Show that the point (3, 13) lies on the curve.

The curve undergoes the following transformations. A reflection in the y-axis, a horizontal translation of 3 units to the right and a vertical stretch scale factor 2.

c Find the new position of:

i the vertex

ii the point (3, 13).

d Use your answer to part **c** to find the equation of the transformed curve.

e Verify your answer by transforming the equation.

4 Consider the functions $f(x) = x^2$ and $g(x) = 5 - (x - 2)^2$.

a Write down a **full geometric description** of the sequence of transformations required to obtain the function $g(x)$ from the function $f(x)$.

b If $g(x)$ is stretched vertically with a scale factor of 2 and then translated by the vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ to become $h(x)$, determine an expression for $h(x)$.

5 Given the function $f(x) = 11 - (x - 2)^2$ in the domain $2 \le x \le 5$.

a **i** Sketch the graph of $f(x)$ in the given domain.

ii Determine the range of $f(x)$ in the given domain.

iii The graph of $f(x)$ can be obtained from the graph of $y = x^2$ using a sequence of transformations. Write down a full geometric description of the transformations.

b The function $f(x)$ undergoes the following sequence of transformations in order to become the function $g(x)$: first a translation 3 units to the right and 1 unit down followed by a stretch by scale factor $\frac{1}{2}$ parallel to the y-axis.

Determine an expression for the function $g(x)$.

6 The function $f(x) = x^2$ becomes the function $g(x) = 2(x + 2)^2 - 15$ under a sequence of transformations. The function $g(x)$, is furthermore translated by the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ to become the function $h(x)$. Determine the full geometric description of the transformations required to take function $h(x)$ **back** to the function $f(x)$.

7 Consider the functions $f(x) = x^2$ and $g(x) = 3(x - 1)^2 + 2$.

a The graph of g can be obtained from the graph of f using a sequence of transformations. Write down a full geometric description of each of these transformations.

b The graph of g is translated 3 units down and reflected about the x-axis to become the graph of the function $h(x)$. Determine an expression for $h(x)$.

8 Consider the function $f(x) = x^2$ and the transformation functions $t(x) = x + 1$, $s(x) = 2x$ and $r(x) = -x$.

Find expressions for the following composite functions giving a full geometric description of the transformations they represent (noting the order).

a $(t \circ f \circ r)(x)$

b $(s \circ f \circ t)(x)$

c $(t \circ s \circ f)(x)$

d $(f \circ s \circ r)(x)$

e For each of the four functions above, determine the coordinates of their vertex using the concepts of transformations.

9 The curve $y = x^2$ is transformed by two sets of transformations:

i a translation of 3 units to the right followed by horizontal stretch of a scale factor of 2.

ii a horizontal stretch scale factor 2, followed by a translation of 6 units to the right.

a Find the image of (1, 1) after each of these sets of transformations.

b Find the image of the curve $y = x^2$ after each set of transformations and explain from the equations why the two sets of transformations result in the same image.

Developing inquiry skills

Return to the chapter opening problem where Oliver throws a basketball through a hoop.

Which three points are you able to find in the path of the basketball?

How could you use these points to model its path?

6.3 Cubic functions and models

Here is a picture of part of a roller coaster.

It is in the shape of the graph of a cubic function.

Reflect How are graphs of cubic functions different from graphs of quadratic functions? Are there any similarities between the graphs of quadratic and cubic functions?

Cubic models

Cubic functions are polynomial functions where the highest power of x is 3.

For example, $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$ and $a, b, c, d \in \mathbb{R}$ is a cubic function.

The fundamental (simplest) cubic function is $f(x) = x^3$.

TOK

Descartes showed that geometric problems could be solved algebraically and vice versa.

What does this tell us about mathematical representation and mathematical knowledge?

Reflect State what type of function $f(x) = ax^3 + bx^2 + cx + d$ would be if $a = 0$.

You can sketch (or draw) the graph of a cubic function by hand with the help of your GDC. In many cases the **domain** of the function that you need to sketch or draw will be explicitly given. In such cases you should only sketch (or draw) the part of the function that is in the given domain.

Make sure that any x- and y-intercepts are in the correct place on the graph. Also, the local maximum and minimum values need to be in the correct place.

You can also use the table of values on the GDC to plot the coordinates of some more points that lie on the curve if necessary.

HINT

A point where the gradient of the curve changes from positive to negative or vice versa is called a local maximum or minimum point. The word local is used because often in the case of cubics there is no actual maximum or minimum.

Example 9

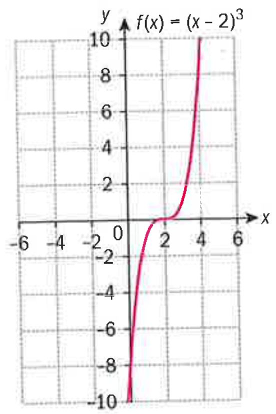
Sketch the graphs of $y = f(x)$ for the following functions.

a $f(x) = (x - 2)^3$, $0 \le x \le 4$ b $f(x) = x^3 - 7x^2 + 4x - 12$, $-2 \le x \le 8$

On your sketch, label the coordinates of points where the graphs intersect the axes, and any local maximum or minimum points.

Continued on next page

a



You can see from your GDC that this curve:

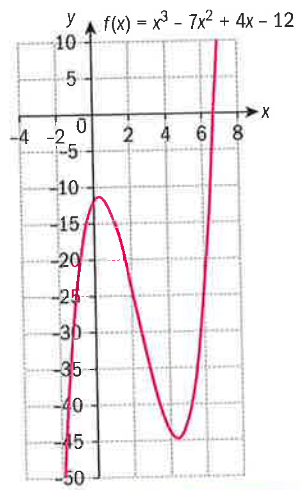
- does not have any local maximum or minimum points
- cuts the y-axis at the point $(0, -8)$
- cuts the x-axis at $(2, 0)$.

Draw suitable axes and mark these points on them.

You can take a few more points from the table of values to help you complete the graph, eg $(1, -1)$, $(3, 1)$ and $(4, 8)$.

Plot your points and draw a smooth line through the points.

b



You can see from your GDC that this curve:

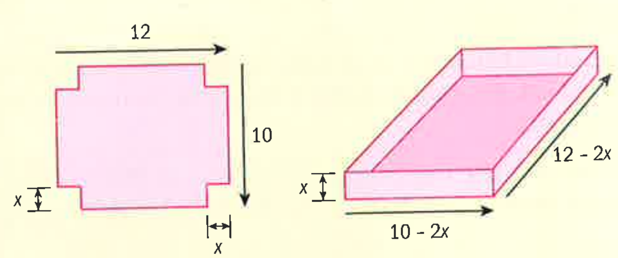
- has a local maximum point at $(0.306, -11.4)$
- has a local minimum point at $(4.36, -44.7)$
- cuts the y-axis at $(0, -12)$
- cuts the x-axis at $(6.67, 0)$.

These four points are probably enough to help you sketch the curve, but you can look at other points from the table of values if you need to.

Draw suitable axes and mark these points on them. Then sketch the curve.

Investigation 3

An open box is made from a piece of card measuring 12 cm by 10 cm, with squares of side x cm cut from each corner.



1 Explain why the width of the box is $(10 - 2x)$ cm, the length is $(12 - 2x)$ cm and the height is x cm.

2 Find an equation for the volume, V, of the open box in terms of x.

3 Use technology to plot a graph of V against x, $0 \leq x \leq 8$.

4 Explain why the function describing the volume of the open box cannot have an inverse under the maximum possible domain and restrict the domain accordingly in order to make the function invertible.

5 What is the difference between a quadratic and a cubic function in trying to isolate the input variable x?

6 Why will it be difficult to find the inverse of the function representing the volume of the box?

7 Find the x-intercepts.

8 What are the upper and lower limits for the value of x in the context of this problem? Explain why.

9 What are the coordinates of the local maximum and local minimum points of the graph?

10 Which of these is not a possible value for the volume of the box? Justify your answer.

11 Given a certain value of x, could you use this model to predict what the volume of the box would be? What limitations would you have?

12 **Conceptual** Using your answer to question **11**, do you think that, in general, cubic models could be used to predict information about real-life situations?

Not all cubic functions have a maximum and a minimum turning point. Some have neither. Also, cubic functions can have one, two or three real roots and have rotational symmetry about a point called the **point of inflexion.**

Exercise 6G

Sketch the graphs of $y = f(x)$ for the following functions.

Write down the coordinates of points where the graphs intersect the axes, and any local maximum and minimum points.

1 $f(x) = (x + 3)^3$

2 $f(x) = x^3 - 2x^2 - x + 3$

3 $f(x) = 2x^3 - 2x^2 - 12x$

4 $f(x) = 3(x + 2)^3 - 4$

5 $f(x) = 3x(x - 4)(x + 1)$

6 Sketch each function and reflect it in the line $y = x$ to find the inverse function. Write down the equation of the inverse function in each case.

 a $f(x) = x^3 + 3$

 b $f(x) = 4x^3$

 c $f(x) = 2x^3 + 1$

7 Sketch the graph of $y = x^3 - 6x^2 + 3x + 10$.

 a Using your GDC, find the coordinates of the x-intercepts and the y-intercept.

 b Find the coordinates of the vertices.

 c The graph is reflected in the y-axis. Write down the equation of the reflected graph.

8 The temperature (T °C) over a 24-hour period beginning at 19:00 on Monday evening is represented by the function $T(x) = 23.5 - 1.72t + 0.2t^2 - 0.0056t^3$, where t is the number of hours that have passed since 19:00.

 a State the highest and lowest temperatures in this 24-hour period.

 b Find the temperature at 05:00 on Tuesday morning.

Regulations state that a hospital should be at a maximum temperature of 22 °C. If the temperature rises above this, the air conditioning system should be switched on.

 c Assuming that the temperature in the hospital is the same as the outside temperature when the air conditioning system is off, find the number of hours over the 24-hour period during which the air conditioner must be switched on. Give your answer in hours, correct to 1 dp.

 d Comment on whether this model would be useful to predict the outside temperature of the hospital at 01:00 on Wednesday morning. Justify your answer.

9 Sketch the graph of the following functions, within the specified domains and hence determine their range.

a $f(x) = -x^3$, for $-3 \le x \le 3$

b $f(x) = x^3 - 2x^2 + 4x - 3$, for $-4 \le x \le 4$

c $f(x) = x^3 - 7x^2 + 11x - 5$, for $0 \le x \le 5$

10 Deana wants to construct an open box from a square piece of cardboard. The cardboard has side length 50 cm. To create the box, she cuts from each corner a smaller square of side length x cm.

a Determine expressions for the dimensions of the box in terms of x.

b Determine an expression for the volume of the box in terms of x.

c State the domain for the expression of the volume of the box.

d Sketch the graph representing the volume of the box.

e Determine the dimensions of the box that has the maximum volume and state the corresponding maximum volume.

11 The maximum monthly temperature in a certain city during the course of one year (from January to December) can be modelled by the equation $T(t) = 0.162t^3 - 3.36t^2 + 18.2t + 1.74$, where t is the month number and $t = 1$ corresponds to January.

a Find the maximum temperature in April.

b Find the month during which the greatest and lowest maximum temperatures occur.

c Find the mean of the two temperatures in part **b**.

d State the month whose maximum temperature is approximately equal to the average you found in part **c**.

e Comment on your findings.

12 A graphic designer has created a function from which to make a tilde symbol (~). The function that produces the tilde is $f(x) = 0.0867x^3 - 0.828x^2 + 2.01x + 1.27$ in the domain $0 \le x \le 6$.

a Sketch the graph representing the tilde.

b Determine the range of values of the function in the given domain.

c Determine the dimensions of the smallest rectangle (in mm) in which the tilde can fit.

13 Violet is designing the roof for a new building. In order to finalize her design, she created a model using a 3D printer.

The function that describes the vertical cross-section of the model roof is

$$f(x) = \frac{1}{81}x^3 - \frac{7}{27}x^2 + \frac{49}{36}x \text{ for } 0 \le x \le 13,$$

where x is the horizontal distance in cm from the leftmost end of the model roof and y is the vertical distance in cm from the bottom of the model roof. The model is made at a scale ratio of 1:200, meaning that a length of 1 cm on the model corresponds to a length of 200 cm on the actual roof.

a Sketch the graph representing the model roof.

b Determine the total horizontal length of the actual roof.

c Determine the maximum height of the model roof above its base at $y = 0$.

Since the roof will be built on top of a building, the function needs to be adjusted so that y represents the distance from the bottom of the building. The building on which the roof is going to be built will be 24 m tall.

d Determine the function that will represent the actual roof on top of the building, and describe a transformation which maps the previous function onto this one.

14 The logo of a GDC manufacturing company is given by $y = x^3 - 6x^2 + 9x$, in the domain $0 \le x \le 4$, where x and y are the coordinates from the centre of the frame enclosing the logo measured in cm. The frame is a square of side length 5 cm.

a Sketch the graph of the function representing the logo on your GDC.

b Determine the coordinates of the y-intercept.

c Determine the coordinates of the maximum and minimum points of the logo.

d Determine the range of the logo.

In order to make the logo more appealing, it was suggested that the design should be altered under a series of transformations. First to reflect the logo about the x-axis, then to stretch it vertically by a scale factor of $\frac{1}{2}$ and finally to translate it 4 units down.

e Write the function representing each of the transformations.

f Use the notation of composite functions to express all the transformations in the appropriate order.

g Write down the function of the new transformed logo in the form $f(x) = ax^3 + bx^2 + cx + d$, stating its domain.

h Sketch the graph of the logo, showing clearly the maximum and minimum points as well as the y-intercept.

i Comparing the two graphs and their corresponding functions, determine the significance of the sign of the coefficient of the x^3 term.

j Comparing the two graphs and their corresponding functions, determine the significance of the value of the constant term.

15 By restricting the domain of the following functions (where necessary), determine the domain on which each function is invertible.

a $f(x) = x^3 - 6x^2 + 9x - 1$

b $f(x) = 2x^3 - 3x^2 - 6x + 5$

c $f(x) = x^3 - 3x^2 + 3x + 2$

d $f(x) = 2x^3 - 3x^2 + 6x + 1$

16 An IB student wants to create some open boxes from pieces of cardboard to use them as flower pots for a biology experiment. The piece of cardboard has dimensions 40 cm by 60 cm. In order to turn it into a box, she cuts squares on each one of the corners of the cardboard and then folds the sides in. After making the first few at random, she realized that she could fill them with different amounts of soil. Determine the dimensions of the squares she cuts out in order to be able to put as much soil as possible in the flower pots?

Cubic regression

As with linear and quadratic regression, you can also model data using cubic regression.

The data in this case needs to depict the features of the graph of a cubic function. In order to choose a cubic model to describe a certain set of data, the data need to have some specific features. These include:

- the data having two vertices: one maximum and one minimum point
- the data being rotationally symmetric about a certain point
- the data having a variable concavity (both concave up and concave down).

The more data you have, the more certain you are about the choice of curve to model them. You need at least four points in order to create a cubic model.

Example 10

The path of a ski jumper is illustrated on the diagram on the right.

In order to determine a function to model this jump, the diagram was processed with graphing software to determine a set of points that the path of the ski jumper goes through, x being the horizontal distance from the starting point and y being the vertical displacement from the starting point. The points are listed in the table on the next page.

x	34.8	58.2	83.1	105.9	120.3	141.9	161.4	180	195	206.1	218.4	228.9	239.4
y	−19.5	−34.2	−49.5	−61.8	−64.8	−64.8	−68.1	−74.4	−82.5	−90.6	−101.1	−112.5	−125.7

a Enter the data into your GDC.

b State the type of function suitable for modelling this set of data points.

c Use your GDC to determine the model function for this set of data.

d Determine the coefficient of determination. Interpret your result in context.

e Plot the model function over the scatter plot and comment on the closeness of fit to the original data.

The straight line that represents the slope of the mountain on which the skier will land passes through the points $(0, 0)$ and $(360, -210)$.

f Determine the equation of the line going through these two points.

g Find the point at which the ski jumper will land on the slope.

From that point on and up to the point where $x = 400$, the skier moves along the straight line describing the slope of the mountain.

h Write in piecewise form the function describing the whole path of the skier.

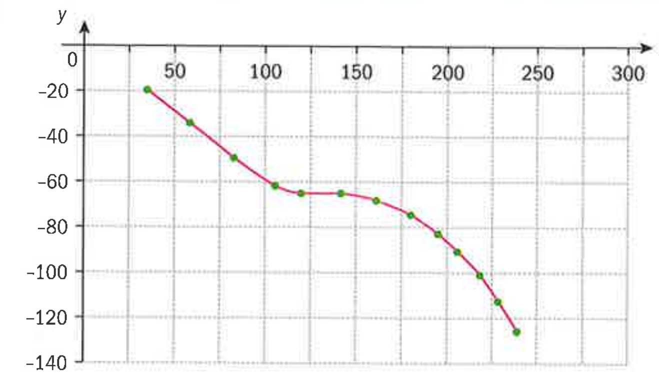
b A cubic function would be suitable to model the ski jumper's path as it is first concave upwards and then concave downwards.

c $f(x) = -0.0000346x^3 + 0.0136x^2 - 1.94x + 35.4$

d $R^2 = 0.995$

The coefficient of determination shows an almost perfect fit, implying a good choice of model.

e



f The slope of the line is $m = \frac{-210-0}{360-0} = -\frac{7}{12}$

$\Rightarrow y - 0 = -\frac{7}{12}(x - 0) \Rightarrow y = -\frac{7}{12}x \approx -0.583x$

g $-0.0000346x^3 + 0.0136x^2 - 1.94x + 35.4 = -0.583x$

$\Rightarrow x = 254 \Rightarrow y = -148$

Equating the two functions or finding their point of intersection graphically.

Notice that we are looking for the point which is further down the slope.

h $f(x) = \begin{cases} -0.0000346x^3 + 0.0136x^2 - 1.94x + 35.4, & 0 \le x \le 254 \\ -0.583x, & 254 < x \le 400 \end{cases}$

Exercise 6H

1 The table below shows the height, h metres, of the tide at time t hours after midnight.

Time (t)	0	1	2	3	4	5	6	7	8
Height (h)	6	5	4	2.5	2	3	3.5	5	6.5

a Using technology, find the equation of the best fit cubic function which models this data.

b Hence, find an estimate for the minimum height of the tide.

c Find a quadratic function that could also be used to model this data.

d State whether a cubic or a quadratic function would best model this data. Justify your answer by considering the coefficient of determination.

2 The temperature, in °C, for 1 day was recorded every 3 hours from 18.00.

Time	0	3	6	9	12	15	18	21	24
Temperature (°C)	12	9	5	4	8	13	17	16	13

a Using technology, plot these points and find the best fit function to model this data.

b Justify your choice of best fit function.

c Explain why it would be appropriate to use your model to give approximations for the times between the recorded hours on the day but not for predicting the temperature for the next day.

3 The number of cases of an infection in a city was recorded each week for 12 weeks.

Week	1	2	3	4	5	6	7	8	9	10	11	12
Number	24	62	103	152	209	245	232	191	146	123	96	58

a Using technology, plot these points and find the best fit function to model this data.

b Justify your choice of best fit function.

c Comment on the problems of your model in predicting the number of future cases.

4 **a** Using technology, plot these points and find the best fit function to model this data.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of fish	63	82	104	91	83	68	52	41	35	45	56	71

b Justify your choice of best fit function.

c Comment on the appropriateness of the model.

5 Consider the following values:

x	1	2	3	4	5	6
y	7	21	27	20	18	8

a For this data find (giving all parameters to 3sf)

i the least squares quadratic regression function

ii the corresponding value for the coefficient of determination

iii the least squares cubic regression function

iv the corresponding value for the coefficient of determination.

b Comment on the better model to estimate non-integer values in the range $0 \le x \le 6$.

It is now given that the values give the path of a particle moving under gravity, where x represents the horizontal displacement from a given point and y the height.

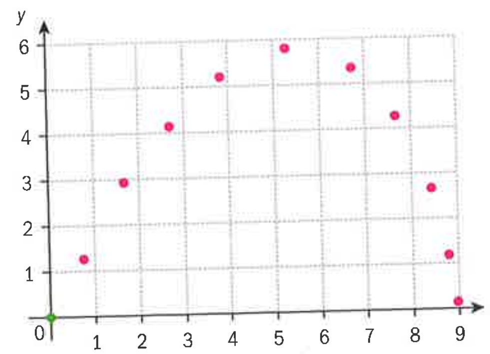
c State one reason why you might prefer the quadratic over the cubic model

6 **a** Find the maximum value for:

i $y = x^2 - x$

ii $y = x - x^3$ between $x = 0$ and $x = 1$.

7 Eliana is modeling the path of a ball for her IA. She plots its path using software and this is shown on the diagram below.



She was expecting to demonstrate that the path of the ball could be modelled by a quadratic equation.

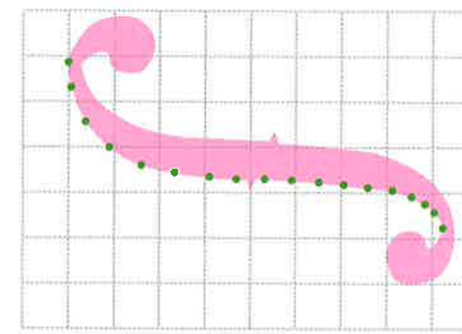
a From the diagram state one reason why a quadratic might be a good model and one reason why it might not.

b Eliana decides that a cubic model might be better. Explain using your answer to part **a** and the diagram above why this might be more appropriate than a quadratic.

8 Violins have specific holes on their top surface called F-holes. These holes plays a very important role in the quality of the sound of the violin.

A violin maker wants to create a function modeling the contour of the F-holes on a violin to be able to carve them with the use of a computer numerical control (CNC) machine.

To do so he put a certain number of points depicting the contour of the bottom side of the F-hole on a graph paper and marked down their coordinates.



The points are the following:

x	0	0.126	0.750	1.80	3.21	4.69	6.22	7.50	8.66
y	9.14	8.62	7.91	6.28	5.26	4.91	4.57	4.13	4.25
x	9.85	11.0	12.1	13.2	14.3	15.1	15.7	16.1	16.5
y	4.73	4.82	4.75	4.36	4.41	3.70	3.54	2.87	2.66

a Explain why a cubic curve would be a good model for the lower side of the F-hole.

b Use cubic regression to find the function that best describes the above set of data stating the domain for which it is valid.

c Comment on the appropriateness of using the model function.

d Determine whether the function modelling the F-holes of the violin has local maximum and minimum points.

9 At Roller Coaster Mania Park, they claim to have the longest vertical roller coaster drop in the world. The previous longest roller coaster drop was 130 m. Part of the sideview of a rollercoaster has roughly the following shape.

This shape can be modelled by the function

$$f(x) = \frac{139x^3 - 14595x^2 + 250200x + 7818500}{62500}$$

$0 \le x \le 75$, where x is the horizontal distance in metres from the start and y is the vertical distance in metres.

a Find the maximum height of the roller coaster in this section.

b Find the minimum height of the roller coaster in this section.

c Calculate the vertical drop of this roller coaster, and comment on your answer

10 The amount of water in the Caniapiscau water reservoir in Quebec, can be modelled by the cubic function $V(t) = 0.1t^3 - 1.8t^2 + 7.2t + 54.3$, where t is time in months after the 1st of January and V is the volume of water in km^3. When the water supplies fall below 48 km^3, a warning is sent out asking people to be more careful in their use of water.

a Sketch the graph representing the volume of water in the reservoir for the course of one year by choosing an appropriate domain.

b Find the value of t at which there is the maximum amount of water in the water reservoir.

c Find the value of t at which there is the minimum amount of water in the water reservoir.

d State the months of the year when the warning about water consumption is active.

Developing inquiry skills

Now that you have learned different ways to model data from real-life situations, what type of function do you think would be the best way to model the path of Oliver's basketball?

Can you find sufficient data points to model the path of the ball?

Can you use the official height of a basket to help find points?

Is it possible to find a general equation that will fit all possibilities?

6.4 Power functions, direct and inverse variation and models

Let d be the diameter, A the circumference and V the volume of a sphere of radius r.

You can write the following equations:

$d = k_1 r$, $A = k_2 r^2$ and $V = k_3 r^3$.

What are the values of k_1, k_2 and k_3 assuming the units of each are the same.

Reflect Can you think of two features common to the graphs of d, A and V for $r \geq 0$.

A **power function** is a function of the form $f(x) = ax^n$, where a and n are the parameters of the function and n is a member of $\mathbb{R}$.

Parameter $a \neq 0$, since $a = 0$ would make the function zero.

- If $n = 0$, the function is constant.
- If $n = 1$, the function is linear.
- If $n = 2$, the function is quadratic.
- If $n = 3$, the function is cubic.

The fundamental (simplest) power function is $f(x) = x^n$.

HINT

When n is not a positive integer the domain might be restricted. For example $f(x) = ax^{0.5}$ has a domain $x > 0$.

Investigation 4

1 Sketch the graphs of the following power functions.

a $f(x) = x^2$ b $f(x) = x^3$ c $f(x) = x^4$

d $f(x) = x^5$ e $f(x) = x^6$ f $f(x) = x^7$

2 Can you group power functions in smaller categories?

3 How are they the same and how are they different from a parabola?

4 How are they the same and how are they different from a cubic graph?

5 How many maximum and minimum points can a power function have?

6 Are power functions symmetric? If yes, what type of symmetry do they possess?

7 What happens to the graph of power functions as the value of the power increases?

8 **Conceptual** What are the distinguishing geometrical features of positive and negative power functions?

HINT

Use a view window of $-3 \leq x \leq 3$ and $-6 \leq y \leq 6$.

The graphs of all **even positive** power functions have a shape similar to a parabola and are symmetric about the y-axis.

The graphs of all **odd positive** power functions have a shape similar to a cubic function and are rotationally symmetric about the origin.

As the index of positive power functions increases, the graphs tend to become flatter around the origin.

Functions

Example 11

A designer wants to create a model function for a bowl that she sketched by hand in order to be able to process it digitally. To do so, she put the sketch over a grid and marked some data points, with the centre of the bottom of the bowl being the origin.

The data points are the following:

Point	A	B	C	D	E	F	G	H	I
x	−16.5	−15.1	−14	−12.8	−11.3	−9.3	−7.5	−5.3	0
y	7	4.9	3.62	2.53	1.54	0.7	0.3	0.07	0

Point	J	K	L	M	N	O	P	Q
x	5.3	7.5	9.3	11.3	12.8	14	15.1	16.5
y	0.07	0.3	0.7	1.54	2.53	3.62	4.9	7

Continued on next page

a Plot the given data on your GDC or other technology.

She first thought of using a quadratic function to model the shape.

b Explain why a quadratic function could be suitable to model this shape.

c Use your GDC to determine the quadratic model function for this set of data.

d Find the coefficient of determination, and comment on your answer.

e Sketch the model function over the scatter plot and comment on the closeness of fit to the original data.

Not being satisfied with the model function she created, she decided to determine a new quartic model function.

f Explain why the designer might have not been satisfied with the model function she created and why a quartic function could be a suitable alternative model.

g Use your GDC to determine the quartic model function for this set of data.

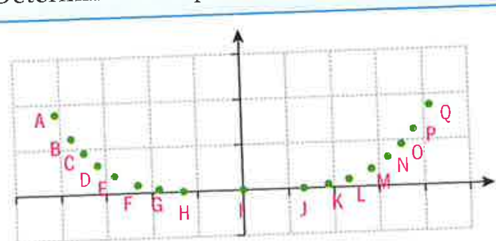
h Find the coefficient of determination, and comment on its significance in relation to the previous model.

i Sketch the model function over the scatter plot and comment on the closeness of fit to the original data and compare it to the previous model.

Using the quartic model, the designer want to create a model for another bowl whose dimensions are double in size to the original.

j Determine the equation of the model function for the larger bowl.

a



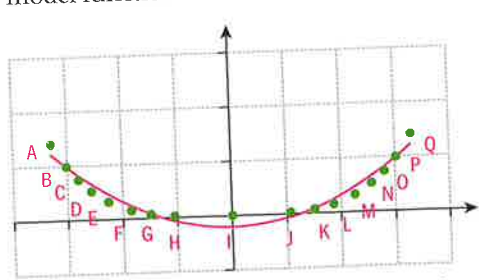
b A quadratic formula would be suitable to model this shape since the data points seem to have vertical symmetry and also because of their constant concavity.

c The best fit parabola is: $f(x) = 0.0263x^2 - 1.15$ — Using quadratic regression in the GDC.

d $R^2 = 0.933$

The coefficient of determination shows that the model function describes the data set very closely.

e



Although the model function goes through (or close to) most points, it still misses the bottom point by a significant amount.

f The quadratic model had some major problems in describing the shape of the bowl, the most important being at the bottom of the bowl.

A quartic function can be a better fit, since in addition to depicting symmetry and constant concavity, it also is flatter close to the origin.

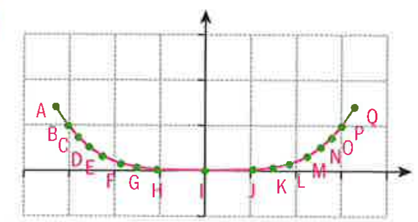
g The best fit quartic function is: — Using quartic regression in the GDC.

$f(x) = 0.0000946x^4 - 0.0000649x^2 + 0.000214$

h $R^2 = 0.999$

The coefficient of determination shows almost perfect fit and is of course significantly better than the quadratic.

i



The model function goes perfectly through all the data points and describes the shape of the bowl in a much better way than the quadratic.

j $g(x) = 2f\left(\frac{x}{2}\right) = 0.0000942x^4 = 0.0000118x^4$ — The enlarged bowl is double the size both in the x and in the y directions.

Direct Variation

Two quantities that are related by a power law of the form $y = ax^n$, $a, n \in \mathbb{R}^+$ are said to be **vary directly** with each other, or to be **directly proportional** to each other.

Because there is only one unknown parameter just one pair of values is required in order to find the relationship, assuming the value of n is known.

Example 12

The rate of the spread of fungus on a petri disk (x) varies directly with the square of the perimeter of the area covered by the fungus (p). If the rate is $2\text{cm}^2\text{s}^{-1}$ when the perimeter is 3.2 cm.

a Find the equation relating the rate of spread to the perimeter.

b Find the perimeter when the rate is $6\,\text{cm}^2\text{s}^{-1}$.

Continued on next page

a $x = ap^2$ $2 = a \times 3.2^2$ $\Rightarrow a \approx 0.195 \Rightarrow x = 0.195p^2$	The general equation for the power function. Substituting the values.
b $6 = 0.195p^2$ $p^2 = 30.72 \Rightarrow p = 5.54\,\text{cm}$	

Power Regression

The power regression function on a GDC will find the best fit curve of the form $y = ax^n$, $a, n \in \mathbb{R}$.

HINT
On most calculators power regression can only be used if all the data points are greater than 0.

Exercise 6I

1 It is known that the resistance to motion of an object falling through a liquid is sometimes best modelled as varying directly with the object's velocity and sometimes as with the square of the velocity.

Emily wishes to test these theories and so releases a weight in a tube of thick liquid. By measuring how quickly it falls, she calculates that when it is travelling at $2\,\text{cm s}^{-1}$ the resistance is 4.2 N.

a Write down an equation linking resistance R with velocity v for each of the two possible models.

To test the models further she collects the following data:

v	3.2	4.0
R	7.1	11.1

b Find the value of R predicted by each of the models for $v = 3.2$ and $v = 4.0$.

c By considering the sum of the square residuals say which model is most likely to best fit the situation.

d Use the power regression function on your GDC and all three data points to find another possible model.

e By considering the sum of squares of residuals for the power function, comment on which model best fits Emily's data.

2 The distance, d metres, that a ball rolls down a slope varies directly with the square of the time, t seconds, it has been rolling. In 2 seconds the ball rolls 9 metres.

a Find an equation connecting d and t.

b Find how far the ball rolls in 5 seconds.

c Find the time it takes for the ball to roll 26.01 metres.

3 The mass of a uniform sphere varies directly with the cube of its radius. A certain sphere has radius 3 cm and mass 113.1 g.

Find the mass of another sphere with radius 5 cm.

4 Sketch the graph of the following functions, within the specified domains and hence determine their range.

a $f(x) = -x^5$, for $-3 \le x \le 3$

b $g(x) = \frac{x^4}{4}$, for $-4 \le x \le 4$

c $h(x) = 2x^{12}$, for $-2 \le x \le 2$

5 Viola is conducting an experiment to demonstrate the relation between the radius and the volume of a sphere. She gathered the following data.

r (cm)	1	2	3	4	5	6	7	8
V (cm^3)	4.19	33.5	113	268	524	905	1440	2140

a Plot the given data on your GDC.

b Use your GDC to determine the power function model for this set of data.

c Comment on the choice of model by determining the coefficient of determination.

d Sketch the model function over the scatter plot and comment on the closeness of fit to the original data.

e Use the model function to determine the volume of a sphere with radius 10 cm.

f Comment on how the value found in part **e** compares with the actual value of the volume of a sphere of radius 10 cm found using the appropriate formula.

6 According to building specifications and physical measurements, the maximum weight that foundation columns of tall buildings can carry varies directly with the fourth power of their perimeter.

In a specific sample check, it was found that a column with a perimeter of 6 m could hold a weight of 55 000 kg.

a Find the formula that relates the perimeter of the column to the weight it can carry.

b Estimate the weight that another column of the same make with a perimeter of 8 m can hold.

c Calculate the perimeter that a column should have in order to be able to hold 200 000 kg?

7 Find the inverse of the following one-to-one functions:

a $f(x) = x^2$ for $x \le 0$

b $f(x) = -x^3$

c $f(x) = 2x^5$

Functions

Inverse variation

Investigation 5

A mathematician drives along the same route to school every day.

1 How is the time that he takes to get to school affected by his driving speed?

2 One day when he was very early there was no traffic so he traveled at twice h is normal average speed. How did the time taken to complete the journey change from normal?

3 On the way home he was caught in heavy traffic so he only managed half his normal average speed. How did the time taken to complete the journey change from normal?

4 **Factual** How could you describe the variation between speed and time?

5 **Conceptual** How would you describe real situations that model asymptotic behavior?

When n is negative and $x > 0$, the function $f(x) = ax^n$ represents inverse variation.

Investigation 6

1 Sketch the graphs of the following inverse variation functions:

a $f(x) = \frac{1}{x^2}$ b $f(x) = \frac{1}{x^3}$ c $f(x) = \frac{1}{x^4}$

d $f(x) = \frac{1}{x^5}$ e $f(x) = \frac{1}{x^6}$ f $f(x) = \frac{1}{x^7}$

2 How can you describe the shape of a inverse variation function?

3 Can you group inverse variation functions in smaller categories?

4 Are inverse variation functions symmetric? If yes, what type of symmetry do they possess?

5 **Factual** Describe what happens to the y-value of the function as

a x becomes very large

b as x approaches 0 from the positive side

c as x approaches 0 from the negative side.

6 What happens to the graph of inverse variation functions as the value of the power becomes more negative?

7 **Conceptual** What are the distinguishing geometrical features of inverse variation functions?

HINT

Use a view window of $-3 \le x \le 3$ and $-6 \le y \le 6$.

TOK

Aliens might not be able to speak an Earth language but would they still describe the equation of a straight line in similar terms? Is mathematics a formal language?

An **asymptote** is a line that a graph approaches but never crosses or touches.

International-mindedness

Asymptote comes from the Greek word "ασύμπτωτη", meaning "to not fall on top of each other".

If a quantity y **varies inversely** with x^n for $x > 0$ then $y = kx^{-n}$ or $y = \frac{k}{x^n}$. In these situations y will decrease as x increases and vice versa.

As with direct variation a single point is needed to find the value of the parameter k.

Example 13

The number of hours N taken to build a wall varies inversely with the number of people x who are available to work on it.

a When three people are available the wall takes two hours to build. Find the time it takes to build the wall when four people are available to work on it.

b Given it takes three hours to build the wall, state how many people worked on it.

a $N = \frac{k}{x}$

$N = 2$ when $x = 3$ so $2 = \frac{k}{3}$

$k = 6$

$N = \frac{6}{x}$

$N = \frac{6}{4} = 1.5$

So, it takes 1.5 hours to build the wall when four people work on it.

For inverse variation, the variable is written as $\frac{1}{x}$ or x^{-1} which is a power function.

Find the value of k using the given information.

Substitute the value of k you found into the equation for N.

Now find N when $x = 4$.

b $3 = \frac{6}{x}$

$3x = 6$

$x = \frac{6}{3} = 2$

So, two people were available to build the wall.

Substitute $N = 3$ into the equation $N = \frac{6}{x}$ which you found in part **a**.

Sometimes the terms direct or inverse variation might not be used but the context will make it clear how to form the required equations.

Investigation 7

1 A local authority pays its workers depending on the number of hours that they work each week. If the workers are paid €22 per hour, complete the following table:

Number of hours	20	25	30	35	40
Pay (€)					

Plot a graph of this information on your GDC.

Describe how a worker's pay varies with the number of hours worked.

2 The local authority has decided to put artificial grass tiles on a football field.

If four people are available to lay the grass tiles, it takes them two hours to complete the work.

Fill in this table showing the number of people available and the number of hours it takes to complete the work.

Number of people	1	2	4	6	8	12
Number of hours			2			

Plot a graph of this information on your GDC.

Factual How do the number of hours to complete the work vary with the number of men available?

3 **Conceptual** For problems which involve direct and inverse variation, how does understanding the physical problem help you to choose the correct mathematical function to model the problem with?

Functions

Any value of x that makes the denominator of a function equal to zero is excluded from the domain of the function. The vertical line represented by this x-value is called a **vertical asymptote** on the graph of the function.

Investigation 8

Consider the function $f(x) = \frac{1}{x}$.

1 Which value of x makes the denominator equal to zero?

2 Which vertical line is the vertical asymptote of the graph of $f(x)$?

Now consider the function $g(x) = f(x-2)$.

3 Which value of x makes the denominator equal to zero?

4 Which vertical line is the vertical asymptote of the graph of $g(x)$?

5 Give a full geometric description of the transformation represented by $g(x)$.

6 What do you notice? How does this transformation affect the vertical asymptote?

Now consider the function $h(x) = f(x) + 3$.

7 To which value does the function tend as x becomes infinitely positive or negative?

8 Which horizontal line is the horizontal asymptote of the graph of $h(x)$?

9 Give a full geometric description of the transformation represented by $h(x)$.

10 What do you notice? How does this transformation affect the horizontal asymptote?

Finally consider the function $j(x) = f(x-h) + k$.

11 Give a full geometric description of the transformation represented by $j(x)$.

12 Hence write down the equations of the asymptotes of $j(x)$.

13 **Conceptual** How do you identify an asymptote?

The function $f(x) = \frac{1}{x-h} + k$ has a vertical asymptote $x = h$ and a horizontal asymptote $y = k$.

Example 14

Consider the function $f(x) = \frac{1}{x}$ when $x \neq 0$. The graph of $y = f(x)$ is transformed to become the following functions. Give a description of the transformations they represent and hence write down the equations of the asymptotes of the transformed function.

a $g(x) = f(x) - 3$

b $h(x) = f(x - 3)$

c $j(x) = f(x + 1) - 2$

a $g(x)$ represents a translation 3 units down. Vertical asymptote $x = 0$. Horizontal asymptote $y = -3$.	The vertical asymptote will remain unaffected. The horizontal asymptote will be translated 3 units down.
b $g(x)$ represents a translation 3 units to the right. Vertical asymptote $x = 3$. Horizontal asymptote $y = 0$.	The vertical asymptote will be translated 3 units to the right. The horizontal asymptote will remain unaffected.
c $g(x)$ represents a translation 1 units to the left and 2 units down. Vertical asymptote $x = -1$. Horizontal asymptote $y = -2$.	The vertical asymptote will be translated 1 units to the left. The horizontal asymptote will be translated 2 units down.

Exercise 6J

1 The volume V of a gas varies inversely with the pressure p of the gas. The pressure is 20 Pa when its volume is 180 m³. Find the volume when the pressure is 90 Pa.

2 A group of children attend a birthday party. The number of pieces of candy c that each child receives varies inversely with the number of children n.

When there are 16 children, each one receives 10 pieces of candy. Find the number of pieces of candy each child receives when there are 20 children.

3 Sketch the graph of the following functions, within the specified domains and hence determine their range.

a $f(x) = \frac{1}{x^5}$, for $-3 \leq x \leq 3 \quad x \neq 0$.

b $g(x) = \frac{4}{x^4}$, for $-4 \leq x \leq 4 \quad x \neq 0$

c $h(x) = \frac{1}{2x^{12}}$, for $-2 \leq x \leq 2 \quad x \neq 0$

4 The density of a sphere of **fixed** mass varies inversely as the cube of its radius.

A sphere with radius 10 cm has a density of 500 kg/m³.

a Find the density of a sphere with radius 5 cm.

b If we require a sphere to have a density of 100 kg/m³, find its radius.

5 Timothy had to choose a topic for his Mathematical Exploration. Being a prospective engineering student, he decided to investigate the inverse-square law. The law states that light intensity from a specific light source is inversely proportional to the square of the distance from the source. To validate the physical law, he performed an experiment where he positioned a light source at various distances to a receiver measuring its intensity. The data he collected appear in the following table.

Functions

d (cm)	10	15	20	25	30	35	40	45
I (lux)	3232	1434	805	513	353	259	201	159

a Plot this data on your GDC.

b Explain why an inverse variation function could be suitable to model this set of data.

c Use your GDC to determine the power function model for this set of data.

d Find the coefficient of determination, and comment on your answer.

e Sketch the model function over the plot of the points and comment on the closeness of fit to the original data.

f Use your model to estimate the light intensity if the light source moves to 50 cm from the receiver.

6 A company selling used cars wants to create mathematical models for the value of cars as a function of their age. Taking as reference one of its biggest selling models, it collected data relating the selling price to the number of years since the car was produced.

The data are shown in the table:

Age (years)	1	2	3	4	5	6	7	8	9	10
Price (€)	21641	17890	14709	12613	12239	10223	10501	9744	8257	6772

a Plot this data on your GDC.

b Explain why a linear function would not be suitable to model this set of data.

c Explain why an inverse variation function could be suitable to model this set of data.

d Use your GDC to determine the power function model for this set of data.

e Find the coefficient of determination, and comment on your answer.

f Sketch the model function over the points and comment on the closeness of fit to the original data.

g According to this model, find the age of the car when its value falls below €4000?

7 Restrict the domain of the following functions (where it is needed) in order to make them invertible.

a $f(x)=\frac{1}{x^4}$, for $x \neq 0$

b $f(x)=\frac{2}{x^3}$, for $x \neq 0$

c $f(x)=-\frac{1}{x^2}$, for $x \neq 0$

8 Find the inverse of the following one-to-one functions.

a $f(x)=\frac{1}{x^4}$, for $x<0$

b $f(x)=\frac{2}{x^3}$, for $x \neq 0$

c $f(x)=\frac{1}{x^2}$, for $x>0$

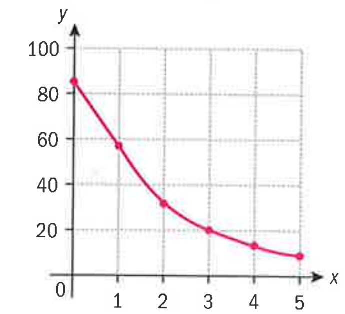
9 **a** Describe the transformations required to convert the function $f(x)=\frac{1}{x}$ to the function $g(x)=3-\frac{2}{x-1}$.

b Determine the asymptotes of both $f(x)$ and $g(x)$.

c State the transformations described in part **a** relate to the relative positions of the asymptotes of the two functions

10 The graph of $g(x)=1-\frac{2}{x-2}$ can be obtained from the graph of $f(x)=\frac{1}{x}$ using a sequence of transformations. Give a full geometric description of each of the transformations required in order for $g(x)$ **to go back** to $f(x)$.

11 The following diagram shows a demand curve for a product. The horizontal axis shows the price (p) a company could sell a product for and the vertical axis shows the percentage (N) of the market who would buy the product at this price.



a **i** Explain why a power function of the form $N=kp^n$ might be a good model for the demand curve, stating a necessary restriction on the possible values of n.

ii State one reason why a function of this form is not a good model for the demand curve.

Two of the points have coordinates $(1, 57)$ and $(4, 13)$.

b Use these points to find possible values for k and n.

Chapter summary

- The standard form of a quadratic function is $f(x)=ax^2+bx+c$.
- The fundamental (simplest) quadratic function is $f(x)=x^2$.
- The shape of the graph of all quadratic functions is called a **parabola** and is symmetric about a vertical line called the **axis of symmetry**, going through its **vertex**.
- Parabolas have **constant** concavity, either being constantly concave upwards or constantly concave downwards.
- The quadratic equation $ax^2+bx+c=0$ has the solution

  $x_{1,2}=\frac{-b\pm\sqrt{\Delta}}{2a}$, where $\Delta=b^2-4ac$.
- In order for any function to have an inverse, its domain has to be restricted in such a way as to make sure that it is a **one-to-one** function.
- Transformations of functions can be expressed in three different ways. One is their mathematical notation, the second is the geometric description and the third is the graphical representation.
  - $y=f(x)+k$: vertical translation k units up, with vector $\begin{pmatrix}0\\k\end{pmatrix}$
  - $y=f(x)-k$: vertical translation k units down, with vector $\begin{pmatrix}0\\-k\end{pmatrix}$

Continued on next page

Functions

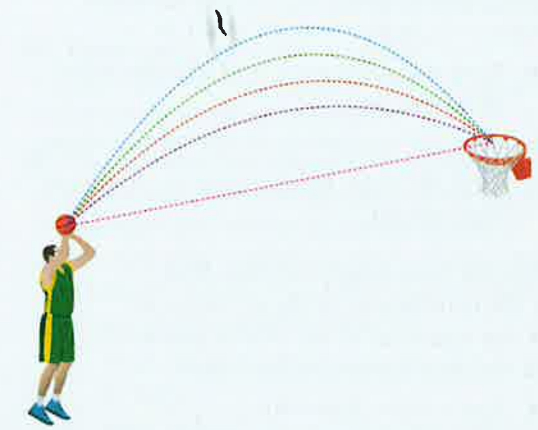
- $y = k \times f(x)$: vertical stretch with scale factor k
- $y = \frac{f(x)}{k} = \frac{1}{k} \times f(x)$: vertical stretch with scale factor $\frac{1}{k}$
- $y = -f(x)$: vertical reflection in the x-axis
- $y = f(x + k)$: horizontal translation k units left, with vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
- $y = f(x - k)$: horizontal translation k units right, with vector $\begin{pmatrix} k \\ 0 \end{pmatrix}$
- $y = f(kx)$: horizontal stretch with scale factor $\frac{1}{k}$
- $y = f\left(\frac{x}{k}\right) = f\left(\frac{1}{k} \times x\right)$: horizontal stretch with scale factor k
- $y = f(-x)$: horizontal reflection in the y-axis

- The standard form of a cubic function is $f(x) = ax^3 + bx^2 + cx + d$.
- The fundamental (simplest) cubic function is $f(x) = x^3$.
- The graphs of all cubic functions are symmetric about a point called the **point of inflexion**.
- The standard form of a power function is $f(x) = ax^n$.
- The fundamental (simplest) power function is $f(x) = x^n$.
- The graph of all even direct variation functions have a shape similar to a parabola and are symmetric about the y-axis. The graph of all odd direct variation functions have a shape similar to a cubic function and are symmetric about the origin.
- The graph of all inverse variation functions have the coordinate axes as asymptotes.
- Any value of x that makes the denominator of a function equal to zero is excluded from the domain of the function. The vertical line represented by this x-value is called a **vertical asymptote** on the graph of $f(x)$.
- If a function tends towards a constant value as the independent variable (x) tends towards positive and/or negative infinity, this constant value represents a horizontal asymptote on the graph of the function.

Developing inquiry skills

How many different trajectories would there be if the ball were to follow a straight line from the hands of the player to the hoop?

How many different trajectories are there in the case that the ball follows a curved line from the hands of the player to the hoop, as in the drawing here?



How would the initial angle at which the ball leaves the hands of the player affect the angle at which it arrives at the hoop?

How would this angle increase the chance of the player making the basket?

Knowing the maximum height of the ball, could we determine a unique model for the trajectory it follows?

How does the height up to which the ball goes increase the chance of the player making the basket?

How can we determine the optimum path for a basketball player to score a basket?

What else would we need to know to answer this question?

What assumptions have you made in the process?

Functions

Chapter review

Click here for a mixed review exercise

1 Sketch the graph of:

  a $f(x) = x^2 - 2$

  b $f(x) = x^2 - 5x + 4$.

2 The perimeter of a picture is 400 cm.

  a If the length of the picture is x cm, find the height in terms of x.

  b Find an expression for the area, A cm^2, of the picture in terms of x.

  c Sketch this graph using a suitable domain and range.

  d Find the x-intercepts and and interpret your results in context.

3 For the graph of the function $f(x) = x^2 + 6x - 7$ find:

  a the coordinates of the y-intercept

  b the coordinates of the x-intercepts

  c the equation of the axis of symmetry

  d the coordinates of the vertex.

4 Anmol throws a stone in the air. The height of the stone, $h(t)$ metres, at time t seconds is modelled by the function $h(t) = -2.2625x^2 + 8.575x + 1.9$.

a Find the y-intercept and explain what this represents.

b Find the maximum height of the stone.

c Find the time when the stone lands on the ground.

5 For the following functions, find
i the coordinates of the x-intercepts
ii the equation of the axis of symmetry
iii the coordinates of the vertex.

a $f(x) = 3(x-2)(x-4)$

b $f(x) = 4(x+1)(x-5)$

6 Sketch the following cubic functions.

a $f(x) = 2x^3 - 1$

b $f(x) = (x-1)(x+1)(x-3)$

7 The number of lilies, N, in a pond from 2004 until 2020 can be modelled using the function $N(x) = -0.04x^3 + 0.9x^2 - 7x + 70$, where x is the number of years after 2004.

a Sketch the graph of $N(x) = -0.04x^3 + 0.9x^2 - 7x + 70$ for $0 \le x \le 20$.

b Find the number of lilies after 5 years.

c Find the number of lilies after 12 years.

d Find the maximum number of lilies and the year in which this occurs.

e Find the minimum number of lilies and the year in which this occurs.

f Find when there are 60 lilies in the pond.

8 The number of miles, m, travelled varies directly with the time, t, for which the train has been travelling.

a If the train travels 100 miles in 1.25 hours, find a relationship connecting m and s.

b Find the number of miles travelled after 2 hours.

c Find how long it takes to travel 300 miles.

9 The line $2y = x + 3$ intersects the curve $f(x) = x^2 + 2x - 5$ at the points A and B. Find the coordinates of A and B.

10 Sketch the graph of the function $f(x) = x^2 - 2x - 3$ for the domain $0 \le x \le 4$.
Find the range of $y = f(x)$ in the given domain.

11 The acceleration a of a rocket is inversely proportional to the square root of the time since take off t.

Its acceleration is $6.2\,\text{m s}^{-2}$ when $t = 5.76\,\text{m s}^{-2}$.

a Find the acceleration of the rocket as a function of time.

b Find the value of t at which the acceleration is equal to $0.5\,\text{m s}^{-2}$.

12 A quadratic curve has x-intercepts at (0, 0) and (6, 0), and the vertex is at $(h, 8)$.

a Find the value of h.

b Find the equation of the curve in the form $y = ax^2 + bx + c$.

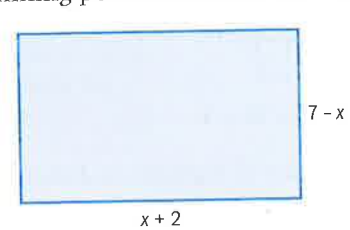
13 The curve $y = kx^n$ passes through the points (2,32) and (4,2). Find the values of k and n:

a without using a GDC

b using the power regression on a GDC.

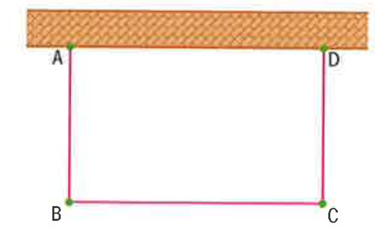
> To do this without a GDC is beyond the scope of the syllabus. This is a good extension exercise. If you are having trouble, the worked solution found in the digital resources may help.

14 The dimensions of a rectangular swimming pool are shown in the diagram.



Determine the set of value(s) that the variable x should take, such that the area of the swimming pool is greater than 16.

15 A farmer wants to make a rectangular pen for his sheep. He needs to put fencing on three sides, with the other side being a brick wall as shown in the diagram.



The total length of fencing that he can use is 60 m.

a Given that the length of side AB is x, determine an expression in terms of x for the length of the side BC.

b Hence determine an expression for the total area of the pen.

c With the aid of a graph, or otherwise, determine the length of side AB (ie the value of x) which makes the area take its maximum value.

16 A ball is thrown vertically upwards into the air. The height, h metres, of the ball above the ground after t seconds is given by $h = -t^2 + 6t + 12$.

a Write down the **initial** height above the ground of the ball (that is, its height at the instant when it is released).

b Show that the height of the ball after one second is 17 metres.

c At a later time the ball is **again** at a height of 17 metres.

i Write down an equation that t must satisfy when the ball is at a height of 17 metres.

ii Solve the equation **algebraically**.

d **i** Find **when** the ball reaches its maximum height.

ii Find the maximum height of the ball.

17 Draw the graph of the function $f(x) = 2x^3 - x^2 - x + 1$, for $-1 \le x \le 1$, and hence determine the range.

18 The monthly precipitation in Oxford in mm during the course of one year (from January to December) can be modelled by the function $P(t) = 0.11t^3 + 1.87t^2 - 6.91t + 45$, where t is the number of the month, and $0 \le t \le 1$ corresponds to January.

a Find the precipitation in Oxford in mm in July.

b Find the months during which the greatest and lowest precipitation occur.

c Find the mean of the two precipitations in part **b**.

d Find the month during which the mean from part **c** occurs.

e Use your results from **a** – **d** to comment on what the position of the month with the mean rainfall tells you about when the majority of the rain falls in Oxford.

19 Sketch the graph of the following functions within the specified domains and hence determine their range.

a $f(x) = -x^4$, for $-3 \le x \le 3$

b $g(x) = \dfrac{x^5}{4}$, for $-4 \le x \le 4$

c $h(x) = 3x^9$, for $-2 \le x \le 2$

20 The average price of a litre of gasoline in France in each year from 2006 to 2017 is given in the following table.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017
Price (€)	29.66	29.39	28.84	28.57	28.51	28.62	28.69	29.52	29.79	30.16	30.31	30.96

a Find a cubic model for this data.

b Comment on the suitability of using this model to predict future gasoline prices.

c Estimate the price in 2021.

21 Sketch the graph of the following functions, within the specified domains and hence determine their range.

a $f(x) = -\frac{1}{x^3}$, for $-3 \le x \le 3$, when $x \ne 0$.

b $g(x) = \frac{1}{x^4}$, for $-4 \le x \le 4$, when $x \ne 0$.

c $h(x) = \frac{1}{2x^5}$, for $-2 \le x \le 2$, when $x \ne 0$.

Exam-style questions

22 P1: The height (s m) of an object which moves freely under gravity is given by $s = c + ut - 5t^2$, where c m is the initial height of the object, u m s^{-1} is the initial velocity in the upward direction, and t s is time the object has been moving for.

Peter releases a model rocket from a platform 1.7 m above the ground. The rocket's initial velocity is 50 m s^{-1}.

a Find the time between the rocket setting off and returning to strike the platform. (2 marks)

b Hence write down how long it takes for the rocket to reach its maximum height. (2 marks)

c Calculate the maximum height of the rocket. (2 marks)

Peter's friend Paul does exactly the same thing, but the height of his platform is 1.8 m.

d By considering the difference in the two displacement-time graphs for the two model rockets, write down the answers to parts **a**, **b** and **c** for Paul's rocket. (4 marks)

23 P2: a Determine the set of values of k for which the quadratic equation $x^2 + kx + 9 = 0, x \in \mathbb{R}$ has

i one repeated root

ii no real roots

iii two distinct roots. (8 marks)

b For $k \in \mathbb{R}$, determine how many roots the equation $-x^2 + kx + 9 = 0, x \in \mathbb{R}$ has. (3 marks)

24 P2: Consider the quadratic function $f(x) = (x - 3)^2 + 7$.

a Write down the coordinates of the minimum point of this quadratic. (2 marks)

b Find the largest possible domain and range of $f(x)$ which include the point $(4, 8)$ such that the inverse function $f^{-1}(x)$ exists. (3 marks)

c Find the inverse function $f^{-1}(x)$, and state its domain and range. (5 marks)

25 P1: The cubic function

$$h(x) = \frac{(x^3 - 105x^2 + 3000x)}{1000}, \quad 0 \le x \le 80$$

models the height of a roller coaster ride, in metres, over the first 80 m of its circuit. x represents the horizontal distance from the start of the ride, and is also measured in metres.

a Find the coordinates of the local minimum point. (3 marks)

b Find the coordinates of the local maximum point. (3 marks)

c Find the set of values of x for which the roller coaster is above 50 m. (3 marks)

26 P1: When a coin is dropped down into an ancient well, the time (t seconds) it takes the coin to fall a distance of d metres can be modelled by $t = \sqrt{5}d^{\frac{1}{2}}$.

a Find how long it takes a coin to fall 10 m. (2 marks)

A coin is dropped into the well, and a splash is heard 8 seconds later as it hits the water at the bottom.

b Find the depth of the well down to the waterline. (2 marks)

If a frog can catch the coin at a depth equal to the time it takes to fall that depth, he changes into a handsome prince.

c Find how far, below the top of the well, a frog should be hanging on, waiting if he wants to become a handsome prince. (2 marks)

27 P1: James has forgotten his 4-digit passcode $abcd$, where a, b, c, d are digits between 0–9. To help him remember it, he has written down a mathematical puzzle which will allow him to work it out without giving it away easily to others:

If he splits the password into two 2-digit integers $x = ab$ and $y = cd$ (for example, if $a = 4$ and $b = 5$ then $x = 45$), then $x + y = 24$ and $xy = 143$.

Find the two possibilities that James' passcode could be. (6 marks)

28 P2: Consider the function $f(x) = x^2$.

The graph of the function is translated vertically to define a new function $g(x) = f(x) + k$.

a Find the value of k such that the graph of $g(x)$ passes through the point $(3, 4)$. (2 marks)

The graph of the function is translated horizontally to define a new function $h(x) = f(x - l)$.

b Find the values of l such that the graph of $h(x)$ passes through the point $(3, 4)$. (3 marks)

The graph of the function is translated to define a new function $m(x) = f(x - r) + s$.

c Find the values of r and s such that the minimum point of $m(x)$ is $(3, 4)$. (3 marks)

The graph of the function is transformed to define a new function $n(x) = af(x - b) + c$.

d Find the values of a, b and c such that the maximum point of $m(x)$ is $(3, 4)$. (4 marks)

29 P2: In an archaeological dig three small red squares are found on the ruin of a mosaic. It is believed that they were part of a red curve. A coordinate system is introduced and the three points have coordinates of $(1, 10)$, $(2, 27)$ and $(4, 115)$.

a Find the equation of a quadratic curve that passes through these three points. (4 marks)

Later a fourth red square is discovered with coordinates of $(5, 198)$.

b Determine if this new point lies on the quadratic curve found in part **a**. (2 marks)

c Find the equation of a cubic function that does pass through all four points. (5 marks)

30 P2: A horizontal suspension bridge has a cable above it. The shape of the cable can be modelled by a quadratic curve. The height of the cable is 10 m above the bridge at each of the two ends of the bridge. The cable is 2 m above the bridge in the middle of the bridge.

Find how far above the bridge the cable is at one quarter of the way along the bridge, from one end. (8 marks)

Hanging around!

Approaches to learning: Thinking skills: Create, Generating, Planning, Producing
Exploration criteria: Presentation (A), Personal engagement (C), Reflection (D)
IB topic: Quadratic Modelling, Using technology

Investigate

Hang a piece of rope or chain by its two ends. It must be free hanging under its own weight. It doesn't matter how long it is or how far apart the ends are.

What shape curve does the hanging chain resemble?

How could you test this?

Import the curve into a graphing package

A graphing package can fit an equation of a curve to a photograph.

Take a photograph of your hanging rope/chain.

What do you need to consider when taking this photo?

Import the image into a graphing package.

Carefully follow the instructions for the graphing package you are using.

The image should appear in the graphing screen.

Fit an equation to three points on the curve

Select three points that lie on the curve.

Does it matter which three points you select?

Would two points be enough?

In your graphing package, enter your three points as x- and y-coordinates.

Now use the graphing package to find the best fit quadratic model to your three chosen points.

Carefully follow the instructions for the graphing package you are using.

Test the fit of your curve

Did you find a curve which fits the shape of your image exactly?

What reasons are there that may mean you did not get a perfect fit?

The shape that a free-hanging chain or rope makes is actually a **catenary** and not a parabola at all. This is why you did not get a perfect fit.

Research the difference between the shape of a catenary and a parabola.

Did you know?

A football field is often curved to allow water to run off to the sides.

International-mindedness

The word "catenary" comes from the Latin word for "chain".

Extension

Explore one or more of the following – are they quadratic?

The curve of a banana.

The cross section of a football field.

The path of a football when kicked in the air – here you would need to be able to use available software to trace the path of the ball as it moves.

A well-known landmark – perhaps the Sydney Harbour Bridge or the arches at the bottom of the Eiffel Tower.

Other objects that look like a parabola – for example, the arch of a rainbow, water coming from a fountain, the arc of a Satellite dish.