7 Modelling rates of change: exponential and logarithmic functions

The reduction in temperature, the curve of a skateboard ramp, the value of an investment under compound interest, and radioactive decay can all be modelled by certain families of curves. This chapter looks at the characteristics of some of these families of curves, and considers how they can be used to model and predict outcomes in real-life situations.

Concepts

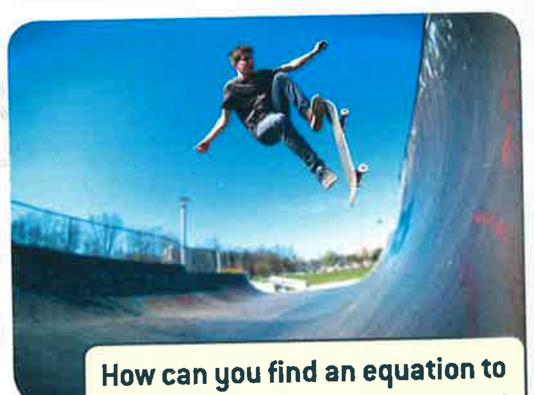
- Change
- Modelling

Microconcepts

- Common ratio, geometric sequences, geometric series and infinite geometric sequences
- Convergence and divergence
- Percentage, interest rate, compound interest, compounding periods, present value, future value, annuity and amortization
- Exponents, exponential growth/decay, inverse functions, logarithms, exponential functions and half-life
- Horizontal asymptotes
- Logistic functions and log-log graphs

How long will it take you to become a millionaire if you receive US$1 the first month, US$2 the second month, US$4 the third month, US$8 the fourth month and so on?

If the temperature of a cup of tea reduces at a given rate, will it ever reach 0°C?



How can you find an equation to model the slope of a skateboard track? Or a ski slope?

Does the same amount of money have the same value today, yesterday and tomorrow?

Look back at the Gapminder graph on p. 45.

1 Describe how the scale on the x-axis increases.
2 Would the x-axis ever reach zero?
3 Describe how the scale on the y-axis increases.
4 What would be the problem in using the same scale on both axes?
5 What type of function could you use to model the general trend of the data points?

Developing inquiry skills

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

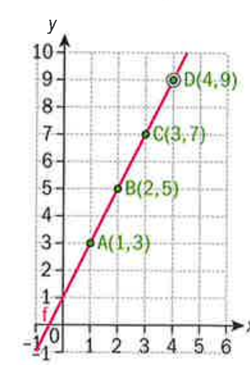
1 Apply the laws of exponents.
eg $x^2 \times x^3 = x^5$

2 Find a specific percentage of a quantity.
eg 6% of 24 $= \frac{6}{100} \times 24 = 1.44$

3 Mappings of the elements of one set to another. Illustration by means of sets of ordered pairs, tables, diagrams and graphs.
eg given the function $f(x) = 2x + 1$ when $x = 3 \Rightarrow f(3) = 2 \times 3 + 1 = 7$
This means that number 3 from the set of the domain is mapped onto number 7 on the set of the range.
Similarly using more x values:

x	1	2	3	4
y	3	5	7	9

Graphically:



4 Sigma notation.

$$S_n = \sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + u_4 + \ldots + u_n$$

eg $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

Skills check

Click here for help with this skills check

1 Find the value of:
a $(x^2)^3$ b $\left(\frac{a}{b}\right)^{-1}$ c $x^{-2} \times \sqrt{x}$
d $\frac{\sqrt{x}}{\sqrt[3]{x}}$ e $\left(\sqrt{x}\right)^5$

2 Find the value of:
a 3% of 24 b 15% of 72
c 28% of 150

3 Consider the function $f(x) = 1 - 3x$.
a Copy and complete the following table based on $f(x)$.

x	−1	0	1	2
y				

b Find the value of $f(12)$.
c Graph the function $f(x)$.

4 Find the value of the following:
a $\sum_{i=1}^{10} 2i + 1$ b $\sum_{i=5}^{10} 2i + 1$ c $\sum_{i=1}^{4} i^3$

7.1 Geometric sequences and series

The Australian Open Men's Singles competition is part of the Grand Slam tennis tournaments. In this competition, 128 players start competing in straight elimination matches until the final, where the champion is declared.

a How many matches are played in the first three rounds of the tournament?

b What do you notice?

c How many matches are be played in the last round (the final round) of the tournament?

d How many matches will be played in total during the complete course of the tournament?

The nth term of a geometric sequence is given by the formula:

$$u_n = u_1 \times r^{n-1}, r \neq 1$$

Investigation 1

1 a One common counting method is decimal counting. This uses units, tens, hundreds, thousands, etc. ie multiples of 1, 10, 100, … Determine the rule for going from one category to the next.

b In a single elimination (or knockout) tournament, the number of remaining participants follows the pattern 64, 32, 16, …. Determine the rule for going from one number to the next.

c The value of a car depreciates because of age. A certain car has the following value at the end of each year after purchase. €40 000, €32 000, €25 600, … Determine the rule for going from one value to the next.

d A pendulum consists of a hanging weight that can swing freely. When moving from side to side, it gets displaced from its rest position. For a certain pendulum the displacements at each end of its oscillation were recorded to be 40, −20, 10, −5, … Determine the rule for going from one value to the next.

What do you notice in the pattern of the terms in all these examples?

Why aren't these sequences arithmetic?

2 Complete the table for each geometric sequence.

	1st term	2nd term	3rd term	4th term	5th term	6th term
a	1	10	100			
b	64	32	16			
c	40 000	32 000	25 600			
d	40	−20	10			

Factual For each sequence, determine whether it is increasing, decreasing or oscillating.

Conceptual When do the terms of a geometric sequence increase? When do they decrease? When do they oscillate?

3 a To get from the first term of a geometric sequence to the second you need to multiply by the common ratio. $u_2 = u_1 \times r = u_1 r$

How many terms after the first term is the second term?

How does this relate to the expression above?

b To get from the second term of a geometric sequence to the third you need to multiply again by the common ratio. $u_3 = u_2 \times r = u_1 r \times r = u_1 r^2$

How many terms after the first term is the third term?

How does this relate to the expression above?

c How many terms after the first term is the eighth term?

How does this relate to an expression for the eighth term u_8 in terms of the first term u_1 and the common ratio r?

d How many terms after the fifth term is the eighth term?

How does this relate to an expression for the eighth term u_8 in terms of the fifth term u_5 and the common ratio r?

e How many terms after the first term is the nth term of the sequence?

How does this relate to an expression for the nth term u_n in terms of the first term u_1 and the common ratio r?

Conceptual How can you find the general term of a geometric sequence?

International-mindedness

The *Elements*, Euclid's book from 300BC, contained geometric sequences and series.

Number and algebra

Functions

A sequence of numbers in which each term can be found by multiplying the preceding term by a **common ratio**, r, is called a geometric sequence.

For a sequence to be geometric, $r \neq 1$.

HINT

A sequence of terms that has a common ratio equal to 1 (a sequence of constant terms) is not a geometric sequence.

Example 1

For each of the following geometric sequences, find the common ratio, the value of the specified term, and an expression for the nth term.

a 4, 12, 36, …, u_8, …

b 2, −6, 18, …, u_5, …

c 2, 2.2, 2.42, …, u_6, …

d $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots, u_7, \ldots$

a $r = \frac{12}{4} = 3$

$u_8 = u_1 \times r^7 = 4 \times 3^7 = 8748$

$u_n = u_1 \times r^{n-1} = 4 \times 3^{n-1}$

In order to determine the common ratio of a geometric sequence you need to divide any term of the sequence by the preceding term.

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b $r = \frac{-6}{2} = -3$

$u_5 = u_3 \times r \times r = 18 \times (-3) \times (-3) = 162$

$u_n = u_1 \times r^{n-1} = 2 \times (-3)^{n-1}$

c $r = \frac{2.2}{2} = 1.1$

$u_6 = u_1 \times r^5 = 2 \times 1.1^5 = 3.22102$
$= 3.22$ (3 s.f.)

$u_n = u_1 \times r^{n-1} = 2 \times 1.1^{n-1}$

d $r = \frac{\left(\frac{2}{3}\right)}{\left(\frac{1}{3}\right)} = 2$

$u_7 = u_1 \times r^6 = \frac{1}{3} \times 2^6 = \frac{64}{3}$

$u_n = u_1 \times r^{n-1} = \frac{1}{3} \times 2^{n-1} = \frac{2^{n-1}}{3}$

Sometimes when the term you are trying to find is a term very close to the ones you already have, instead of using the formula you can just multiply by the common ratio as many times as is necessary until you reach the required term.

Exercise 7A

1 For each one of the following geometric sequences, find the common ratio, the value of the specified term and an expression for the nth term.

a 5, 10, 20, …, u_7, …

b −3, 15, −75, …, u_8, …

c $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \ldots, u_6, \ldots$

d $\frac{3}{2}, 1, \frac{2}{3}, \ldots, u_4, \ldots$

e 2, 20, 200, …, u_{10}, …

2 For each one of the following geometric sequences, find the common ratio and an expression for the nth term.

a $u_1 = 3$ and $u_4 = 24$

b $u_1 = 32$ and $u_6 = 243$

c $u_1 = 1$, $u_5 = 81$ and all terms are positive.

d $u_1 = \frac{1}{2}$ and $u_4 = \frac{4}{27}$

e $u_1 = -2$, $u_7 = -1458$ and the sequence has both positive and negative terms.

f $u_3 = 13.5$ and $u_6 = 367.5$

3 There is a flu epidemic in Cozytown. On the first day, 2 people have the flu. On the second day, 10 people have the flu. On the third day, 50 people have the flu.

a Show that the number of people with the flu forms the first of three terms in a geometric sequence.

b Calculate how many people have the flu after 1 week (7 days).

c Use your model to calculate how many people will have the flu after 1 year. Comment on the reasonableness of your answer.

Investigation 2

Censuses have taken place in the United Kingdom every ten years since 1801. Studies of the findings on the population have shown that it is increasing by 0.6% each year. The population in the last census at the beginning of 2011 was found to be 63.2 million.

Assuming the population continues to increase at the same rate:

1 Explain why the population at the beginning of 2012 can be found by multiplying 63.2 million by 1.006.

2 The population (in millions) at the beginning of 2017 can be given by $1.006^n \times 63.2$. State the value of n.

3 If a population increases at a rate p% per year write down an expression for the population after n years (P_n) in terms of the original population (P_0).

4 Explain how this formula is different from the usual formula for a geometric series.

The population of Japan is decreasing on average by 0.2% per year. Its population in 2018 is 127 million.

Assuming the rate of decrease remains the same:

5 Write down an expression for the population of Japan in 2025 and calculate this value.

6 In which year would the population drop below 120 million?

7 If a population decreases at a rate p% per year write down an expression for the population after n years (P_n) in terms of the original population (P_0).

8 **Conceptual** What type of sequence models a situation with constant percentage change?

9 **Conceptual** How do geometric sequences model real-life situations?

Percentage change is a form of a geometric sequence.

When terms **increase** by p% from the preceding term, their common ratio corresponds to $r = 1 + \frac{p}{100}$.

When terms **decrease** by p% from the preceding term, their common ratio corresponds to $r = 1 - \frac{p}{100}$.

TOK

Why is proof important in mathematics?

Example 2

Costis bought a car for €16 000. The value of the car depreciates by 10% each year.

a Find the value of the car at the end of the first year.

b Find the value of the car after 5 years.

c Calculate after how many years the value of the car fall below half its original value.

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a 10% of €16 000 is $\frac{10}{100}\times 16000 = €1600$

$V_1 = 16000 - 1600 = €14400$

or $r = \frac{100-10}{100} = 0.9$ and

$V_1 = 16\,000 \times 0.9 = €14\,400$

A decrease of 10% is found by multiplying by 0.9.

b $V_5 = 16\,000\times 0.9^5 = €9447.84$

c Half of the original value is €8000

$16\,000\times 0.9^n < 8000$

$n > 6.58 \Rightarrow n \geq 7$

After 7 years the value of the car will first drop below half of its original value and become $V_7 = 16\,000\times 0.9^7 = €7652.75$

This equation can be solved directly using the table or graphing function on a GDC.

Example 3

The number of Snapchat daily users is found to be increasing by 2.5% per month. In May of 2018, the number of Snapchat daily users is 187 million.

a Determine the number of daily users in December 2018.

b Find the month of which year when the number of daily users first exceeds 300 million.

c The company aims to reach 300 million daily users during May of 2019. Determine the monthly percentage increase required to achieve this goal.

a $V_{Dec} = V_{May}\times r^7 \Rightarrow$

$V_{Dec} = 187\times 1.025^7 = 222$ million daily users.

From May (5th month) to December (12th month) the number of months that will pass is $12 - 5 = 7$ months and so the power to which the common ratio is raised is 7.

b $V = V_{May}\times r^n \Rightarrow 187\times 1.025^n > 300$

$\Rightarrow n > 19.1 \Rightarrow n \geq 20 \Rightarrow n_{min} = 20 \Rightarrow$

after 20 months the number of daily users will exceed 300 million, ie in November of 2020.

The common ratio is $r = \frac{100+2.5}{100} = 1.025$

c $V_{May2019} = V_{May2018}\times r^{12} \Rightarrow 300 = 187\times r^{12}$

$\Rightarrow r = 1.040 \Rightarrow \frac{100+p}{100} = 1.040$

$\Rightarrow p = 4.0\%$

From May of 2018 until May of 2019, the number of months that will pass is 12 and so the power to which the common ratio is raised is 12.

The exponent n to which the common ratio r is raised will always be the number of time periods that have elapsed since the growth or decay begins.

Exercise 7B

1 At the end of 2016 the population of a city was 200 000. At the end of 2018 the population was 264 500.

a Assuming that these end of year figures follow a geometric sequence, find the population at the end of 2017.

b Calculate the population at the end of 2020.

c Comment on whether this increase will continue.

2 One kilogram of tomatoes costs \$2.20 at the end of 2015. Prices rise at 2.65% per year. Find the cost of a kilogram of tomatoes at the end of 2019.

3 Petra buys a camper van for €45 000. Each year the camper van decreases in value by 5%. Find the value of the camper van at the end of six years.

4 Beau spends €15 000 buying computer materials for his office. Each year the materials depreciate by 12%.

a Find the value of the materials after three years.

b Find how many years it takes for the materials to be worth €5000.

5 Find the common ratio for the geometric sequence representing the following percentage changes:

a increase by 12.5% **b** decrease by 7.3%

c becomes 89% **d** increase by 0.1%

6 Determine the number of times that the % change is applied in the following cases and determine their value.

The diameter of sequoia trees increases by 5% each year.

a If the diameter of a sequoia tree today is 1.2 m, find what it will be in 7 years.

b If the diameter of the first annual measurement made was 1.2 m, determine the 7th measurement.

c If the diameter at the beginning of 2010 was 1.2 metres, find the diameter at the **end** of 2017.

7 The world population at the beginning of 2019 is 7.7 billion people. Assume the annual rate of increase throughout the time periods considered in the question is 1.72%.

a Find the world population after one decade.

b Determine the world population 7 years ago.

c Determine the world population at the **end** of 2040.

d Determine when the world population has doubled compared to the beginning of 2019?

e Calculate the annual percentage increase in order for the world population to become 10 billion in 2090.

Investigation 3

Let S_n denote the sum of the first n terms of a geometric series, so

$$S_5 = \sum_{i=1}^{5} u_i = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4$$

1 Write down an expression for rS_5.

2 Hence find an expression in terms of u_1 and r for:

a $rS_5 - S_5$ **b** S_5.

The result obtained in part **b** can be generalized to $S_n = \frac{u_1(r^n - 1)}{r-1}$.

Continued on next page

3 Explain why the formula can also be written as $S_n = \dfrac{u_1(1-r^n)}{1-r}$ and why this form might be easier to use when $r < 1$.

According to a story, Sissa ibn Dahir, invented the game of chess for King Shihram to play. The king was so pleased with the game that he asked Sissa what reward he wanted for this great invention. Sissa said that he would take this reward: the king should put one grain of wheat on the first square of a chessboard, two grains of wheat on the second square, four grains on the third square, eight grains on the fourth square, and so on, doubling the number of grains of wheat

4 Find how many grains of wheat the King would have to give Sissa.

5 How does this compare with the annual world production of wheat consisting of approximately 1.2×10^{16} grains?

The sum of the first n terms of a geometric sequence is called a geometric series and is given by the formula:

$$S_n = \sum_{k=1}^{n} u_k = \frac{u_1\times(r^n-1)}{r-1} = \frac{u_1\times(1-r^n)}{1-r},\ r\neq 1$$

Example 4

The first term of a geometric sequence is equal to 3 and the common ratio is equal to 2.

a Find the sum of the first 10 terms of the sequence.

b Find the sum of the next 10 terms of the sequence.

a $S_{10} = \dfrac{3\times(2^{10}-1)}{2-1} = 3069$	
b $S_{20} = \dfrac{3\times(2^{20}-1)}{2-1} = 3\,145\,725$ Thus the sum of the next 10 terms of the sequence is $S_{20}-S_{10} = 3\,145\,725 - 3069 = 3\,142\,656$	The sum of the next ten terms of the geometric series is actually the sum $u_{11}+u_{12}+u_{13}+\ldots+u_{20}$. You can think of this as the sum of the first 20 terms minus the sum of the first 10 terms. $S_{20} = u_1+u_2+u_3+\ldots+u_{10}+u_{11}+\ldots+u_{20}$ $S_{10} = u_1+u_2+u_3+\ldots+u_{10}$ $\Rightarrow S_{20}-S_{10} = u_{11}+u_{12}+\ldots+u_{20}$

Example 5

The students in a school decided to raise money in order to install hammocks in the campus. They have 10 days to raise the required money of €300.
The money raised on the first day was €50. The money that they raise on each subsequent day is 15% less than the previous.

a Calculate the amount of money they expect to raise in total. Comment on whether this will be enough to purchase the hammocks.

b Calculate the number of full days they would need to fundraise on if they are to raise enough money to purchase the hammocks.

c Find the maximum daily percentage decrease in the money they raise if they are to reach their goal of raising €300 in 10 days.

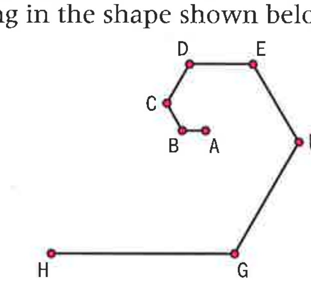
a $S_{10} = \dfrac{50(1-0.85^{10})}{1-0.85} = €267.71$ The amount will not be enough in order to purchase the hammocks.	This is a geometric series with $u_1 = 50$ and $r = \dfrac{100-15}{100} = 0.85$ Use the second form of the formula for the sum of a geometric series as $r < 1$.
b $S_n = \dfrac{50(1-0.85^n)}{1-0.85} \geq 300 \Rightarrow n \geq 14.2$ $\Rightarrow n \geq 15$. They would need at least 15 days in order to collect the amount.	The students will need to fundraise for n days, such that the sum to n of the geometric progression is greater than or equal to 300.
c $S_{10} = \dfrac{50(1-r^{10})}{1-r} = 300 \Rightarrow r = 0.879$ $\Rightarrow \dfrac{100-p}{100} = 0.879 \Rightarrow p = 12.1\%$ There should only be a decrease of 12.1% each day in order to achieve the goal of raising €300 in 10 days.	In order to raise 300 in 10 days, the amounts must form a geometric progression with $u_1 = 50$ and $S_{10} = 300$. Find r as a percentage, p.

TOK

Is mathematics a language?

Exercise 7C

1 Maria, a jewellery designer, is designing an earring in the shape shown below



The straight-line segments from which the earring is made follow a geometric sequence with common ratio 1.4. The first and smallest segment is 1 cm long.

a Find the length of the longest segment of the earring.

b Find the total length of material required for this earring.

Number and algebra

Functions

2 A biologist is running an experiment with a certain colony of bacteria in a petri dish. Every hour the number of new bacteria that are created is 10% more than the previous hour. There were approximately 72 million bacteria at the beginning of the experiment and during the first hour the number of bacteria that were created was approximately 12 million.

- **a** Find the number of bacteria that were created during the 6th hour of the experiment.
- **b** Find the total number of bacteria that were created during the first 6 hours.
- **c** Find the total number of bacteria in the petri dish 10 hours later.
- **d** Determine how long it will take for the number of bacteria to exceed 1 billion.

3 The numbers of email users increase by 6.2% each year. The number of email users in 2008 was approximately 1.7 billion.

- **a** Find the common ratio of the sequence.
- **b** Find the number of email users in 2019.
- **c** What is the percentage increase from 2008 to 2019?
- **d** Find the first year in which the number of email users will exceed 3 billion.
- **e** Comment on why this sequence is likely to have a limited lifetime.

4 The sum of the first 10 terms of a geometric sequence is 33 times the sum of its first five terms. The first term of the sequence is 3.

- **a** Find the common ratio of the series.
- **b** Find the greatest term in the sequence which is less than 1000.
- **c** Find the smallest value of n such that the sum of the first n terms of the series is greater than 1000.
- **d** Given that the sum of the first k terms of the series is 33 825 times the sum of the first five terms, find the value of k.

5 Find the value of the following geometric series.

- **a** $2 + 10 + 50 + \ldots + 781\,250$
- **b** $6.4 + 9.6 + 14.4 + \ldots + 164.025$
- **c** $\frac{8}{3} + \frac{2}{3} + \frac{1}{6} + \ldots + \frac{1}{6144}$

6 Anna Louisa is practicing for the interschool swimming championships. She records the best time from every day's practice in order to monitor her performance. She noticed that her times go down by 0.2% after each day of practice. She started with a time of 30.4 seconds. The record of the competition is 29.3 seconds and the competition is in 20 days from the day she started practicing.

- **a** What is the common ratio of the geometric sequence?
- **b** What time is she expected to swim after five days of practice?
- **c** Will she manage to break the record time on the day of the championship?

Investigation 4

1 For a geometric sequence with $u_1 = 1$ and $r = 1.1$ find u_{10}, u_{50} and u_{100}.

2 Repeat for the sequence with $u_1 = 1$ and $r = 0.9$.

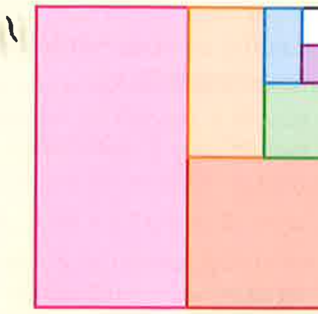
3 State the main differences between the results of **1** and **2** and justify your answer using the formula $u_n = u_1 r^{n-1}$.

If as n increases a sequence, u_n, approaches a value but never quite reaches it, we say it displays **asymptotic behavior** and can talk about its limit as n **tends to infinity**, which is written as as $\lim_{n\to\infty}(u_n)$.

4 When $-1 < r < 1$ write down $\lim_{n\to\infty}(u_n)$ and justify your answer.

5 Why does it make no sense to talk about $\lim_{n\to\infty}(u_n)$ when $r > 1$ or $r < -1$ (we can also write $|r| > 1$)?

Consider a piece of paper of area 1 m² which is divided as shown in the diagram. First one half is shaded, then one half of the remainder is shaded, then one half again.



Let u_n be the area shaded on the nth "shading". Hence $u_1 = \frac{1}{2}$, $u_2 = \frac{1}{4}$ etc.

6
- **a** Find an expression for u_n.
- **b** State the value of r for the sequence.
- **c** In the context of the diagram explain what is meant by S_n.
- **d** From the diagram write down $\lim_{n\to\infty}(S_n)$.
- **e** Use the formula $S_n = \frac{u_1(1-r^n)}{1-r}$ to justify your answer to **6d**.
- **f** Hence right down a formula for the limit as n tends to infinity for the sum of a geometric series giving the condition on r for it to be valid.

This limit is referred to as the sum to infinity of the series and is written as S_∞.

7 Use your formula to find the sum to infinity of the following geometric sequences.

- **a** 1, 0.1, 0.01, 0.001, … **b** 8, −4, 2, −1, …

8 **Conceptual** When does an infinite sum of a geometric series converge to a finite sum?

The **sum to infinity** of a geometric sequence is:

$$S_\infty = \frac{u_1}{1-r}, \text{ for } |r| < 1$$

TOK

How do mathematicians reconcile the fact that some conclusions conflict with intuition?

Example 6

Sheldon is carrying out an experiment that involve adding decreasing amounts of a chemical to a collection of test solutions.

He adds 60 ml to the first and 50 ml to the second. The amounts added form a geometric sequence.

- **a** Find the amount added to the fifth solution.
- **b** Find which solution will be the first to have less than 10 ml added.
- **c** Given Sheldon has 400 ml of the chemical, prove that he will have enough however many test solutions he has included in his experiment.

Continued on next page

a $r = \frac{5}{6}$

Let the amount added on the nth experiment be u_n.

$u_5 = 60 \times \left(\frac{5}{6}\right)^4 = 28.9\,\text{ml}$

b $u_n < 10 \Rightarrow u_1 r^{n-1} < 10$

$\Rightarrow 60 \times \left(\frac{5}{6}\right)^{n-1} < 10$

$n = 11$

As a whole number answer is required this can most easily be solved using the table function in the GDC. Alternatively the graphing or numerical solving function can be used to first show $n > 10.8$

c Maximum needed can be given by the sum to infinity.

Using $S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{60}{1-\frac{5}{6}} = 360\,\text{ml}$

So Sheldon will have enough of the chemical.

Exercise 7D

1 Determine whether the following infinite series possess a finite sum. If they do, determine the value of the infinite sum.

  a $0.001 + 0.002 + 0.004 + \ldots$

  b $1\,000\,000 + 500\,000 + 250\,000 + \ldots$

  c $1 - 3 + 9 - 27 + \ldots$

  d $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

2 The sum of an infinite geometric sequence is seven times the value of its first term.

  a Find the common ratio of the sequence.

  b Find the least number of terms of the sequence that must be added in order for the sum to exceed half the value of the infinite sum.

3 The value of an infinite series is 24. The first term of the sequence is 4.

  a Find the common ratio of the sequence.

  b Find the first three terms of the sequence.

  Now all the terms of the sequence are squared.

  c Find the first three terms of the new series.

  d Find the common ratio of the new sequence. What do you notice?

  e Find the infinite sum of the new sequence.

  f Find the ratio of the infinite sum of the new sequence to the square of the infinite sum of the original sequence. Comment on your result.

4 In Physics class, Zoe learned that because of energy losses, when a basketball is dropped to the ground, it would bounce to 75% of its original height.

  If the ball is dropped from an initial height of 2 m:

  a Determine the height it would rise after the first bounce.

  b Determine the height it would rise after the second bounce.

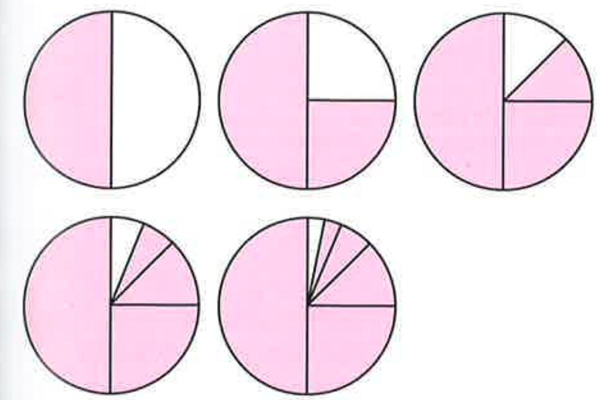
  c Determine the height it would rise after the third bounce.

  d Determine the common ratio of the sequence of bounce heights.

  e Comment on whether or not the ball would ever stop bouncing.

  f If the bouncing continues indefinitely, calculate the total distance covered by the ball.

5 In a HL Mathematics class, the teacher gave the following project to his students. He gave them a piece of paper with an empty circle of radius 4 cm drawn in the middle. He then asked each student to shade half of the empty shape, determine in terms of pi the area of the whole shape that has been shaded and give it to the next student. These are the first few shaded circles:



  a Find the area shaded by the first student.

  b Find the area shaded by the second student.

  c Find the area shaded by the first two students.

  d Find the area shaded by the fifth student.

  e Find the area shaded by the first five students.

  f Find the number of students required to shade 99% of the circle.

  g If the process was continued indefinitely, find what area of the circle would eventually be shaded. Verify your answer using the formula for the sum of an infinite series.

6 A mathematics teacher has taken his daughter to the playground. While she was on the swing, he noticed that the amplitude of each successive swing was significantly less than the previous and he had to give her a push every now and then.

  Going back home he decided to analyze the motion of the swing. The amplitude of the first oscillation was 1 m and every subsequent oscillation was 89% of the previous one.

  The swing needs a push whenever the amplitude falls below 0.5 m.

  a Find the amplitude of the third oscillation.

  b Find after how many oscillations he will need to push the swing again.

  c Find the total distance travelled up to the point that he needs to push the swing again.

  d If he left the swing to oscillate indefinitely, calculate the total distance covered.

Developing inquiry skills

Look back to the opening problem at the start of the chapter.

You should now be able to answer questions 1–5.

1 What do you notice about the scaling of the axes?

2 Describe how the scale on the x-axis increases.

3 Would the x-axis ever reach zero?

4 Describe how the scale on the y-axis increases.

5 What would be the problem in using the same scale on both axes?

7.2 Financial applications of geometric sequences and series

As soon as she was born, Zaira's parents decided to put some money into an account for her to have on her 18th birthday. They were offered the following investment schemes from the different banks that they approached:

- Scheme A: To invest 10 000 MAD at a simple interest rate of 5% per annum for 18 years.
- Scheme B: To invest 10 000 MAD at a compound interest rate of 4% per annum compounded annually for 18 years.
- Scheme C: To invest 10 000 MAD at a compound interest rate of 4% per annum compounded monthly for 18 years.
- Scheme D: To invest 10 000 MAD and get back double this amount at the end of the 18 years.

How much interest would they receive under each one of the schemes?

Compound interest

If the interest paid is **simple interest**, then the interest remains the same for each year that you have your money in the bank.

If the interest paid is **compound interest**, then the interest is added to the original amount and the new value is used to calculate the interest for the next period.

Compound interest is not always calculated per year, it could also be calculated per month, per week, etc.

It is also possible to have yearly interest with several **compounding periods** within the year, for example monthly compounding.

In the previous section you saw that when any quantity increases by $p\%$ per time period then the value at the end of n time periods can be given as $u_n = u_0\left(\frac{100+p}{100}\right)^n = u_0\left(1+\frac{p}{100}\right)^n$ where u_0 is the initial amount.

In this section we shall consider the case where p represents the interest rate received on an investment which can be added at varying intervals. When a yearly interest rate is given it can be referred to as $p\%$ per annum (year)

Investigation 5

Eric decides to put his money into a compound interest scheme in a bank. He is going to invest €10 000. The interest rate is 5% per annum compounded yearly.

1. How much interest will he receive on the amount of his investment during the first year?
2. What is the total amount that he will have at the end of the first year?
3. Write down an expression for the total amount he will have in the account at the end of n years.
4. Hence find the total amount he will have in his account at the end of the 10th year.
5. **Conceptual** How can you model compound interest?

Another bank pays the same annual interest rate but compounded monthly (ie the interest is added each month rather than just at the end of the year).

Assume that the monthly interest rate is one twelve of the annual interest rate.

6. State the number of times interest is added in the first year (these are the compounding periods).
7. Hence find how much money he will have in the account after
   i one year
   ii ten years.
8. By considering by how much his investment has increased in the first year, give the actual percentage interest he has received from the bank.

The interest rate given assuming that the interest is added just once is called the **nominal** rate; the interest rate which takes into account the compounding is called the **effective** rate.

9. What do you notice about the two investment schemes and the interest that they pay?

As the number of compounding periods within a year increase, the interest rate per compounding period decreases and can be calculated as $\frac{1}{k}\times r\%$, where $r\%$ is the nominal annual interest rate and k is the number of compounding periods within a year.

If the amount is invested for n years, the times (periods) at which the investor will receive interest increase and become $k \times n$.

If PV is the present value, FV the future value, r the nominal interest rate, n the number of years and k the compounding frequency, or number of times interest is paid in a year (ie $k=1$ for yearly, $k=2$ for half-yearly, $k=4$ for quarterly and $k=12$ for monthly, etc), then the general formula for finding the future value is

$$FV = PV \times \left(1+\frac{r}{100k}\right)^{k\times n}$$

EXAM HINT

In this course, you should be able to calculate compound interest using technology, including the financial packages of your GDC. When answering exam questions, you should show your working clearly, including the information from the financial package.

Example 7

1 Rafael invests BRL 5000 (Brazilian real) in a bank offering 2.5% interest compounded annually.

a Calculate the amount of money he has after five years.

After the five years, Rafael withdraws all his money and puts it in another bank that offers 2.5% interest per annum compounded monthly.

b Calculate the amount of money that he has in the bank after three more years.

2 Alexis invests RUB 80 000 (Russian ruble) in a bank that offers interest at 3% per annum compounded quarterly.

a Calculate how much money Alexis has in the bank after six years.

b Calculate how long it takes for his original amount of money to double.

1 a $FV = 5000\left(1+\frac{2.5}{1\times 100}\right)^{1\times 5}$ = BRL 5657.04	$PV = 5000$, $r = 2.5$, $k = 1$, $n = 5$
b $FV = 5657.04\left(1+\frac{2.5}{12\times 100}\right)^{12\times 3}$ = BRL 6097.16	$PV = 5657.04$, $r = 2.5$, $k = 12$, $n = 3$
Or, using the Finance app on your GDC:	
a N = 5 I% = 2.5 PV = −5000 PMT = 0 FV = PpY = 1 CpY = 1 Move the cursor back to FV and press enter to get the answer.	PV is usually negative because you have given it to the bank. PMT is periodic money transfers (also called annuity payment) and you don't need it here. PpY is periods in the year; when you are dealing with years, this is always 1. CpY is compounding periods: 1, 2, 4 or 12 depending on how often interest is paid.
b N = 3 I% = 2.5 PV = −5657.04 PMT = 0 FV = PpY = 1 CpY = 12 Move the cursor to FV and press enter.	
2 a $FV = 80\,000\left(1+\frac{3}{4\times 100}\right)^{4\times 6}$ = RUB 95 713.08	N = 6 I% = 3 PV = −80 000 PMT = 0 FV = PpY = 1 CpY = 4 Move the cursor back to FV and press enter.
b $160\,000 = 80\,000\left(1+\frac{3}{4\times 100}\right)^{4\times n}$ Using the Finance app: $n = 23.19$ So it would take 23 years for his money to double.	Double the original amount is RUB 160 000 N = I% = 3 PV = −80 000 PMT = 0 FV = 160 000 PpY = 1 CpY = 4 Move the cursor back to N and press enter.

Example 8

Zoe deposits €25 000 in a bank offering 2.4% annual interest rate compounded quarterly. From this account she wishes to pay her rent, which is €600 per month. Find how long it will take until the account can no longer be used to pay the rent.

Using the finance app on your GDC: N = I% = 2.4 PV = −25000 PMT = 600 FV = 0	In this case, money is also withdrawn from the account, so PMT = 600. The Future Value is set to zero, since you are looking for the value of n at the time when the account will have no more money.
PpY = 12	PpY is set to 12 since Zoe has to pay the rent 12 months per year.
CpY = 4 Move the cursor back to N and press enter to get the answer. N = 43.5 months, meaning that she will be able to pay the rent from this account for 43 complete months.	CpY is set to 4 since the interest she receives is compounded 4 times per year.

Inflation measures the rate that prices for goods increase over time and, as a result, how much less your money can buy.

This means that if inflation is at i% an investment which receives r% interest compounded annually will actually have its **real value** increased by only $(r - i)$%.

Therefore when adjusting for inflation to find the real value of an investment replace r by $(r - i)$ in the compound interest formula.

TOK

"Debt certainly isn't always a bad thing. A mortgage can help you afford a home. Student loans can be a necessity in getting a good job. Both are investments worth making, and both come with fairly low interest rates" – Jean Chatzky

Do all societies view investment and interest in the same way?

What is your stance?

Example 9

Kathryn would like to buy a new house in five year's time. The average price of houses in the area she is considering is €120 000. She has €110 000 in an investment account, which is earning 5.2% interest per year compounded yearly. House inflation is expected to be 3.1% per year. Will she be able to afford an average price house in five year's time?

Investment will be worth $110\,000\left(1+\frac{5.2-3.1}{100}\right)^5$ = €122 045 So she will be able to afford it	This can also be done using the financial application on your GDC entering the interest as 2.1%.

Exercise 7E

1 Oswald and Martha both have €5000 to invest. Oswald invests his money in a bank that offers to pay 3.7% nominal annual interest compounded annually. Martha invests her money in a bank that offers to pay 3.5% nominal annual interest compounded monthly.

Calculate who has the most money at the end of 15 years.

2 Adam invests 50 000 NIS (New Israeli Sheqel) in a bank that pays 3.2% nominal annual interest compounded annually.

a Calculate the amount he will have in the bank after 10 years.

b Find how many complete years it will take for his money to double?

c Calculate the interest rate he should ask for if he wants to double his investment in 20 years.

3 Celia plans to invest £4500 in a bank that offers r% nominal annual interest compounded monthly. After 5 years, she has £5803.94 in the bank.

a Find the interest rate.

b Find how many years it will take for Celia to increase her initial amount by 50%.

4 Tony wants to buy a scooter that costs $1500. He deposits $1000 in a bank that pays 7.5% interest compounded quarterly. Determine after how many years he will be able to buy the scooter.

5 Leon invests an amount in euro in a bank that offers 1.2% interest compounded half-yearly. 10 years later his account has €6762.56. Calculate the original amount of his investment.

6 I invest a sum of money at a nominal rate of 6% per year. Find the effective yearly interest rate if the interest is compounded

a quarterly

b monthly.

7 Celia is planning to invest $20 000 in the United States. The inflation rate is projected to be 2.1%. She found an investment scheme in a bank that pays a nominal annual interest rate of 2.4% compounded monthly.

a Determine the real interest rate she will receive next year.

b Determine the worth of her investment after one year.

8 The inflation rate in Ecuador is negative (deflation) meaning that the price of basic commodities decreases every year. The projection for the next year is −0.5%.

Juliana is planning to invest an amount of $2000 at an interest rate of 0.1%.

a Determine the real interest rate that she will receive.

b Determine the worth of her investment at the end of one year.

Annuities and amortization

When a constant investment, P, is made for n periods always compounded with the same interest rate, r%, it is called an **annuity**.

Investigation 6

Laura deposits $3000 at the beginning of each year for 10 years in a pension scheme.

The rate of interest is fixed at 6% per annum. Interest is compounded yearly.

1 Write down an expression for the final value (after 10 years) of the $3000 invested at the beginning of the first year.

2 Write down an expression for the final value of the $3000 invested at the beginning of the second year.

3 Similarly, write down an expression consisting of the sum of ten terms for the **total** value of the pension fund at the end of 10 years.

4 What type of series does your expression represent?

5 How much will the deposit be worth after 10 years?

6 **Conceptual** How does an annuity represent compounded growth?

The formula for working out an annuity is:

$$FV = A\frac{(1+r)^n - 1}{r}$$

where FV is the future value, A is the amount invested each year, r is the interest rate and n is the number of years.

When a constant payment, P, to repay a loan is made for a certain number of n periods always compounded with the same interest rate, r%, it is called **amortization**.

Investigation 7

Pim borrows \$1000 from a bank that charges 4% interest compounded annually. He wants to pay the loan back in six months in monthly instalments. The bank informs Pim that he must pay \$168.62 back each month.

1 Work out how much he has to pay in total.

If the interest is 4% compounded annually, then that would be 2% for half a year.

2 Work out 2% of \$1000.

3 Explain why the amount that Pim pays back in total is less than \$1000 + 2% of \$1000.

Pim sets up a spreadsheet to monitor his payments.

He pays \$168.62 each month, but the bank also charges interest each month on the amount owed.

4 If the interest rate is 4% per annum, work out what it is per month.

5 Calculate the interest for the first month.

6 Describe how you worked out how much is paid off the loan each month.

7 The spreadsheet shows the first two lines of Pim's payments. Fill in the next few lines.

Amount owed	Payment	Interest	Payment – Interest	Remaining loan
1000	168.62	3.33	165.29	834.71
834.71	168.62	2.78	165.84	668.87

After six months, the remaining loan should be \$0.

Example 10

Jack receives a loan of \$5000 from a bank at an annual interest rate of 7.5% compounded monthly. It is to be repaid in monthly installments within a five-year period.

a Determine the monthly installments in order to repay the loan on time.

b Jack starts repaying the \$5000 loan with the monthly installments calculated in part **a**. Calculate the amount he still owes after the 10th installment.

a N = 60 as the number of installments is 60 I% = 7.5 ie the annual interest rate PV = −5000, the loan received PMT = blank as you are looking for it FV = 0, to totally repay the loan $P_pY = 12$, for 12 payments per year $C_pY = 12$ The GDC gives the amount of the monthly installments (PMT) to be \$100.19.	Use your GDC to find the PMT (ie the required monthly installments). The number of repayments will be $12 \times 5 = 60$
b PM2 = 10, which is the month of the last payment I% = 7.5 ie the annual interest rate PV = −5000, the loan received PMT = 100.189743 (found in part **a**) $P_pY = 12$, for 12 payments per year $C_pY = 12$ The GDC gives the amount of amount of principal that remains to be repaid (BAL) to be \$4290.89.	Use your GDC to find the BAL (ie the balance on the debt of a loan received).

Exercise 7F

1 Brandt receives a loan of \$10 000 from a bank at an annual interest rate of 6% compounded monthly to be repaid in monthly installments within a 10-year period.

a Determine the monthly installments in order to repay the loan on time.

b Find how much she still owes after the fifth year.

2 Spyro takes out a loan of 40 000 AED from a bank at an annual interest rate of 6.3% compounded quarterly to be repaid in quarterly installments within a six-year period.

a Determine the monthly installments in order to repay the loan on time.

b Find how much he still owes after the end of the 3rd year.

3 Maude decides to invest in a 25-year private pension scheme, where she will have to deposit TRY1000 every month. The rate of interest is fixed at 8% per annum. Interest is compounded monthly.

a Determine the future value of her investment at the end of the 25 years.

After the 25-year period, when she will have retired, she will receive (according to the pension scheme) a monthly pension of TRY1200.

b Calculate how long it will be until she breaks even with the amount that she invested?

4 Donny received a loan of €11,000 at an interest rate of 5.19% compounded monthly. He agreed to pay back the loan in 72 equal monthly instalments over the next six years.

a Determine the amount of each one of the 72 equal monthly installments.

Five years after he started repaying his loan, Donny decided to repay the rest in a final single installment.

b Determine the amount still owing at the end of the five years.

5 Larry wants to make a decision on a six-year amortized loan for him to buy a new car. The value of the car is \$33 560. The annual interest rate of the loan is 14.06% compounded monthly. He can afford to pay up to \$700 monthly in order to repay the loan. Determine if it is enough for him to agree on this contract.

Developing inquiry skills

- Scheme A: Investing 10 000 MAD at a simple interest rate of 5% per annum for 18 years.
- Scheme B: Investing 10 000 MAD at a compound interest rate of 4% per annum compounded annually for 18 years.
- Scheme C: Investing 10 000 MAD at a compound interest rate of 4% per annum compounded monthly for 18 years.
- Scheme D: Investing 10 000 MAD and getting back double this amount at the end of the 18 years.

Which scheme is the most rewarding?

TOK

"A government's ability to raise and lower short-term interest rates is its primary control over the economy" – Alex Berenson

How can knowledge of mathematics result in individuals being exploited or protected from extortion?

7.3 Exponential functions and models

The marathon is a 42 kilometre race that has been held in the Olympics since 1896. Originally, it was run by a Greek messenger reporting the victory of the Battle of Marathon to Athens. The winning time has dropped considerably over the years: from around 3 hours down to almost 2 hours.

Can you predict the shortest time in which an athlete will ever be able to run a Marathon?

Investigation 8

Helen is examining the rate of growth of the bacteria lactobacillus acidophilus. At the beginning of her experiment she introduced 200 bacteria into a petri dish. Under a fixed environment she measured that after one hour the bacteria had doubled. This rate of growth remained the same as the bacteria doubled in number again within the next hour.

a Copy and complete the table below showing the number of bacteria at the specified number of hours after the beginning of the experiment

Hours after	0	1	2	3	4
Number of bacteria	200				

b What sequence does the trend of the number of bacteria follow?

c What is the formula of the nth term of the above sequence of numbers?

Assume that we now want to determine the number of bacteria 1.5 hours after the experiment begun.

d Why is a sequence insufficient to determine this value?

If the same relation is used with the independent variable being any positive real number it now becomes a function.

e Use this function in order to determine the number of bacteria present after 1.5 hours.

f Sketch the graph of the function representing the number of bacteria.

g What do you notice? What is the domain and range of this function?

The function found in Investigation 8 is an example of an exponential function, one in which the independent variable is the **exponent** of a number, called the **base**.

The basic exponential function is of the form $f(x) = a^x$, $a > 0$. Investigation 9 looks at some properties of all such exponential functions.

Investigation 9

1. Use technology to sketch the graphs of these functions.
   a $f(x) = 2^x$ **b** $f(x) = 3^x$ **c** $f(x) = 0.5^x$
   d $f(x) = 0.2^x$ **e** $f(x) = 2^{-x}$
2. Explain why the exponential equation $f(x) = \left(\frac{1}{2}\right)^x$ can be written as $f(x) = 2^{-x}$.
3. Give two different conditions on the base and the exponent for an exponential function to be a decreasing function.
4. How are these functions different from power functions?
5. How can you describe the shape of these graphs? Do they have any asymptotes? Are they increasing or decreasing?
6. How are they the same and how are they different?
7. What happens as x tends to $+\infty$ and $-\infty$?
8. How many zeros do these functions have and why?
9. Could you sketch the graph of the function $f(x) = (-2)^x$? If not, why?
10. **Conceptual** Describe the main features of the graphs of all exponential functions of the form $f(x) = a^x$, $a > 0$.

International-mindedness

The term "power" was used by the Greek mathematician Euclid for the square of a line.

The independent or input variable is the exponent.

The general equation is of the form $f(x) = a \times b^x + c$, $b > 0$, $a \neq 0$.

The simplest form of an exponential function is $f(x) = b^x$, $b > 0$.

Since any positive power raised to any power is still a positive number, the range of these functions is $f(x) > 0$.

International-mindedness

The word "exponent" was first used in the 16th century by German monk and mathematician Michael Stifel. Samuel Jeake introduced the term "indices" in 1696.

Number and algebra

Functions

All exponential functions have a **horizontal asymptote**.

Exponential functions of the form $f(x) = b^x$ have the line $y = 0$ as a horizontal asymptote. Unlike inverse proportion functions, the asymptotic behavior occurs either as $x \to \infty$ or as $x \to -\infty$, but not both.

The general form $f(x) = ab^x + c$, has a horizontal asymptote at $y = c$ since the exponential term ab^x can never become equal to zero.

Investigation 10

1 Consider the transformation of the graph of $y = 2^x$ onto the graph of each of the following functions.

For each function:

i state the sequence of transformations which maps $y = 2^x$ onto that function

ii sketch the graph of the new function

iii write down the equation of the asymptote

iv write down the value of the y-intercept.

a $y = 3 \times 2^x$ b $y = 2^x - 3$ c $y = 2^{-x}$ d $y = -2^x$

e $y = 3 \times 2^x - 4$ f $y = 2^{-x} + 4$

2 Describe the key features of the function $f(x) = ka^x + c$ for $a > 1$.

3 State the difference in the curve when $0 < a < 1$ or when the function is rewritten as $f(x) = ka^{-x} + c$, $a > 1$.

4 Write the following two functions in the form $f(x) = ka^x + c$.

a $f(x) = 3^{x+2} + 5$ b $f(x) = 3^{2x} + 7$

5 Hence explain why all exponential functions of the form $f(x) = k_1 a_1^{bx+d} + c$ can be written in the form $f(x) = ka^x + c$ and state the values of k and a in terms of k_1 and a_1.

6 **Conceptual** How do the parameters of the exponential function affect the graph of the function?

The exponential function $f(x) = ka^x + c$ has the following properties:

The straight line $y = c$ is a horizontal asymptote .

It crosses the y-axis at the point $(0, k + c)$.

Is increasing for $a > 1$ and decreasing for $0 < a < 1$.

The exponential function $f(x) = ka^{-x} + c$ has the following properties:

The straight line $y = c$ is a horizontal asymptote as $x \to \infty$.

It crosses the y-axis at the point $(0, k + c)$

Is decreasing for $a > 1$ (exponential decay)

Functions of the form $f(x) = k_1 a_1^{bx+d} + c$ can also be written in the form $f(x) = ka^x + c$ where $k = k_1 \times a_1^d$ and $a = a_1^b$. Either form is acceptable, and the choice will often depend on the context.

Example 11

For each of the following functions:

i find the equation of the horizontal asymptote

ii find the coordinates of the point where the curve cuts the y-axis.

iii state if the function is increasing or decreasing.

a $f(x) = 3^x + 2$

b $f(x) = 5 \times 0.2^x - 3$

a i The horizontal asymptote has equation $y = 2$.	2 is the value for c.
ii The curve cuts the y-axis at the point $(0, 3)$.	$k = 1$ and $c = 2$, so, $1 + 2 = 3$
iii The function is increasing because 3 is greater than 1.	
b i The horizontal asymptote has equation $y = -3$.	-3 is the value for c.
ii The curve cuts the y-axis at the point $(0, 2)$.	$k = 5$ and $c = -3$, so $5 - 3 = 2$
iii The function is decreasing because 0.2 lies between 0 and 1.	

Exercise 7G

1 For the graphs of the following equations:

i find the y-intercept

ii find the equation of the horizontal asymptote

iii state whether the function shows growth (increasing) or decay (decreasing).

a $f(x) = 4^x + 1$ b $f(x) = 0.2^x - 3$

c $f(x) = 5^{2x}$ d $f(x) = 3^{0.1x} + 2$

e $f(x) = 3(2)^x - 5$ f $f(x) = 4(0.3)^{2x} + 3$

g $f(x) = 5(2)^{0.5x} - 1$ h $f(x) = 2(2.5)^{-x} - 1$

2 Write the following functions in the form $f(x) = ka^x + c$.

a $f(x) = 2(4)^{2x} + 5$

b $f(x) = 7(0.5)^{-3x} + 2$

3 John is a photographer and he is practicing the 19th technique called the "Collodion process" producing ambrotypes. In this process he has to be very precise with the timing of the use of the various chemicals.

In order to perfect his technique, he created several mathematical models relating the time that he applies the various chemicals to the sharpness of his images.

The model function that he created for Ethyl ether is $S(t) = 12 + 10 \times 1.2^{-t}$.

a Find the initial value of the sharpness.

b Find how long it will take for the sharpness to drop to 15.

For another chemical Ethyl alcohol, the sharpness is 2 units higher than with Ethyl ether.

c Use graph transformations to determine the function for Ethyl alcohol.

For another chemical, nitrocellulose the sharpness is the same as with Ethyl ether but takes double the time.

d Use graph transformations to determine the function for nitrocellulose.

For a final chemical that he uses, iodide, the sharpness is double that of Ethyl ether.

e Use graph transformations to determine the function for iodide.

TOK

The phrase "exponential growth" is used popularly to describe a number of phenomena.

Do you think that using mathematical language can distort understanding?

Investigation 11

A bank is offering to pay 100% interest per year on a special account.

1 If the initial investment is \$$P$ find in terms of P the value of the investment after one year if the interest is compounded

a yearly **b** monthly **c** weekly **d** daily.

2 Comment on the values obtained.

3 If k is the number of compounding periods explain why the value of the investment after one year is $P\left(1+\frac{1}{k}\right)^k$

4 Use your GDC or other technology to investigate $\lim\limits_{k\to\infty}\left(1+\frac{1}{k}\right)^k$

The limit is an irrational number which is denoted by the letter e. Like π it is a number that appears often in many unexpected places. In the sciences most exponential equations use e as their base when the growth is continuous.

5 Suppose now that the interest rate is R%. Let $r=\frac{R}{100}$

a Show that the value of the investment after one year is $P\left(1+\frac{r}{k}\right)^k$

b By writing $\left(1+\frac{r}{k}\right)^k$ as $\left(1+\frac{1}{\left(\frac{k}{r}\right)}\right)^k$ show that the $\lim\limits_{k\to\infty}\left(1+\frac{r}{k}\right)^k=e^r$

If a quantity is growing at a rate of r per time period, where r is the proportion of the original added on rather than the final amount (eg the interest rate in our example), then after one time period it will have increased by a factor of e^r. After t time periods it will have increased by a factor of $\left(e^r\right)^t=e^{rt}$

6 **Conceptual** What is the connection between e and rate of growth of a quantity?

Exercise 7H

1 Determine the missing entries in the following table with the value of each function at the specified x-values:

		y-intercept	Horizontal Asymptote	Growth or Decay	Range
a	$f(x)=e^x+3$				
b	$f(x)=2e^{-x}+4$				
c	$f(x)=2e^{0.3x}-2$				
d	$f(x)=5-2e^{-3x}$				

2 Sketch the graph of each of the functions from Question 1, clearly showing all the information that you found.

3 A cup is filled with hot water and left to stand. The temperature T of the water after t minutes have elapsed is modelled by $T(t)=24.5+55.9\,e^{-0.0269t}$, for $t\geq 0$.

a Determine the y-intercept of the graph of function.

b State a contextual interpretation of the y-intercept.

c State whether the function displays growth or decay.

d State how the growth or decay relates to the cooling of the water.

e Determine the horizontal asymptote of the function.

f State how the horizontal asymptote relates to the physics of the cooling of the water.

g Determine the range of the function in the given domain and explain its significance.

h Sketch the graph of the function.

4 Find the value of a if the following piecewise functions are continuous.

a $f(x)=\begin{cases}ax^3 & x<3\\ 2e^x & x\geq 3\end{cases}$

b $f(x)=\begin{cases}ax^2+8 & x<4\\ 2^{x+1} & x\geq 4\end{cases}$

Exponential modelling

Example 12

The function $M(t)=85.7\times 0.966^t$ models the amount (M) in grams of a radioactive material t years from its production.

a Determine the original mass of the radioactive material

b Determine the mass of the radioactive material after one decade?

c Calculate the complete number of years it would take for the radioactive material to reduce below 55 grams.

d Determine the half-life of the material

a At $t=0 \Rightarrow M(0)=85.7$ grams **b** At $t=10 \Rightarrow M(10)=60.6$ grams **c** $M(t)<55 \Rightarrow t>12.8 \Rightarrow t=13$ years **d** $M(t)=\frac{85.7}{2} \Rightarrow t=20$ years	All these values should be found directly from the GDC.

HINT

The half-life of a material is how long it takes for the material to reduce to half its original amount.

Example 13

The function that determines the value of an investment in a fund is $V(t) = V_0\,e^{kt}$, where t is measured in years.

a Given that the initial amount in the account was €5000, determine the value of V_0.

b The amount in the account increased by 25% after 6 years. Determine the value of k correct to 3 significant figures.

c Using the parameters found in parts **a** and **b**, determine the value of the investment after exactly eight and a half years.

d Calculate the number of complete years for the investment to double.

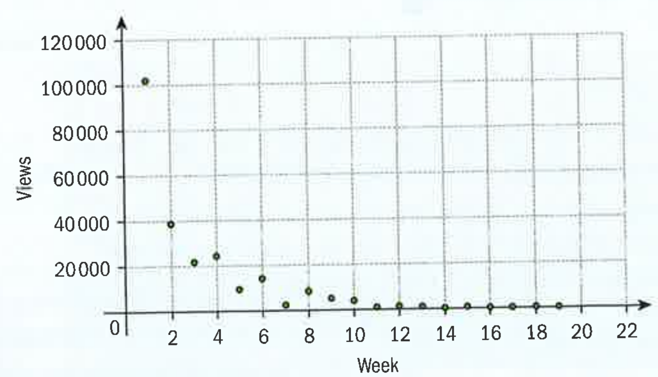
a $5000 = V_0\,e^{k0} \Rightarrow V_0 = €5000$	Using $t = 0$ and $V = €5000$
b $6250 = 5000\,e^{k6} \Rightarrow k = 0.0372$	Using $t = 6$ and $V = 5000 \times 1.25 = €6250$
c $V(8.5) = 5000e^{0.0372 \times 8} = €6859.52$	
d $5000\,e^{0.0372t} > 10000 \Rightarrow t > 18.6$ $\Rightarrow$ the investment will double after 19 complete years.	As a whole number of years is being looked for the value 19 can also be found directly from the table function of a GDC without the need to find 18.6.

Example 14

A rock band the VKs published a new video on their YouTube channel. The weekly views up to and including the 20th week of publication were as follows:

Week	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Views	102 365	38 716	21 617	24 305	9321	14 148	2103	8285	5098	3777
Week	11th	12th	13th	14th	15th	16th	17th	18th	19th	20th
Views	831	1007	834	34	378	204	6	42	54	31

A scatter plot of the data is shown below:



a Explain why an exponential model would be suitable to model this set of data.

b Use exponential regression to determine the best fit exponential model.

c Determine R^2 the coefficient of determination of the model function.

d Sketch a graph showing the data points with the model function.

e For their next video, the band wants to start advertising it as soon as it falls below 1000 views per week. Assuming it follows the same pattern as the first video, predict after how many **days** they should do so.

a The data shows the sales decaying at a diminishing rate continuous decay, which is a major characteristic of exponential decay functions.

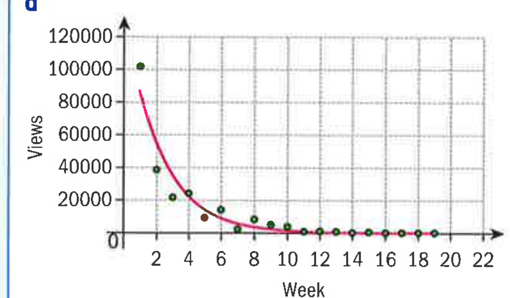
Also the data have a physical horizontal asymptote as the number of views can never fall below zero.

b The exponential function that models the data is $f(x) = 129\,000 \times 0.642^x$.

Exponential regression will give the function in the form abx. The next section will explain how to convert to base e if required.

c $R^2 = 0.888$

d



e $129\,000 \times 0.642^x = 1000$

$\Rightarrow x = 10.97$ weeks.

$\Rightarrow d = 7 \times 10.97 = 76.8$ days.

They have to start advertising their video again after 77 days.

Exercise 7I

1 The number of bacteria, n, in a dish, after t minutes is given by $n = 8e^{kt}$.

a Given that the number of bacteria after 23 minutes is 252, determine the value of k giving your answer to two decimal places.

b Find the value of n after one hour giving your answer to the nearest integer.

c After T minutes, the value of n becomes greater than 100 000. Find the least value of T, where $T \in \mathbb{Z}$.

2 The mass m in kg of a radioactive substance at time t hours is given by $m(t) = 952e^{-\frac{t}{3}}$.

a Find the initial mass of the radioactive substance.

b Find the amount of radioactive substance after 2.5 hours.

c Find the half-life of the radioactive substance.

d Find how long would it take for the radioactive substance to reduce to 10% of its original mass.

3 Consider the function $f(x) = 3^{\frac{x}{2}} - 3$.

a Determine the equation of the horizontal asymptote.

b Sketch the graph of $f(x)$. In your graph, show clearly the horizontal asymptote and the exact coordinates of any intersections with the coordinate axes.

c State clearly the domain and range of the inverse function.

4 A tank initially contains 5000 ml of water. At $t = 0$, a tap is opened, and water starts flowing out of the tank. The volume of liquid, V ml, which remains in the tank after t minutes is given by the function $V(t) = ae^{bt}$.

a Determine the exact value of a.

b Given that the volume after 35 seconds is 151 ml, find the value of the constant b, giving your answer correct to 1 decimal place.

c After T minutes, the volume of water becomes less than 100 ml. Find the least value of T, where $T \in \mathbb{Z}$.

5 Jaroslav is trying to determine a pattern for the real estate prices in his area for similar houses to his. After collecting monthly data for the past year, he displayed them in the following table (1 accounting for the 1st month he collected data and so on).

Month	1st	2nd	3rd	4th	5th	6th
Value (CZK) millions	7.20	7.27	7.31	7.22	7.35	7.38
Month	7th	8th	9th	10th	11th	12th
Value (CZK) millions	7.24	7.27	7.47	7.47	7.33	7.47

a Plot the data on an appropriate set of axes.

b Explain why an exponential model would be suitable to model this set of data.

c Determine the best fit exponential model.

d Determine R^2 the coefficient of determination of the model function.

e Sketch a graph showing the data points with the model function.

Jaroslav wants to sell his house when its price reaches 9 million CZK.

f Assuming that the values of the houses in this area keep on following the same pattern, find how long he will need to wait until he sells his house.

6 The health officer in a new district swimming pool is responsible for maintaining the bacterial levels in the water within health and safety specifications. Specifically, the heterotrophic plate count (HPC) should not exceed 100 colonies per 1 milliliter of water. To test the swimming pool, he took a sample of water every day since he first put in the disinfectant. The data are shown in the following table:

Day	1	2	3	4	5	6	7	8	9	10
HPC	3.5	2.2	16.8	11.5	17.5	41.0	73.9	114.1	181.7	279.4

a Plot the data on an appropriate set of axes on your GDC.

b Explain why an exponential model would be suitable to model this set of data.

c Use exponential regression to determine the best fit exponential model.

d Determine R^2 the coefficient of determination of the model function.

e Plot a graph showing the data points with the model function.

The health officer wants to prevent the bacterial levels even reaching close to the safety levels and has thus decided to introduce disinfectants whenever the level of HPC reaches half the safety levels. In order not to take measurements every day, he decided to follow the model calculated above.

f Find how often, in days, he should disinfect the swimming pool for HPC

7 The number of bacteria, n, in a petri dish, after t minutes is given by $n = 8e^{kt}$.

a Given that the number of bacteria after 23 minutes is 252, determine the value of k giving your answer to two decimal places.

b Find the value of n after one hour giving your answer to the nearest integer.

c After T minutes, the value of n becomes greater than 1000. Find the least value of T, where $T \in \mathbb{Z}$.

8 Initially a tank contains 5000 ml of liquid. At the time $t = 0$ seconds a tap is opened, and liquid then flows out of the tank. The volume of liquid, V ml, which remains in the tank after t minutes is given by $V = ae^{bt}$.

a Determine the value of a.

b Given that the volume after 35 minutes is 151 ml, find the value of the constant b, giving your answer correct to 1 decimal place.

c After T minutes, the volume of liquid becomes less than 100 ml. Find the least value of T, where $T \in \mathbb{Z}$.

9 The number of bacteria in a culture N, varies according to the function $N = 1000 \times 2^{k \times t}$, for $t \geq 0$, where t is measured in days and k is a constant.

a Find the initial number of bacteria.

b If the number of bacteria is 4000 after 4 days find the exact value of k.

c Find the exact number of bacteria after 8 days.

d Determine how long it will take for the number of bacteria to grow to 32 000.

Developing inquiry skills

Now return to the question posed at the start of this section. The following table shows what fraction of one hour was shaved off the original men's world record for running a marathon at various years after the original record was set.

Time since first record (years)	0	5	25	45	50	56	70	80	100	110
Reduction in world record time (hours)	0.98	0.74	0.56	0.41	0.35	0.33	0.25	0.21	0.17	0.13

a What sequence do the "Reduction in world record times" approximately follow?

b What would be the formula of the nth term of these times?

You want to determine the record time 63.2 years since records were first recorded.

c Why is a sequence insufficient to determine this value?

If the same relation is used with the independent variable being any positive real number, it now becomes a function.

d Use this function in order to determine the record time 63.2 years since records were first recorded.

e Sketch the graph of the function representing the record time since records were first recorded.

f What do you notice? What is the domain and range of this function?

g Can you predict the shortest time in which an athlete will ever be able to run a marathon?

h What information will you need and what assumptions will you make?

i How will you choose an appropriate model? Will it be different for men and women?

7.4 Laws of exponents – laws of logarithms

Investigation 12

a Consider the exponential equation $2^x = 8$. How could you describe the solution in words? What is the exact solution?

b Consider the exponential equation $2^x = 5$. How could you describe the solution in words? Why can't you determine the exact numerical solution without the use of technology? Sketch the graph $f(x) = 2^x$ and the horizontal line $y = 5$ in order to solve the equation.

c Consider the exponential equation $2^x = -2$. Does this equation have a solution?

d Consider the equation $a^x = b$. Can you describe the solution in words?

e Do all exponential equations have a solution? How can you find the solution to an exponential equation?

Some mathematical equations do not have an exact numerical solution. For this reason, mathematicians have "invented" symbols in order to describe the exact solution.

In general, the solution of the exponential equation $a^x = b$ is the **logarithmic function** $x = \log_a b$, where $a, b > 0$ and $a \neq 1$.

Here a is called the **base** of the logarithm. Any positive number can be used as a base for logarithms, but the only ones you need to know about for this course are 10 and Euler's number e.

You can write $\log_{10} x$ as $\log x$.

You can write $\log_e x$ as the **natural logarithm** $\ln x$.

Some examples follow.

Two fundamental exponential equations with their solutions are:

- $10^x = c \Rightarrow x = \log c$
- $e^x = c \Rightarrow x = \ln c$

and two fundamental logarithmic equations are:

- $\log x = c \Rightarrow x = 10^c$
- $\ln x = c \Rightarrow x = e^c$

On your GDC, $\log_{10}$ is called "LOG" and $\log_e$ is called "LN".

Example 15

Find the exact value of x for each of the following equations.

1 $10^x = 5$ **2** $e^{2x} = 12$ **3** $\log x = 3$ **4** $3\ln x = 7$

1 $x = \log 5$	By the definition of log or by using the result above.
2 $2x = \ln 12 \Rightarrow x = \frac{1}{2}\ln 12$	
3 $x = 10^3 = 1000$	
4 $\ln x = \frac{7}{3} \Rightarrow x = e^{\frac{7}{3}}$	It is important to isolate the log term before applying the rule.

Investigation 13

1 Consider the equations $10^x = 1$ and $e^x = 1$ or even the general case $a^x = 1$.
Find the value of x.
What is the solution in terms of logs?

2 Consider the equations $10^x = 10$ and $e^x = e$ or even the general case $a^x = a$.
Find the value of x.
What is the solution in terms of logs?

3 Consider the equations $10^x = 10^n$ and $e^x = e^n$ or even the general case $a^x = a^n$, where x is the unknown variable and n is a constant parameter.
Find the value of x.
What is the solution in terms of logs?

4 Use your GDC to copy and complete the following table, giving your answers to three significant figures.

log 2		log 3		log 6	
log 3		log 4		log 12	
ln 5		ln 7		ln 35	

What do you notice? What can you conjecture about $\log_a x + \log_a y$?

5 Use your GDC to copy and complete the following table, giving your answers to three significant figures.

log 12		log 2		log 6	
log 15		log 3		log 5	
ln 11		ln 7		$\ln \frac{11}{7}$	

What do you notice? What can you conjecture about $\log_a x - \log_a y$?

Continued on next page

International-mindedness

Logarithms do not have units but many measurements use a log scale such as earthquakes, the pH scale and human hearing.

Number and algebra

Functions

6 Use your GDC to copy and complete the following table, giving your answers to three significant figures.

$\log(3^2)$		$\log 3$		$2\log 3$	
$\ln\sqrt{2}$		$\ln 2$		$\frac{1}{2}\log 2$	

What do you notice? What can you conjecture about $\log_a(x^n)$?

$\log 1 = 0, \ln 1 = 0$ and in general, $\log_a 1 = 0$
$\log 10 = 1, \ln e = 1$ and in general, $\log_a a = 1$
$\log 10^n = n, \ln e^n = n$ and in general, $\log_a a^n = n$

Laws of logarithms:

- $\log_a x + \log_a y = \log_a xy$
- $\log_a x - \log_a y = \log_a \frac{x}{y}$
- $\log_a (x)^n = n\log_a x$

Example 16

Write each of the following as a single logarithm:

a $3\log x$ **b** $\frac{\log x}{2}$ **c** $2\log x + \log y$ **d** $\log x - 2\log y$
e $-\ln x$ **f** $1 + \ln x$ **g** $\ln x + \ln y - \ln z$

a $3\log x = \log(x^3)$

b $\frac{\log x}{2} = \frac{1}{2}\log x = \log\left(x^{\frac{1}{2}}\right) = \log\sqrt{x}$

c $2\log x + \log y = \log(x^2) + \log y = \log(x^2 y)$

d $\log x - 2\log y = \log x - \log(y^2) = \log\left(\frac{x}{y^2}\right)$

e $-\ln x = -1\ln x = \ln(x^{-1}) = \ln\left(\frac{1}{x}\right)$

f $1 + \ln x = \ln e + \ln x = \ln(ex)$

g $\ln x + \ln y - \ln z = \ln(xy) - \ln z = \ln\left(\frac{xy}{z}\right)$

Exercise 7J

1 Find the value of each of the following logarithms.

a $\log 100$ **b** $\log 0.1$ **c** $\ln e$
d $\ln\frac{1}{e}$ **e** $\ln e^2$ **f** $\log\sqrt{10}$
g $\ln\left(\frac{1}{\sqrt{e}}\right)$

2 Find the solution to the following exponential equations giving the answer as a logarithm where appropriate.

a $10^x = 10$ **b** $10^x = 100$
c $10^x = 38$ **d** $e^x = e^2$
e $e^x = 3$ **f** $e^x = 0.3$
g $e^x = 1$

3 Write each of the following as a single logarithm:

a $2\log x$ **b** $\frac{\log x}{3}$
c $3\log x + 2\log y$ **d** $\log x - 3\log y$
e $-2\ln x$ **f** $2 + \log x$
g $\ln x - \ln y - \ln z$

4 Find the exact value of a if the following piecewise functions are continuous for $x > 0$

a $f(x) = \begin{cases} 4x^3 - 3 & x \le 1 \\ 2e^{ax} & x > 1 \end{cases}$

b $f(x) = \begin{cases} 3x^2 - 4 & x \le 2 \\ 2\ln ax & x > 2 \end{cases}$

TOK

The phrase "exponential growth" is used popularly to describe a number of phenomena.

Do you think that using mathematical language can distort understanding?

Investigation 14

1 **a** Verify that if $f(x) = \log x$ and $g(x) = 10^x$ then $f \circ g(x) = g \circ f(x) = x$.

b Hence state the connection between the two functions f and g.

2 Use the result of **1 a** to sketch the graphs $y = 10^x$ and $y = \log x$ on the same axes.

3 State the domain and range of $f(x) = \log x$.

4 **a** Find the inverse of $h(x) = e^x + 2$.

b Sketch the graph of $y = h(x)$ and its inverse on the same axes.

5 State the domain and range of h and h^{-1}.

6 **Factual** What are the domain and range of the function $f(x) = \log_a x$?

7 **Conceptual** What is the relationship between the logarithmic function and the exponential function?

The fuctions $y = 10^x$ and $y = \log x$ are inverse functions and so their composition gives x.
Similarly, the same thing holds for the functions $y = e^x$ and $y = \ln x$.
$10^{\log x} = x$ and $e^{\ln x} = x$

Exercise 7K

1 Solve the following exponential equations using logarithms. Give your answers as a logarithm.

a i $10^x = 3$, ii $\frac{10^x}{5} = 15$ iii $e^x = 5$

b i $3e^x - 1 = 14$, ii $\frac{10^x}{2} + 3 = 5$,

iii $3(e^x - 1) = 5$

2 Solve the following exponential equations using logarithms, given that all parameters are positive real numbers and $a \neq 1$.

a $a^x - c = 0$ b $be^x = 2$, c $2\times 10^x = k$

3 Simplify the following:

a $10^{3\log x}$ b $e^{\ln x - \ln y}$

c $10^{2\log x - \log y}$ d $e^{-2\ln x}$

4 Given the functions $f(x) = \ln(x + 2)$ and $g(x) = 2e^x$

a Find $(g \circ f)(x)$, giving your answer in the form $ax + b$.

b For the function $f(x)$, determine the equation of any possible asymptotes and the exact coordinates of any possible intersections with the coordinate axes.

c Hence, sketch the curve of $f(x)$.

d Determine the inverse function of $f(x)$and sketch it on the same set of coordinate axes, indicating any possible asymptotes and intercepts with the coordinate axes.

5 Consider the function $f(x) = 10^x - 3$.

a Find its inverse function $f^{-1}(x)$.

b Sketch the graph of $f^{-1}(x)$. On your graph, show clearly any possible asymptotes with their equations and the exact coordinates of any possible intersections with the coordinate axes.

c State clearly the domain and range of the inverse function $f^{-1}(x)$.

6 Consider the function $f(x) = 2e^x - 6$.

a Sketch the graph of $f(x)$. On your graph, show clearly any possible asymptotes with their equations and the exact coordinates of any possible intersections with the coordinate axes.

b Find its inverse function $f^{-1}(x)$.

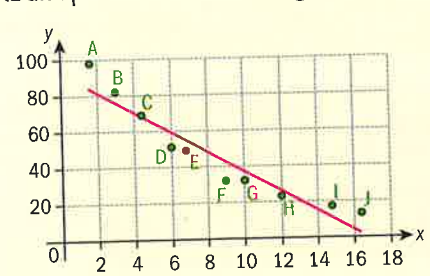
c State clearly the domain and range of the inverse function $f^{-1}(x)$.

Investigation 15

Kim recorded the battery remainder indication on her laptop during the course of a day with respect to the time of the day and gathered the following data, where t is the time in hours since she started the experiment and B is the battery left on her laptop.

t	1.6	3	4.4	6	6.8	9	10	12	14.8	16.4
B	96	79	65	52	47	34	30	23	15	12

She plotted the data and produced the following scatter diagram:



1 Find the PMCC for this data.

The scatter diagram she created did not have strong enough linear correlation for her to conclude that the relation between her variables is linear.

Her physics teacher suggested that in fact the relationship was more likely to be exponential of the form $B = ae^{ct}$ and suggested she draw a semi-log graph of $\ln B$ against t.

2 By taking the natural log of both sides of the equation, explain why a graph of $\ln B$ against t for this function would be a straight line and give expressions for of the gradient and the y-intercept of this line.

3 a Copy and completed the table below

t	1.6	3	4.4	6	6.8	9	10	12	14.8	16.4
$\ln B$										

b Create a scatter diagram based on the table in part **a** with t on the horizontal axis and $\ln B$ on the vertical axis. This is called a semi-log graph—one varible plotted against the logarithm of the other variable.

c Find the PMCC for this data and compare your result with the one for a linear model.

d Use your values for the gradient and for the y-intercept to estimate the parameters for the equation $B = ae^{ct}$

Her chemistry teacher suggested that in fact the relationship was more likely to be of the form $B = at^c$ and suggested she draw a log-log graph of $\ln B$ against $\ln t$.

4 By taking the natural log of both sides of the equation, explain why a graph of $\ln B$ against $\ln t$ for this function would be a straight line and give an expression for the gradient of the line and for the y-intercept.

5 a Copy and complete the table below

$\ln t$										
$\ln B$										

b Create a scatter diagram based on the table in part **a** with $\ln t$ on the horizontal axis and $\ln B$ on the vertical axis. This is called a **log-log graph** – the logarithm of one variable plotted against the logarithm of the other variable.

c Find the PMCC for this data and compare your result with the previous two models.

d Use your values for the gradient and for the y-intercept to estimate the parameters for the equation $B = at^c$.

6 By considering the value of B when $t = 0$ which of the two non-linear models might you prefer irrespective of the values of the PMCC?

7 **Conceptual** How does logarithmic scaling of graphs help you identify the relationship between variables?

TOK

"One reason why mathematics enjoys special esteem, above all other sciences, is that its propositions are absolutely certain and indisputable"–Albert Einstein.

How can mathematics, being after all a product of human thought which is independent of experience, be appropriate to the objects of the real world?

A power function of the form $y = ax^b$ can be **linearized** as $\ln y = b\ln x + \ln a$.

A graph of $\ln y$ against $\ln x$ will have a gradient of b and a y-intercept of $\ln a$.

An exponential function of the form $y = ae^{bx}$ can be **linearized** as $\ln y = bx + \ln a$.

A graph of $\ln y$ against x will have a gradient of b and a y-intercept of $\ln a$.

EXAM HINT

In most cases the parameters for the function can also be found by using either power or exponential regression on your GDC. But if instructed to find them using linearization techniques you must show sufficient method to demonstrate you have found them this way.

Some GDCs will give the exponential regression function in the form $y = ab^x$. This can be converted into base e by writing $b = e^r \Rightarrow r = \ln b$ and hence $y = a(e^r)^x = ae^{rx}$ or $y = ae^{x\ln b}$.

Investigation 16

a Sketch the following data on a scatter diagram.

x	3	5	7	9	11	13	15	17
y	95	1071	9812	103 201	1 100 458	9 526 983	107 247 384	993 485 029

b Write down the domain and the range of the data. What do you notice? What is the problem with viewing this scatter diagram?

c Complete the following table to include the logarithm of the y-values.

y	95	1071	9812	103 201	1 100 458	9 526 983	107 247 384	993 485 029
$\log y$								

d Sketch the following data on a scatter diagram.

x	3	5	7	9	11	13	15	17
$\log y$								

e Write down the domain and the range of the new set of data. What do you notice? How was the problem resolved?

Exercise 7L

1 In a set of data, for reasons of manageable scaling the logarithm (in base 10) of the measurements have been found and recorded. The new set of data is as follows.

Value	A	B	C	D	E	F	G	H	I	J
$\log y$	1.7	3.2	4.1	4.7	5.2	6	6.6	7.2	8	10.1

a Find the value of y for measurement B and measurement G. How many times is G greater than B?

b State the range of values in this set of data

c Comment on the advantages of using a log scaling of the original data

2 Data linking the life expectancy in countries of the European Union to their gross national income (GNI) index are shown below.

Country	AUT	BEL	BGR	HRV	CZE	CNK	EST	FIN	FRA
GNI	42 080	39 270	13 980	19 330	24 280	42 300	20 830	38 500	35 650
L.E.	80.7	80.2	74.2	77.1	77.8	79.9	76.3	80.3	81.6
Country	DEU	GRC	HUN	IRL	ITA	LVA	LTU	LUX	NLD
GNI	39 970	26 090	20 260	33 230	32 710	17 820	19 690	64 410	43 260
L.E.	80.5	80.5	75	80.6	82	74.6	74.3	81.5	80.9
Country	POL	PRT	ROU	SVK	SVN	ESP	SWE	GBR	
GNI	20 480	24 480	15 140	22 230	26 960	31 660	42 200	35 940	
L.E.	76.5	80.2	74.5	76.1	79.9	82	81.7	80.5	

a Calculate the log of both sets of data and include your results in an enlarged table.

Let the value of GNI be x and the value of L.E. be y.

It is thought that the relation ship between the two variables is of the form $y = ax^b$

b Find the line of best fit for $\log y$ against $\log x$.

c Hence find estimates for a and b.

d Use the power regression function on the GDC to verify your values of a and b.

3 Frances suspects that the time taken (t minutes) to answer one question in a multiple-choice quiz increases with the total time (q minutes) that she has spent working on the quiz.

She times herself over different periods and collects the following data.

Total time working on the quiz, q (minutes)	5	20	30	40	50
Time taken to answer a question, t (minutes)	2.4	2.8	3.4	4.0	5.1

Frances thinks t and q are related by an equation of the form $q = ae^{bt}$.

a Find the values of ln q for the values of q given in the table.

b Plot a graph of ln q against t.

c Either by measuring values from your graph, or by using the linear regression function on your GDC, find the values of b and ln a.

d Hence, suggest a suitable model for the time taken to answer questions.

e Use the model to estimate how long it would take Frances to answer a question after working on the quiz for one hour in total.

4 Yenni owns a small factory that manufactures shirts. She wants to model the relationship between the daily total cost of manufacturing shirts ($\$s$) and the number of shirts that are produced (n).

She believes the model is of the form $s = an^b + c$

She knows that, even if she were making no shirts, the daily cost of running the factory would be \$1000.

a Find the value of c.

Yenni collects data and estimates the following values

n	50	100	150	200
s	2000	2800	3400	3800

b Write down a linear equation of $\ln(s - c)$ against $\ln n$.

c Plot a graph of $\ln(s - c)$ against $\ln n$.

d Find values for b and $\ln a$ using your graph or linear regression.

e Hence find a model for the cost of producing the shirts.

f Yenni would like to produce 500 shirts a day. Use the model from part **e** to estimate the total daily cost.

g Describe the dangers of using this model to estimate the cost of producing 500 a day.

h **i** Describe how you could enter data into your GDC to find the model using the power regression function.

ii Hence, verify your answer to part **e**.

5 The velocity (v) of a falling body is recorded at different distances (d) from the point at which it was launched. Consideration of the physics involved leads to the following model being proposed:

$v = a\sqrt{d} + b$

Data for v and d is collected and shown below.

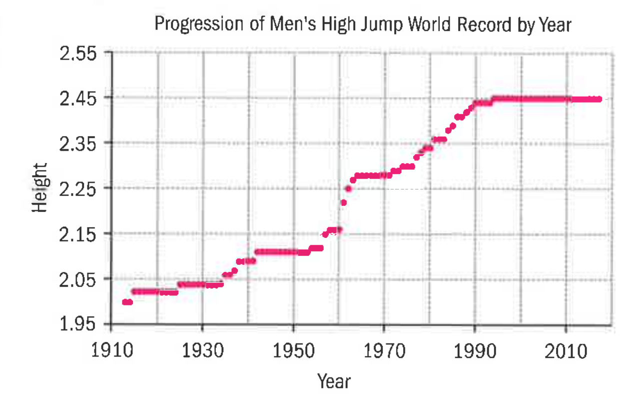
d	1	2	3	4
v	8.4	10.3	11.7	12.9

a Plot a graph of v against $\sqrt{d}$.

b Estimate values of a and b.

7.5 Logistic models

The progression of the world record for men's high jump at the beginning of each year since 1912 is shown on the scatter diagram.



a How could you describe the basic features of the progression of the world record?

b How is it different from a linear graph?

c How is it different from an exponential graph?

d Where else do you encounter this shape?

The logistic function is used to model situations where there is a restriction on the growth. For example, population on an island, bacteria in a petri dish, or the increase in height of a person or seedling.

Logistic functions have a horizontal asymptote at $f(x) = L$.

The equation of a logistic function has the standard form $f(x) = \dfrac{L}{1 + Ce^{-kx}}$, where L, C and k are the parameters of the function and e is Euler's number.

TOK

Mathematics is all around us in patterns, shapes, time and space.

What does this tell you about mathematical knowledge?

Investigation 17

a Sketch the graph of the following functions.

i $f(x) = \dfrac{10}{1 + e^{-x}}$ **ii** $f(x) = \dfrac{20}{1 + e^{-x}}$ **iii** $f(x) = \dfrac{30}{1 + e^{-x}}$

What is the relation of the value of the numerator to the graph of the function?

How does this link with graph transformations?

How does the rate of change vary at different points on the graph?

At what point does the rate of change seem greatest?

Give the equations of any asymptotes.

b Sketch the graph of the following functions:

i $f(x) = \dfrac{10}{1 + e^{-x}}$ **ii** $f(x) = \dfrac{10}{1 + e^{-2x}}$ **iii** $f(x) = \dfrac{10}{1 + e^{-3x}}$

What is the relation of the parameter multiplying x to the graph of the function?

How does this link with graph transformations?

How does the rate of change vary at different points on the graph?

HINT

In a logistic equation $f(x) = \dfrac{L}{(1 + Ce^{(-kx)})}$, $f(x) = L$ is an asymptote to the curve and L is referred to as the carrying capacity for the model being considered.

Logistic functions have a varying rate of change. The most usual shape of a logistic function starts with the data being almost constant (close to zero rate of change), then the data show a rapid increase and finally become almost constant again (close to zero rate of change again). Thus the data show two different horizontal asymptotes, one for small data values and one for large data values.

HINT

In some real-life situations the change will begin at a certain time and so the lower horizontal asymptote will not be part of the function.

Exercise 7M

1 The function that models the percentage of people globally with access to broadband Internet with respect to time is given by the logistic function $P(t) = \frac{100}{1+10e^{-0.5t}}$, $t \geq 0$.

a Sketch the graph of the function.

b Write down the range of values of the function and interpret their significance.

c Use the model to predict how long it will take for half the people to have access to broadband Internet.

d Calculate the percentage of people who will have access to broadband Internet in 20 years.

2 Data for the population of the Virgin Islands from 1950 to 2015 have produced the following logistic function model.

$P(t) = \frac{107000}{1+4e^{-0.135t}}$, where t is the number of years that have passed since 1950.

a Estimate the population of the Virgin Islands in 1950.

b Estimate the population of the Virgin Islands in 2015.

c Assuming that this model will also be valid in the future, state the largest population that the Virgin Islands will ever be able to accommodate.

3 A flu epidemic is spreading throughout Europe. An estimated 120 million people are susceptible to this particular strain and it is predicted that eventually all of them will get infected. There are 10000 people already infected when $x = 0$ and it is projected that the number of people who will have been infected will double in the next two weeks.

The logistic function $f(x) = \frac{L}{1+Ce^{-kx}}$ models the number of people infected where x is measured in weeks.

a Determine the approximate values of the parameters L, C and k.

b Use the model to determine how many people will be infected for after four weeks.

c The infection is considered terminated when 90% of the people have been infected. Use the model to determine how long it will be until the infection can be considered terminated.

Chapter summary

- A sequence of numbers in which each term can be found by multiplying the preceding term by a **common ratio**, r, is called a geometric sequence. For a sequence to be geometric, $r \neq 1$.
- The nth term of a geometric sequence is given by the formula:

$$u_n = u_1 r^{n-1}, r \neq 1$$

- Percentage change is a form of a geometric sequence.
- When terms **increase** by p% from the preceding term, their common ratio corresponds to $r = \frac{100+p}{100}$.
- When terms **decrease** by p% from the preceding term, their common ratio corresponds to $r = \frac{100-p}{100}$.
- The sum of the first n terms of a geometric sequence is called a geometric series and is given by the formula:

$$S_n = \frac{u_1(r^n-1)}{r-1} = \frac{u_1(1-r^n)}{1-r}, r \neq 1$$

- The **sum to infinity** of a geometric sequence is:

$$S_\infty = \frac{u_1}{1-r}, \text{ for } |r| < 1$$

- If PV is the present value, FV the future value, r the interest rate, n the number of years and k the compounding frequency, or number of times interest is paid in a year (ie $k = 1$ for yearly, $k = 2$ for half-yearly, $k = 4$ for quarterly and $k = 12$ for monthly, etc), then the general formula for finding the future value is

$$FV = PV\left(1+\frac{r}{100k}\right)^{kn}$$

- When a constant investment of \$$P$ is made for a certain number of n periods always compounded with the same interest rate, it is called an annuity.
- All exponential functions have a **horizontal asymptote**.
- Exponential functions of the form $f(x) = b^x$ have the line $y = 0$ as a horizontal asymptote. Unlike inverse proportion functions, the asymptotic behavior occurs either as $x \to \infty$ or as $x \to -\infty$, but not on both.
- The exponential function $f(x) = ka^x + c$ has:
  - The straight line $y = c$ as a horizontal asymptote.
  - It crosses the y-axis at the point $(0, k+c)$.
  - Is increasing for $a > 1$ and decreasing for $0 < a < 1$.
- In general, the solution of the exponential equation $a^x = b$ is the **logarithmic function** $x = \log_a b$, where $a, b > 0$ and $a \neq 1$.
- Here a is called the **base** of the logarithm. Any positive number can be used as a base for logarithms, but the only ones you need to know about for this course are 10 and Euler's number e.
- You can write $\log_{10} x$ simply as $\log x$.
- You can write $\log_e x$ simply as the **natural logarithm** $\ln x$.

Continued on next page

Number and algebra

Functions

- Two fundamental exponential equations with their solutions are:
  - $10x = c \Rightarrow x = \log c$
  - $e^x = c \Rightarrow x = \ln c$

  and two fundamental logarithmic equations are:
  - $\log x = c \Rightarrow x = 10^c$
  - $\ln x = c \Rightarrow x = e^c$

Laws of logarithms:

- $\log_a x + \log_a y = \log_a xy$
- $\log_a x - \log_a y = \log_a \frac{x}{y}$
- $\log_a (x^n) = n\log_a x$

The functions $y = 10^x$ and $y = \log x$ are inverse functions and so their composition gives x.

Similarly, the same thing holds for the functions $y = e^x$ and $y = \ln x$.

$10^{\log x} = x$ and $e^{\ln x} = x$

The equation of a logistic function has the standard form $f(x) = \frac{L}{1 + Ce^{-kx}}$, where L, C and k are the parameters of the function and e is Euler's number. The logistic function is used in situations where there is a restriction on the growth. L is referred to as the carrying capacity.

Developing inquiry skills

If you keep doubling a number, how long will it take before it becomes more than 1 million times its original value?

Can something that grows indefinitely never exceed a certain value?

What is the equation that models a skateboard quarter pipe ramp?

Does the same amount of money have the same value today, yesterday and tomorrow?

How can you decide on which is the best investment plan?

Will an ice cube left outside of the freezer or a boiling cup of water that has been removed from heat ever reach room temperature?

Chapter review

Click here for a mixed review exercise

1 The first four terms of a geometric sequence are 2, 5, 12.5, 31.25.

a Find the common ratio.

b Find the 8th term.

c Find the sum of the first eight terms.

2 Saul bought a new bicycle for US$350. Every year the value of the bicycle decreased by 12%. Find the value of the bicycle at the end of five years.

3 Molly bought a flat. At the beginning of the third year, the value of Molly's flat had increased to 363 000 euros. At the beginning of the fifth year, the value had increased to 439 230 euros. Assume that the value of the flat increases by a constant rate each year.

a Calculate the rate of increase as a percentage of the value of the house.

b Calculate how many euros Molly originally paid for the flat.

c Find how much the flat is worth at the beginning of the ninth year.

4 A finite geometric sequence has terms 2, 6, 18, …, 118 098.

a Calculate the number of terms in this sequence.

b Find the sum of the sequence.

5 The school fees increase each year by the rate of inflation. When Barnaby joined the school, the fees were UK£9500. At the end of the first year the rate of inflation was 1.16%.

a Find the cost of the school fees for the second year.

The following year the rate of inflation was 1.14%.

b Find the cost of the school fees for the third year.

6 Wei invested SGD 3000 (Singapore dollars) in a bank that paid 2.35% interest per year compounded monthly.

a Find how much Wei had in the bank after six years.

b Find the number of years before he had SGD 5000 in the bank.

7 Omar invested JOD 4500 (Jordanian dinar) in a bank that paid interest per annum compounded quarterly. After six years he has JOD 5179.27 in the bank. Find the interest rate.

8 Silvia is left an annuity of US$4000 in her uncle's will. The annuity is for five years at 4% per annum and is to be paid out monthly. Find the monthly payments.

9 Nathalie borrows 6500 euros for a motorbike. The loan is for five years at 2.5% interest per annum. Find how much Nathalie's monthly payments are to clear the loan.

10 A population of rabbits can be modelled by the function $f(x) = 24\,000 \times 1.12^x$, where x is the time in years.

a Find the number of rabbits after five years.

b Calculate how long it will take for the population of rabbits to double.

11 The temperature, T°C, of a cup of soup can be modelled by the equation $T(x) = 21 + 74 \times (1.2)^{-x}$, where x is the time in minutes.

a Find the initial temperature.

b Find the temperature after 10 minutes.

c Find how many minutes it takes for the soup to reach 40°C.

d Write down the room temperature.

12 An exponential function is $f(x) = 4 \times 2.3^x + 16$.

a Write down the equation of the horizontal asymptote.

b Write down the coordinates of the point where the curve cuts the y-axis.

13 The spread of a disease can be modelled by the equation $y = 4 + e^x$, where x is the time in days from the time the disease was first detected.

a Find the number of people with the disease after seven days.

b Find the number of days it takes for 25 000 people to be infected.

14 The height, in metres, of a runner bean plant increases each week according to the model function $h(t) = 0.25 + \log(2t - 0.6)$, $t > 0$, where t is the time in weeks.

a Find the height of the plant after four weeks.

b Find the number of weeks it takes for the plant to reach a height of 2 m.

15 **a** Find the value of the following in the most simplified form: $\sum_{r=1}^{n}(5\times 2^r)$.

b Find the value of the following in the most simplified form: $\sum_{r=1}^{2n}(5\times 2^r)$

16 \$900 is invested at the beginning of each year into an account earning 4% per annum compounded annually. Find the total value of the investments at the end of the tenth year.

Exam-style questions

17 **P1:** Maria's parents will not let her have a pet animal, so instead she has a pet rock. She keeps it in her pocket and strokes it, but sadly this wears it away. Its mass is given by the equation

$m = 0.1e^{-0.4t}$, where m is mass in kg and t is time in years.

a Write down the initial mass of Maria's pet rock. (1 mark)

b Find how long it takes for the mass of the rock to be half the original mass. (3 marks)

When the mass of the pet rock is less than 0.01 kg it becomes too small for Maria to find.

c Find how long it takes for this sad event to occur. (3 marks)

18 **P2:** Let $\log x = p$ and $\log y = q$. Express the following in terms of p and q.

a $\log xy$ (1 mark)

b $\log \frac{x}{y}$ (1 mark)

c $\log \sqrt{x}$ (1 mark)

d $\log x^2y^5$ (2 marks)

e $\log(x^y)$ (3 marks)

f $\log 0.01x^3$ (3 marks)

19 **P1:** The Martian high jump record height, h m, is given by $h = \frac{a}{b+e^{-0.1t}}$, where t is the time in years from when records were first kept. The first recorded height was 2 m. As t increases, the record tends to a limit of 3 m.

a Find the values of a and b. (4 marks)

b Calculate the record height after 10 years. (2 marks)

c Find the value of t for which $h = 2.8$. (3 marks)

20 **P1:** Carbon dating can be used to estimate the age of a piece of wood. Scientists measure the ratio r of carbon 14 in the sample to carbon 14 in living wood. They can then estimate the time (t years) since the tree was cut down using the model $t = \frac{-6000}{\ln 2}\ln(r)$ where $r < 1$.

a Calculate t if

i $r = 0.5$ **ii** $r = 0.25$. (3 marks)

If $r < 0.01$ then there is insufficient carbon 14 for the test to be reliable.

b Find the age of the oldest piece of wood for which the test would just cease to be reliable, giving your answer to the nearest 10 000 years. (2 marks)

c Express r in terms of t. (2 marks)

d Hence find r if $t = 10\,000$. (2 marks)

21 **P1:** A geometric sequence has 3rd term 18 and 6th term of 486.

a Find the first term a and the common ratio r. (3 marks)

b Find:

i the 8th term

ii the sum of the first 8 terms. (3 marks)

c Find the smallest value of n such that the nth term is greater than 10^6. (2 marks)

22 **P1:** A geometric sequence has first term a and common ratio r, where a and r are both positive.

a Write down an expression in terms of a and r for

i the nth term, u_n

ii the sum S_N of the first N terms of u_n. (2 marks)

A new sequence is defined by $v_n = \log u_n$.

b By writing v_n in terms of a and r, determine what type of sequence v_n defines, and find an expression for the nth term v_n. You must clearly show your working and justify your answer. (4 marks)

c Find an expression for the sum T_N of the first N terms of v_n (2 marks)

d Determine whether or not $T_n = \log S_n$. (2 marks)

23 **P1:** **a** If the sum to infinity of a geometric sequence is 3 times the first term, find the common ratio. (2 marks)

b If the sum to infinity of a geometric sequence is $\frac{2}{3}$ times the first term, find the common ratio. (2 marks)

c Determine whether the sum to infinity of a geometric sequence can be $\frac{1}{3}$ times the first term. Justify your answer. (3 marks)

24 **P2:** In this question all monetary values are to be given to 2 decimal places.

Anna invests her money in a bank that gives her 3% compound interest per year.

a Calculate how much money Anna would have to deposit in the bank now, for it to be worth £100 in one year's time. (2 marks)

b Calculate how much money Anna would have to deposit in the bank now, for it to be worth £100 in ten years' time. (2 marks)

Anna decides to invest ten instalments of £M each, with each instalment being deposited in the bank on 1st January each year. On the 31st of December of final year (when her last instalment will have been invested for one year) she wishes her investment to be worth £1000.

c Find the value of M. (6 marks)

25 **P1:** Bonjana has 5000 euros that she wishes to invest for five years. There are two schemes that she can choose between.

Scheme 1 offers 4% annual compound interest, compounded yearly.

Scheme 2 offers 3.8% annual compound interest, compounded monthly.

a If Bonjana wants to gain much interest as possible, determine which option she should choose. (4 marks)

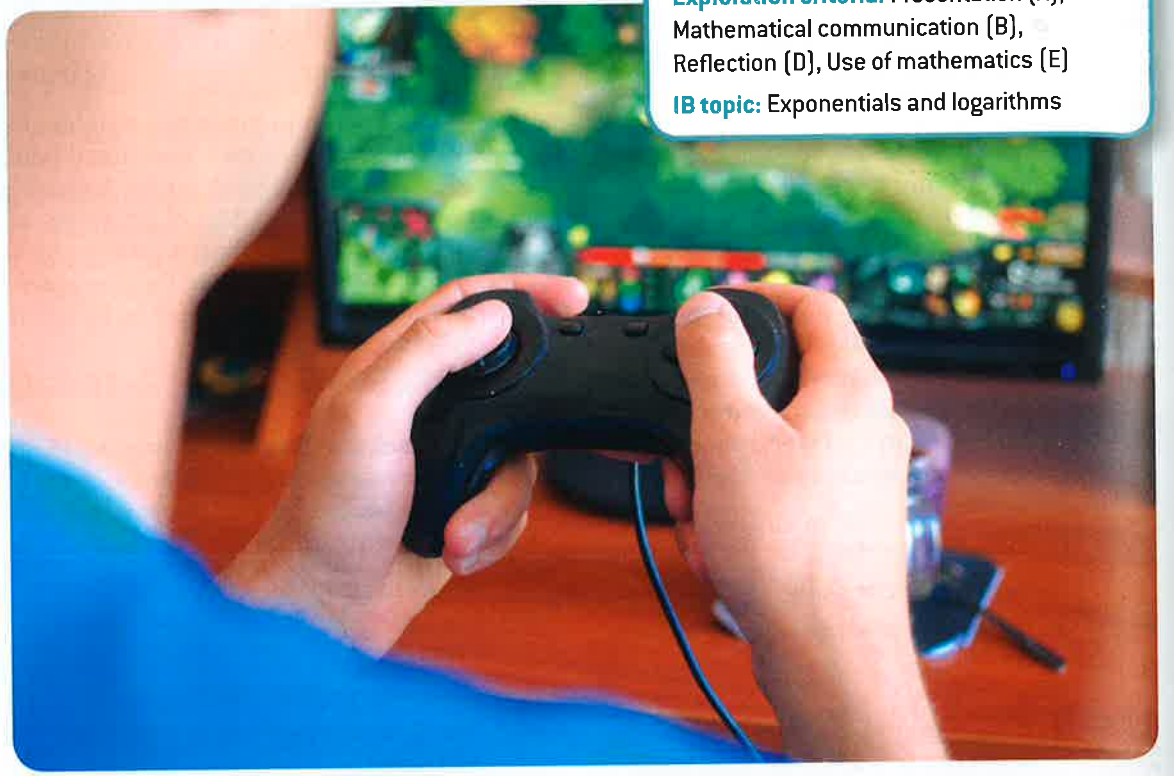
b For the scheme you identified in part **a**, calculate the interest that Bonjana will earn over five years. Give your answer correct to 2 decimal places. (2 marks)

c Determine whether your answer to part **a** would change if Bonjana had a different amount to invest, for a different number of complete years. (3 marks)

Click here for further exam practice

A passing fad?

Approaches to learning: Communication, Research
Exploration criteria: Presentation (A), Mathematical communication (B), Reflection (D), Use of mathematics (E)
IB topic: Exponentials and logarithms



Look at the data

Fortnite was released by Epic Games in July 2017 and quickly grew in popularity.

Here are some data for the total number of registered players worldwide from August 2017 to June 2018:

Date	August 2017	November 2017	December 2017	January 2018	June 2018
Months since launch, t	1	4	5	6	11
Number of registered players, P (million)	1	20	30	45	125

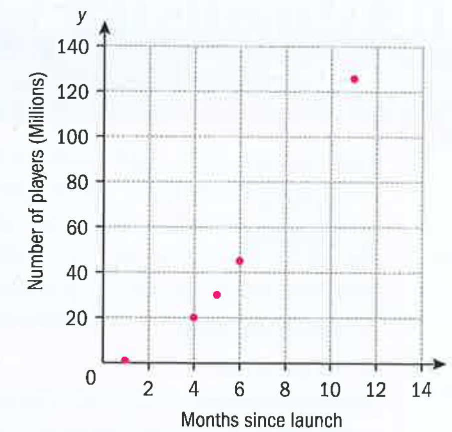
The data is taken from the press releases of the developers, Epic Games.

Are these data reliable?

Are there any potential problems with the data that has been collected?

What other data might be useful?

Plotting these data on your GDC or other graphing software gives this graph:



Describe the shape of the graph.

What could be some possible explanations for this growth?

Model the data

Why might it be useful to find a model that links the number of players of Fortnite to the number of months since the launch of the game?

Who might this be useful for?

Let t be the number of months since July 2017.

Let P be the number of players in millions.

Assume that the data is modelled by an exponential function of the form $P = a.b^t$ where a and b are constants to be found.

Use the techniques from the chapter or previous tasks to help you.

Consider different models **by hand** and **using technology**.

How do these models differ? Why?

Which one is preferable? Why?

What alterations could be made to the model?

Use the model to predict the number of users for the current month.

Do you think this is likely to be a reliable prediction?

How reliable do you think the models are at predicting how many Fortnite players there are now? Justify your answer.

Research the number of players there are in this current month who play Fortnite.

Compare this figure with your prediction based on your model above. How big is the error?

What does this tell you about the reliability of your previous model?

Plot a new graph with the updated data and try to fit another function to this data.

Will a modified exponential model be a good fit?

If not, what other function would be a better model that could be used to predict the number of users now?

Extension

Think of another example of data that you think may currently display a similar exponential trend (or exponential decay).

Can you collect reliable and relevant data for your example?

Find data and present it in a table and a graph.

Develop a model or models for the data (ensure that your notation is consistent and your variables are defined) – you could use technology or calculations by hand.

For how long do you think your model will be useful for making predictions?

Explain.