8 Modelling periodic phenomena: trigonometric functions and complex numbers

Concepts

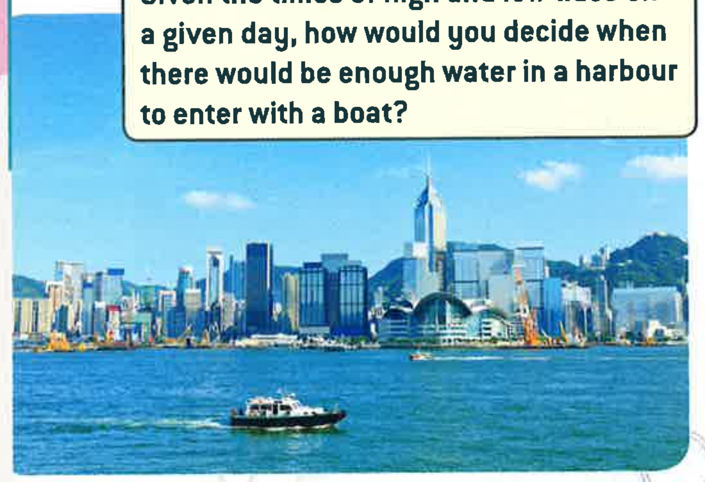
- Systems
- Modelling

Microconcepts

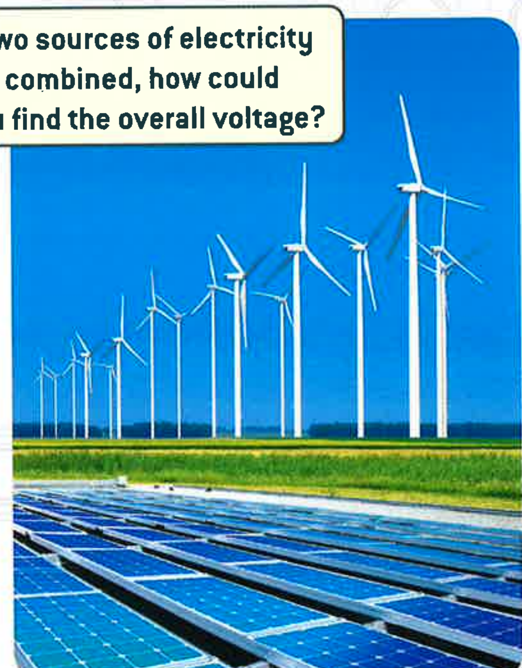
- Radian measure
- Length of arc
- Area of sector
- Period
- Amplitude
- Phase shift
- Principal axis
- Sinusoidal models: $f(x) = a\sin(b(x-c)) + d$
- Cartesian form of complex numbers
- Polar form of complex numbers
- Exponential form of complex numbers
- Modulus
- Argument
- Powers
- Complex numbers
- The complex plane.

Many natural, and human-made, phenomena repeat themselves regularly over time. Tides rise and fall, the sun rises and sets, the moon changes its appearance, a clock's pendulum swings, crystals vibrate when an electrical current is applied. All of these things happen in a predictable fashion.

Given the times of high and low tides on a given day, how would you decide when there would be enough water in a harbour to enter with a boat?



If two sources of electricity are combined, how could you find the overall voltage?

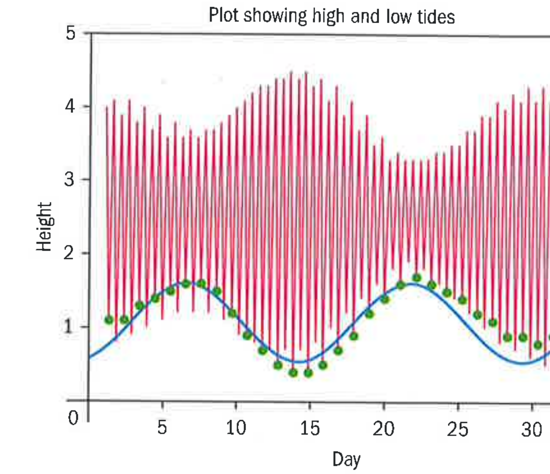


If there is a full moon tonight, how can we predict when the next new moon will occur?

How can we predict the number of hours of daylight on a particular date?

The harbour in Aberdeen, Scotland has the depth of water varies with the tide. As a result, boats can only enter or leave at certain times.

On 1 January 2017, high tides were at 3:02 and 15:09 where the water had a depth of 4m. The low tide occurs at 8:54 where the depth of water is 1.1m. A boat requires 2m of water to be able to move. What was the latest a boat could leave the harbour on that day?



The water depth over the course of January is shown on the graph with red lines. The height of alternate low tides is marked with a green dot. An attempt has been made to approximate that to a function, in blue.

- In what ways is the model a good fit?
- In what ways does the model not fit the data well?
- What features can you identify about the model that might enable you to find its equation?
- What other information is shown on the graph?

Developing inquiry skills

What other climate phenomena could have a repeating pattern? What questions could you ask for your country? What data would you need?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

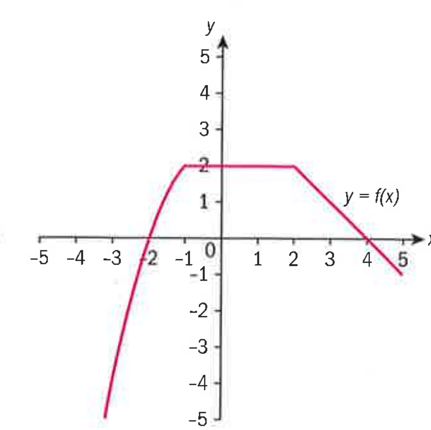
1 Transform graphs of functions.
e.g. The transformations needed to transform $y = x^2$ onto $y = 2(3x-1)^2 + 4$ are:

- A translation of $\begin{pmatrix}1\\0\end{pmatrix}$
- Then a stretch parallel to the y-axis scale factor $\frac{1}{3}$
- Then a stretch parallel to the x-axis scale factor 2
- Then a translation of $\begin{pmatrix}0\\4\end{pmatrix}$

Skills check

Click here for help with this skills check

1 The graph shows $y = f(x)$.



Draw the graphs of:

a $y = 2f(x) + 1$ **b** $y = f(2x+1)$

8.1 Measuring angles

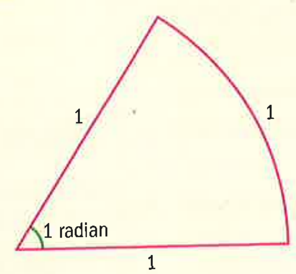
Units of measurement are often varied and equally often quite arbitrary in how they have been developed. Lengths may be measured in feet, metres, chains, etc, all of which have been established in a certain manner and have their different uses. Equally there are ways to convert between different measures.

The measurement of angles is the same, and there have been many measures that have been adopted over the years. We are probably most comfortable with using degrees for the measurement of angles and we know that a full circle is comprised of 360 degrees. However, in some more advanced mathematics we use a mathematical unit of measurement called the **radian**.

TOK

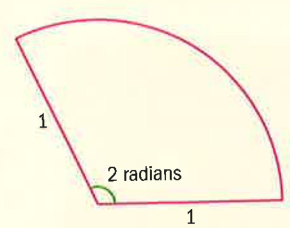
Which is a better measure of angle: radian or degree? What are the "best" criteria by which to decide?

A radian is defined as the angle subtended at the centre of a circle of radius 1 by an arc of length 1 as shown in the diagram.



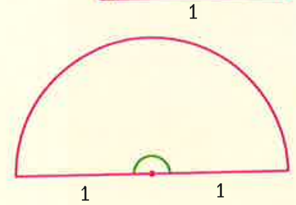
Investigation 1

1 If a sector has radius 1 and angle 2 radians, what will be the length of the arc?



It is clear that if the angle is doubled, then the length of the arc will be doubled too.

We can consider a semicircle to establish the conversion between degrees and radians.



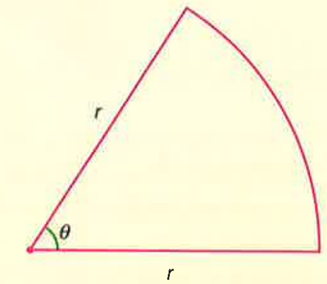
2 What is the angle shown in degrees?

3 **Factual** What is the length of the arc? Hence state the size of the angle in radians.

We can see that 180 degrees are equivalent to π radians or approximately 3.14 radians. Note the difference between a numerical value and an exact value.

4 **Factual** What would be the exact radian equivalence of 90°? 45°? 120°?

In degrees the arc length of a sector subtended by an angle θ is given by the formula $\frac{\theta}{360} \times 2\pi r$



5 a Explain how this formula can be derived.

b Give the formula for arc length when θ is measured in radians.

c Explain why this matches the definition for one radian given at the start of the investigation.

6 a Write down the formula for the area of a circle with radius r.

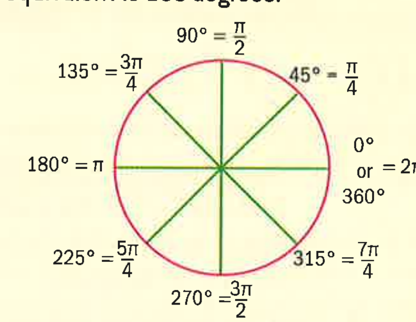
b Hence give the formula for the area of the sector of a circle with radius r subtended by an angle θ, when θ is measured in radians.

7 **Conceptual** Why might you choose to use radians rather than degrees when finding the lengths of arcs and areas of sectors?

International-mindedness

Why are there 360 degrees in a complete turn?

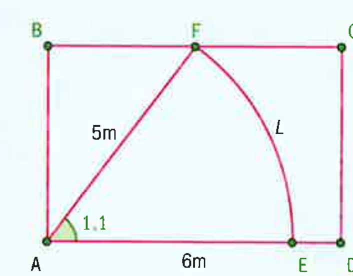
- π radians are equivalent to 180 degrees.



- The length of arc, L, of a sector radius r, is given by $L = r\theta$, if θ is in radians.
- The area, A, of a sector radius r, is given by $A = \frac{1}{2}r^2\theta$, if θ is in radians.

Example 1

The diagram below shows a sector of a circle of radius 5m, subtended by an angle of 1.1 radians inside a rectangle with a length 6m.



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Number and algebra

Functions

Geometry and trigonometry

a Find the length, L, of the arc FE.

The rectangle ABCD represents a garden of length 6m. A security light is placed at point A which has a range of 5m.

b Find the length BF.

c Hence find the area of the garden which is covered by the security light.

a $L = 5 \times 1.1 = 5.5$ m	Using the formula $L = r\theta$
b BF $= 5\cos(1.1) = 2.27$ m	The trig buttons on the calculators are used in the normal way, but if inputting an angle in radians the calculator needs to be set to radians mode.
c Area of sector = $\frac{1}{2} \times 5^2 \times 1.1 = 13.75\text{m}^2$	Using the formula $A = \frac{1}{2}r^2\theta$ Area of yard covered by the security light is the area of the sector plus the area of the triangle ABF.
Area of triangle ABF = $\frac{1}{2} \times 2.27 \times 5 \times \sin(1.1) = 5.05\,\text{m}^2$	Using Area of a triangle $= \frac{1}{2}ab\sin C$
Total area = 13.75 + 5.05 = 18.8 m²	

International-mindedness

Seki Takakazu calculated π to ten decimal places in 17th century Japan.

Exercise 8A

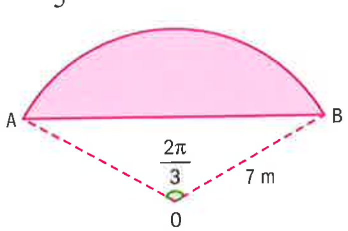
1 State the exact radian equivalent for each of the following angles.

a 30° **b** 165°
c 270° **d** 300° **e** 210°

2 State the degree equivalent to each of the following radian measures.

a $\frac{\pi}{3}$ **b** $\frac{4\pi}{3}$
c $\frac{3\pi}{5}$ **d** 3π **e** 1

3



a Find the length of the arc AB.

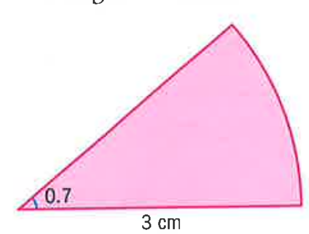
b Find the area of sector OAB.

c Use an appropriate formula to find the area of the triangle OAB.

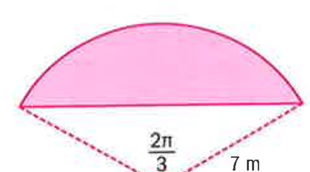
d Hence find the area of the segment, cut off by the chord AB.

4 Find the perimeter and area of each of the shaded regions below:

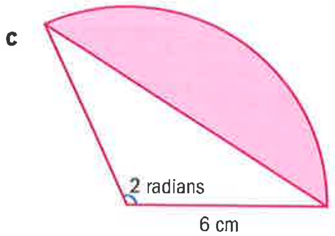
a



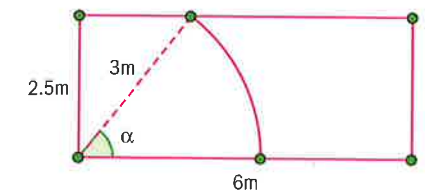
b



c



5 A goat is attached by a rope of length 3 m in the corner of a small, fenced rectangular field. The dimensions of the field are 6 m by 2.5 m.

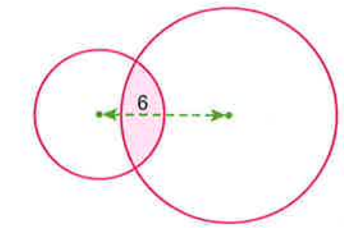


a Find the value of α.

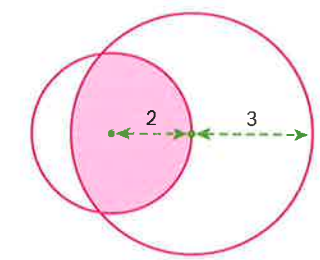
b Hence find the area of grass that can be reached by the goat.

6 The diagram below shows two circles with radii 3 cm and 5 cm and centres 6 cm apart. For the shaded region, find:

a the perimeter **b** the area.

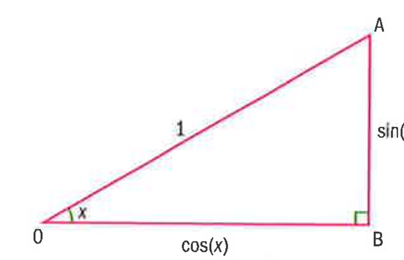


7 Find the area of the shaded region in the diagram below.



8.2 Sinusoidal models: $f(x) = a\sin(b(x-c)) + d$

In Chapter 1 you saw how a right-angled triangle with hypotenuse of 1 unit length had other sides of lengths $\sin x$ and $\cos x$ if one of its angles was x.



This gave us a way to define the sine and cosine of acute angles. What about other angles? For example, a carriage on a Ferris wheel rotates 360 degrees around its centre point.

How would we extend the idea of sine and cosine for angles from 0 to 360 and beyond? What would these tell us about the position of the Ferris wheel car?

When dealing with angles greater than 90°, it is useful to have a more general definition for sine and cosine.

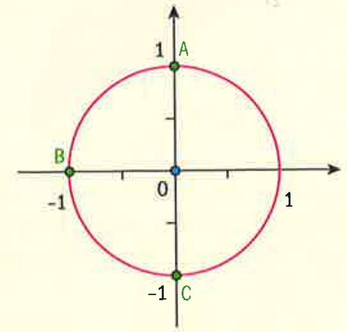
The definition used is based on the idea of the **unit circle**, which is a circle with radius of 1 unit, centred on the origin.

Angles on a unit circle are always measured clockwise, from the positive x-axis.

Number and algebra
Functions
Geometry and trigonometry

Investigation 2

On the diagram of the unit circle below, the points A, B and C represent angles of 90°, 180° and 270°.



1 Write 90°, 180° and 270° in radians.

2 Where might you place

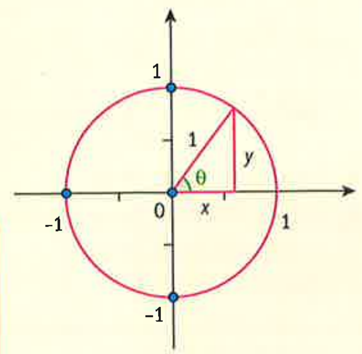
a 225° **b** 360° **c** 3π?

3 Where might you place

a $-\dfrac{\pi}{2}$ **b** −45° **c** −400°?

4 **a** From the unit circle explain why $\cos\theta = x$ and $\sin\theta = y$.

b Give the value of $\tan\theta$ in terms of x and y and hence in terms of $\sin\theta$ and $\cos\theta$.



The results of question 4 can be extended to all angles, both positive and negative.

On the unit circle the position of an angle $-\theta$ is found by moving clockwise around the unit circle through an angle θ.

In general a negative angle represents a clockwise rotation. A rotation of −45° and a rotation of 45° **clockwise** define the same movement.

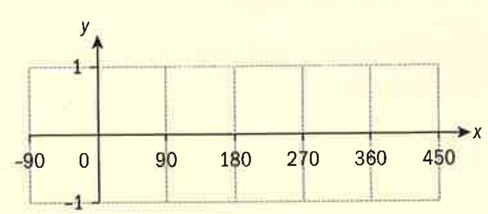
For all values of θ:

- $\sin\theta$ is the y-coordinate of the point on the unit circle that represents an angle θ
- $\cos\theta$ is the x-coordinate of the point on the unit circle that represents the angle θ.
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

5 Use the definitions above to complete the following table.

θ	−90°	0°	90°	180°	270°	360°	450°
sin θ							
cos θ							

6 Copy the axes below and on them plot the points found above. Join these points to form a smooth curve for $y = \sin\theta$ and $y = \cos\theta$.



7 **Conceptual** How do we describe the sine function beyond the first quadrant?

8 **Conceptual** How do we describe the cosine function beyond the first quadrant?

9 Draw the graph of $y = \sin x$ on your GDC with your GDC set to:

a **i** degrees **ii** radians.

b Comment on the differences and the similarities.

10 **a** With your GDC set to either degrees or radians, draw the graph of $y = \sin^2 x + \cos^2 x$. What do you notice?

b Use the unit circle definition of sine and cosine to explain why this must always be true.

11 **a** Use the unit circle to explain why $\cos(-\theta) = \cos\theta$.

b Find an expression for $\sin(-\theta)$.

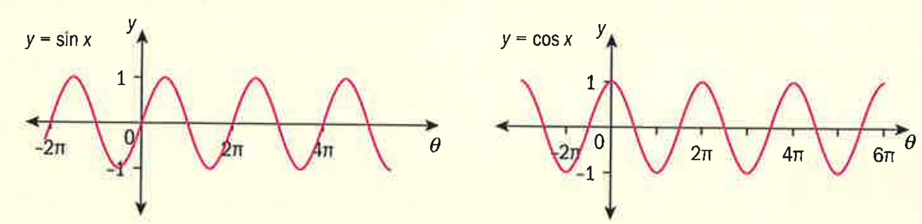
c Justify the answers for parts **a** and **b** from the graphs of the functions shown below.

You should know the following trigonometric identities:

$\sin^2 x + \cos^2 x \equiv 1$ $\tan x \equiv \dfrac{\sin x}{\cos x}$

$\sin(-\theta) \equiv -\sin\theta$ $\cos(-\theta) \equiv \cos\theta$

The graphs of $y = \sin x$ and $y = \cos x$ repeat every 360° or 2π radians.



If solving for $\sin x = a$ or $\cos x = a$ you should always consider the multiple possible values for x.

Example 2

Use your GDC to find all values of x, $-\pi \le x \le 2\pi$ for which:

a $\sin x = \dfrac{1}{2}$ **b** $\cos x = -0.2$.

a 0.524, 2.62	Using the button for the inverse of sine on your GDC will only give one result. You should always plot a graph to ensure you find all the solutions in the range required by the question.
b −1,77, 1,77, 4.51	This time there are three solutions.

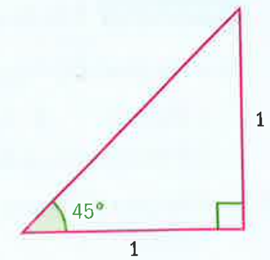
Example 3

a Use the triangle shown to find an exact expression for:

i $\sin 45°$ **ii** $\cos 45°$.

b Hence write down the exact values of the following:

i $\sin(-45°)$ **ii** $\cos\left(-\frac{\pi}{4}\right)$



a $\sqrt{1^2+1^2}=\sqrt{2}$ **i** $\sin 45° = \frac{1}{\sqrt{2}}$ **ii** $\cos 45° = \frac{1}{\sqrt{2}}$	Using Pythagoras' theorem to find the hypotenuse, and then right-angled trigonometry to find sine and cosine.
b **i** $\sin(-45°) = -\frac{1}{\sqrt{2}}$	Using $\sin(-\theta) = -\sin\theta$.
ii $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$	Using $\cos\left(-\frac{\pi}{4}\right) = \cos(-45°)$ and $\cos(-\theta) = \cos\theta$.

Exercise 8B

1 From your knowledge of the unit circle or the graphs $y = \sin x$ and $y = \cos x$ write down the values of:

a $\sin 0$ **b** $\cos 0$ **c** $\sin\pi$

d $\cos\left(\frac{\pi}{2}\right)$ **e** $\sin\left(-\frac{\pi}{2}\right)$ **f** $\cos(-\pi)$

g $\sin\left(\frac{3\pi}{2}\right)$ **h** $\cos\left(-\frac{\pi}{2}\right)$

2 **a** For each of the following equations, find all solutions for $0 \le x \le 720°$.

i $\sin x = \sin 10°$ **ii** $\sin x = 0.3$

iii $\cos x = \cos 200°$

b For each of the following equations, find all solutions for $0 \le x \le 3\pi$.

i $\sin x = \sin\left(\frac{2\pi}{5}\right)$ **ii** $\cos x = -0.1$

iii $\cos x = 0$

3 **a** Find the coordinates of all the intersection points for the graph of $y = \sin x$ and:

i $y = 1 - x$ **ii** $y = 1 - 0.2x$.

b In each case justify the fact that there are no further solutions.

4 In Chapter 1 you considered the ambiguous case of the sine rule by looking at the geometric properties of the two possible triangles that could be drawn. In this question you will approach it using the sine function.

a From consideration of the unit circle explain why $\sin\theta = \sin(180 - \theta)$.

b **i** Use your GDC to find a value of θ in degrees for which $\sin\theta = \frac{1}{2}$.

ii Use the result of part **a** to write down another solution to $\sin\theta = \frac{1}{2}$.

Consider the triangle ABD such that AB = 5 cm, $B\hat{A}C = 20°$ and BC = 3 cm.

c **i** Show that $\sin B\hat{C}A = \frac{5\sin 20°}{3}$.

ii Use your GDC to find a value for $B\hat{C}A$.

iii Use the result of part **a** to write down a second value for $B\hat{C}A$.

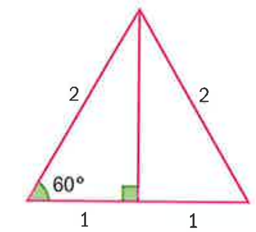
iv Sketch the triangle ABC for both these angles.

5 A triangle PQR is such that $\hat{P} = 25°$, PQ = 7cm, and QR = 5cm. Find the two possible values for PR.

6 **a** Use the equilateral triangle shown to find an exact expression for

i $\sin 60°$ **ii** $\cos 60°$

iii $\sin 30°$ **iv** $\cos 30°$.



b Hence find the exact values of

i $\cos(-60°)$ **ii** $\sin\frac{\pi}{6}$

iii $\cos\left(-\frac{\pi}{6}\right)$ **iv** $\sin\left(-\frac{\pi}{3}\right)$.

7 It is given that $\cos x = \frac{1}{3}$, $0 \le x \le \frac{\pi}{2}$.

a Use the identities $\sin^2 x + \cos^2 x \equiv 1$ and $\tan x \equiv \frac{\sin x}{\cos x}$ to find the exact values of:

i $\sin x$ **ii** $\tan x$.

b Write down the exact values of

i $\cos(-x)$ **ii** $\sin(-x)$.

8 It is given that $2\sin x = \cos x$, $0 \le x \le \frac{\pi}{2}$

a Write down the value of $\tan x$.

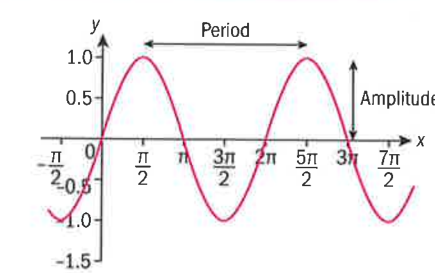
b Find the exact value of

i $\sin x$ **ii** $\cos x$.

c Why is it not possible that $-\frac{\pi}{2} \le x \le 0$?

The graph of $y = \sin x$ is shown:

The curve is **periodic**, as it repeats itself continually, and it oscillates either side of its **principal axis**.



The **period** of a function is defined as the distance along the horizontal axis between a point and the corresponding point in the next cycle.

For $y = \sin x$ the period is 2π or 360°.

The **amplitude** of a periodic function is defined as the distance from the principal axis to the maximum or minimum.

For $y = \sin x$ the amplitude is 1.

TOK

Sine curves model musical notes and the ratios of octaves. Does this mean that music is mathematical?

Investigation 3

In this investigation you will consider the graph of $y = a\sin(b(x - c)) + d$ for different values of the parameters a, b, c and d.

1 **a** Use your GDC to draw $y = \sin x$ and $y = \sin 2x$ on the same axes and comment on your results.

b Use your knowledge of the transformations of functions covered in Chapter 6 to write down the transformation that takes the graph $y = f(x)$ to $y = f(bx)$.

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c Hence give the period for the functions with graphs:

i $y = \sin 2x$ **ii** $y = \cos 4x$.

2 **Factual** How does changing the parameter b change the graph?

3 Deduce the period of $y = \sin bx$.

4 **a** Write down the transformation that takes the graph $y = f(x)$ onto the graph $y = af(x)$.

b Draw graphs of $y = 2\sin x$, $y = 3\sin x$, $y = \frac{1}{2}\sin x$, $y = -\sin x$.

Find the amplitude of each of the functions.

5 **Factual** How does changing the parameter a change the graph?

6 Deduce the amplitude of $y = a\sin x$.

7 Two transformations will map the graph of $y = \sin x$ onto the graph of $y = 2\sin x + 3$.

Describe the two transformations, taking care to state the order of the transformations. Illustrate the two steps using technology.

8 **Factual** How does changing the value of d affect the graph?

9 Write down the transformation that will take $y = \sin x$ onto $y = \sin(x - c)$

10 The curve $y = \cos x$ undergoes two transformations:

i a stretch parallel to the x-axis, scale factor $\frac{1}{2}$

ii a horizontal translation of 1 unit.

a Sketch the curve after each of these transformations indicating the coordinates of the first maximum with $x \geq 0$.

b Explain why the equation of this curve is $y = \cos(2(x - 1))$.

11 The curve $y = \cos x$ undergoes two different transformations:

i a horizontal translation of 2 units

ii a stretch parallel to the x-axis, scale factor $\frac{1}{2}$.

a Sketch the curve after each of these transformations indicating the coordinates of the first maximum.

b Write down the equation of this curve.

c Verify that the equations in parts **10b** and **11b** are equal. Explain why the translations are different depending on the order of the transformations.

12 **Factual** How does changing the value of the parameter c affect the graph?

The phase shift is the horizontal translation of the curve, which takes place after the horizontal stretch. In the equation $y = a\sin(b(x - c)) + d$ the phase shift is c.

Conceptual What is the difference of having values inside the sine function and outside?

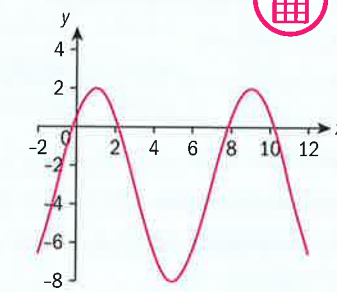
13 **Conceptual** How does changing the parameters generally alter a trigonometric graph?

- The graph of $f(x) = a\sin x + d$ can be obtained from the graph of $f(x) = \sin x$ by stretching vertically by a scale factor of a (giving an amplitude of a), then translating d units upwards (in the positive y direction). If $a < 0$ then the curve has also been reflected in the x-axis. In this case the amplitude is $|a|$

- The graph of $f(x) = \sin(b(x + c))$ can be obtained from the graph of $f(x) = \sin x$ by stretching horizontally by a scale factor of $\frac{1}{b}$ (giving a period of $\frac{2\pi}{b}$), then translating c units to the left (in the negative y direction).
- Similarly the graph of $f(x) = \sin(bx + c)$ can be obtained from the graph of $f(x) = \sin x$ by translating c units to the left then stretching horizontally by a scale factor of $\frac{1}{b}$.
- The graph of $f(x) = -\sin x$ can be obtained from the graph of $f(x) = \sin x$ by reflecting the graph in the x-axis.
- The graph of $f(x) = \sin(-x)$ can be obtained from the graph of $f(x) = \sin x$ by reflecting the graph in the y-axis.

Example 4

For the function $f(x) = a\sin(b(x + c)) + d$, the graph of $y = f(x)$ is drawn. The first maximum point shown has coordinates $(1, 2)$ and the first minimum point has coordinates $(5, -8)$.



a State the equation of the principal axis.

b Hence find the value of d.

c Find the amplitude of the function.

d Hence find the value of a.

e Find the period of the function.

f Hence find the value of b.

g Find the smallest positive value for c.

h State the values of $f(x)$ for $x = 0, 1, 5, 8, 9$ and use those data points to verify a sinusoidal regression calculation on your GDC gives the same result.

a $y = -3$	The graph reaches maximum points of 2, and minimum points of −8. Halfway between the two is −3.
b $d = -3$	The principal axis would normally be $y = 0$ so it has been translated down by −3.
c 5	$2 - (-3) = -3 - (-8) = 5$
d 5	The amplitude would normally be 1 so it has stretched by a scale factor of 5.
e 8	The two maximums occur when $x = 1$ and $x = 9$. The distance between is 8.

Continued on next page

f $b = \frac{\pi}{4}$

$\frac{2\pi}{b} = 8 \Rightarrow b = \frac{\pi}{4}$

g $c = 1$

Substituting the point (1, 2) and solving the equation $2 = 5\sin\left(\frac{\pi}{4}(1+c)\right) - 3$.

h $f(0) = f(8) = 5\sin\frac{\pi}{4} - 3$

$f(1) = f(9) = 2$

$f(5) = -8$

The equation is
$y = 5\sin(0.785x + 0.785) - 3$.

Exercise 8C

1 For the following curves state

i the amplitude

ii the equation of the principal axis

iii the period

iv the phase shift

a $y = 2\sin(3(x-4)) - 5$

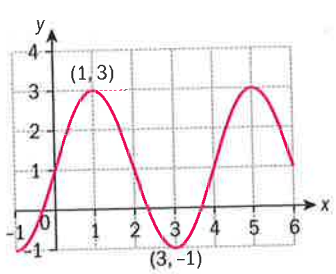
b $y = -3\cos(\pi(x+1)) + 3$

c $y = 4\sin(3x - 6)$

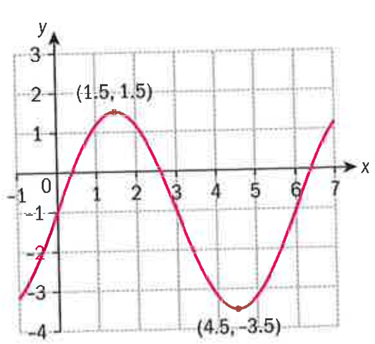
b $y = \cos(2x + 5) - 1$.

2 Find the equations of the following curves in the form $y = a\sin(bx) + d$.

a

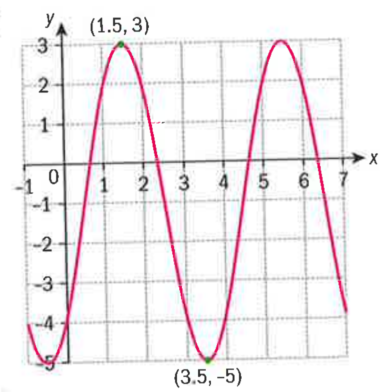


b

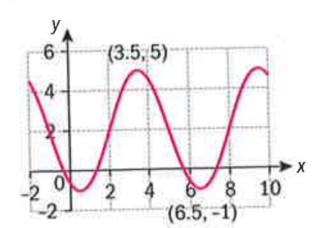


3 **a** Find the equations of the following curves in the form $y = a\sin(b(x-c)) + d$.

i



ii



b What would be the equation of the curves if they were to be given as $y = a\cos(b(x-c)) + d$?

c The transformation that takes the graph of the curve $y = a\sin(b(x-c)) + d$ onto the graph of $y = a\cos(b(x-c)) + d$ is a horizontal translation of $k \times$ the period.

State the value of k.

4 The times of sunrise in London for every other Monday of 2018 are given. You are required to provide a model for the times.

Date	Sunrise	Day-number	Hours after midnight	Date	Sunrise	Day-number	Hours after midnight
01/01/2018	08:06	1	8.1	02/07/2018	04:49	183	4.817
15/01/2018	08:00	15	8	16/07/2018	05:03	197	5.05
29/01/2018	07:44	29	7.733	30/07/2018	05:22	211	5.367
12/02/2018	07:21	43	7.35	13/08/2018	05:43	225	5.717
26/02/2018	06:53	57	6.883	27/08/2018	06:06	239	6.1
12/03/2018	06:22	71	6.367	10/09/2018	06:28	253	6.467
26/03/2018	06:50	85	6.833	24/09/2018	06:50	267	6.833
09/04/2018	06:19	99	6.317	08/10/2018	07:13	281	7.217
23/04/2018	05:49	113	5.817	22/10/2018	07:37	295	7.617
07/05/2018	05:23	127	5.383	05/11/2018	07:01	309	7.017
21/05/2018	05:01	141	5.017	19/11/2018	07:25	323	7.417
04/06/2018	04:48	155	4.8	03/12/2018	07:47	337	7.783
18/06/2018	04:44	169	4.733	17/12/2018	08:01	351	8.017
				31/12/2018	08:06	365	8.1

a Explain why a daynumber will need to be used rather than the date.

b The hours after midnight are used instead of the time. Explain how they were calculated.

c Use the sine regression function on your GDC to establish a model for the time of sunrise against the daynumber.

d Use your model to predict the time of sunrise for the 2 February 2019.

e Comment on the reliability of your prediction.

5 If the period of a wave is P (seconds) then its frequency (Hz) is defined as $f = \frac{1}{P}$.

An electronic signal has amplitude 4 and frequency, $f = 500$Hz.

The strength of the signal (S) at time t seconds is given by the equation $S = 4\sin(bt)$.

a Find the value of b.

A second signal of equal strength is sent out with a time delay of 1 millisecond. Its equation is $S = 4\sin(b(t-c))$.

b State the time delay in seconds and hence write down the equation for the strength of the second signal at time t.

The strength of the signal given by the two curves interacting is given by $S_T = 4\sin(b(t-c)) + 4\sin(bt)$.

c Use your GDC to plot the graph of $y = S_T$ and comment on your result with reference to the graphs of the two signals.

6 The strength of two sound waves with the same frequency but different amplitudes and phase can be represented by the equations $S_1 = 3\sin(200t)$ and $S_2 = 2\sin(200(t - 0.001))$.

The combined effect of the two waves is given by $S_T = S_1 + S_2$.

a Plot the graph of $y = S_T$ for $0 \le t \le 0.1$.

b Find the coordinates of the first maximum and minimum points.

c Find the value of **i** the amplitude **ii** the period of the combined waves.

d Hence find the equation for S_T in the form $S_T = a\sin(b(t-c)) + d$.

7 The heights of the water in Aberdeen were measured every two hours on 1st January 2017. The high and low tides on that day were also recorded. All the data is shown on the table below.

Time (hours after midnight)	0	2	3.033	4	6	8	8.9	10	12
Height (m)	2.48	3.8	4	3.84	2.57	1.24	1.1	1.15	2.41
Time (hours after midnight)	14	15.15	16	18	20	21.32	22	24	—
Height (m)	3.76	4.1	3.88	2.65	1.28	.8	1.02	2.33	—

It is assumed that the height of water $h(t)$ can be modelled against time, t, by a function $h(t) = a\sin(bt + c) + d$.

a One possible way to estimate a and d might be to take the mean height for high tide and the mean height for low tide as the maximum and minimum values of the function. Find a and d using this method.

b In a similar way an approximation for the period of the function might be to find the mean of the time difference between the two high tides and between the two low tides. Using this method show that your estimate of b would be 0.512.

c Use the value of t and h at the first high tide to find an approximate value for c. Hence write down an equation to model the height of the tides.

d Use the sine regression function on your GDC to find a model for the height of the water. Compare this answer with the one obtained in part **c**.

e The rowing club shows the times when it is possible to row.

Unrowable Start	Unrowable End	Unrowable Start	Unrowable End
07:24	10:24	19:49	22:49

Use your model to estimate the depth of water necessary to be able to row.

Developing inquiry skills

Could you now make a better estimate for the earliest that the boat in the opening section would be able to leave Aberdeen harbour?

8.3 Completing our number system

Number sets

When you first encountered numbers, you were only aware of counting numbers. As your mathematical awareness grew, so new sets of numbers had to be introduced, fractions, negative numbers, irrational numbers all gradually needed to make sense of your expanding mathematical world. The historical development of mathematics has followed a similar course. When you were very

young, and were only aware of counting numbers, you were told things like "you can't take 5 from 2" or "3 cannot be split into 5 parts", but with the introduction of negative integers, and fractions, you found these things are possible. The same is true of the next step in our mathematical development. Until now you have probably understood that there are no solutions to $x^2 = -4$ or $x^2 + x + 2 = 0$. We shall see that with the introduction of the next level of sophistication in our number system that there are.

The number sets we have been introduced to, in increasing complexity, are:

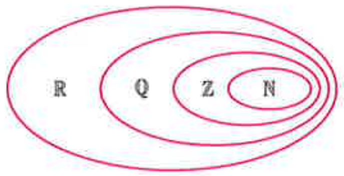
$\mathbb{N}$ – Natural numbers – counting numbers. 1, 2, 3, ...

$\mathbb{Z}$ – Integers – positive and negative whole numbers and zero. ... −3, −2, −1, 0, 1, 2, 3 ...

$\mathbb{Q}$ – Rational numbers – any number that can be expressed as a fraction with integer numerator and denominator.

$\mathbb{R}$ – Real numbers – up to now any number you have used. Includes irrational numbers such as π, e, $\sqrt{2}$.

Each new number set contains all the previous sets. Real numbers include all rational numbers which include all integers, etc.



Complex numbers, $\mathbb{C}$, is the next set and contains all real numbers in addition to the inclusion of an "imaginary" concept, i, where $i^2 = -1$. Complex numbers provide solutions to previously unsolvable equations; for example, i is one of the solutions to the equation $x^2 = -1$.

Investigation 4

The quadratic equation $ax^2 + bx + c = 0$ can be solved using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (note – this is in your formula book).

1 Use the quadratic formula to find exact solutions of the equations:

a $2x^2 + 2x - 1 = 0$ **b** $x^2 - 5x + 3 = 0$.

2 What happens when you try to solve the following equations?

a $x^2 - 8x + 16 = 0$ **b** $x^2 + 2x + 5 = 0$

The part of the quadratic formula within the square root is called the **discriminant**, often written as a capital *delta*, Δ.

Continued on next page

3 If $i^2=-1$, use your knowledge of the laws of indices to find: $(2i)^2$, $(-i)^2$, $(-2i)^2$, $(5i)^2$, $(-5i)^2$

4 Use your answers to conclude the values of x which satisfy: $x^2=-4$, $x^2=-9$, $x^2=-2$, $x^2=-18$

A complex number is written as $a+bi$ where a and b are real numbers.

5 We now return to $x^2+2x+5=0$.

Write down the two complex solutions to this equation.

Factual When does a quadratic equation have complex roots?

6 Show that each of the following also have two solutions, and write both the solutions in the form $a+bi$.

$x^2-6x+13=0$, $x^2+2x+26=0$, $x^2-4x+6=0$, $x^2-10x+26=0$.

7 Comment on the solutions of each equation.

8 **Factual** Given one complex solution of a quadratic equation, how could you find the other?

9 **Conceptual** Why do we need complex numbers?

For the equation $ax^2+bx+c=0$, $\Delta=b^2-4ac$

- When $\Delta>0$ the equation has two real roots.
- When $\Delta=0$ the equation has one real root.
- When $\Delta<0$ the equation has no real roots.

TOK

Solving an equation has given you an answer in mathematics, but how can an equation have an infinite number of solutions?

A **complex number** has two parts, and can be written in the form $a+bi$ where $a, b\in\mathbb{R}$. a is known as the **real part** and b is known as the **imaginary part**.

The notation used is: $\text{Re}(a+bi)=a$ and $\text{Im}(a+bi)=b$.

The **conjugate** of a complex number $a+bi$ is defined as $a-bi$.

Complex solutions to quadratic equations with real coefficients always occur in conjugate pairs.

The arithmetic of complex numbers

As ever, when a new set of numbers is introduced, the associated arithmetic also needs to be developed. When you were introduced to the concepts of fractions or of negative numbers, you very soon had to develop the ability to add, subtract, multiply and divide those different types of numbers. Complex numbers are no different, but since you already have algebraic skills, those can be simply applied to the arithmetic of complex numbers.

Investigation 5

1 Collect like terms to simplify $2+a-3+3b+4a$.

Apply the same logic to simplify $3+2i+4+5i$.

It should follow that the sum or difference of any complex numbers can be found easily:

If $a=2+3i$, $b=3-i$ and $c=-4+2i$, calculate $a+b$, $b-c$ and $2a$.

If $z_1=x_1+y_1i$ and $z_2=x_2+y_2i$ deduce expressions for z_1+z_2, z_1-z_2, pz_1+qz_2.

2 **Conceptual** What rules of algebra do we use to be able to add or subtract complex numbers?

3 Show that $\left(2-3\sqrt{2}\right)\left(1+2\sqrt{2}\right)=-10+\sqrt{2}$.

Apply the same logic to simplify $(2-i)(3+2i)$.

If $a=2+3i$, $b=3-i$ and $c=-4+2i$, calculate ab, bc and a^2.

4 a Find the product of $(1+2\sqrt{2})(1-2\sqrt{2})$.

b Show that $\dfrac{2-3\sqrt{2}}{1+2\sqrt{2}}=-2+\sqrt{2}$.

Apply the same logic to simplify $\dfrac{2-i}{3+2i}$

If $a=2+3i$, $b=3-i$ and $c=-4+2i$, calculate $\dfrac{a}{b}$, $\dfrac{b}{c}$ and $\dfrac{ac}{b^2}$

5 **Conceptual** How do we simplify and manipulate complex numbers?

6 Make sure you know how to find i on your GDC. Check the answers you have found in the investigation.

HINT

i^2 should be simplified to -1.

Number and algebra | Functions | Geometry and trigonometry

Example 5

Let $z_1=2+3i$ and $z_2=3-i$.

a Calculate each of the following, writing your answers in the form $a+bi$.

i $2z_1+3z_2$; ii z_1z_2 iii $\dfrac{z_1+z_2}{z_1-z_2}$

b If $\text{Im}(z_1+pz_2)=0$ find p.

a i $13+3i$ ii $9+7i$ iii $\dfrac{3-22i}{17}$	These values can be calculated using the rules given above or directly on a GDC.
b 3	$z_1+pz_2=(2+3p)+(3-p)i$ If imaginary part is zero then $3-p=0\Rightarrow p=3$

Example 6

One solution of the equation $x^2 + px + q = 0$ is $2 - 3i$. Find the value of p and the value of q.

$p = -4$, $q = 13$	If one solution is $2 - 3i$ then the other is $2 + 3i$. The equation is therefore: $(x - (2 - 3i))(x - (2 + 3i)) = 0$ Multiplying out gives: $x^2 - (2 - 3i + 2 + 3i) + (2 - 3i)(2 + 3i) = 0$ $x^2 - 4x + 13 = 0$

Exercise 8D

1 Find the exact solutions to the following equations:

a $2x^2 - 4x + 1 = 0$ **b** $x^2 - 4x + 5 = 0$

c $4x^2 - 8x + 5 = 0$ **d** $x^2 + 10 = 0$

2 If $a = 2 + i$, $b = 3 - 2i$ and $c = 1 - i$, find:

a $2a - 3b$ **b** ab **c** $\frac{a}{b}$

d b^2 **e** c^3 **f** $\frac{a^4}{b}$

You should find each calculation without a GDC and check your answers with the GDC.

3 Find the solutions to $x^2 - 4x + 5 = 0$. Hence write $x^2 - 4x + 5$ as the product of two linear factors.

4 One solution of $x^2 + px + q = 0$ is $3 - i$. Find the value of p and the value of q, assuming both are real.

8.4 A geometrical interpretation of complex numbers

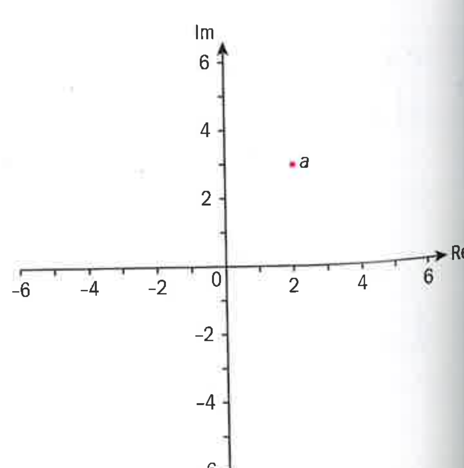
International-mindedness

Jean-Robert Argand was an 18th-century amateur mathematician born in Geneva, Switzerland.

The complex plane

Much of our understanding about numbers to date has been enhanced by placing the numbers on a number line. Complex numbers have two parts, a real and an imaginary part, and so can be thought of as two dimensional numbers. Much understanding about complex numbers can be developed by placing them on the complex plane, also called an **Argand diagram**.

Traditionally we have the horizontal axis as the Real axis and the vertical axis as the Imaginary axis. Consequently the number $a = 2 + 3i$ can be represented by the point shown.



$|z|$, the **modulus** of a complex number, z, is defined as its distance from O. $\arg z$, the **argument** of a complex number, z, is defined as the angle between the positive real axis and the line from O to z. The argument is normally given in radians with $-\pi < \arg z \le \pi$.

TOK

Why is it called the Argand plane and not the Wessel plane?

Example 7

Let $z_1 = 2 + 2i$, $z_2 = 3 - 4i$ and $z_3 = -\sqrt{3} + i$.

a Calculate the modulus of: **i** z_1 **ii** z_2 **iii** z_3.

b Calculate the argument of: **i** z_1 **ii** z_2 **iii** z_3, giving your answers in radians.

c Find the area of the triangle with vertices at z_1, z_2 and z_3.

a i $\sqrt{8}$ **ii** 5 **iii** 2	**i** $\sqrt{2^2 + 2^2} = \sqrt{8} \left(= 2\sqrt{2}\right)$ **ii** $\sqrt{3^2 + 4^2} = 5$ **iii** $\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$
b i $\frac{\pi}{4}$	From the diagram it is clear that the argument will be 45° and hence $\frac{\pi}{4}$. The numerical answer 0.785 is equally acceptable.
ii -0.927 (3 s.f.)	When looking for the argument it is often best to ignore any negative values and use right angled trigonometry to find the angle made with the real axis. The actual value for the argument can then be calculated from the position of the point in the Argand plane. In this example the GDC gave 0.927 and the argument is -0.927.
iii 2.62	Find $\arctan\left(\frac{1}{\sqrt{3}}\right) = 0.524$. The required angle is therefore $\pi - 0.524 = 2.62$. If working in degrees the GDC would have given the exact answer $30° = \frac{\pi}{6}$, so $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ would also be acceptable.

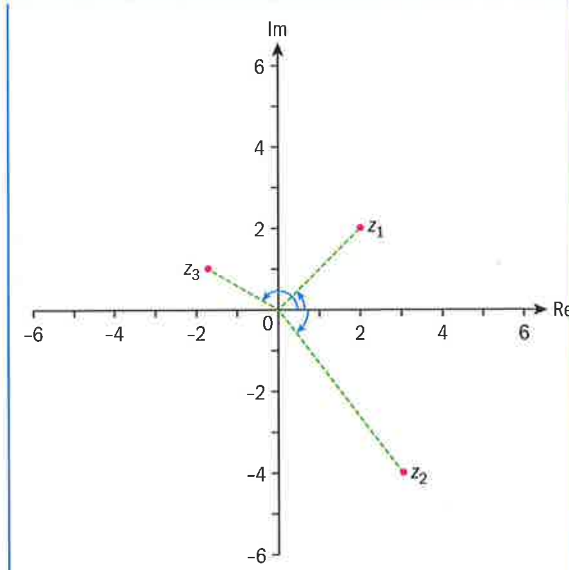
Continued on next page

c $A_1 = \frac{1}{2} \times \sqrt{8} \times 5 \sin\left(\frac{\pi}{4} + 0.927\right) = 2.73$

$A_2 = \frac{1}{2} \times 5 \times 2 \sin(2\pi - 0.927 - 2.62) = 1.96$

$A_3 = \frac{1}{2} \times \sqrt{8} \times 2 \sin\left(2.62 - \frac{\pi}{4}\right) = 7$

Total area = 11.7 (3s.f.)

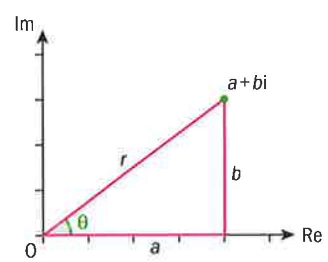


The total area can be found by adding the area of each of the 3 triangles, found using the formula $A = \frac{1}{2} ab \sin C$.

The required angles can be easily found from the arguments of the points.

If the complex number $a + bi$ has a modulus of r and an argument of θ we can use right-angled trigonometry to write down $a = r \cos \theta$ and $b = r \sin \theta$.

Hence $a + bi = r \cos \theta + r \sin \theta i = r(\cos \theta + \sin \theta i)$. This expression is often abbreviated to $a + bi = r \text{ cis } \theta$.



HINT

This form has many names, including **modulus-argument** form, **polar** form or **cis** form.

The form $a + bi$ is called **Cartesian** form (after René Descartes (1596–1650)) or **rectangular** form.

The definition of sine and cosine on the unit circle ensures that $a + bi = r \text{ cis } \theta$ is also valid when $\theta < 0$ and $\theta > \frac{\pi}{2}$.

Example 8

a Write the complex number 2 + 3i in modulus argument form.

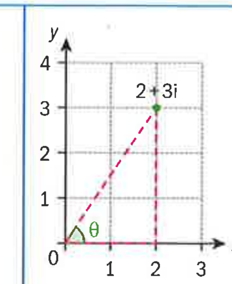
b Find in the form $a + bi$ the complex numbers with the following modulus (r) and argument (θ) values.

i $r = 3, \theta = 0.4$ **ii** $r = 5, \theta = 3.4$

a modulus $= \sqrt{13}$

argument $= \arctan\left(\frac{3}{2}\right) = 0.983$

$2 + 3i = \sqrt{13} \text{ cis}(0.983)$



b **i** $a + bi = 3(\cos 0.4 + \sin 0.4 i)$
$= 2.76 + 1.17i$

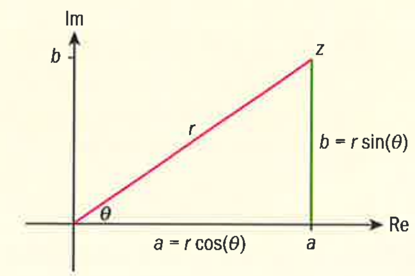
ii $a + bi = 5(\cos 3.4 + \sin 3.4i)$
$= -4.83 - 1.28i$

Applying the formula, ensuring the GDC is set to radian mode.

$3.4 > \pi$ so the complex number must lie in the fourth quadrant and so we expect a and b to be negative, which is what is given by the formula.

A complex number, z, can be expressed in different forms:

- In Cartesian form, $z = a + bi$
- In polar form, $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$. This is often abbreviated to $z = r \text{cis}\theta$.



Exercise 8E

1 Plot the following complex numbers on an Argand diagram and hence give their values in modulus and argument form.

a 8i **b** −7 **c** 12 **d** −5i

2 Find the modulus and argument of the following complex numbers, giving your answers to 3 significant figures.

a 2 − 3i **b** 2 + 5i **c** 3 − i
d −4 − 2i **e** − 5 + 2i **f** 1− 3i

3 Given that

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $\tan\frac{\pi}{4} = 1$ and $\tan\frac{\pi}{3} = \sqrt{3}$, find exact values for the modulus and argument for each of the following complex numbers. Hence write the numbers in the form $r\text{cis}\theta$.

a $4 + 4i$ **b** $2 + 2\sqrt{3}i$
c $3\sqrt{3} - 3i$ **d** $-\sqrt{3} - i$
e $-5 + 5i$ **f** $7 - 7\sqrt{3}i$

4 Use the expression $a + bi = r \cos \theta + r \sin \theta i$ to find in the form $a + bi$ the complex numbers with the following modulus (r) and argument (θ).

a $r = 3, \theta = 60°$
b $r = 4, \theta = 120°$
c $r = 2, \theta = -150°$
d $r = 5, \theta = 0.4$
e $r = 2.4, \theta = 1.9$
f $r = 3.8, \theta = -0.6$

TOK

Imagination is one of the ways of knowing in TOK. How does this relate to imaginary numbers?

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Around 1740 the mathematician Leonard Euler proved the following link between exponential and trigonometric functions:

$e^{i\theta} = \cos\theta + i\sin\theta$ which is known as Euler's formula.

A result of this formula is that the complex number $a + bi = r\cos\theta + r\sin\theta i = r(\cos\theta + \sin\theta i)$ can also be written as $a + bi = re^{i\theta}$.

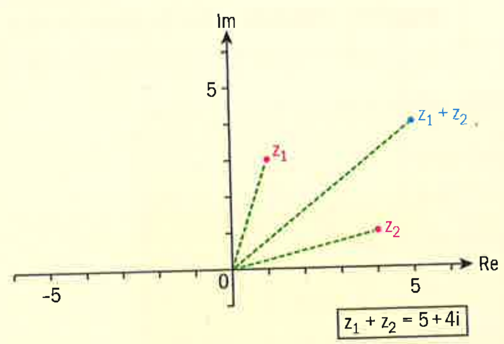
This form is referred to as **exponential** or **Euler** form, and like cis it is also an example of polar or modulus-argument form as the parameters are the modulus and the argument.

HINT

Most GDCs will convert from Cartesian to modulus-argument form (rectangular to polar form) and vice versa. When they do they are likely to use the exponential form for the modulus-argument form.

Investigation 6

The diagram shows the complex numbers $z_1 = 1 + 3i$, $z_2 = 4 + i$ and their sum.



1 If the points represented by $O, z_1, z_1 + z_2$ and z_2 were joined, what would be the shape formed?

Experiment with other values of z_1 and z_2.

What conclusion can you reach about the geometrical interpretation of addition of two complex numbers?

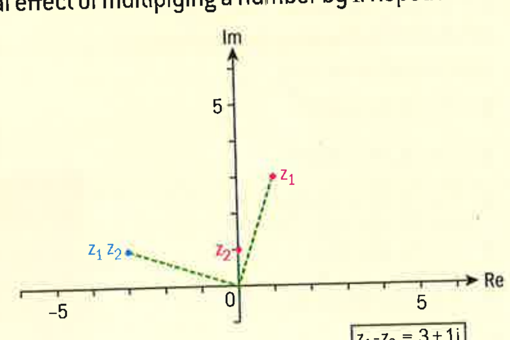
How can the addition/subtraction be visualized geometrically?

2 By examining different values of z_1 and z_2 conclude a similar geometrical interpretation of subtraction of two complex numbers.

3 **Conceptual** How can the addition and subtraction of complex numbers compare to the addition and subtraction of vectors?

4 The effect of multiplying $1 + 3i$ by i is shown.

Experiment multiplying other numbers by i and draw a conclusion as to the geometrical effect of multiplying a number by i. Repeat for multiplying by $2i$.



5 Use the definition of $e^{i\theta} = \cos\theta + i\sin\theta$ to show $|e^{i\theta}| = 1$.

6 a State the modulus and argument of i and hence write i in the form $re^{i\theta}$.

A general complex number is written as $re^{i\theta}$.

b Use your answer to part **6a** to find $i \times re^{i\theta}$ in exponential form. Does this confirm your conjecture in question **4**?

7 Two complex numbers z_1 and z_2 are written in exponential form as $z_1 = r_1 e^{\theta_1 i}$ and $z_2 = r_2 e^{\theta_2 i}$. Find expressions for

a $z_1 z_2$ b $\frac{z_1}{z_2}$.

8 Describe the geometrical effect of multiplying the complex number z by the complex number $z_1 = re^{\theta i}$, in terms of a stretch and a rotation.

HINT

Transformations are covered in detail in Chapter 9. It is sufficient for now to refer to an increase in the modulus of a complex number by a factor of a as a stretch with scale factor a.

9 a Find $3i \times (\sqrt{2} + \sqrt{2}i)$.

b Show $3i$, $\sqrt{2} + \sqrt{2}i$ and $3i \times (\sqrt{2} + \sqrt{2}i)$ on an Argand diagram.

c Write $\sqrt{2} + \sqrt{2}i$ in exponential form. Use the diagram drawn in part **b** to verify your conjecture in question **8**.

Factual What is the exponential form of a complex number?

Conceptual What will be the geometrical effect of multiplying z_1 by z_2 when $|z_2| = r$ and $\arg z_2 = \theta$?

Powers of complex numbers

The exponential form provides an easy way to find powers of complex numbers.

If $z = re^{i\theta}$ then $z^n = (re^{i\theta})^n = r^n e^{in\theta}$ If z is raised to the power n then its modulus is also raised to the power n and the argument is multiplied by n.

This can also be written expressed in cis form:

If $z = r\text{cis}\theta$ then $z^n = r^n\text{cis}(n\theta)$.

Example 9

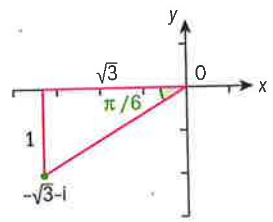
a Given $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ find the modulus and argument of $z = -\sqrt{3} - \text{i}$.

b Find an expression for z^n and hence find the smallest value of n for which $\text{Im}(z^n) = 0$ and for this value of n give z^n in Cartesian form.

a $|z| = \sqrt{1+3} = 2$

$\arg z = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

The modulus and argument can be found using the diagram below:



An alternative solution is:

$\arg z = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

z^n can easily be found using exponential form.

b $z^n = 2^n \text{e}^{\frac{-5\pi}{6}n\text{i}}$

$\frac{-5\pi}{6}n = k\pi$, where $k \in \mathbb{Z}$.

Hence smallest value of n is 6

$z^6 = 2^6 \text{e}^{-5\pi\text{i}} = -64$

If $\text{Im}(z^n) = 0$ then the argument has to be a multiple of π so the point lies on the real axis.

An argument of -5π will place the number on the negative real axis.

This could also be done on a GDC.

Movement in the Argand plane

Investigation 7

Complex numbers can be used to define curves on the Argand plane.

For example as t varies the points given by $z = 2\text{e}^{t\text{i}}$ will form a circle, centre $(0, 0)$ with a radius of 2.

What curve will the points given by $z = t\text{e}^{t\text{i}}$ form as t varies?

To see the curve enter the expression into suitable software

Investigate the effect on the curve of changing r and a and write down your conclusions.

How can complex numbers be used to create spiral curves?

Exercise 8F

1 **a** Given that $a = 1 + \sqrt{3}\,\text{i}$, $b = -1 + \text{i}$ and $c = \sqrt{3} - \text{i}$, find a, b and c in modulus-argument form using the values $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $\tan\frac{\pi}{4} = 1$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

b Hence find each of the following in Cartesian form:

i ac **ii** $\frac{a}{c}$ **iii** b^4

iv $\frac{a^3}{b^2}$ **v** $\frac{a^2 b}{c^2}$

2 If $z = 3 + \text{i}$, find the values of n for which $\text{Im}(z^n) = 0$.

3 **a** Write the number $\sqrt{2} + \sqrt{2}\text{i}$ in exponential form.

b Draw the point $3 + 3\text{i}$ on an Argand diagram.

c Without calculating the values plot the approximate positions of u, v and s where:

$u = (3+3\text{i}) + (\sqrt{2} + \sqrt{2}\text{i})$

$v = (3+3\text{i}) \times (\sqrt{2} + \sqrt{2}\text{i})$ $\quad s = \frac{(3+3\text{i})}{(\sqrt{2} + \sqrt{2}\text{i})}$

d Describe the geometrical transformation(s) that will take $3 + 3\text{i}$ onto of u, v and s.

4 Let $z = r\text{e}^{\text{i}\theta}$. Write z^* (the conjugate of z) in exponential form and hence prove $\text{Im}(zz^*) = 0$.

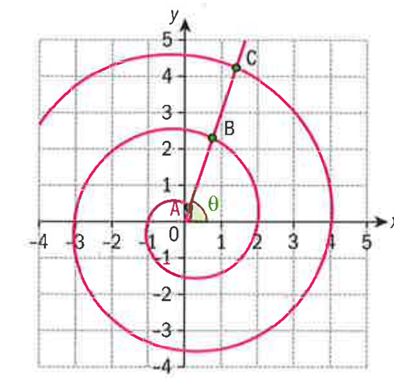
5 Use Euler's formula $\text{e}^{\text{i}\theta} = \cos\theta + \text{i}\sin\theta$ to obtain Euler's identity $\text{e}^{\text{i}\pi} + 1 = 0$.

TOK

This formula is widely regarded as the most beautiful in mathematics as it combines the five fundamental numerical constants into a simple formula.

6 **a** Describe the line formed on the Argand plane by all the points, z, for which $\arg z = \frac{\pi}{4}$

The diagram below shows the complex numbers given by the expression $z = 0.5t\text{e}^{\frac{\pi}{2}t\text{i}}$, along with those points, z, for which $\arg z = \theta$.



b Find the values of t at which the curve first crosses:

i the imaginary axis

ii the real axis.

c Give the values of z for each of these points in Cartesian form.

The curve and line shown intersect at points, A, B and C.

d Find in exponential form and in terms of θ, the complex numbers, z_1, z_2 and z_3 which are at the points A, B and C.

e Write down $|z_2| - |z_1|$ and $|z_3| - |z_2|$.

Would you expect this pattern to continue? Justify your answer.

Number and algebra

Functions

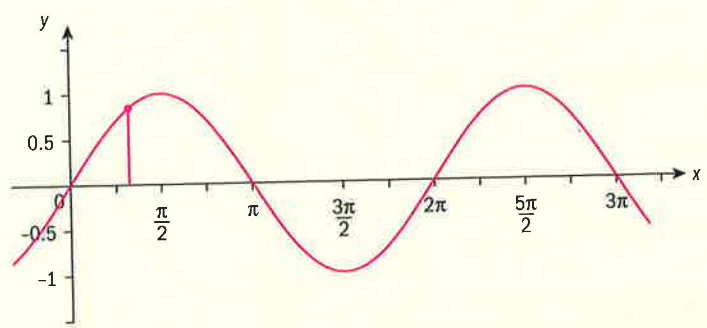
Geometry and trigonometry

8.5 Using complex numbers to understand periodic models

It is surprising that complex numbers, that have been developed in a very abstract context, turn out to be very important in practical applications where sinusoidal functions are found. They have a particular importance in electronics where currents or voltages are following sinusoidal waves.

Investigation 8

1 The diagram shows the output of the "sine function"



It shows how we can plot the imaginary part of a complex number z, with fixed modulus (r_1), as a function of its argument (angle), x.

Show that $f(\theta)=\text{Im}(r_1\,\text{cis}\,\theta)$ is a sinusoidal function.

2 **Conceptual** What effect will changing the modulus of z (ie r_1) have on the function?

3 Describe the function $f(\theta)=\text{Im}\left(r_1\,\text{cis}\left(\theta+\frac{\pi}{6}\right)\right)$.

4 Show that the function $f(x)=a\sin(x+c)$ can be written $f(x)=\text{Im}(a\text{e}^{xi}\text{e}^{ci})$.

If two functions are defined $f(x)=\sin(x)$ and $g(x)=2\sin\left(x+\frac{\pi}{2}\right)$, you can use your knowledge of complex numbers to understand the nature of $h(x)=f(x)+g(x)$.

5 a Use the result from question 4 to write $f(x)$ and $g(x)$ in the form $\text{Im}(a\text{e}^{xi}\text{e}^{ci})$.

b Hence show that $h(x)=\text{Im}\left(\text{e}^{ix}\left(1+2\text{e}^{\frac{\pi}{2}i}\right)\right)$

c Use your GDC to write $1+2\text{e}^{\frac{\pi}{2}i}$ in exponential form.

d Hence show that $h(x)=2.24\sin(x+1.11)$.

e Verify your answer by using your GDC to plot both $h(x)=\sin x+2\sin\left(x+\frac{\pi}{2}\right)$ and $h(x)=2.24\sin(x+1.11)$.

f State the amplitude of the combined trigonometric functions.

6 If two functions are defined $f(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ and $g(x)=2\sin\left(x+\frac{2\pi}{3}\right)$, show that $h(x)=\text{Im}\left(\text{e}^{ix}\left(\sqrt{2}\text{e}^{i\frac{\pi}{4}}+2\text{e}^{2i\frac{\pi}{3}}\right)\right)$

and hence that $h(x)=\text{Im}\left(\text{e}^{ix}\left(a\text{e}^{bi}\right)\right)$, $a, b\in\mathbb{R}$.

a Find the values of a and b.

b Deduce $h(x)$ can be written as $h(x)=2.73\sin(x+1.57)$.

7 **Conceptual** What can you deduce about the sum of two sine functions with the same frequency and amplitudes r_1 and r_2 and phase shifts α_1 and α_2?

8 If $f(x)=r_1\sin(ax+\alpha_1)$ and $g(x)=r_2\sin(ax+\alpha_2)$, show that $f(x)+g(x)=\text{Im}\left(\text{e}^{iax}\left(r_1\text{e}^{i\alpha_1}+r_2\text{e}^{i\alpha_2}\right)\right)$.

Hence show that if $f(x)+g(x)=r\sin(ax+\alpha)$, then $r=\left|r_1\text{e}^{i\alpha_1}+r_2\text{e}^{i\alpha_2}\right|$ $\alpha=\arg\left(r_1\text{e}^{i\alpha_1}+r_2\text{e}^{i\alpha_2}\right)$.

HINT

When adding sine functions of the form $\sin(b(x+c))$ or $\sin(bx+bc)$ it is normal to refer to the phase shift as bc rather c. In exams the questions will always make it clear which definition is being used.

Example 10

The voltage of an AC electrical source can be modelled by the equation $V=a\sin(bt+c)$, where c is the phase shift. Two AC sources with equal frequencies are combined. One has a maximum voltage of 60V and the other of 80V. The amplitude of the sine function gives the maximum voltage of each electrical source. The first electrical source has a phase shift of 30° and the other of 120°. Find the maximum voltage and the phase difference of the combined source.

amplitude = $\left\|60\text{e}^{30i}+80\text{e}^{120i}\right\|=100$	These are found using the formula derived in the investigation. A fuller solution is below.
Phase shift = $\arg\left(60\text{e}^{30i}+80\text{e}^{120i}\right)=83.1°$	The two voltages can be written as $V_1=60\sin(bt+30)$ $V_2=80\sin(bt+120)$ $V_1+V_2=\text{Im}(\text{e}^{bti}(60\text{e}^{30i}+80\text{e}^{120i}))$ $=\text{Im}(\text{e}^{bti}(100\text{e}^{83.1i}))$ $=100\sin(bt+83.1°)$

HINT

Some GDCs will only work in radians for some complex number operations. If you are not obtaining the answers shown here, then you should convert the phase shift to radians before calculating.

Exercise 8G

1 Show that if two AC electrical sources each output a maximum of 110V and have a phase difference of 60° then the combined output will have a maximum of $110\sqrt{3}$ volts.

2 A three phase electrical supply has three sources, each with a maximum 110V output and a phase difference of $\frac{2\pi}{3}$ between each.

a Show that if the three phases are added together the output will be zero.

b If one of the three sources is connected in reverse the value of the voltage is the negative of the previous value. Explain why that would be equivalent to adding an extra π to the phase shift.

c Show that the new output in this case would be 220V.

3 Two electrical sources have maximum outputs 6V AC and 10V AC and a phase difference of 40°. Find the output of the combined sources.

4 Two functions are defined as $f(x) = 3\sin\left(3x + \frac{\pi}{12}\right)$ and

$g(x) = 4\sin\left(3x + \frac{3\pi}{4}\right)$

Find r and α if
$f(x) + g(x) = r\sin(3x + \alpha)$.

5 In Exercise 8C, question 1 you found that the times of sunrise, in hours after midnight, was given by $f(t) = 2.14\sin(0.0165t + 1.81) + 5.97$ where t is the number of days after midnight on 31 December.

In a similar way, the times of sunset can be modelled by the function
$g(t) = 2.19\sin(0.0165t - 1.23) + 18.0$.

a Show that the length of day can be modelled by the function
$h(t) = 2.19\sin(0.0165t - 1.23) + 2.14\sin(0.0165t + 4.95) + 12.03$

b Hence find the length of the longest and shortest days according to the model.

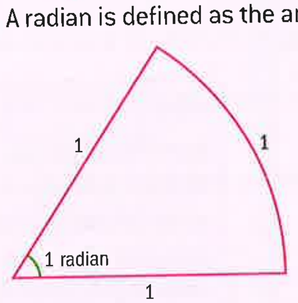
c According to the model, determine the dates on which the longest and shortest days fall.

TOK

Is there always a trade-off between accuracy and simplicity?

Chapter summary

- A radian is defined as the angle subtended at the centre by an arc of length 1 and radius 1.

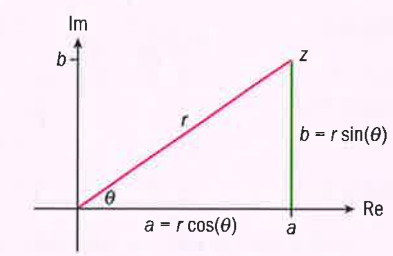


- π radians are equivalent to 180 degrees.
- The length of arc, L, of a sector radius r, is given by $L = r\theta$, if θ is in radians.
- The area, A, of a sector radius r, is given by $A = \frac{1}{2}r^2\theta$, if θ is in radians.
- The **period** of a function is defined as the length to complete one cycle before repeating. For $y = \sin x$ the period is 2π.

- The **amplitude** of a function is defined as the distance from the principal axis to the maximum or minimum. For $y = \sin x$ the amplitude is 1.
- The **phase shift** of a periodic function is defined as the distance the graph has moved to the left.
- The graph of $f(x) = a\sin x + d$ can be obtained from the graph of $f(x) = \sin x$ by stretching vertically by a scale factor of a (giving an amplitude of a), then translating d units upwards (in the positive y direction).
- The graph of $f(x) = \sin b(x + c)$ can be obtained from the graph of $f(x) = \sin x$ by stretching horizontally by a scale factor of $\frac{1}{b}$ (giving a period of $\frac{2\pi}{b}$), then translating c units to the left (in the negative y direction).
- Similarly the graph of $f(x) = \sin(bx + c)$ can be obtained from the graph of $f(x) = \sin x$ by translating c units to the left then stretching horizontally by a scale factor of $\frac{1}{b}$.
- The graph of $f(x) = -\sin x$ can be obtained from the graph of $f(x) = \sin x$ by reflecting the graph in the x-axis.
- The graph of $f(x) = \sin(-x)$ can be obtained from the graph of $f(x) = \sin x$ by reflecting the graph in the y-axis.

A **complex number** has two parts, and can be written in the form $a + bi$ where $a, b \in \mathbb{R}$. a is known as the **real part** and b is known as the **imaginary part.** The notation used is: $\text{Re}(a + bi) = a$ and $\text{Im}(a + bi) = b$

- The **conjugate** of a complex number $a + bi$ is defined as $a - bi$.
- Complex solutions to quadratic equations with real coefficients always occur in conjugate pairs.



$|z|$, the **modulus** of a complex number, z, is defined as its distance from O. $\arg z$, the **argument** of a complex number, z, is defined as the angle between the positive real axis and the line from O to z, measured anticlockwise.

A complex number, z, can be expressed in different forms:

- In Cartesian form, $z = a + bi$.
- In polar form, $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$. This is often abbreviated to $z = r\text{cis}\,\theta$.
- In exponential form (also known as Euler form), $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg z$.

Developing inquiry skills

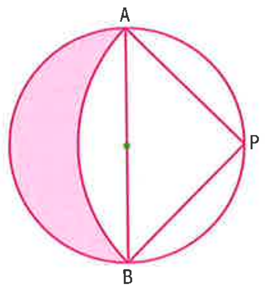
Return to the opening problem. You were given a graph showing high and low tides in the harbour at Aberdeen.

How has what you have learned in this chapter changed the way you understand this problem?

Chapter review

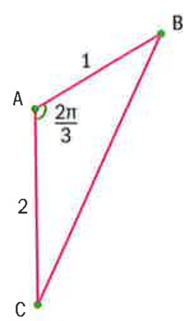
Click here for a mixed review exercise

1 A circle has radius R and [AB] is a diameter. The point P is the midpoint of the arc AB and the angle APB is 90°. A circle with centre P passes through A and B to give the shaded region shown in the diagram.



Find the area of the shaded region in terms of R.

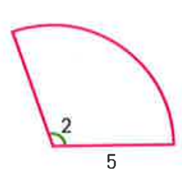
2 In the triangle drawn AB = 1, AC = 2 and angle A is $\frac{2\pi}{3}$.



a Find the area of the triangle.

b Find BC.

3 Find the perimeter and area of the sector of radius 5 cm and internal angle 2 radians shown here.

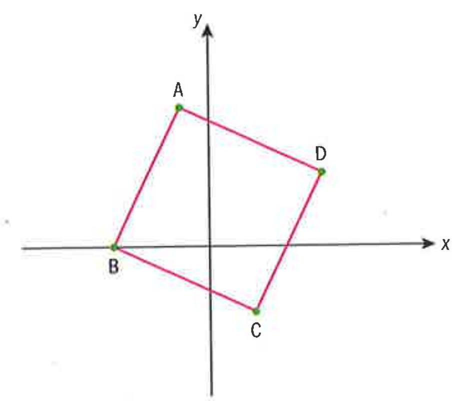


4 Calculate the argument of the complex number $\left(\sqrt{3}+i\right)^{10}$ giving your answer in radians between $-\pi$ and π.

5 Given that $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ find $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ in modulus-argument(polar) form. Hence express, in polar form:

a $z_1 z_2$ **b** $\frac{z_1}{z_2}$ **c** $\frac{z_1^3 z_2^3}{i}$

6 In the following Argand diagram the point A represents the complex number $-1 + 4i$ and the point B represents the complex number $-3 + 0i$. The shape of ABCD is a square. Determine the complex numbers represented by the points C and D.



7 **a** If $w = 2 + 2i$, find the modulus and argument of w.

b Given $z = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$, find in its simplest form, $w^4 z^6$.

8 The complex numbers z_1 and z_2 have arguments between 0 and π radians. Given that $z_1 z_2 = -\sqrt{3} + i$ and $\frac{z_1}{z_2} = 2i$, find the modulus and argument of z_1 and of z_2.

9 Find the amplitude of $f(x) = 2\cos(2x + 60°) + 3\sin(2x + 30°)$.

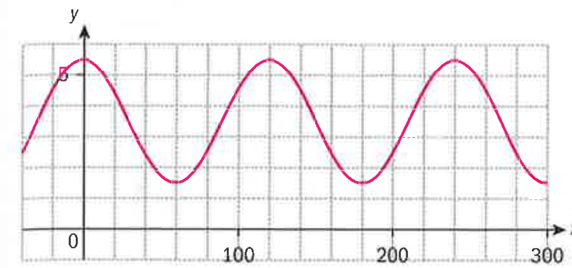
10 An electric source of 20 amps has another source of 10 amps and a phase difference of 60° added to it. Find the total current of the combined power source. If the second source is connected in reverse, determine the combined current?

HINT

If one of the three sources is connected in reverse the value of the voltage is the negative of the previous value. Explain why that would be equivalent to adding an extra π to the phase shift.

Exam-style questions

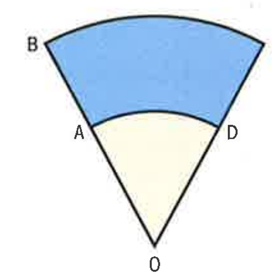
11 **P2:** The following shows a portion of the graph of $y = p + q\cos(rx)$ (where x is given in degrees).



a Determine the values of the constants p, q and r. (6 marks)

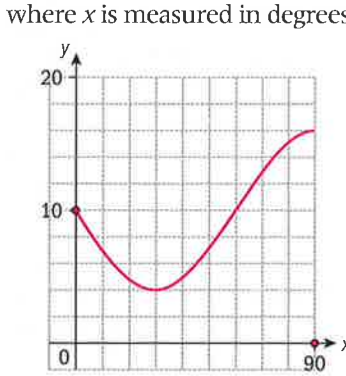
b Hence, using technology, solve the inequality $p + q\cos(rx) < q + p\sin(rx)$ for $0° \le x \le 180°$ (4 marks)

12 **P1:** The following diagram shows the sector of a circle OBC, where O is the centre of the circle, and length OA = 10 cm. AD is the arc of a smaller circle, centre O. $\text{A\^{O}D} = \left(\frac{6}{5}\right)^c$.



Given that the blue and yellow shaded areas are equal, determine the perimeter of ABCD. Give your answer in the form $a\sqrt{2} + b$, where a and b are integers. (8 marks)

13 **P1:** The following shows a portion of the graph of $y = p - q\sin(rx)$ $(p > 0, q > 0, r > 0)$, where x is measured in degrees.



a On the same axes, sketch the graph of $y = p + \frac{q}{2}\sin(rx)$. (2 marks)

b Determine the values of the constants p, q and r. (7 marks)

14 **P1:** Two sources of electrical alternating current (AC) have voltages of 100 V and 180 V respectively. The first source has a phase shift of 45° and the second has a phase shift of 135°.

The two sources of current are combined.

a Find the voltage of the combined source. (2 marks)

b Find the phase shift of the combined source. (3 marks)

15 **P2:** Given $z_1 = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$ and $z_2 = 3\,\text{cis}\left(\frac{5\pi}{6}\right)$, find expressions for the following. In each case, give your answers in the form $a + bi$, $a,b \in \mathbb{R}$.

a $z_1 z_2$ (3 marks)

b $\left(\frac{z_1}{z_2}\right)^3$ (5 marks)

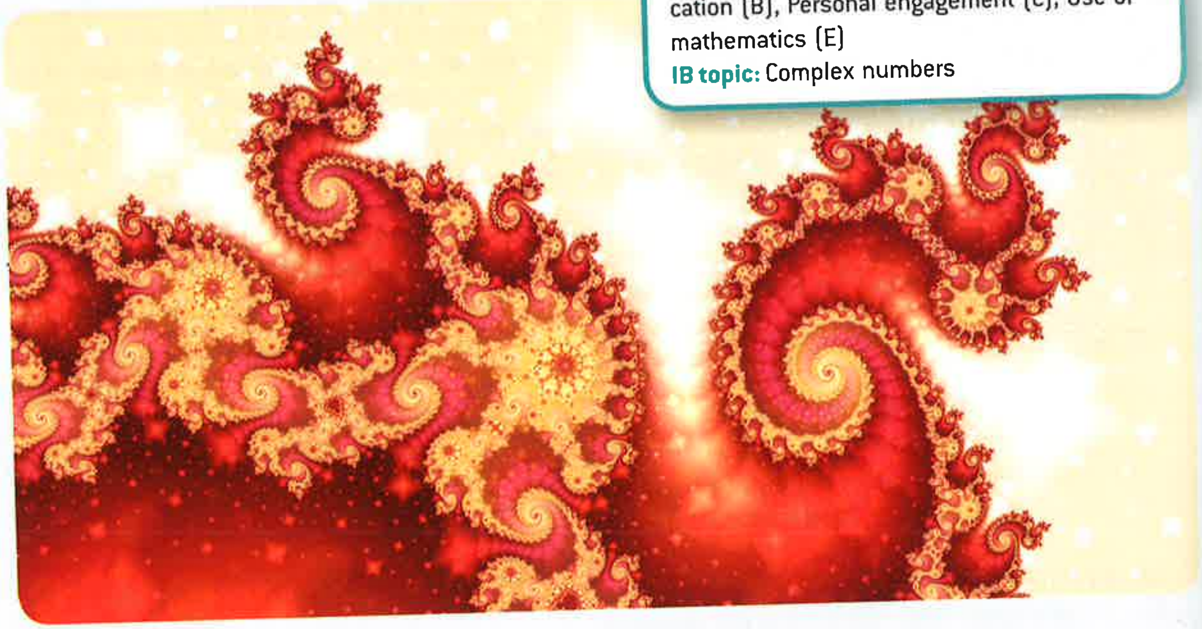
c $\left(z_1^2\right)^*$ (3 marks)

16 **P1:** Two functions are given such that $f(x) = \sin\left(2x + \frac{\pi}{3}\right)$ and $g(x) = 2\sin\left(2x + \frac{\pi}{4}\right)$.

Given that $f(x) + g(x) = r\sin(2x + \alpha)$ $(r > 0, \alpha > 0)$, determine possible values for r and α, giving your answers to 3 significant figures. (8 marks)

Making a Mandelbrot!

Approaches to learning: Critical thinking, Communication,
Exploration criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)
IB topic: Complex numbers



Fractals

You may have heard about fractals.

This image is from the Mandelbrot set, one of the most famous examples of a fractal:

This is not only a beautiful image in its own right. The Mandelbrot set as a whole is an object of great interest to mathematicians. However, as yet there have been no practical applications found!

This image appears to be very complicated, but is in fact created using a remarkably simple rule.

Exploring an iterative equation

Consider this iterative equation:

$Z_{n+1} = (Z_n)^2 + c$

Consider a value of $c = 0.5$, so you have $Z_{n+1} = (Z_n)^2 + 0.5$.

Given that $Z_1 = 0$, find the value of Z_2.

Now find the value of Z_3.

Repeat for a few more iterations.

What do you think is happening to the values found in this calculation?

A different iterative equation

Now consider a value of $c = -0.5$, so you have $Z_{n+1} = (Z_n)^2 + 0.5$.

Again start with $Z_1 = 0$.

What happens this time?

The Mandelbrot set

The Mandelbrot set consists of all those values of c for which the sequence starting at $Z_1 = 0$ does **not** escape to infinity (those values of c where the calculations zoom off to infinity).

That is only part of the story however.

The value of c in the function does not need to be real.

It could be complex of the form $a + bi$.

If you pick a complex number for c, is it in the Mandelbrot set or not?

Consider starting with a value of $Z_1 = 0 + 0i$, rather than just "0".

Consider, for example, a value of c of $1 + i$, so you have $Z_{n+1} = (Z_n)^2 + (1 + i)$.

Find Z_2.

Repeat to find Z_3, Z_4, etc.

Does this diverge or converge? (Does it zoom off to infinity or not?)

Repeat for a few more values of c.

Try, for example:

a $c = 0.2 - 0.7i$

b $c = -0.25 + 0.5i$

What happens if $c = i$?

Try some values of your own.

You may find a GDC will help with these calculations as it can be used for complex numbers. When the number gets very big, the calculator will not be able to give a value. You may also notice that some numbers get "big" a lot quicker than others.

It is clearly a time-consuming process to try all values!

There are programs that can be used to calculate the output after several iterations when you input a number for c. This will indicate which values of c belong to the Mandelbrot set and which don't.

The Mandelbrot set diagram is created by colouring points on an Argand diagram of all those values of c which do not escape to infinity in one colour (say black) and all those that remain bounded in another colour (say red).

HINT

One way to check is to plot the different values of Z_1, Z_2, Z_3, Z_4, etc, on an Argand diagram and to see whether the points remain within a boundary square, or you could calculate the modulus of each value and see whether this is increasing.

Extension

- Try to construct your own spreadsheet or write a code that could do this calculation a number of times for different chosen values of c.
- Explore what happens as you zoom in to the edges of the Mandelbrot set.
- What is the relationship between Julia sets and the Mandelbrot set?
- What is the connection to Chaos Theory?
- What are Multibrot sets?
- How could you find the area or perimeter of the Mandelbrot set?