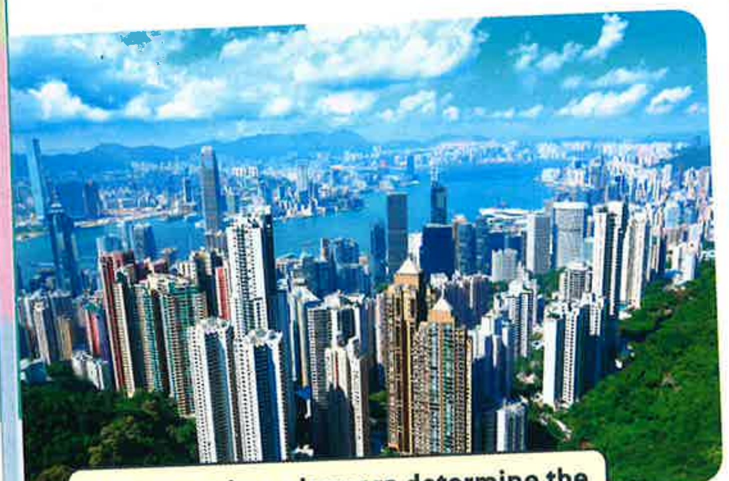
9 Modelling with matrices: storing and analysing data

Concepts

- Systems
- Modelling

Microconcepts

- Matrix
- Array
- Order of matrix
- Element
- Dimensions of matrix
- Eigenvalue
- Eigenvector
- Fractal
- Determinant
- Inverse
- Identity
- Linear system
- Transformations
- Iterative function systems



How can urban planners determine the population of different cities if people are constantly moving between them?

How can athletes ensure their diet provides all the nutrients they need?

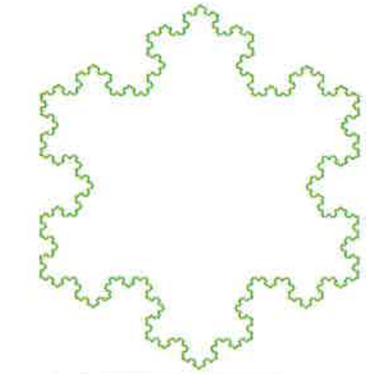
How do we send messages securely?

How do computers simulate movement?

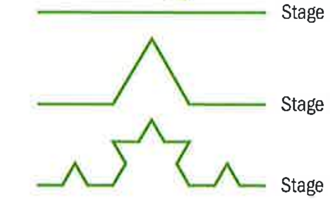
Is it possible to create a snowflake using mathematical formulae?

The Koch snowflake is an attempt to describe the patterns like those found in an ice crystal.

When an ice crystal is magnified we see that smaller versions of the ice crystal can be found at each level of magnification. This property is called self-similarity and structures that display it are called *fractals*. By identifying and describing the smallest piece in this pattern with a mathematical process we can create the entire structure simply by carrying out this process repeatedly.

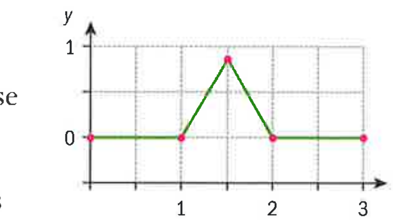


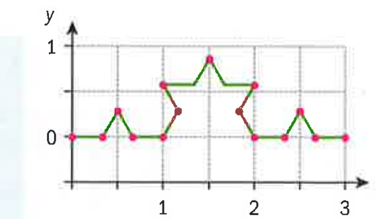
Each stage of the Koch snowflake is described by the iterative process shown.



We begin by dividing the line segment in stage 0 into thirds and creating an equilateral triangle in the middle third of the segment.

- Write down step-by-step instructions describing geometrically how you could use the shape in stage 1 to create the shapes in stages 2, 3 and 4.
- Construct graphs of stages 1 and 2 of the Koch snowflake on square millimetre graph paper using the scale 1 unit = 30mm like the one shown. Determine the exact coordinates of each vertex and endpoint for each figure.
- Is there a pattern in the values of the coordinates of each of these points that you could use to determine the coordinates of the points on stage 3?





Save your work as you will need the results to check your answers to the investigation at the end of the unit.

Developing inquiry skills

Where else do fractals appear in the real world?

Can we use fractal formulae to model real-life situations? How might these be useful?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1. Identify and write the general term of a geometric sequence.
   eg The general term of $4, -8, 16, -32, \ldots$ is $u_n = 4 \times (-2)^{n-1}$
2. Find the sum of geometric series.
   eg The sum of 8 terms of the series $12 + 6 + 3 + 1.5 + \ldots$ is
   $$S_8 = \frac{12(1-0.5^8)}{1-0.5} = 23.9.$$

Skills check

Click here for help with this skills check

1. Find a formula for the nth term in the sequence: $\left\{18, -12, 8, -\frac{16}{3}, \ldots\right\}$
2. Determine the sum of the first 15 terms of each sequence in question 2.
3. Write an expression for S_n for the series $6 + 6\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right)^3 + \cdots$ then use it to find S_{12}.

9.1 Introduction to matrices and matrix operations

Computers store large amounts of data using a programming structure called an array. Understanding how data is stored and manipulated leads you to a study of matrices and matrix operations.

A **matrix** A with dimensions $m \times n$ is a rectangular array of real (or complex) numbers containing m rows and n columns where $a_{i,j}$ refers to the element located in row i and column j. Such a matrix is said to have order $m \times n$.

$$\begin{array}{cccc} & \text{Column 1} & \text{Column 2} & & \text{Column n} & \\ A = & \left(\begin{array}{c} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{array}\right. & \begin{array}{c} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \\ \cdots \end{array} & \left.\begin{array}{c} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{array}\right) & \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \\ \leftarrow \text{row m} \end{array} \end{array}$$

International-mindedness

English mathematician and lawyer, James Sylvester, introduced the term "matrix" in the 19th century and his friend, Arthur Cayley, advanced the algebraic aspect of matrices.

For example, matrix $\boldsymbol{P} = \begin{pmatrix} -1 & 3 & 4 & 2 \\ 1 & 5 & 0 & 7 \\ -3 & 4 & -2 & 6 \end{pmatrix}$ has dimensions 3×4 where $a_{2,3}$ is the element located in the 2nd row and 3rd column and has a value of 0. Note that matrix $\boldsymbol{P}$ contains $3 \times 4 = 12$ elements.

If $m = n$ (ie the number of columns is equal to the number of rows) then we say that $\boldsymbol{P}$ is a **square matrix of order n.**

Consider the 3×5 matrix $\boldsymbol{C}$ showing the cost (in euros) of manufacturing 3 different products for 5 different manufacturers.

$$\begin{array}{lccccc} & \text{Manufacturer 1} & \text{Manufacturer 2} & \text{Manufacturer 3} & \text{Manufacturer 4} & \text{Manufacturer 5} \\ & \left(\begin{array}{c}\\ \\ \\ \end{array}\right. & & & & \\ \end{array}$$

$$C = \begin{array}{l} \text{Product 1} \\ \text{Product 2} \\ \text{Product 3} \end{array} \begin{pmatrix} 3.50 & 4.25 & 3.82 & 4.10 & 3.95 \\ 8.25 & 10.50 & .75 & 9.45 & 7.80 \\ 10.50 & 9.75 & 8.95 & 11.00 & 10.80 \end{pmatrix}$$

For example, the cost of producing product 3 with manufacturer 4 is given by element $a_{3,4}$ and is €11.00.

Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if the matrices have the same dimensions and their corresponding elements are equal.

Matrix addition

Three models of a particular brand of television undergo two manufacturing processes each carried out in different factories. The elements of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ represent the shipping and manufacturing costs (in USD) for each of the three models at each of two factories.

$$A = \begin{array}{cc} \text{Manufacturing} & \text{Shipping} \\ \end{array}\begin{pmatrix} 125 & 35 \\ 275 & 40 \\ 180 & 55 \end{pmatrix}\begin{array}{l} \text{Model 1} \\ \text{Model 2} \\ \text{Model 3} \end{array} \quad \text{and} \quad B = \begin{array}{cc} \text{Manufacturing} & \text{Shipping} \end{array}\begin{pmatrix} 300 & 50 \\ 210 & 35 \\ 325 & 65 \end{pmatrix}\begin{array}{l} \text{Model 1} \\ \text{Model 2} \\ \text{Model 3} \end{array}$$

Adding $a_{1,1} + b_{1,1} = 425$ indicates that the total manufacturing cost of Model 1is $425. Similarly, adding $a_{3,2} + b_{3,2} = 120$ and indicates that the total shipping cost of Model 3 is $120.

Thus,

$$A+B=\begin{pmatrix} 125 & 30 \\ 275 & 40 \\ 180 & 55 \end{pmatrix} + \begin{pmatrix} 300 & 50 \\ 210 & 35 \\ 325 & 65 \end{pmatrix} = \begin{pmatrix} 425 & 85 \\ 485 & 75 \\ 505 & 120 \end{pmatrix}$$

To add or subtract two or more matrices, they must be of the same order. You add or subtract corresponding elements.

TOK

"There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world." – Nikolai Lobatchevsky

Where does the power of mathematics come from? Is it from its ability to communicate as a language, from the axiomatic proofs or from its abstract nature?

The zero matrix $\boldsymbol{0}$ is the matrix whose entries are all zero.

For 2×2 matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero matrix, for 2×3 matrices it is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

For any matrix $\boldsymbol{A}$, $\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A}$ and $\boldsymbol{A} - \boldsymbol{0} = \boldsymbol{A}$.

Scalar multiplication for matrices

Consider the situation where manufacturer A increases both its shipping and manufacturing costs by 10%. That is, each new cost is 110% of its old cost. Therefore, multiplying each value by 1.10 gives a matrix representing the new costs.

$$1.10A = \begin{pmatrix} 125(1.10) & 35(1.10) \\ 275(1.10) & 40(1.10) \\ 180(1.10) & 55(1.10) \end{pmatrix} = \begin{pmatrix} 137.50 & 38.50 \\ 302.50 & 44 \\ 198 & 60.5 \end{pmatrix}$$

In this way, we define **scalar multiplication** of matrices.

Given a matrix $\boldsymbol{A}$ and a real number k then $k\boldsymbol{A}$ is obtained by multiplying every element of $\boldsymbol{A}$ by k where k is referred to as a scalar.

HINT

A scalar is a quantity that is one-dimensional. Some examples of scalars are temperature and weight.

Example 1

Find a and b if $2\boldsymbol{P} - 5\boldsymbol{Q} = \boldsymbol{0}$, $\boldsymbol{P} = \begin{pmatrix} 1 & 2b \\ 3c & 1 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} a & -1 \\ -1 & 0.4 \end{pmatrix}$

$2\begin{pmatrix} 1 & 2b \\ 3c & 1 \end{pmatrix} - 5\begin{pmatrix} a & -1 \\ -1 & 0.4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	The zero matrix is a matrix all of whose elements are 0.
$\begin{pmatrix} 2-5a & 4b+5 \\ 6c+5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	
$2 - 5a = 0 \Rightarrow a = \frac{2}{5}$	If two matrices are equal then their corresponding elements are equal.
$4b + 5 = 0 \Rightarrow b = \frac{-5}{4}$	
$6c + 5 = 0 \Rightarrow c = \frac{-5}{6}$	

Exercise 9A

1 Given that

$$\boldsymbol{A} = \begin{pmatrix} 2 & -3 & 5 \\ 4 & -2 & 6 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix},$$

and $\boldsymbol{C} = \begin{pmatrix} 8 & 5 & 2 \\ -4 & 0 & 3 \\ 1 & -1 & 7 \end{pmatrix}$, write down the values of $a_{1,2}$, $a_{2,3}$, $b_{3,1}$, $c_{2,2}$ and $c_{3,1}$.

2 The matrix $\boldsymbol{Q}$ describes the number of units of each of three products that are produced by each of four manufacturers where $q_{i,j}$ represents the amount of product i produced by manufacturer j.

$$\boldsymbol{Q} = \begin{pmatrix} 420 & 250 & 145 & 0 \\ 340 & 575 & 420 & 100 \\ 200 & 375 & 425 & 235 \end{pmatrix}$$

a Write down the dimensions of $\boldsymbol{Q}$.

b Write down the value of $q_{2,4}$ and describe its meaning.

c Find a 4×1 matrix $\boldsymbol{T}$ representing the total number of products produced by each manufacturer.

3 Consider the matrices $\boldsymbol{R}, \boldsymbol{S}, \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$, and $\boldsymbol{X}$ below.

$$\boldsymbol{R} = \begin{pmatrix} 0 & -1 \\ 3 & 7 \\ -5 & 4 \end{pmatrix} \quad \boldsymbol{S} = \begin{pmatrix} 3 & -2 & 1 \\ 4 & 1 & -7 \\ -9 & 6 & 2 \end{pmatrix}$$

$$\boldsymbol{T} = \begin{pmatrix} -3 & -9 \\ 6 & 12 \end{pmatrix} \quad \boldsymbol{U} = \begin{pmatrix} 8 & -3 & -2 \\ 6 & 7 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{V} = \begin{pmatrix} -2 & 0 \\ -4 & -8 \end{pmatrix} \quad \boldsymbol{W} = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} 7 & -2 \\ -6 & 4 \end{pmatrix}$$

Find each of the following if possible. If not possible, explain why. Check your answers using technology.

a $3\boldsymbol{W}$ **b** $\boldsymbol{R} - \boldsymbol{W}$ **c** $4\boldsymbol{U} + \boldsymbol{S}$

d $\frac{1}{3}\boldsymbol{T} - \frac{1}{2}\boldsymbol{V}$

4 The matrix $\boldsymbol{P} = \begin{pmatrix} 32450 \\ 18725 \\ 24175 \\ 19250 \end{pmatrix}$ describes the prices (in USD) of four types of cars. The sales tax rate is 7.5% of the purchase price.

a Write down $\boldsymbol{T} = k\boldsymbol{P}$ where $\boldsymbol{T}$ represents the amount of tax paid for each car.

b The matrix $\boldsymbol{C}$ represents the cost of each car including tax. Write a matrix equation that describes $\boldsymbol{C}$ in terms of $\boldsymbol{P}$.

5 A column vector can be regarded as a 2×1 matrix. The point $P = (-3, 6)$ is written in matrix form as $\boldsymbol{X} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$.

a Determine the points:

i $2\boldsymbol{X}$

ii $\frac{1}{3}\boldsymbol{X}$

iii $-\boldsymbol{X}$.

b Show each of these the points on a graph.

c Make a conjecture about the set of points described by $k\boldsymbol{X}$.

6 Find real values of μ and λ if

$\mu\boldsymbol{A} + \lambda\boldsymbol{B} = \begin{pmatrix} -8 & 12 \\ -10 & 4 \end{pmatrix}$, where $\boldsymbol{A} = \begin{pmatrix} -9 & 12 \\ -6 & 3 \end{pmatrix}$

and $\boldsymbol{B} = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}$.

9.2 Matrix multiplication and properties

Consider a matrix $\boldsymbol{F}$ whose elements represent the number of each of three types of medium size drinks (frappé, cappuccino, and iced coffee) sold at a café on each day of the first week in June.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Frappé} & \text{Cappuccino} & \text{Iced coffee} \end{array} \\ \boldsymbol{W} = \begin{array}{r} \text{Sun} \\ \text{Mon} \\ \text{Tues} \\ \text{Wed} \\ \text{Thurs} \\ \text{Fri} \\ \text{Sat} \end{array} & \begin{pmatrix} 28 & 32 & 16 \\ 34 & 51 & 8 \\ 40 & 31 & 15 \\ 37 & 24 & 20 \\ 75 & 47 & 29 \\ 38 & 29 & 19 \\ 47 & 34 & 12 \end{pmatrix} \end{array}$$

The price for a frappe is €4.00, cappuccino is €3.50 and iced coffee is €2.50.

The following calculation can be used to determine the revenue $\boldsymbol{R}$ for each day:

day	revenue	=	# frappes sold	×	price	+	# cappuccinos sold	×	price	+	# iced coffee	×	price
Sunday	$264	=	28	×	4.00	+	32	×	3.50	+	16	×	2.50

Similarly, the revenue for each day is obtained by summing the products of the number of each type of drink sold and the corresponding prices for each drink on each day of the week. That is,

$$R=\begin{pmatrix}28(4.00)+32(3.50)+16(2.50)\\34(4.00)+51(3.50)+8(2.50)\\40(4.00)+31(3.50)+15(2.50)\\37(4.00)+24(3.50)+20(2.50)\\75(4.00)+47(3.50)+29(2.50)\\38(4.00)+29(3.50)+19(2.50)\\47(4.00)+34(3.50)+12(2.50)\end{pmatrix}=\begin{pmatrix}264\\334.50\\306\\282\\537\\301\\337\end{pmatrix}$$

A more compact way of expressing R would be to represent the price of each drink as a 3×1 matrix $P=\begin{pmatrix}4.00\\3.50\\2.50\end{pmatrix}$.

so that $$R=WP=\begin{pmatrix}28&32&16\\34&51&8\\40&31&15\\37&24&20\\75&47&29\\38&29&19\\47&34&12\end{pmatrix}\begin{pmatrix}4.00\\3.50\\2.50\end{pmatrix}=\begin{pmatrix}264\\334.50\\306\\282\\537\\301\\337\end{pmatrix}$$

Sunday's revenue $=28(4.00)+32(3.50)+16(2.50)$

$$=\begin{pmatrix}28&32&16\end{pmatrix}\begin{pmatrix}4.00\\3.50\\2.50\end{pmatrix}=264$$

Friday's revenue $=38(4.00)+29(3.50)+19(2.50)$

$$=\begin{pmatrix}38&29&19\end{pmatrix}\begin{pmatrix}4.00\\3.50\\2.50\end{pmatrix}=301$$

If $P=AB$ then each element of P (named as $P_{i,j}$) is found by summing the products of the elements in row i of A with the elements in column j of B.

For example, if $A=\begin{pmatrix}1&5&-3\\-1&4&2\end{pmatrix}$ and $B=\begin{pmatrix}-1&2\\-3&4\\2&1\end{pmatrix}$ then

$$P=\begin{pmatrix}1&5&-3\\-1&4&2\end{pmatrix}\begin{pmatrix}-1&2\\-3&4\\2&1\end{pmatrix}$$

$$=\begin{pmatrix}1(-1)+5(-3)-3(2)&1(2)+5(4)-3(1)\\-1(-1)+4(-3)+2(2)&-1(2)+4(4)+2(1)\end{pmatrix}=\begin{pmatrix}-22&19\\-7&16\end{pmatrix}.$$

Investigation 1

Consider the matrices $A=\begin{pmatrix}2&4\\-1&1\end{pmatrix}$, $B=\begin{pmatrix}2&-1&3\\0&-3&4\end{pmatrix}$, $C=\begin{pmatrix}1\\2\end{pmatrix}$,

$D=\begin{pmatrix}1&2&-2\\0&3&1\\-1&-3&1\end{pmatrix}$, and $E=\begin{pmatrix}-2\\1\\-1\end{pmatrix}$.

1 **Factual** Find the products of AB, AC, BD, DE, and BE.

2 Complete the table below to determine the dimensions (order) of the products AB, AC, BD, DE, and BE.

Product	Dimensions of first matrix	Dimensions of second matrix	Dimensions of product
AB	2 x 2	2 x 3	
AC			
BD			
DE			
BE			

3 **Factual** Based upon the results in question **1** explain how you could determine the dimensions of the product of two matrices using the dimensions of the two matrices being multiplied?

4 Explain why the products BA, CB, and DB cannot be determined.

5 **Conceptual** When does the product of two matrices exist?

If $\mathbf{A}$ has dimensions $m\times n$ and $\mathbf{B}$ has dimensions $p\times q$ then

- the product AB is defined only if the number of columns in $\mathbf{A}$ is equal to the number of rows in B (that is, $n=p$)
- and when the product does exist, the dimensions of the product is $m\times q$.

TOK

How are mathematical definitions different from definitions in other areas of knowledge?

How are mathematical definitions different from properties, axioms, or theorems?

Example 2

A diet research project consists of adults and children of both sexes. The number of participants in the survey is given by the matrix:

$$A=\begin{array}{c}\\ \\ \end{array}\begin{array}{cc}\text{Adults}&\text{Children}\\ \end{array}$$

$$A=\left(\begin{array}{cc}75&180\\110&250\end{array}\right)\begin{array}{l}\text{Male}\\\text{Female}\end{array}$$

Continued on next page

The number of daily grams of protein, fat, and carbohydrates consumed by each child and adult is given by the matrix:

$$B = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccl} \text{Protein} & \text{Fat} & \text{Carbohydrate} & \\ \end{array}$$

$$B = \begin{pmatrix} 15 & 20 & 25 \\ 8 & 16 & 20 \end{pmatrix} \begin{array}{l} \text{Adult} \\ \text{Child} \end{array}$$

a Determine AB. **b** Explain the meaning of $AB_{3,2}$ **c** Explain why BA does not exist.

a $AB = \begin{pmatrix} 75 & 180 \\ 110 & 250 \end{pmatrix}\begin{pmatrix} 15 & 20 & 25 \\ 8 & 16 & 20 \end{pmatrix}$

$= \begin{pmatrix} 75(15)+180(8) & 75(20)+180(16) & 75(25)+180(20) \\ 110(15)+250(16) & 110(20)+250(16) & 110(25)+250(20) \end{pmatrix}$

$= \begin{pmatrix} 2565 & 4380 & 5475 \\ 5650 & 6200 & 7750 \end{pmatrix}$

Multiply the matrices.

b $AB_{2,3} = 110(25) + 250(20) = 7750$ and represents the total number of carbohydrates consumed by females in the project.

c BA is not possible since the number of columns in B is not equal to the number of rows in A.

Total carbohydrates consumed by the 110 adult females is 110(25) and the total carbohydrates consumed by 250 child females is 250(20).

Investigation 2

Use the matrices $A = \begin{pmatrix} -2 & -3 \\ 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 5 \\ 6 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 1 \\ 3 & -4 \end{pmatrix}$ to answer each question.

1. Multiplication of real numbers a and b is commutative since $ab = ba$, for example $5 \times 4 = 4 \times 5 = 20$. Determine AB and BA. Is matrix multiplication commutative?

   Multiplication of real numbers is distributive since $a(b+c) = ab + ac$ and $(b+c)a = ab + ac$.
2. Determine each of the following then describe your observations.

   i $A(B+C)$ ii $(B+C)A$ iii $AB+AC$ iv $BA+CA$
3. Based upon your observations is matrix multiplication distributive? Is $A(B+C) = AB+BC$? Is it correct to write $(B+C)A = AB + CA$?
4. Multiplication of real numbers is associative which means that $a(bc) = (ab)c$. Is matrix multiplication associative? You can answer this by showing whether or not $A(BC) = (AB)C$.
5. Consider the marix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   a Find AI and IA **b** Write down what you notice.
6. **Conceptual** How do matrix and matrix multiplication work in terms of the commutative and associative properties?

Properties of multiplication for matrices

For matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

- Non-commutative $AB \neq BA$
- Associative property $A(BC) = (AB)C$
- Distributive property $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$

These properties only hold when the products are defined.

- In $A(B+C)$, $B+C$ is pre-multiplied by A.
- In $(B+C)A$, $B+C$ is post-multiplied by A.

Multiplicative identity for matrices

The multiplicative identity for real numbers is 1 since $a \times 1 = 1 \times a = a$ for any real number a.

If A is any square matrix and I is the identity matrix, then $A \times I = I \times A = A$. In other words, if any square matrix is pre- or post-multiplied by the identity matrix, then the answer is the original matrix.

The multiplicative identity of a square $n \times n$ matrix A is given by

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Note that both $A \times I_n = A$ and $I_n \times A = A$ hold only in the case where A is a square matrix.

Unless needed for clarity, n is not normally written, and I is used alone to denote the identity matrix, whatever the size.

Exercise 9B

1. Using the matrices

   $A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -4 & 1 \\ -1 & 0 & 3 \end{pmatrix}$,

   $C = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ -2 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 1 & -2 & -1 \\ 3 & 0 & -4 \\ -3 & 2 & 1 \end{pmatrix}$,

   $E = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, and $F = \begin{pmatrix} -3 & -1 \\ 2 & 4 \\ 5 & -2 \end{pmatrix}$,

   find each of the following matrices if possible. If it is not possible, state the reason.

   a $(C+F)A$ **b** DE **c** ABC
   d CF **e** $2A - 3I_2$
2. Given that P is a $4 \times m$ matrix, Q is a $2 \times n$ matrix, determine the values of m and n if PQ is a 4×3 matrix.
3. Find a matrix that has the effect of summing the entries in every row of a 3×3 matrix. That is, if $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ find a matrix B such that $AB = \begin{pmatrix} (a+b+c) \\ (d+e+f) \\ (g+h+i) \end{pmatrix}$.

4 Consider the matrix $A = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 5 & -6 \end{pmatrix}$.

a Find a matrix B such that $AB = A$.

b Find a matrix C such that $CA = A$.

c Explain why A does not have a multiplicative identity.

5 A manufacturer makes three types of products P_1, P_2, and P_3 at each of its four plant locations L_1, L_2, L_3, and L_4. Matrix A gives the daily amount (in kilograms) of carbon monoxide, sulfur dioxide, and nitric oxide produced during the manufacturing process of each product.

$$A = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{carbon monoxide} & \text{sulfur dioxide} & \text{nitric oxide} \end{array} \\ \begin{pmatrix} 20 & 35 & 15 \\ 18 & 28 & 12 \\ 45 & 72 & 32 \end{pmatrix} \end{array}$$

New federal laws require that the manufacturer reduce its daily emissions of carbon monoxide, sulfur dioxide, and nitric oxide by 60%, 20%, and 40% of its current levels. The manufacturer takes corrective measures to reduce emissions to meet the minimum daily standards.

a Let matrix N be a matrix whose entries are the total daily number of kilograms of pollutants released for each product after the corrective measures. Write down the matrix R such that $N = AR$ then find N.

The daily cost (in USD) of removing each kilogram of carbon monoxide, sulfur dioxide, and nitric oxide at each of the four plants is given by the matrix C:

$$C = \begin{array}{c} \begin{array}{cccc} L_1 & L_2 & L_3 & L_4 \end{array} \\ \begin{pmatrix} 10 & 8 & 12 & 8 \\ 3 & 2 & 4 & 3 \\ 7 & 9 & 6 & 11 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{carbon monoxide} \\ \text{sulfur dioxide} \\ \text{nitric oxide} \end{array}$$

b Find AC and describe the meaning of its entries.

Investigation 3

Consider the matrix $A = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1 Using the fact that $A^2 = A \times A$ show that $A^2 \neq \begin{pmatrix} 2^2 & (-3)^2 \\ 4^2 & 7^2 \end{pmatrix}$.

2 Find B^2, C^2, and $(I_2)^2$.

3 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ under what conditions is $A^2 = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$? Does this same condition apply to all $n \times n$ matrices? Explain.

4 Determine D^2, D^3, and D^4. Write a formula for D^k where $k \in \mathbb{Z}^+$.

5 Without using technology write your own definition for the *zeroth power of a square matrix* A^0. Explain your reasoning. How does the GDC interpret A^0 and is this result consistent with your definition?

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ factors of } A}$.

In the case where $b = c = 0$ then $A^k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & d^k \end{pmatrix}$.

Exercise 9C

Complete each exercise using the definitions and properties you learned in this section then check your answers using technology.

1 If $A = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$. Find each if possible. If not possible, explain why.

a AB b BA

c A^2 d B^2

Use the following matrices to answer questions 2–6.

$R = \begin{pmatrix} 0 & -1 \\ 3 & 7 \\ -5 & 4 \end{pmatrix}$, $S = \begin{pmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ -7 & 5 & 2 \end{pmatrix}$,

$T = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, $U = \begin{pmatrix} 4 & -3 & -2 \\ 5 & -4 & 2 \\ 0 & 1 & -1 \end{pmatrix}$,

$W = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 7 & -2 \\ -6 & 4 \end{pmatrix}$

2 Calculate each of the following if possible. If not possible, explain why.

a SR b TR c $WR + X$ d $W(S - U)$

e $(S - U)W$

3 Use the properties of matrices to show that $UR + R = (U + I)R$. Verify your answer.

4 In algebra, $(a + b)(a - b) = a^2 - b^2$ and $(a + b)^2 = a^2 + 2ab + b^2$ where $a, b \in \mathbb{R}$. Expand $(T + X)(T - X)$ and $(T + X)^2$ using properties of matrices. Explain why $(T + X)(T - X) \neq T^2 - X^2$ and $(T + X)^2 \neq T^2 + 2TX + X^2$

5 Show that $(SU)^2 \neq S^2U^2$. Explain why using matrix multiplication.

6 Show that $2(RT) = (2R)T = R(2T)$. What does this suggest about scalar multiplication?

7 Consider the matrices $A = \begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -5 \\ -1 & -3 \end{pmatrix}$.

a Show that $AB = BA = I_2$.

b Determine $(2AB)^{10}$.

8 Michael makes the following conjectures for all square matrices A and B.

a $(AB)^n = A^nB^n$ b $(kA)^n = k^nA^n$

If you agree with Michael's claims, prove them. If you disagree, write down counter-examples.

9.3 Solving systems of equations using matrices

One of the many useful applications of matrix multiplication is that it gives you an efficient way to express and solve systems of linear equations. Any system containing n variables and n equations can be expressed as a matrix equation.

For example,

$\begin{cases} 10x - 5y = 35 \\ -3x + 7y = 23 \end{cases}$ is equivalent to $\begin{pmatrix} 10 & -5 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 35 \\ 23 \end{pmatrix}$.

$\begin{cases} 4s - 3t - 2z = 0 \\ 2s + 2t + 3z = -6 \\ 6s + t - z = 2 \end{cases}$ is equivalent to $\begin{pmatrix} 4 & -3 & -2 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{pmatrix}\begin{pmatrix} s \\ t \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}$.

In general, an $n \times n$ system of equations can be expressed in the form $AX = B$ where A is an $n \times n$ matrix containing the coefficients, B is a $n \times 1$ matrix containing the constants, and X is an $n \times 1$ matrix containing the variables.

Since you are interested in the values of the variables that solve all equations in the system, our goal is to determine X. At first glance you notice that $AX = B$ looks similar to the more familiar linear equation such as $2x = 10$, where you would simply divide both sides by 2. However, solving the matrix equation by dividing both sides by the matrix A would not lead you to $X = \frac{B}{A}$. Unfortunately, matrix division is not defined in the same way as other matrix operations. The key to unlocking the answer to this problem utilizes the multiplicative inverse property.

Recall that if a is a real number then $\frac{1}{a}$ is referred to as the multiplicative inverse of a since $a \times \frac{1}{a} = 1$ and $\frac{1}{a} \times a = 1$. For example, $-\frac{3}{2}$ and $-\frac{2}{3}$ are multiplicative inverses since $\left(-\frac{3}{2}\right)\left(-\frac{2}{3}\right) = 1$ and $\left(-\frac{2}{3}\right)\left(-\frac{3}{2}\right) = 1$. Note that there are two conditions for which a number has a multiplicative inverse. First, both the number and its inverse must have a product equal to 1 and second, this product must satisfy the commutative property of multiplication.

Similarly, $n \times n$ matrices A and B are called multiplicative inverses if $AB = I_n$ and $BA = I_n$ where I_n is the identity matrix. For example

$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ are multiplicative inverses since

$$AB = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, the central question is, given a square matrix A how do you find its multiplicative inverse? The following investigation will guide you through this process for a 2×2 matrix.

International-mindedness

Matrix methods were used to solve simultaneous linear equations in the second century BC in China in the book "Nine Chapters on the Mathematical Art" written during the Han Dynasty

Investigation 4

Determine the multiplicative inverse of $A = \begin{pmatrix} 8 & -6 \\ -5 & 4 \end{pmatrix}$.

a **i** Let $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be the multiplicative inverse of A. Use the fact that $AB = I$ to show that $\begin{Bmatrix} 8w - 6y = 1 \\ -5w + 4y = 0 \end{Bmatrix}$ and $\begin{Bmatrix} 8x - 6z = 0 \\ -5x + 4z = 1 \end{Bmatrix}$.

ii Solve each system in **i** to find the values of w, x, y, and z. Check your answer by verifying that $AB = I$ and $BA = I$.

b Consider that you are given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c and d are real numbers and you wish to determine the multiplicative inverse of A.

Let $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be the multiplicative inverse of A.

i Use the fact that $AB = I$ to show that $\begin{Bmatrix} aw + by = 1 \\ cw + dy = 0 \end{Bmatrix}$ and $\begin{Bmatrix} ax + bz = 0 \\ cx + dz = 1 \end{Bmatrix}$.

ii Solve each system in **i** to find the values of w, x, y, and z in terms of a, b, c, and d.

Verify that $AB = I$ and $BA = I$.

The multiplicative inverse of a 2 × 2 matrix

Investigation 4 reveals that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the inverse of A is given by:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ where } ad - bc \neq 0$$

You can thus determine the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ using the following steps.

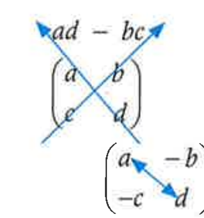
1 Subtract the product of the values on the minor diagonal from the product of the numbers on the main diagonal.

2 Interchange the elements on the main diagonal and change the sign of the numbers on the minor diagonal.

3 Divide each element of $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ by $(ad - bc)$.

International mindedness

Carl Gauss first used the term "determinant" in 1801.



$$A^{-1} = \begin{pmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$$

You will see in later sections of this chapter as well as in future chapters that the value $(ad - bc)$ has geometric significance. For this reason, this value is given a name and is defined as the **determinant of matrix** A and is written as det A or $|A|$. Restating the formula for A^{-1} you now have:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ where } ad - bc \neq 0$$

In the case where $|A| = 0$, A^{-1} does not exist and A is said to be a **singular matrix or non-invertible**.

EXAM HINT

Finding the multiplicative inverse for square matrices beyond a 2×2 requires using technology, and you will need to know how to do this for your exams.

Example 3

Find the inverse of $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$.

$\dfrac{1}{1\times(-4)-(-2)\times 3}\begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$ $=\dfrac{1}{2}\begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}=\begin{pmatrix} -2 & 1 \\ -1.5 & 0.5 \end{pmatrix}$	Either of these two forms can be used, choose whichever is most appropriate for the context.

Example 4

Use technology to determine the inverse of $P=\begin{pmatrix} 4 & -3 & -2 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{pmatrix}$.

Verify that $PP^{-1}=P^{-1}P=I_3$.

$$P^{-1}=\begin{pmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{3} & -\frac{2}{15} & \frac{4}{15} \\ \frac{1}{6} & \frac{11}{30} & -\frac{7}{30} \end{pmatrix}$$

$$PP^{-1}=\begin{pmatrix} 4 & -3 & -2 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{3} & -\frac{2}{15} & \frac{4}{15} \\ \frac{1}{6} & \frac{11}{30} & -\frac{7}{30} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I_3$$

$$PP^{-1}=\begin{pmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{3} & -\frac{2}{15} & \frac{4}{15} \\ \frac{1}{6} & \frac{11}{30} & -\frac{7}{30} \end{pmatrix}\begin{pmatrix} 4 & -3 & -2 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I_3$$

EXAM HINT

In examinations you should be able to demonstrate the use the formula $A^{-1}=\dfrac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ to compute the inverse of a 2×2 matrix but are expected to use the matrix utility using technology to find the inverse of a matrix that is larger than a 2×2.

You are now ready to unlock the solution to the problem proposed at the beginning of section 9.3, which was to solve the matrix equation $AX=B$ for X.

$$AX=B$$
$$A^{-1}AX=A^{-1}B \quad (\text{pre-multiplying both sides by } A^{-1})$$
$$I_nX=A^{-1}B \quad (\text{multiplicative inverse property})$$
$$X=A^{-1}B \quad (\text{Multiplicative identity property})$$

TOK

Do you think that one form of symbolic representation is preferable to another?

Number and algebra

Geometry and trigonometry

Example 5

Solve the systems of equations by first forming a matrix equation.

a $\begin{cases} 10x-5y=35 \\ -3x+7y=23 \end{cases}$ **b** $\begin{cases} 4s-3t-2z=0 \\ 2s+2t+3z=-6 \\ 6s+t-z=2 \end{cases}$

a $\begin{cases} 10x-5y=35 \\ -3x+7y=23 \end{cases} \Leftrightarrow \begin{pmatrix} 10 & -5 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 35 \\ 23 \end{pmatrix}$	Rewriting the system in the form $AX=B$
$\begin{pmatrix} 10 & -5 \\ -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} 10 & -5 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 10 & -5 \\ -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} 35 \\ 23 \end{pmatrix}$	Pre-multiplying both sides by A^{-1}
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\dfrac{1}{70-15}\begin{pmatrix} 7 & 5 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 35 \\ 23 \end{pmatrix}$	$AA^{-1}=I_2$
$\begin{pmatrix} x \\ y \end{pmatrix}=\dfrac{1}{55}\begin{pmatrix} 360 \\ 335 \end{pmatrix}$	$I_2X=X$
$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{72}{11} \\ \frac{67}{11} \end{pmatrix}$	The calculation $\begin{pmatrix} 10 & -5 \\ -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} 35 \\ 23 \end{pmatrix}$ could have been done directly on a GDC. Use GDC to obtain the result.
The solution to the system is $x=\dfrac{72}{11}$ and $y=\dfrac{67}{11}$	
b $\begin{matrix} 4s-3t-2z=0 \\ 2s+2t+3z=-6 \\ 6s+t-z=2 \end{matrix} \Rightarrow$ $\begin{pmatrix} 4 & -3 & -2 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{pmatrix}\begin{pmatrix} s \\ t \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}$	

Continued on next page

$$\begin{pmatrix} s \\ t \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & -2 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ -\frac{8}{3} \end{pmatrix}$$

The solution to the system is

$s = -\frac{1}{3}$, $t = \frac{4}{3}$, and $z = -\frac{8}{3}$

Applications to cryptography

Protecting sensitive electronic information such as a password relies upon a process called data encryption. Data encryption protects data by translating it into an unrecognizable form using an algorithm called a *cipher.*

To encrypt your bank account password "CharLie578Sam*!" you assign a number to each lowercase and uppercase letter along with the special characters and the digits 0 through 9 as shown below.

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52
!	@	#	$	%	^	&	*	"	?	:	;	/	\	0	1	2	3	4	5	6	7	8	9	space	
53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	

Using table above converts the password "CharLie578Sam*!":

Original "word"	C	h	a	r	L	i	e	5	7	8	S	a	m	*	!
Converted word	29	8	1	18	38	9	5	72	74	75	45	1	13	60	53

Next, you decide upon the length of the block you will use to divide the password. In this case you will use a block of three which will divide the word into five blocks giving the following 3 × 5 matrix M.

HINT

If the word length was not a factor of 5 then the remaining spaces could simply be filled with zeros.

$$\begin{pmatrix} 29 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 18 \\ 38 \\ 9 \end{pmatrix}, \begin{pmatrix} 5 \\ 72 \\ 74 \end{pmatrix}, \begin{pmatrix} 75 \\ 45 \\ 1 \end{pmatrix}, \begin{pmatrix} 13 \\ 60 \\ 53 \end{pmatrix} \Leftrightarrow M = \begin{pmatrix} 29 & 18 & 5 & 75 & 13 \\ 8 & 38 & 72 & 45 & 60 \\ 1 & 9 & 74 & 1 & 53 \end{pmatrix}$$

To encode the message, you can choose any 3 × 3 matrix for our cipher. For example, if you choose $C = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$ then our encrypted message E is

$$E = CM = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \begin{pmatrix} 29 & 18 & 5 & 75 & 13 \\ 8 & 38 & 72 & 45 & 60 \\ 1 & 9 & 74 & 1 & 53 \end{pmatrix} = \begin{pmatrix} 220 & 211 & 253 & 616 & 264 \\ 23 & 105 & 142 & 134 & 127 \\ -57 & 80 & 125 & -47 & 95 \end{pmatrix}$$

That is, the bank password is stored in the document in matrix form as

$$\begin{pmatrix} 220 & 211 & 253 & 616 & 264 \\ 23 & 105 & 142 & 134 & 127 \\ -57 & 80 & 125 & -47 & 95 \end{pmatrix}.$$

To decrypt the password requires you to solve $E = CM$ for M:

$E = CM \Leftrightarrow C^{-1}E = C^{-1}CM \quad \therefore M = C^{-1}E$

$$M = C^{-1}E = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 220 & 211 & 253 & 616 & 264 \\ 23 & 105 & 142 & 134 & 127 \\ -57 & 80 & 125 & -47 & 95 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix} \begin{pmatrix} 220 & 211 & 253 & 616 & 264 \\ 23 & 105 & 142 & 134 & 127 \\ -57 & 80 & 125 & -47 & 95 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 18 & 5 & 75 & 13 \\ 8 & 38 & 72 & 45 & 60 \\ 1 & 9 & 74 & 1 & 53 \end{pmatrix}$$

So the decrypted password is

29	8	1	18	38	9	5	72	74	75	45	1	13	60	53
C	h	a	r	L	i	e	5	7	8	S	a	m	*	!

Exercise 9D

1 Determine the multiplicative inverse of each matrix using $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Verify your answer by showing that $AA^{-1} = I$

a $\begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & -3 \\ -1 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$ **d** $\begin{pmatrix} 0.25 & 0.8 \\ 0.75 & 0.2 \end{pmatrix}$

2 Determine the multiplicative inverse of each matrix using technology expressing the elements as exact values. Verify your answer by showing that $AA^{-1} = I$.

a $\begin{pmatrix} 2 & -1 & 4 \\ 1 & 2 & 0 \\ -1 & 1 & -2 \end{pmatrix}$ **b** $\begin{pmatrix} 1.2 & -2.5 & 3 \\ 0.8 & -1 & 4 \\ 1.5 & -0.75 & 1 \end{pmatrix}$

c $\begin{pmatrix} \frac{2}{3} & -\frac{3}{4} & \frac{1}{2} \\ 1 & \frac{1}{3} & -1 \\ \frac{5}{2} & 0 & 1 \end{pmatrix}$

3 Write each system in the form of $AX = B$ then solve the system using $X = A^{-1}B$.

a $\begin{cases} x + y + 3z = 30 \\ 3x + 2y - z = 20 \\ 2x + y + z = 10 \end{cases}$ **b** $\begin{cases} 5x - 7y = 12 \\ -2x + 5y = 20 \end{cases}$

c $\begin{cases} 2a + b - c + 4d = -2 \\ 5b - 3c = 4 \\ 4a - 3b + d = 1.75 \\ a + 2c - 8d = -0.5 \end{cases}$

For questions 4–6 create a system of equations that represents each situation then solve the system using matrices. Be sure to define all variables.

4 In 2016, Sonya invested a total of \$175,000 in three different index funds F_1, F_2, and F_3. After one year, the combined value of all of her investments was \$181,615. Data collected on each of these investments showed that each investment made average annual gains of 2.5%, 4.8%, and 3.5% respectively during the year. If Sonya invested twice as much money in F_2 than in F_1, calculate the amount she invested in each fund.

5 A manufacturer wishes to produce four different products A, B, C, and D. The table below shows the number of minutes required on each of four machines to produce each product.

Machine	Product A	Product B	Product C	Product D
I	2	1	1	3
II	1	3	2	4
III	2	1	2	2
IV	3	4	1	2

The maximum amount of time available for each machine I, II, III, and IV is 240 minutes, 380 minutes, 280 minutes, and 400 minutes respectively. Calculate how many of each product can be manufactured if each machine uses all of its available time.

6 **a** Jonathan makes the conjecture that $(AB)^{-1} = A^{-1}B^{-1}$. Show that Jonathan's conjecture does not hold using the matrices $A = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$.

b Michelle claims that $(AB)^{-1} = B^{-1}A^{-1}$ does hold using the matrices A and B above.

i Verify that Michelle's claim is correct

ii Choosing your own 2×2 matrices for A and B show that $(AB)^{-1} = B^{-1}A^{-1}$.

iii Prove that $(AB)^{-1} = B^{-1}A^{-1}$ using the fact that $(AB)(AB)^{-1} = I_2$

7 Using the conversion table on page 386 decrypt the famous quote by James Joseph Sylvester [James Joseph Sylvester (1814–1897) – an English mathematician who made fundamental contributions to matrix theory, invariant theory, number theory, partition theory, and combinatorics.]

Encrypted message: $\begin{pmatrix} 224 & 288 & 84 & 192 & 279 & 75 & 38 & 158 & 103 & 96 & 83 & 220 \\ 144 & 202 & 50 & 109 & 182 & 49 & 28 & 80 & 71 & 60 & 50 & 150 \\ 83 & 163 & 49 & 89 & 174 & 44 & 15 & 79 & 51 & 51 & 47 & 131 \end{pmatrix}$

Encryption matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

8 Develop your own cipher using matrices to encrypt a sensitive piece of information. Design your own conversion table and encryption matrix by assigning different numbers to each of the characters and numbers. Check your work by having a friend decrypt your message by giving them your encrypted message, encryption key, and conversion table.

9.4 Transformations of the plane

For this section you need to be familiar with the ideas of rotation, reflection and enlargement, as illustrated in the example below.

Example 6

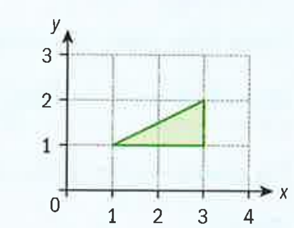
Draw the image of the triangle shown after the following transformations:

a rotation of 90° counter clockwise (or anticlockwise) about (0, 0)

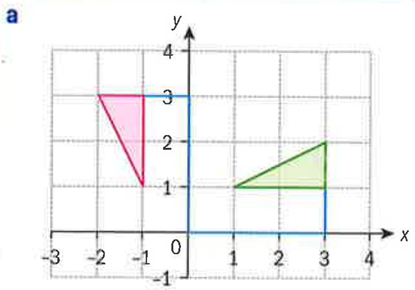
b reflection in the line $x = 0$

c enlargement scale factor 2 centre (0, 0)

d a stretch parallel to the x-axis, scale factor 2.

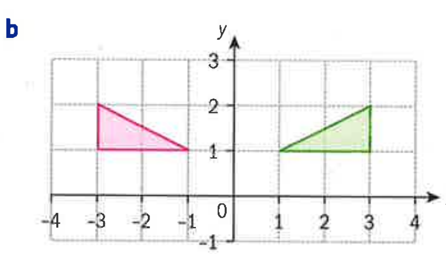


a



The rotation can be done using tracing paper or drawing an L shape from the origin to the point and rotating the L shape the required angle.

b



Reflection in the line $x = 0$ means all points are transformed to a position an equal distance the other side of the **mirror line**.

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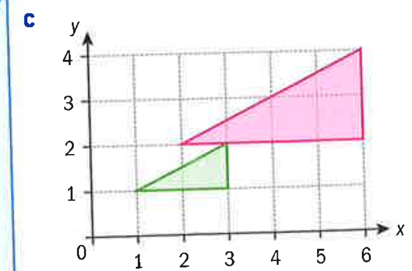
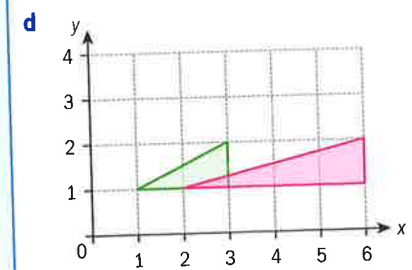
c



An enlargement scale factor 2 centre (0, 0) means the **image** of each point moves twice as far from the origin as before. Their new positions can easily be calculated by multiplying all their coordinates by 2.

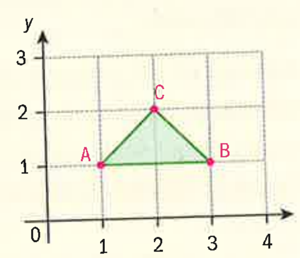
d



A stretch scale factor 2 parallel to the x-axis (or horizontally) just multiplies all the x-coordinates by 2 and leaves the y-coordinates unchanged. Similarly for a stretch parallel to the y-axis.

Investigation 5

Points in the plane can be represented by their position vectors. In the diagram below, for example, the position vector of A is $\begin{pmatrix}1\\1\end{pmatrix}$.



The position vectors of the vertices of the triangle ABC can be put in a single matrix $\begin{pmatrix}1 & 2 & 3\\1 & 2 & 1\end{pmatrix}$.

1 Find the product of $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\1 & 2 & 1\end{pmatrix}$.

Let the columns of the new matrix be the position vectors of the **image** of triangle ABC under the transformation represented by the matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

2 On a copy of the diagram above draw the triangle ABC and its image after the transformation.

What is the transformation?

3 In the same way use the matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ to find the image of (1, 0) and (0, 1) under this transformation.

What do you notice about the image matrix?

4 Test your conjecture by considering the image of (1, 0) and (0, 1) under this transformation represented by the matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$.

5 By considering the images of (1, 0) and (0, 1) suggest a matrix that represents an enlargement scale factor 2, centre (0, 0).

Verify your answer by multiplying the points from the triangle above by this matrix and seeing if all the coordinates are multiplied by two.

How can you use the points (1, 0) and (0, 1) to find a transformation matrix?

All standard transformations of the xy-plane, including rotations, reflections, stretches and enlargements can be represented by 2 × 2 matrices provided the point (0, 0) is invariant. These are often referred to as **linear transformations.**

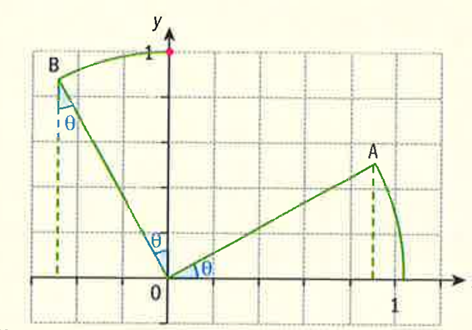
A matrix representing a linear transformation can be written $\begin{pmatrix}a & c\\b & d\end{pmatrix}$ where (a, b) is the image of $(1, 0)$ and (c, d) the image of $(0, 1)$.

6 Find the matrix that represents a rotation of 90° clockwise about (0, 0).

7 Find the matrix that represents a stretch parallel to the x-axis with a scale factor of 2 and the y-axis invariant.

Using the same method some general formulae can be found.

8 **a** In the diagram below A and B are the images of (1, 0) and (0, 1) under a counter clockwise rotation of θ about (0, 0). Use the diagram to show this rotation is represented by the matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$.

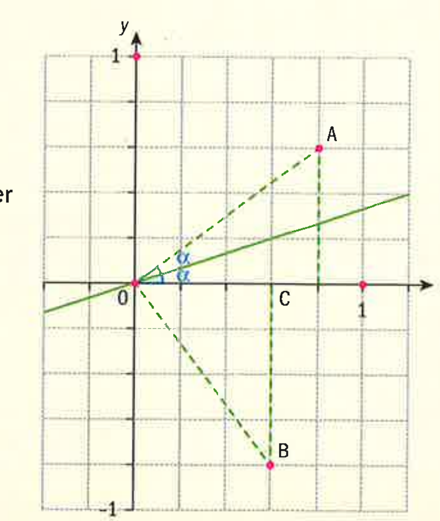


b What will be the matrix for a clockwise rotation of magnitude θ?

c Hence write down the matrix that represents a rotation of 60° clockwise about (0, 0).

9 **a** The line $y = mx$ can be written as $y = (\tan\alpha)x$, where α is the angle made with the x-axis.

In the diagram below A and B represent the images of (1, 0) and (0, 1) respectively under a reflection in the line $y = (\tan\alpha)x$.



a Explain why the image of (1, 0) has coordinates $(\cos2\alpha, \sin2\alpha)$.

b By finding $O\hat{B}C$ in terms of α find the image of (0, 1) under the transformation.

c Hence verify that the matrix that represents a reflection in the line $y = (\tan\alpha)x$ is $\begin{pmatrix}\cos2\alpha & \sin2\alpha\\\sin2\alpha & -\cos2\alpha\end{pmatrix}$.

d By first finding α, determine the matrix that represents a reflection in the line $y = \sqrt{3}x$.

10 By considering the images of (1, 0) and (0, 1) find the general matrices for:

a a one-way stretch, parallel to the x-axis, scale factor k

b an enlargement scale factor k, centre (0, 0).

HINT

These formulae are all in the formula book

Example 7

a Write down the matrix that represents a rotation of 45° anticlockwise about the origin given that $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$.

b Write down the matrix that represents a horizontal stretch of scale factor $\sqrt{2}$.

c Hence find the image of (2, 3) after a rotation of 45° anticlockwise about (0, 0) followed by a horizontal stretch of scale factor $\sqrt{2}$.

d Write down the single matrix that represents a rotation of 45° anticlockwise about (0, 0) followed by a horizontal stretch of scale factor $\sqrt{2}$.

a $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	This is using the formula for an anticlockwise rotation.
b $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$	This is using the formula for a one-way stretch. Both these formulae are in the formula book.
c $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$ $= \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{5}{\sqrt{2}} \end{pmatrix}$	The point is first rotated and then its image is stretched.
d $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$	

If a transformation represented by a matrix A is followed by a transformation represented by matrix B then the single matrix that represents both transformations is BA.

Example 8

Find the 2 × 2 matrix that will transform the point (2, 1) to (1, 4) and the point (1, −3) to (4, 9).

Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.	
Therefore, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Leftrightarrow 2a + b = 1$ and $2c + d = 4$	Write down the matrix equation that describes the transformation.
$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \Leftrightarrow a - 3b = 4$ and $c - 3d = 9$	Develop two 2 × 2 systems of equations and solve each system using elimination or technology.
$\left.\begin{cases} 2a + b = 1 \\ a - 3b = 4 \end{cases}\right\}_{\times(-2)} \Leftrightarrow 7b = -7 \quad \therefore b = -1$ and $a = 1$	
and $\left.\begin{cases} 2c + d = 4 \\ c - 3d = 9 \end{cases}\right\}_{\times(-2)} \Leftrightarrow 7d = -14 \quad d = -2$ and $c = 3$	
Therefore, $T = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	

Investigation 6

Consider the triangle ABC with vertices A(0, 0), B(2, 2), C(−1, 5).

1. Show that ΔABC is a right-angled triangle and find its area.
2. Triangle ABC is enlarged by a factor of 3 to obtain triangle A′B′C′.
   - **a** Use matrix multiplication to determine the coordinates of A′B′C′.
   - **b** Determine the area of triangle A′B′C′. By what factor has the area of triangle ABC changed? By what factor will the area change if it is enlarged by a factor of k?
3. Triangle ABC is transformed by $T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.
   - **a** Find the coordinates of the image points of triangle ABC.
   - **b** Show that the new triangle is a right-angled triangle and determine its area. By what factor has the area of triangle ABC changed under T?
   - **c** Calculate $\det(T)$ and show that the area of triangle ABC enlarged by a factor of $|\det(T)|$.

Continued on next page

Number and algebra

Geometry and trigonometry

4 Consider triangle ABC under the transformation $T = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

a Use graphing software to show that area of A′B′C′ = 42. Is ABC a right triangle? How do you know?

b Show that this area is equal to $|\det(T)| \times$ Area of triangle ABC.

5 Consider the transformation of triangle ABC under $T = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Does the relationship $|\det(T)| \times$ Area of triangle ABC hold for T? Explain why.

In the investigation above you discovered a very useful relationship between the area of a figure's pre-image and the area of its image under a linear transformation T.

Area of the image $= |\det(T)| \times$ Area of the pre-image

Exercise 9E

1 Find the transformation matrix for:

a A reflection in the x-axis

b An anticlockwise rotation about the origin of 45°

c A clockwise rotation of $\frac{\pi}{2}$

d A reflection in the line $y = -x$

e A vertical stretch of scale factor 3

f An enlargement scale factor 2, centre (0, 0)

g A reflection in the line $y = \sqrt{3}x$.

2 a Write down the transformation matrix for an enlargement scale factor 4, centre (0, 0).

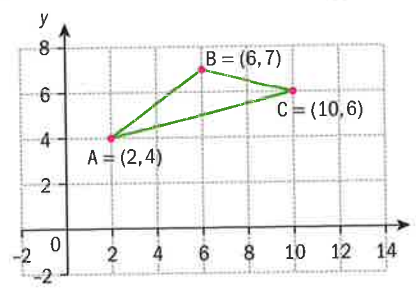
b Write down the transformation matrix for a rotation of 180° about (0, 0).

A rotation of 180° about the origin followed by an enlargement of scale factor k where $k > 0$ is often referred to as an enlargement with scale factor $-k$.

c Find the product of the two matrices found in parts **a** and **b** and hence explain why this definition is justified.

3 Consider the triangle ABC whose vertices are described by the matrix

$$P = \begin{pmatrix} 2 & 6 & 10 \\ 4 & 7 & 6 \end{pmatrix}.$$



a Use matrix multiplication to determine the image of triangle ABC when it is reflected across the x-axis to obtain triangle A′B′C′. Sketch the triangle and its image on the same axis.

Triangle A′B′C′ is reflected across the y-axis to obtain A″B″C″.

b Write a single matrix equation to find the image of P under the composition of both reflections. Sketch triangle A″B″C″ on the same axis.

c Show that if triangle ABC is reflected across the x-axis then again across the x axis that the resulting image is triangle ABC.

d What transformation is equivalent to reflection in the x-axis followed by reflection in the y-axis.

4 a Write down the transformation matrix for a reflection in the line $y = (\tan\alpha)x$.

b Verify that a reflection followed by a second reflection in the same line is equivalent to the identity matrix.

5 a Write down the exact matrix R for a rotation of 45° clockwise about the origin, given that $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$.

b Find the value of R^8 and interpret your result.

6 a Find the area of the triangle with vertices at (0, 5), (4, 5) and (2, 7).

b Use the result, area of image $= |\det(T)| \times$ area of object to find the area of the image of the triangle:

i after a vertical stretch with scale factor 2, followed by an enlargement, centre (0, 0), scale factor 3

ii after undergoing the transformation represented by the matrix $\begin{pmatrix} 2 & 2 \\ -3 & 5 \end{pmatrix}$.

7 Prove that neither transformation will change the area of an object by considering the determinant for the general matrix for:

a a reflection   b a rotation.

8 Let T be a linear transformation such that $T = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$. Find:

a the coordinates of the image of the point $P(14, 12)$ under T

b the coordinates of the point having an image of (−12, 12).

9 Let T be a linear transformation such that $T = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$. Find:

a the coordinates of the image of the point $P(-3, 4)$ under T

b the coordinates of the point having an image of (−12, 12).

10 Find the 2 × 2 matrix that will transform the point (2, 1) to (1, 4) and the point (1, −3) to (4, 9).

11 A triangle with vertices (1, 1), (3, 1) and (3, 3) is rotated $\frac{\pi}{2}$ clockwise about (0, 0) and then reflected in the line $y = -x$. Let R represent the rotation and T represent the reflection.

a Write down the matrices R and T.

b Find the single matrix that represents R followed by T.

c Hence find the image of the triangle after both these transformations.

d From a sketch or otherwise describe the single transformation represented by RT.

12 Let E be a matrix representing an enlargment scale factor 0.5, centre (0, 0).

a Write down the matrix E.

b Find the single matrix that represents a series of n enlargements scale factor 0.5, centre (0, 0).

13 The point $P(-2, 5)$ undergoes two rotations R_1 followed by R_2.

a Given that R_1 is a counterclockwise rotation of 20° and R_2 is a counterclockwise rotation of 40°. Determine the coordinates of the image of P.

b Show that

$$\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 20° \cos 40° - \sin 20° \sin 40° & -\cos 40° \sin 20° - \sin 40° \cos 20° \\ \sin 40° \cos 20° + \cos 40° \sin 20° & -\sin 40° \sin 20° + \cos 40° \cos 20° \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

c Given that R_1 is a counterclockwise rotation of α and R_2 is a counterclockwise rotation of θ, show that

$\cos(\alpha + \theta) \equiv \cos\alpha\cos\theta - \sin\alpha\sin\theta$ and $\sin(\alpha + \theta) \equiv \sin\theta\cos\alpha + \cos\alpha\sin\theta$.

Number and algebra

Geometry and trigonometry

Affine transformations

Points in the plane can also be transformed using vectors (**translations**). For example, the point (3, 2) under the translation $\begin{pmatrix}1\\-3\end{pmatrix}$ will have position vector $\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}$ or coordinates (4, −1).

An Affine transformation consists of a linear transformation (represented by a matrix) and a translation and is of the form $\boldsymbol{Ax}+\boldsymbol{b}$ where $\boldsymbol{x}$ is the point being acted on. Affine transformations are often used to describe or create objects for which the whole object is the same (or approximately the same) as part of itself (self-similar), many fractals are self-similar. A famous example is the Barnsley Fern.

Example 9

The square PQRS with vertices P(−2, 4), Q(−6, −3), R(1, −7), S(5, 0) undergoes a transformation described by

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}.$$

a By multiplying two transformation matrices verify that $\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ is equivalent to an enlargement of scale factor 0.5 followed by a reflection in the y-axis.

b Determine the coordinates of the vertices of the image of PQRS.

c Draw the square PQRS and its image on the same axes.

a $\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-0.5 & 0\\ 0 & 0.5\end{pmatrix}$

b $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\times\begin{pmatrix}-2\\4\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$

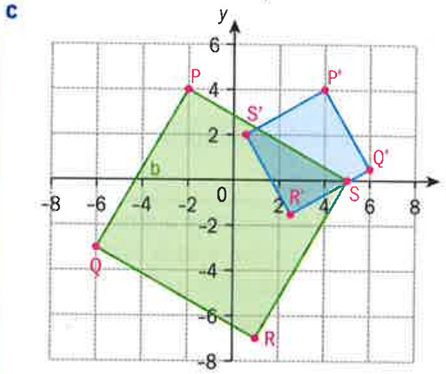
Take each point in turn and find its image. The full working for the point (−2, 4) is shown.

$(-6,-3)\rightarrow(6,0.5)$

$(1,-7)\rightarrow(2.5,-1.5)$

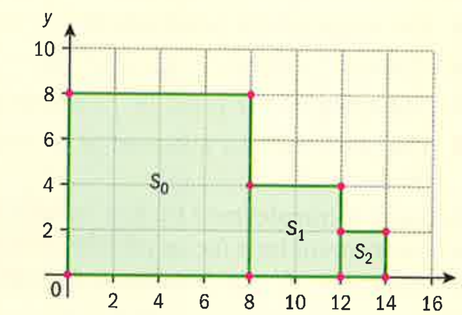
$(5,0)\rightarrow(0.5,2)$

c



Investigation 7

Consider the series of squares S_0, S_1, S_2, … where the sides of each successive square is one-half of the previous square, as shown. Each successive square is formed by a transformation of the form $\boldsymbol{AX}+\boldsymbol{b}$ where $\boldsymbol{A}$ is a 2×2 matrix and $\boldsymbol{b}$ is a 2×1 column vector.



Let (x_{n-1}, y_{n-1}) be a point in S_{n-1} and (x_n, y_n) be the image of this point in S_n such that

$$\begin{pmatrix}x_n\\y_n\end{pmatrix}=\boldsymbol{A}\begin{pmatrix}x_{n-1}\\y_{n-1}\end{pmatrix}+\boldsymbol{b}.$$

1 By considering the points (0, 0), (0, 8), and (8, 0) in S_0 and their corresponding images in S_1 write down and solve a system of equations to determine $\boldsymbol{A}$ and $\boldsymbol{b}$.

2 Using the coordinates of a point in the interior of S_1 along with the coordinates of its image in S_2 verify your answer in question 1.

3 Determine the coordinates of the image of (6, 6) in S_3.

4 a Apply the transformation to find (x_1, y_1) in terms of (x_0, y_0).

b Apply the transformation again to find (x_2, y_2) in terms of (x_0, y_0).

c Hence show that $\begin{pmatrix}x_3\\y_3\end{pmatrix}=\begin{pmatrix}\frac{1}{8}x_0+8+4+2\\ \frac{1}{8}y_0\end{pmatrix}$.

d Use the formula for a geometric series to find an expression for (x_n, y_n) in terms of (x_0, y_0).

5 Use your conjecture to determine the coordinates of the image of (6, 2) in S_4.

Exercise 9F

1 The points A(−3, 4), B(2, −3), C(0, 4), and D(5, 8) are transformed to A′, B′, C′, D′ according to a linear transformation defined by $X = \begin{pmatrix} -4 & 0 \\ 1 & -4 \end{pmatrix} X + \begin{pmatrix} 6 \\ -2 \end{pmatrix}$. Find:

a the coordinates of the image points A′, B′, C′, D′

b the coordinates of the point whose image is P(−6, −3).

2 Under a transformation T the image (x', y') of a point (x, y) is obtained by the matrix equation

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Find

a the image of the point (−5, 8)

b the point with an image of (3, 6)

c the image of the point (a, a) where $a \in \mathbb{R}$

d the point with an image of (a, a) where $a \in \mathbb{R}$.

3 A shape is transformed by first undergoing a enlargement by a factor of 3 followed by a rotation of 90° anti-clockwise then lastly translated by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

a Determine the 2 × 2 matrix T and the 2 × 1 column vector b that maps the pre-image point (x, y) to its image point (x', y') such that $\begin{pmatrix} x' \\ y' \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix} + b$

The image point (x', y') is now reflected across the origin to obtain the image point (x'', y'')

b Determine the 2 × 2 matrix A and the 2 × 1 column vector c that maps the pre-image point (x, y) to its image point (x'', y'') such that $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + c$.

4 a Find the 2 × 2 transformation matrices P, Q, R, and S given that

i matrix P represents a reflection in the x-axis

ii matrix Q represents a clockwise rotation of 135°

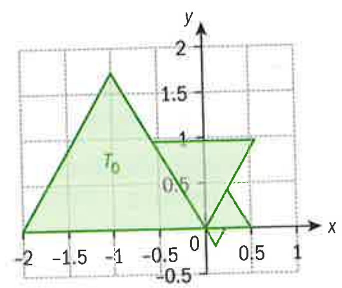
iii matrix R represents a reflection in the line $y = \sqrt{3}x$

iv matrix S represents a stretch of factor 2 parallel to the x-axis and a stretch of factor 4 parallel to the y-axis.

b Find a single transformation in the form of $AX + b$ that transforms a point (x, y) by Q followed by P followed by R followed by S.

c Find a single transformation in the form of $AX + b$ when a point (x, y) is translated by the vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ followed by R.

5 A sequence of equilateral triangles (T_n) are shown below. T_n is formed from T_{n-1} by rotating the triangle 60° clockwise about (0, 0) and enlarging by a factor 0.5 (centre (0, 0).



T_0 is the triangle shown with two vertices at (−2, 0) and (0, 0).

The perimeter of the shape formed from these triangles is the distance around the outside edge.

a Find the perimeter of the shape shown made up of triangles T_0 to T_3.

b Explain why the maximum perimeter is reached once T_5 is added to the diagram and find this value.

Let E be the enlargement matrix and R the rotation matrix.

c i Find det(E).

ii Find the area of T_0.

iii Hence find the total area covered by T_0 to T_6.

d Write down the matrix R.

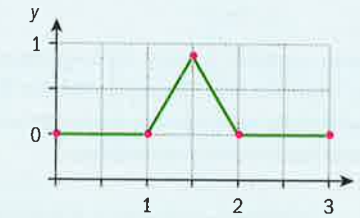
e Find ER.

f Use your GDC to find the coordinates of all three vertices in T_8.

Developing inquiry skills

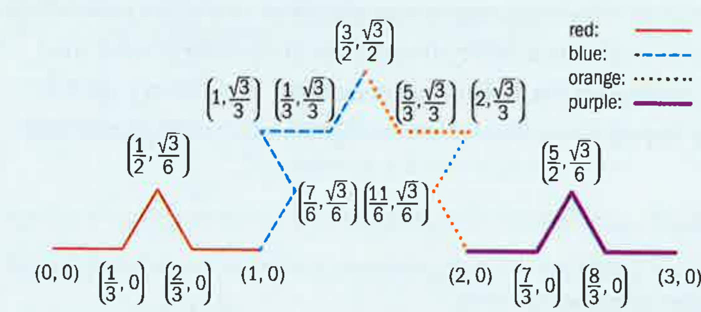
In the chapter opener, you found, using geometry, the exact values of the coordinates of each vertex and endpoint of stages 1 and 2 of the Koch Snowflake.

Stage 1



You may have noticed that the basic building block of the Koch Snowflake is given by stage 1 – that is, using a series of transformations on stage 1 you can create each colored piece of the snowflake in stage 2.

Stage 2



1 Describe the series of transformations that are necessary to map the red piece in stage 1 to the red piece in stage 2. Write your answer in the form of $AX + b$ where A is a 2 × 2 matrix, b is a 2 × 1 column vector, and X is a 2 × 1 column vector of the vertices and endpoints of stage 1. Name this transformation T_1. By applying T_1 to each vertex and endpoint in stage 1, determine the image points of the red piece in stage 2.

2 Repeat the steps in question **1** to determine the transformations of the form $AX + b$ that maps the vertices in stage 1 onto the blue, orange, and purple pieces of stage 2. Name these transformations T_2, T_3, and T_4 respectively. By applying these transformations determine the coordinates of each vertex and endpoint in each stage.

9.5 Representing systems

Some applications of probability consider sequences of events in which an outcome depends on the previous outcome. For example, the probability of our inheriting a particular genetic trait is dependent on the genes of our parents. Whether or not today sees heavy rainfall may be affected by the type of weather from the previous day.

Investigation 8

Sequences A and B are simulations of weather data over 21 days.
W represents the event "It was wet today" and D represents the event "It was dry today".

A	D	D	D	D	W	D	W	W	D	D	W	D	W	D	W	W	W	D	D	D	D
B	W	D	D	W	W	D	W	W	W	W	W	W	W	D	D	D	W	D	W	W	W

You can investigate this data to see if there is any evidence of the weather tomorrow being dependent on the weather today as follows.

In sequence A there are 8 wet days and 13 dry. You only consider the first 20 days however since you do not have information about the weather on the 22nd day. From the first 20 days you have this data: {WD, WW, WD, WD, WD, WW, WW, WD} to represent the days on which the weather was wet and then the weather on the next day. From this set you can find these experimental conditional probabilities:

$$P(\text{Wet tomorrow} \mid \text{Wet today}) = \frac{3}{8} \text{ and } P(\text{Dry tomorrow} \mid \text{Wet today}) = \frac{5}{8}$$

Show that for sequence B an estimate of the conditional probabilities are:

$$P(\text{Wet tomorrow} \mid \text{Dry today}) = \frac{4}{7} \text{ and } P(\text{Dry tomorrow} \mid \text{Dry today}) = \frac{3}{7}$$

One sequence was generated by a computer simulation programmed with these conditional probabilities:

$$P(\text{Wet tomorrow} \mid \text{Dry today}) = 0.4 \text{ and } P(\text{Dry tomorrow} \mid \text{Dry today}) = 0.6 \text{ and}$$
$$P(\text{Wet tomorrow} \mid \text{Wet today}) = 0.5 \text{ and } P(\text{Dry tomorrow} \mid \text{Wet today}) = 0.5$$

The other sequence was simulated by flipping a coin. Discuss in a small group which method may have generated each sequence

1 How did you decide?

2 Why is it difficult to be sure?

3 **Conceptual** How do you describe independent events?

The **states** of this system are "Dry" and "Wet" and the sequences A and B show the system changing state from one day to another. In this system, you cannot predict the outcome of a given trial with absolute certainty, but you can use probabilities to quantify the likely state of the system in any given trial.

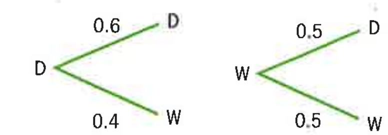
A **Markov Chain** is a system in which the probability of each event depends only on the state of the previous event.

TOK

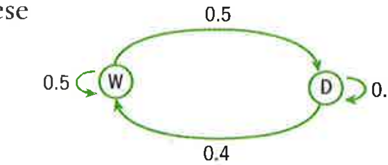
How do "believing that" and "believing in" differ?

How does belief differ from knowledge?

You could represent the probabilities in the computer simulation given in investigation 8 in two tree diagrams as below. These are called **transition probabilities.**



Another convenient way to summarise and represent these probabilities is in this **transition state diagram:**



You can also represent the same information in a **transition matrix** in which the column headings refer to the **current state** of the system, shown here as the matrix T and the row headings the next (future) state.

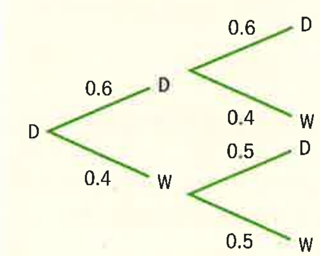
The entries in the transition matrix show the probability of the transition from the current state to the future state.

Current state

$$T = \begin{array}{cc} & \begin{array}{cc} D & W \end{array} \\ \text{Future state} \begin{array}{c} D \\ W \end{array} & \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \end{array}$$

In the following investigation you will find out how to interpret repeated multiplication of a transition matrix.

Investigation 9



Coninuing with the situation described above You can use this tree diagram to find the probability that if is **dry** today, it is also dry two days from now by finding

$$0.6 \times 0.6 + 0.4 \times 0.5 = 0.56$$

and the probability that if it is **dry** today it will be wet two days from now by finding

$$0.6 \times 0.4 + 0.4 \times 0.5 = 0.44$$

1 Apply a tree diagram to find the probability that if is **wet** today, it is also wet two days from now.

2 Apply a tree diagram to find the probability that if is **wet** today, it is dry two days from now.

3 Given $T = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix}$, Find T^2.

4 How do you interpret the values in the matrix T^2?

5 **Factual** What is more efficient to find the probabilities in T^2- matrix multiplication or drawing the two tree diagrams?

6 **Factual** How do you interpret the values in the matrix T^3?

7 **Factual** How can you interpret our results when multiplying by the transition matrix?

8 **Factual** What does n represent in T^n?

9 **Conceptual** How can the probabilities of Markov chains be represented?

A transition matrix is a square matrix which summarizes a transition state diagram. It describes the probabilities of moving from one state in a system to another state.

The sum of each column of the transition matrix is 1.

The transition matrix is extremely useful when there are more than two states in the system and when you wish to predict states of the system after several time periods.

International mindedness

Matrices play an important role in the projection of three-dimensional images into a two-dimensional screen, creating the realistic seeming motions in computer-based applications.

Example 10

Dockless bicycle company Mathbike hires bicycles in a city through a mobile phone app. Users can unlock a bicycle with their smartphone, ride it to their destination then lock the bicycle. Mathbike divides the city into three zones: Inner (I), Outer (O), and Central business district (C). By tracking their bicycles with GPS over several weeks, the company finds that at the end of each day:

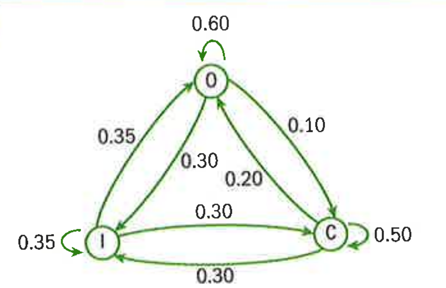
- 50% of the bicycles rented in zone C remained in zone C, 30% were left in Zone I, and 20% were left in Zone O
- 60% of the bicycles rented in zone O remained in zone O, 30% were left in zone I, and 10% were left in zone C.
- 35% of bicycles rented in zone I remained in zone I, 35% were left in zone O, and 30% were left in zone C.

a Show this information in a transition state diagram.

b Show this information in a transition matrix.

c Determine the probability that after three days, a bicycle that started in C is now in O.

a



The diagram gives a quick way to check that the relationships given in the question are put in the right places.

b

Current location

Location at the end of the day

$$\begin{array}{cc} & \begin{array}{ccc} \text{C} & \text{I} & \text{O} \end{array} \\ \begin{array}{c} \text{C} \\ \text{I} \\ \text{O} \end{array} & T = \begin{pmatrix} 0.50 & 0.30 & 0.10 \\ 0.30 & 0.35 & 0.30 \\ 0.20 & 0.35 & 0.60 \end{pmatrix} \end{array}$$

Be careful to make the column headings represent the current state of the system and the row headings the future state.

c

$$T = \begin{pmatrix} 0.50 & 0.30 & 0.10 \\ 0.30 & 0.35 & 0.30 \\ 0.20 & 0.35 & 0.60 \end{pmatrix} \quad \begin{pmatrix} 0.50 & 0.30 & 0.10 \\ 0.30 & 0.35 & 0.30 \\ 0.20 & 0.35 & 0.60 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0.307 & 0.280 & 0.243 \\ 0.316 & 0.316 & 0.316 \\ 0.377 & 0.405 & 0.441 \end{pmatrix}$$

The most efficient method is to store the matrix in a GDC as T and find T^3.

The probability that a bicycle starting in the central business district at the end of three days is in the outer zone is 0.441.

Read off the number in the C column and the O row, which is 0.441 and write a conclusion that interprets this number.

Reflect From what you have learned in this section, how can you best represent a system involving probabilities?

For **visualising** a system, would you choose a tree diagram, a transition state diagram or a transition matrix?

For **calculating probabilities**, would you choose a tree diagram, a transition state diagram or a transition matrix?

Exercise 9G

1 For these transition matrices, construct the corresponding transition diagrams:

a

Current state

Future state

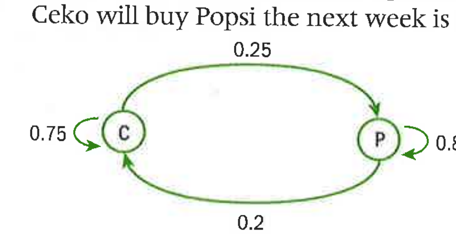
$$\begin{array}{cc} & \begin{array}{cc} \text{X} & \text{Y} \end{array} \\ \begin{array}{c} \text{X} \\ \text{Y} \end{array} & \begin{pmatrix} \frac{5}{11} & \frac{1}{5} \\ \frac{6}{11} & \frac{4}{5} \end{pmatrix} \end{array}$$

b

Current location

$$\begin{array}{cc} & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & T = \begin{pmatrix} 0.2 & 0 & 0.47 \\ 0.5 & 0.9 & 0 \\ 0.3 & 0.1 & 0.53 \end{pmatrix} \end{array}$$

2 This transition state diagram shows the findings from market research of the buying habits of consumers who shop weekly and buy either soft drink brand Ceko or Popsi. For example, the probability that a person buying Ceko will buy Popsi the next week is 0.25.



a Show this information in a transition matrix T by writing down the blank entries.

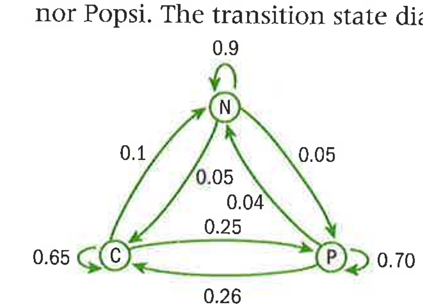
Current location

Future state

$$\begin{array}{cc} & \begin{array}{cc} \text{C} & \text{P} \end{array} \\ \begin{array}{c} \text{C} \\ \text{P} \end{array} & T = \begin{pmatrix} & 0.2 \\ 0.25 & \end{pmatrix} \end{array}$$

b Calculate T^3 and hence find the probability that a person who buys Popsi now will change to Ceko three weeks from now.

3 Upon further research, it is found that a more realistic model is to create a third state N to represent a person buying neither Ceko nor Popsi. The transition state diagram is



a Complete the transition matrix:

$$\begin{array}{c} \text{Drink purchased next time} \end{array} \quad \begin{array}{c|ccc} & \multicolumn{3}{c}{\text{Current drink purchased}} \\ & C & P & N \\ \hline C & & & \\ P & & & \\ N & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} C & P & N \end{array} \\ \begin{array}{c} C \\ P \\ N \end{array} & T = \begin{pmatrix} 0.65 & & \\ & & 0.05 \\ & 0.04 & \end{pmatrix} \end{array}$$

b Find the probability that a person buying neither Ceko nor Popsi now will buy Popsi five weeks from now.

4 A game begins in state 1 with two white coins in one box and two blue in another. A coin is chosen at random from each box and put into the other then the process is repeated many times.

State 1 State 2 State 3

a Construct the transition matrix to show the probabilities of transitioning from one state to another.

b The game is made more sophisticated by starting with three white coins in one box and three blue in another, represented as 3W3B. State the new number of states. the transition matrix to show the probabilities of transitioning from one state to another.

c Investigate the transition matrices for other situations such as 4W4B, 5W5B, 3W2B etc.

5 Two candidates A and B are running for an elected government office. Statistics gathered from a survey taken over several weeks prior to the election showed that after each week approximately 4.1% of the candidates who were planning to vote for candidate A changed their mind and decided to vote for candidate B and 3.2% of those who were planning to vote for candidate B changed their mind and decided to vote for candidate A.

Five weeks before the election a final survey showed that 45 520 people were planning to vote for candidate A and 38 745 people were planning to vote for candidate B.

a Write a matrix equation that describes the number of voters supporting each candidate n weeks after the final survey.

b Assuming that the trends revealed in the survey continue up until the day of the election determine which candidate will win. Calculate the margin of victory.

9.6 Representing steady state systems

You have seen in section 9.5 that if T is an $m \times m$ transition matrix and if

$$T^n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}$$

TOK

If we can find solutions of higher dimensions, can we reason that these spaces exist beyond our sense perception?

then a_{ij} is the probability that the system transitions from state j to state i *after* n stages. The sequence of transition matrices T^n, $n \in \mathbb{Z}^+$ behaves in different ways according to T, just as the geometric sequence $u_1 r^n$, $n \in \mathbb{Z}^+$ behaves in different ways according to the value of r.

Investigation 10

1 Use technology to find these powers of each transition matrix T.

	T	T^2	T^3	T^4	T^{20}	T^{50}	Term
a	$\begin{pmatrix} \frac{5}{11} & \frac{1}{5} \\ \frac{6}{11} & \frac{4}{5} \end{pmatrix}$						
b	$\begin{pmatrix} 0.2 & 0 & 0.47 \\ 0.5 & 0.9 & 0 \\ 0.3 & 0.1 & 0.53 \end{pmatrix}$						
c	$\begin{pmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 1 \\ 0 & 0.5 & 0 \end{pmatrix}$						
d	$\begin{pmatrix} 0 & 0 & 0 & 0.8 \\ 0.3 & 1 & 0 & 0.1 \\ 0.3 & 0 & 1 & 0.1 \\ 0.4 & 0 & 0 & 0 \end{pmatrix}$						
e	$\begin{pmatrix} 0.25 & 0 & 0.65 \\ 0 & 0 & 0.35 \\ 0.75 & 1 & 0 \end{pmatrix}$						

2 What patterns are there in these sequences of matrices?

Each sequence above is classified by one of these terms: absorbing, regular, periodic.

An absorbing state is one that when entered, cannot transition to another.

3 What would a definition of the other terms be? Drawing a transition state diagram may help you distinguish between the different types.

4 Identify the appropriate term for each.

5 **Conceptual** What happens to a regular Markov chain with transition matrix T as n tends to infinity?

A **regular** transition matrix T has the property that there exists $n \in \mathbb{Z}^+$ such that all entries in T^n are greater than zero. For high powers of n a regular transition matrix converges to a matrix in which all the columns have the same values. In the rest of this chapter you only consider regular transition matrices.

You can use this property to make predictions about the state of a system after many time periods. In practical applications you often need to consider a population in which a certain number are in either state at the beginning. For example, consider the transition matrix which represents the probabilities of transitioning between buying Ceko or Popsi each week:

$$T = \begin{array}{c} \\ \text{Future state} \end{array} \begin{array}{c} \\ \text{C} \\ \text{P} \end{array} \begin{array}{c} \text{Current location} \\ \begin{array}{cc} \text{C} & \text{P} \end{array} \\ \begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{pmatrix} \end{array}$$

A survey of 126 consumers showed that 81 bought Ceko and 45 Popsi.

The **initial state vector** S_0 shows the initial state of the system.

$$S_0 = \begin{array}{c} \\ \text{C} \\ \text{P} \end{array} \begin{array}{c} \text{Number of consumers} \\ \begin{pmatrix} 81 \\ 45 \end{pmatrix} \end{array}$$

You can predict the market share after one week by finding

$$S_1 = TS_0 = \begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{pmatrix}\begin{pmatrix} 81 \\ 45 \end{pmatrix} = \begin{pmatrix} 69.75 \\ 56.25 \end{pmatrix}$$

Hence you can predict that after one week, between 69 and 70 of these 126 consumers will purchase Ceko. Remember that some of these customers will have initially bought Ceko and some will have changed from Popsi. Also, some customers who originally bought Ceko will have changed to Popsi. The multiplication $S_1 = TS_0$ quantifies and represents all the transitions.

Similarly, you can multiply by T successively to predict the states in subsequent weeks, and look for patterns in our predictions. $S_2 = TS_1 = T^2S_0$. Repeating this process establishes the equation $S_n = T^nS_0$

Investigation 11

1 Using the Ceko and Popsi matrix and vector above, use technology to complete the table.

n = no. of weeks	Algebra representation of S_n	Matrix representations of S_n	Vector representation of S_n
0	S_0	$\begin{pmatrix} 81 \\ 45 \end{pmatrix}$	$\begin{pmatrix} 81 \\ 45 \end{pmatrix}$
1	TS_0	$\begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{pmatrix}\begin{pmatrix} 81 \\ 45 \end{pmatrix}$	$\begin{pmatrix} 69.75 \\ 56.25 \end{pmatrix}$
2	$TS_1 = T^2S_0$	$\begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{pmatrix}\begin{pmatrix} 69.75 \\ 56.25 \end{pmatrix} = \begin{pmatrix} 0.6125 & 0.31 \\ 0.3875 & 0.69 \end{pmatrix}\begin{pmatrix} 81 \\ 45 \end{pmatrix}$	$\begin{pmatrix} 63.5625 \\ 62.4375 \end{pmatrix}$
3	$TS_2 = T^3S_0$	$\begin{pmatrix} 0.75 & 0.2 \\ 0.25 & 0.8 \end{pmatrix}\begin{pmatrix} 63.5625 \\ 62.4375 \end{pmatrix} = \begin{pmatrix} 0.536875 & 0.3705 \\ 0.463125 & 0.6295 \end{pmatrix}\begin{pmatrix} 81 \\ 45 \end{pmatrix}$	$\begin{pmatrix} 60.1594 \\ 65.8406 \end{pmatrix}$
4			
10			
20			
49			

2 **Factual** What are the converging **steady state** patterns in your results?

3 **Conceptual** Generalise these convergent patterns using the terms **long-term probability matrix** and **steady state vector**.

4 What interpretations can you make of these patterns regarding the long-term market share of each brand?

5 What assumptions are made?

6 **Conceptual** How can you use powers of the transition matrix to make predictions?

If P is a square regular transition matrix, there is a unique vector q such that $Pq = q$. This q is called the **steady-state vector** for P.

Example 11

Dockless bicycle company Mathbike open their business by distributing a number of bicycles in a city according to the initial state vector $S_0 = \begin{pmatrix} 110 \\ 80 \\ 50 \end{pmatrix}$.

Continued on next page

a Calculate the likely number of bicycles in each zone of the city after five days using the transition matrix given in example 10.

b Find the likely steady state of Mathbikes.

c Comment on how the owners of Mathbikes should reflect on their choice of the initial state vector S_0. State the limitations of this model.

a $\begin{pmatrix} 0.50 & 0.30 & 0.10 \\ 0.30 & 0.35 & 0.30 \\ 0.20 & 0.35 & 0.60 \end{pmatrix}^5 \begin{pmatrix} 110 \\ 80 \\ 50 \end{pmatrix} = \begin{pmatrix} 65.746 \\ 75.789 \\ 98.465 \end{pmatrix}$	Find $S_5 = T^5 S_0$ with a GDC
There would be approximately 66 Mathbikes in zone C, 76 in Zone I and 98 in Zone O.	Make sure to interpret your findings.
b $\begin{pmatrix} 0.50 & 0.30 & 0.10 \\ 0.30 & 0.35 & 0.30 \\ 0.20 & 0.35 & 0.60 \end{pmatrix}^{50} \begin{pmatrix} 110 \\ 80 \\ 50 \end{pmatrix} = \begin{pmatrix} 65.263 \\ 75.789 \\ 98.947 \end{pmatrix}$	Use your GDC to investigate higher powers of the transition matrix and look for convergence. As with geometric sequences, the rate of convergence depends on the values you are given.
So, the system quickly reaches an equilibrium at 65 bikes in C, 76 in I and 99 in O.	A steady state matrix is reached earlier than this, after ten days.
c This shows that Mathbike's original distribution of bikes to zone I was a good guess, but they would have met the needs of the customers better by placing 50 bikes in C and 110 in O at the start.	Think critically about the context.
This model does not account for broken or stolen bikes. Nor does it account for changing habits amongst the customers due to weather patterns or competition from other companies.	Write a complete sentence that presents your findings.

Investigation 12

1 By investigating the sequence T^n show that if $T = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$, then the long-term probability matrix is $X = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$.

2 Fill in the table with examples of your own initial state vectors and find values for the steady state population vector and the steady state probability vector.

S_0	Total population (p) represented by S_0	$XS_0 = v$	$u = \frac{1}{p}v$ (the steady state probability vector)
$\begin{pmatrix} 63 \\ 18 \end{pmatrix}$	81	$\begin{pmatrix} 27 \\ 54 \end{pmatrix}$	
$\begin{pmatrix} 568 \\ 123 \end{pmatrix}$			
$\begin{pmatrix} 56358 \\ 2 \end{pmatrix}$			
...			

Then repeat with your own 2×2 regular transition matrix T.

3 **Factual** Does the steady state probability vector depend on the initial state vector?

4 What relationship exists between the steady state probability vector and the long-term probability matrix?

Let $T = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$ with long-term probability matrix $X = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$, and

let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be the steady state probability vector. Hence,

$$Tu = u \Rightarrow \begin{cases} 0.6u_1 + 0.2u_2 = u_1 \\ 0.4u_1 + 0.8u_2 = u_2. \\ u_1 + u_2 = 1 \end{cases}$$

5 Explain why each of the three equations hold and show that the solution of this system is $u = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

6 With your own regular 2×2 matrix T, find the long-term probability matrix by repeated multiplication and then form three equations to solve $Tu = u$ using technology.

7 Repeat with your own regular 3×3 matrix.

8 **Factual** What do you notice?

9 **Conceptual** How can you find the steady state probabilities? Give two different methods.

Exercise 9H

1 Find the long-term probability matrices for the Markov processes with these transition matrices:

a $\begin{pmatrix} 0.13 & 0.71 \\ 0.87 & 0.29 \end{pmatrix}$ **b** $\begin{pmatrix} 0.24 & 0 & 0.31 \\ 0.42 & 0.85 & 0.05 \\ 0.34 & 0.15 & 0.64 \end{pmatrix}$

c $\begin{pmatrix} 0.1 & 0.6 & 0.2 & 0.7 \\ 0.05 & 0.2 & 0.2 & 0.05 \\ 0.4 & 0.05 & 0.2 & 0.2 \\ 0.45 & 0.15 & 0.4 & 0.05 \end{pmatrix}$

2 A regular transition matrix T in a given Markov process is defined as $T = \begin{pmatrix} 0.81 & 0.1 & 0.08 \\ 0.09 & 0.15 & 0.5 \\ 0.1 & 0.75 & 0.42 \end{pmatrix}$.

Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ be the steady state probability vector such that $Tu = u$.

a Find u by solving a system of equations.

b Demonstrate your answer for part **a** is correct by finding the long-term probability matrix.

3 Three electricity providers A, B and C compete in a market. A survey of consumers has determined the probabilities that a person using an electricity provider in one year will either stay with that provider or will switch to another next year. The probabilities are given in this transition matrix:

$$\begin{array}{cc} & \text{Current provider} \\ & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \text{Provider after one year} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & T = \begin{pmatrix} 0.8 & 0.05 & 0.06 \\ 0.12 & 0.92 & 0.1 \\ 0.08 & 0.03 & 0.84 \end{pmatrix} \end{array}$$

At the present time, 45% of the customers are provided electricity by A, 25% by B and 30% by C.

a Find the percentage of the customers provided by each electricity provider after two years.

b Calculate the long-term percentage of customers who will be provided electricity by B. State your assumptions.

c By comparing the matrices T, T^2, T^5 and T^{20}, comment critically on the likelihood that the transition probabilities remain constant in the long term.

4 A hire car owned by Mathcar is in one of four possible states: A: functioning normally, B: functioning despite needing a minor repair, C: functioning despite needing a major repair, D: broken down. Mathcar only takes a car out of service if it breaks down.

The transition matrix

$F = \begin{pmatrix} 0.9 & 0 & 0 & 0.96 \\ 0.03 & 0.85 & 0 & 0 \\ 0.02 & 0.05 & 0.6 & 0 \\ 0.05 & 0.1 & 0.4 & 0.04 \end{pmatrix}$ represents the probabilities of the car transitioning from one state to another after a period of a month.

a Interpret the zeros in the matrix in the context of the problem, and the entry in the fourth row fourth column.

b Verify C that F is a regular Markov matrix.

c Mathcar begins business with a fleet of 500 cars in state A, 25 in B, 0 in C and 0 in D. Find how many cars are in each state after 10 months.

d Comment on how Mathcar could change their practices so that changes are made to F which improve the availability of their cars.

9.7 Eigenvalues and eigenvectors

Investigation 13

Consider the transformation $T = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ applied to the points on the line

$L_1: y = 2x$.

i Write down the coordinates of any five points on L_1 then show that images of these five points lie on the same line.

Take a general point on the line with coordinates $(x, 2x)$ and find its image.

ii Based upon your observations in part **i** write down the value of λ for which $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \lambda \begin{pmatrix} x \\ 2x \end{pmatrix}$. What does this tell you about how T transforms the points on L_1 geometrically?

iii $T = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ also maps the points on line $L_2: y = x$ to image points that are also on $y = x$. Determine the scale factor of the stretch that describes this mapping by solving the equation $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \lambda \begin{pmatrix} x \\ x \end{pmatrix}$.

TOK

The multi-billion dollar eigenvector.

Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix

How ethical is it to create mathematics for financial gain?

The investigation above reveals a special type of linear transformation that maps points on a line back onto the same line and is described as a stretch by a factor λ. In this section you will learn how this idea along with our knowledge of matrix algebra will allow you to determine large powers of matrices. These results lay the foundation for our study of modelling and analyzing dynamical systems and networks in chapters 12 and 14.

HINT

When a transformation matrix T maps each point on one line to a point on a different line we say that the lines are invariant but the points themselves are not.

Eigenvalues and eigenvectors

Geometrically, you say that the lines $y = 2x$ and $y = x$ are invariant under the transformation $\boldsymbol{T}$.

In general, you are interested in determining the equations of the invariant lines and their associated scale factors, λ, for a given transformation matrix $\boldsymbol{A}$. That is, you wish to solve the equation

$\boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ or expressed more simply $\boldsymbol{Ax} = \lambda \boldsymbol{x}$. The values of λ are referred to as the **eigenvalues** of matrix $\boldsymbol{A}$. Each eigenvalue of $\boldsymbol{A}$ is associated with a particular line that is invariant under the transformation $\boldsymbol{A}$ whose equation is described by the **eigenvector** $\boldsymbol{x}$.

Consider the problem of determining the eigenvalues and eigenvectors of the transformation matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

International-mindedness

The prefix "eigen-" is German, meaning "to own" or "is unique to" indicating that that an eigenvalue is a unique value associated with a unique vector named the eigenvector. Eigenvalues describe the characteristics of a transformation and for this reason are also referred to as characteristic values and eigenvectors referred to as characteristic vectors.

You would like to find the eigenvalues λ and their associated eigenvectors such that $\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\boldsymbol{x} = \lambda\boldsymbol{x}$. Proceeding with the matrix algebra yields:

$$\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0} \qquad \text{(addition property of equality)}$$

$$\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\boldsymbol{x} - \lambda\mathbf{I}_2\boldsymbol{x} = \mathbf{0} \qquad \text{(identity property)}$$

$$\left(\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\boldsymbol{x} = \mathbf{0} \qquad \text{(distributive property)}$$

$$\begin{pmatrix}1-\lambda & 4\\2 & 3-\lambda\end{pmatrix}\boldsymbol{x} = \mathbf{0} \qquad \text{(matrix addition)}$$

When the matrix $\begin{pmatrix}1-\lambda & 4\\2 & 3-\lambda\end{pmatrix}$ has an inverse then both sides of the equation can be multiplied by the inverse matrix to give the solution $\boldsymbol{x} = \begin{pmatrix}0\\0\end{pmatrix}$. If looking for additional solutions they must occur when the inverse does not exist which happens when the matrix $(A - \lambda I)$ is singular and therefore its determinant is equal to zero.

Thus:

$$\begin{vmatrix}1-\lambda & 4\\2 & 3-\lambda\end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\therefore \lambda_1 = -1 \text{ and } \lambda_2 = 5$$

Now, returning to the original equation $\begin{pmatrix}1-\lambda & 4\\2 & 3-\lambda\end{pmatrix}\boldsymbol{x} = \mathbf{0}$ to find the corresponding eigenvector for each eigenvalue gives you:

$$\lambda_1 = -1 \Leftrightarrow \begin{pmatrix}2 & 4\\2 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{cases}2x+4y=0\\2x+4y=0\end{cases} \text{both equations give } y = -\frac{1}{2}x$$

$$\lambda_2 = 5 \Leftrightarrow \begin{pmatrix}-4 & 4\\2 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{cases}-4x+4y=0\\2x-2y=0\end{cases} \text{both equations give } y = x$$

The eigenvectors associated with the eigenvalue of $\lambda_1 = -1$ are described by the position vectors of the points on the line $y = -\frac{1}{2}x$

International-mindedness

Belgian/Dutch mathematician Simon Stevin use vectors in his theoretical work on falling bodies and his treatise "Principles of the art of weighing" in the 16th century.

and therefore have the form $\boldsymbol{x}_1 = \begin{pmatrix}t\\-\frac{t}{2}\end{pmatrix}$ where t is any real number.

That is, there are an infinite number of eigenvectors associated with the eigenvalue of $\lambda_1 = -1$. One particular eigenvector (when $t = 2$) is $\boldsymbol{x}_1 = \begin{pmatrix}2\\-1\end{pmatrix}$. Similarly, there are an infinite number of eigenvectors associated with the eigenvalue of $\lambda_2 = 5$ described by the points on the line $y = x$ and these vectors have the form $\boldsymbol{x}_2 = \begin{pmatrix}t\\t\end{pmatrix}$ where t is any real number. One particular eigenvector (when $t = 1$) is $\boldsymbol{x}_2 = \begin{pmatrix}1\\1\end{pmatrix}$.

If $\boldsymbol{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ then the real number λ is called an eigenvalue of $\boldsymbol{A}$ if there exists a nonzero vector $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ where $\boldsymbol{x}$ is the associated eigenvector with the eigenvalue λ.

The eigenvalues of A are given by the solutions of $\begin{vmatrix}a-\lambda & b\\c & d-\lambda\end{vmatrix} = 0$ referred to as the **characteristic equation.**

HINT

The process for finding the steady state vector of a Markov chain, ie solving $\boldsymbol{Tu} = \boldsymbol{u}$, is equivalent to finding the eigenvector for an eigenvalue of 1.

Matrix algebra provides you with a tool for modelling, analyzing, and predicting the long-term behaviour of systems that change over time (called dynamical systems).

Example 12

- Find the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix}0 & 1\\3 & -2\end{pmatrix}$.

Solve $\begin{vmatrix}-\lambda & 1\\3 & -2-\lambda\end{vmatrix} = 0$	
$-\lambda(-2-\lambda) - 3 = 0$ $\lambda = -3$ or 1 When $\lambda = -3$	This can be expanded or solved directly using technology.

Continued on next page

Number and algebra

Geometry and trigonometry

$\begin{pmatrix}3 & 1\\3 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$	Using $(A - \lambda I)\,x = 0$
$\Rightarrow 3x + y = 0$ or $y = -3x$	
Similarly when $\lambda = 1$ $\begin{pmatrix}-1 & 1\\3 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$	An alternative equation to find the eigenvectors is $\begin{pmatrix}0 & 1\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=-3\begin{pmatrix}x\\y\end{pmatrix}$
$x + y = 0$ or $y = x$	
Possible eigenvectors are therefore $\begin{pmatrix}1\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$	In the first case any vector which has the y-coordinate $-3\times$ the x-coordinate will do.

Exercise 9I

1 Determine the characteristic equation of each matrix in the form $\lambda^2 + p\lambda + q = 0$ then determine its eigenvalues.

a $A=\begin{pmatrix}-4 & -2\\3 & 1\end{pmatrix}$ b $B=\begin{pmatrix}5 & 0\\-3 & -5\end{pmatrix}$

c $C=\begin{pmatrix}-2 & 1\\-1 & 2\end{pmatrix}$

2 Determine whether or not $\lambda = -1$ an eigenvalue of $\begin{pmatrix}3 & 2\\3 & 8\end{pmatrix}$.

3 Find the eigenvalues and eigenvectors of each matrix.

a $C=\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$ b $Q=\begin{pmatrix}2 & 4\\5 & 3\end{pmatrix}$

c $I_2=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ d $P=\begin{pmatrix}0 & -3\\-3 & 0\end{pmatrix}$

e $T=\begin{pmatrix}0.4 & 0.7\\0.6 & 0.3\end{pmatrix}$

4 Consider the transition matrix

$T=\begin{pmatrix}a & b\\1-a & 1-b\end{pmatrix}$ where $0 \le a \le 1$ and $0 \le b \le 1$.

a Show that $\lambda = 1$ is an eigenvalue of T and determine the other value of λ in terms of a and b.

b Determine eigenvectors for each eigenvalue of T in terms of a and b.

Diagonalization and powers of a matrix

Investigation 14

Consider the matrix $A=\begin{pmatrix}5 & -3\\-6 & 2\end{pmatrix}$.

1 Show that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 8$ and that the corresponding eigenvectors are

$x_1=\begin{pmatrix}1\\2\end{pmatrix}$ and $x_2=\begin{pmatrix}1\\-1\end{pmatrix}$

2 Let P be a matrix containing the eigenvectors x_1 and x_2 of A and $D=\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}$. That is, $P=\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$ and $D=\begin{pmatrix}-1 & 0\\0 & 8\end{pmatrix}$.

a Find AP and show that $AP = PD$.

b Explain using the definition of eigenvalues and eigenvectors why this will always be the case.

3 a Use matrix algebra to show that if $AP = PD$ then $A = PDP^{-1}$

b Use the result in part **a** to show that $A^3 = PD^3P^{-1}$

c How did you use the fact that matrix multiplication is associative?

d How did you use properties of inverses?

4 a For the value of D given in question **2** find the value of D^3.

b Can you generalize your result for D^n.

5 a Determine A^3 by finding PD^3P^{-1}. Verify the result using the GDC to calculate A^3.

b Can you generalize your result for A^n?

Let $A=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ containing real number entries and having distinct real eigenvalues λ_1 and λ_2 then you say that A is *diagonalizable* and can thus be written in the form of $A = PDP^{-1}$ where $P = (X_1\ X_2)$ is the matrix of eigenvectors and $D=\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}$.

EXAM HINT

In an exam, matrices will always have distinct real eigenvalues.

Let A be a diagonalizable 2×2 matrix expressed in the form of $A = PDP^{-1}$ then $A^n = PD^nP^{-1}$ where $P = (X_1\ X_2)$ and $D=\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}$.

International-mindedness

Vectors developed quickly in the first two decades of the 19th century with Danish-Norwegian Caspar Wessel, Swiss Jean Robert Argand and German Carl Friedrich Gauss.

Example 13

a Find the diagonalization of $A=\begin{pmatrix}1 & 2\\3 & 0\end{pmatrix}$.

b Hence find an expression for A^4 in the form PD^4P^{-1}.

c Find an expression for A^4 as a product of 3 matrices with no exponents.

$\lvert A-\lambda I\rvert = 0 \Leftrightarrow \begin{vmatrix}1-\lambda & 2\\3 & -\lambda\end{vmatrix} = 0 \Leftrightarrow \lambda^2 - \lambda - 6 = 0$ $\lambda_1 = 3$ and $\lambda_2 = -2$	Eigenvalues are the solutions of the characteristic equation $\lvert A - \lambda I\rvert = 0$

Continued on next page

Number and algebra

Geometry and trigonometry

$$\begin{pmatrix}1&2\\3&0\end{pmatrix}x_1 = 3x_1 \Leftrightarrow \begin{pmatrix}1&2\\3&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 3\begin{pmatrix}x\\y\end{pmatrix} \Leftrightarrow y = x$$

$$x_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1&2\\3&0\end{pmatrix}x_2 = -2x_2 \Leftrightarrow \begin{pmatrix}1&2\\3&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -2\begin{pmatrix}x\\y\end{pmatrix} \Leftrightarrow y = -\frac{3}{2}x$$

$$x_2 = \begin{pmatrix}2\\-3\end{pmatrix}$$

Therefore,

$$P = \begin{pmatrix}1&2\\1&-3\end{pmatrix}, D = \begin{pmatrix}3&0\\0&-2\end{pmatrix}, P^{-1} = -\frac{1}{5}\begin{pmatrix}-3&-2\\-1&1\end{pmatrix}$$

Therefore:

$$A = -\frac{1}{5}\begin{pmatrix}1&2\\1&-3\end{pmatrix}\begin{pmatrix}3&0\\0&-2\end{pmatrix}\begin{pmatrix}-3&-2\\-1&1\end{pmatrix}$$

$$\text{or } A = \begin{pmatrix}1&2\\1&-3\end{pmatrix}\begin{pmatrix}3&0\\0&-2\end{pmatrix}\begin{pmatrix}0.6&0.4\\0.2&-0.2\end{pmatrix}$$

b $A^4 = \begin{pmatrix}1&2\\1&-3\end{pmatrix}\begin{pmatrix}3&0\\0&-2\end{pmatrix}^4\begin{pmatrix}0.6&0.4\\0.2&-0.2\end{pmatrix}$

c $A^4 = \begin{pmatrix}1&2\\1&-3\end{pmatrix}\begin{pmatrix}81&0\\0&16\end{pmatrix}\begin{pmatrix}0.6&0.4\\0.2&-0.2\end{pmatrix}$

There are an infinite number of eigenvectors of the form $x_1 = \begin{pmatrix}t\\t\end{pmatrix}$ and $x_2 = \begin{pmatrix}2t\\-3t\end{pmatrix}$ where $t \in \mathbb{R}$ you arbitrarily choose $t = 1$ (for simplicity.)

Using $P^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d&-c\\-b&a\end{pmatrix}$.

The $\frac{1}{5}$ can be taken out as a factor or placed inside one of the matrices, either is acceptable.

Return to transition matrices

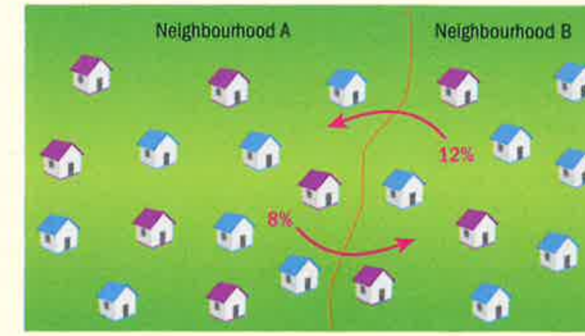
In section 9.5 you saw how to find the steady state of a dynamical system by considering high powers of the transition matrices or by solving the equation $Tu = u$ where u is the steady state probability vector.

This section will show how the same results can be obtained using diagonalization. Though this is a initially a longer process, once the diagonalised form has been found it is much easier for you (or for computers) to calculate high powers using this than by performing successive multiplications.

An additional benefit is that it also gives you an easy way to find a formula for the state of the system after n transitions.

Investigation 15

Consider the situation where people move between two neighbourhoods in a particular city. Each year since 2015, 8% of people currently living in neighbourhood A move to neighbourhood B and 12% of people currently living in neighbourhood B move to neighbourhood A. You may assume any movement other than between the two neighborhoods exactly balances those arriving with those leaving.



What will the population of each neighbourhood be in 2020 assuming the migration rates remain fixed?

1 Let T be the transition matrix where T_{ji} represents the probability of a person moving from neighbourhood i to neighbourhood j. Explain why the transition matrix is

$$T = \begin{array}{cc} & \begin{array}{cc}\text{A} & \text{B}\end{array} \\ \begin{array}{c}\text{A}\\\text{B}\end{array} & \begin{pmatrix}0.92 & 0.12\\0.08 & 0.88\end{pmatrix}\end{array}$$

2 The populations of neighbourhoods A and B are 24,500 and 45,200 respectively at the beginning of 2015. After 1 year the population can be expressed as the matrix $P_1 = TP_0 = \begin{pmatrix}0.92&0.12\\0.08&0.88\end{pmatrix}\begin{pmatrix}24500\\45200\end{pmatrix}$

What is the population of neighbourhood A after 1 year? Neighbourhood B?

Let $S_0 = \begin{pmatrix}24500\\45200\end{pmatrix}$. The population after n years is given by the expression T^nS_0.

3 Find the eigenvalues and eigenvectors for T.

4 Diagonalize T and use the relationship $T^n = PD^nP^{-1}$ to show that

$$T^n = \begin{pmatrix}3&1\\2&-1\end{pmatrix}\begin{pmatrix}1^n&0\\0&0.8^n\end{pmatrix}\begin{pmatrix}0.2&0.2\\0.4&-0.6\end{pmatrix}.$$

5 As $n \to \infty$, what will $D^n = \begin{pmatrix}1^n&0\\0&0.8^n\end{pmatrix}$ approach? What will T^n approach?

6 Hence find the long-term population of the neighbourhoods.

The administration of both neighborhoods would like a formula to tell them how many residents are expected to be in each community n years after 2015.

7 Multiply out your expression found in question 3 to find a suitable formula for each of the school boards.

8 Use your formula to write down the populations of neighborhood A in 2020.

Exercise 9J

1 Given that eigenvalues of $R = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$ are $\lambda_1 = -4$ and $\lambda_2 = -1$ write R in the form $R = PDP^{-1}$.

2 Consider $A = \begin{pmatrix} 16 & -35 \\ 6 & -13 \end{pmatrix}$.

a Determine the eigenvalues and eigenvectors of A. State why A is diagonalizable.

b Express A in the form of $A = PDP^{-1}$.

c Find a general expression for A^n in terms of n.

d Use your expression in part **c** to find A^4. Verify your result using the matrix utility on the GDC.

3 Consider matrix $T = \begin{pmatrix} 0.4 & 0.75 \\ 0.6 & 0.25 \end{pmatrix}$.

a Express T in the form $T = PDP^{-1}$.

b Find T^4 using your answer in part **a**. Verify your result by finding $\begin{pmatrix} 0.4 & 0.75 \\ 0.6 & 0.25 \end{pmatrix}^4$ using the matrix utility on the GDC.

c Using the result from part **b** determine the long-term behavior of T.

4 Suppose only two rival companies, R and S, manufacture a certain product. Each year, company R keeps $\frac{1}{4}$ of its customers while $\frac{3}{4}$ of them switch to company S. Each year, company S keeps $\frac{2}{3}$ of its customers while $\frac{1}{3}$ of them switch to company R. At the beginning of the 2005, company R had 6500 customers while company S had 5200 customers.

a Write down a transition matrix T representing the proportion of the customers moving between the two companies.

b Find the distribution of the market after two years. Describe the change in this distribution from the 2005.

c Write T in the form $T = PAP^{-1}$

d Show that $T^n = \frac{1}{17}\begin{pmatrix} 8+9p^n & 8-9p^n \\ 9-9p^n & 9+8p^n \end{pmatrix}$ where $p = -\frac{5}{12}$.

e Hence, find an expression for the number of customers buying from R after n years.

f Verify your formula by finding how many customers are purchasing from R after two years.

g Find the long-term number of customers buying from R.

Chapter summary

- To add or subtract two or more matrices, they must be of the same order. You add or subtract corresponding elements.
- Given a matrix A and a real number k then kA is obtained by multiplying every element of A by k where k is referred to as a scalar.
- If $P = AB$ then each element of P (named as $P_{i,j}$) is found by summing the products of the elements in row i of A with the elements in column j of B.

- If A has dimensions $m \times n$ and B has dimensions $p \times q$ then
  - the product AB is defined only if the number of columns in A is equal to the number of rows in B (that is, $n = p$)
  - and when the product does exist, the dimensions of the product is $m \times q$.
- **Properties of multiplication for matrices**

  For matrices A, B, and C:
  - Non-commutive $AB \neq BA$
  - Associative property $A(BC) = (AB)C$
  - Distributive property $A(B + C) = AB + AC$ and $(B + C)\, A = BA + CA$

  These properties only hold when the products are defined.
- The multiplicative identity of a square $n \times n$ matrix A is given by $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$. Note that both $A \times I_n = A$ and $I_n \times A = A$ hold only in the case where A is a square matrix.
- If A is a square matrix then $A^k = \underbrace{A \times A \times \cdots \times A}_{\text{k factors of A}}$.
- In the case where $b = c = 0$ then $A^k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 0 & d^k \end{pmatrix}$.
- The matrices which represent rotations, reflections, enlargements and one way stretches are all given in the IB formula book.
- A **Markov Chain** is a system in which the probability of each event depends only on the state of the previous event.
- A transition matrix is a square matrix which summarizes a transition state diagram. It describes the probabilities of moving from one state to another in a stochastic system.
  - The sum of each column of the transition matrix is 1.
  - The transition matrix is extremely useful when there are more than two states in the system and when you wish to predict states of the system after several time periods.
- A **regular** transition matrix T has the property that there exists $n \in \mathbb{Z}^+$ such that all entries in T^n are greater than zero.
- The **initial state vector** S_0 shows the initial state of the system.
- If P is a square regular transition matrix, there is a unique probability vector q such that $Pq = q$. This q is called the **steady-state vector** for P.
- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ containing real number entries and having distinct real eigenvalues λ_1 and λ_2 then you say that A is diagonalizable and can thus be written in the form of $A = PDP^{-1}$ where $P = (x_1\ x_2)$ is the matrix of eigenvectors and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
- Let A be a diagonalizable 2×2 matrix expressed in the form of $A = PDP^{-1}$ then $A^n = PD^nP^{-1}$ where $P = (x_1\ x_2)$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Chapter review

Click here for a mixed review exercise

1 If $A=\begin{pmatrix}5\\-2\end{pmatrix}$, $B=\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $C=\begin{pmatrix}-1&3\\-2&4\end{pmatrix}$, $D=\begin{pmatrix}-3&1\\0&-5\end{pmatrix}$, $E=\begin{pmatrix}-1&2&1\\-2&1&0\\-3&-1&4\end{pmatrix}$, $F=\begin{pmatrix}1&2&-3\\-1&4&-2\end{pmatrix}$ find each of the following without using the matrix utility on the GDC. If it does not exist state why it does not.

a $2C+3D$ **b** $A-B$ **c** DF **d** FE
e E^2 **f** $(C+D)^2$ **g** $EB+B$ **h** C^{-1}
i CDA **j** $C-2I_2$

2 Determine the values of a and b such that

$$\begin{pmatrix}3a&2\\-1&2b\end{pmatrix}+4\begin{pmatrix}-1&-2b\\1&-\frac{a}{2}\end{pmatrix}=\begin{pmatrix}8&6\\3&-9\end{pmatrix}.$$

3 Solve each equation for X.

a $3X-\begin{pmatrix}2&-1\\1&-2\end{pmatrix}=\begin{pmatrix}3&5\\-7&0\end{pmatrix}$

b $\begin{pmatrix}1&3\\-1&2\end{pmatrix}X=\begin{pmatrix}-5&2\\1&-10\end{pmatrix}$

c $X\begin{pmatrix}3&-1\\-4&2\end{pmatrix}+3I_2=\begin{pmatrix}5&4\\2&6\end{pmatrix}$

4 Write each system as a matrix equation then solve the system using matrix algebra (if possible). Write your answers in exact form.

a $\begin{aligned}7y&=4x-20\\3x&=2y+10\end{aligned}$ **b** $\begin{aligned}w&=3+2y+2z\\3y-4z&=6-w\\4w+5y-3&=2z\end{aligned}$

c $\begin{aligned}6x&=5y+18\\27-9x&=-7.5y\end{aligned}$

5 Parallelogram PQRS with vertices P(−4, −2), Q(−1, 7), R(8, 4), T(5, −5) is translated by the vector $\begin{pmatrix}-3\\5\end{pmatrix}$ then enlarged by a factor of $\frac{1}{2}$ then reflected across the x-axis.

a Determine a single transformation of the form of $AX+b$ that maps PQRS to its image P′Q′R′S′.

b Determine the coordinates of P′, Q′, R′, and S′.

c Area of P′Q′R′S′ $=k\times$ Area of PQRS. Determine the value of k.

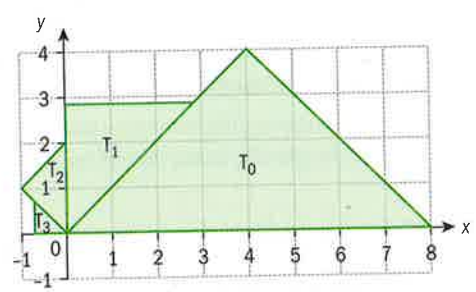
6 The linear transformation $\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}-3&1\\-1&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}$ maps the points (x,y) to (x',y'). Find

a the coordinates of the image of (−3, 5)

b the coordinates of the point who image is (−7, 0)

c the coordinates of the image of $(a, 2a)$ where $a\in\mathbb{R}$.

7 A series of triangles T_0, T_1, T_2,… are formed by rotating each consecutive triangle anti-clockwise by $\theta°$ then enlarging the triangle by a factor of k.



a Given that the series of transformations can be described by $\begin{pmatrix}x_n\\y_n\end{pmatrix}=A\begin{pmatrix}x_{n-1}\\y_{n-1}\end{pmatrix}$ determine

i the 2 × 2 matrix A

ii the exact coordinates of the vertices of T_3.

b Given that $\begin{pmatrix}x_n\\y_n\end{pmatrix}=\mathbf{C}_n\begin{pmatrix}x_0\\y_0\end{pmatrix}$

i write down $\mathbf{C}_1$

ii find $\mathbf{C}_2$, $\mathbf{C}_3$, and $\mathbf{C}_4$

iii find the sum of the areas of T_0, T_1, T_2,…

Exam-style questions

8 P1: Given $A=\begin{pmatrix}3&-2\\2&-3\end{pmatrix}$ and $B=\begin{pmatrix}4&0\\1&-2\end{pmatrix}$, find the following matrices:

a $2B$ (2 marks)
b $A-B$ (2 marks)
c AB (2 marks)
d $(A+B)^2$ (2 marks)

9 P1: Given that the matrix $A=\begin{pmatrix}x&-1\\4&3-x\end{pmatrix}$ is singular, determine the possible value(s) of x. (5 marks)

10 P1: $P=\begin{pmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$ and $Q=\begin{pmatrix}\frac{\sqrt{3}}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{\sqrt{3}}{2}\end{pmatrix}$.

a Find the matrix $(PQ)^3$. (3 marks)

b Hence, find the smallest value of n such that $(PQ)^n=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$. (2 marks)

11 P2: a Show that the matrix $A=\begin{pmatrix}4&1\\3&6\end{pmatrix}$ satisfies the equation $A^2-10A+21I=0$, where I is the 2 × 2 identity matrix. (4 marks)

b Hence express A^3 in the form $pA+qI$. (3 marks)

c Hence express A^4 in the form $rA+sI$. (3 marks)

12 P1: The matrix $A=\begin{pmatrix}3&2&-1\\0&1&2\\1&1&3\end{pmatrix}$ and $B=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}$

a Find A^{-1}. (2 marks)

b Hence find the matrix C such that $AC=B$. (3 marks)

13 P1: Consider the following system of equations:

$$\begin{aligned}5x+3z&=23\\x-2y+5z&=23\\3y+7z&=122\end{aligned}$$

a Write the system in the form $AX=B$, where A, X and B are matrices. (2 marks)

b Find the matrix A^{-1}. (2 marks)

c Hence solve the original system of equations. (3 marks)

14 P1: At a particular opening night screening of Galaxy Wars 2, one cinema had a full attendance of 750 people. It charged £8 for adults, £5 for children, and £4 for OAPs.

The number of OAPs attending was one-fifth of the total attendance.

The total amount received in entrance tickets was £4860.

Use a matrix method to determine the number of children that attended the screening. (9 marks)

15 P2: Consider the matrix $A=\begin{pmatrix}4&-2\\-2&4\end{pmatrix}$

a Find the eigenvalues and corresponding eigenvectors of A. (7 marks)

b Hence find a matrix P and a matrix D such that $D=P^{-1}AP$. (2 marks)

c Find a general expression for A^n in terms of n. (5 marks)

MarComm phones and Markov chains

Approaches to learning: Communication, Critical thinking

Exploration criteria: Mathematical communication (B), Use of mathematics (E)

IB topic: Markov chains, Matrices

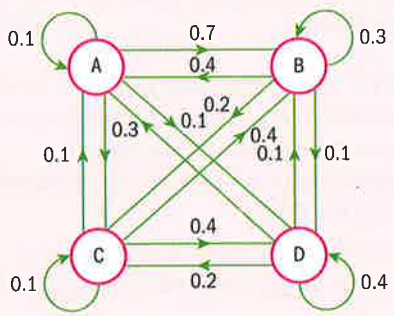
Example

A phone company, MarComm, has four package options, A, B, C and D.

A customer is initially allocated to package A for a year.

After this, every year, customers can either continue with their present option or change to one of the other options.

The probabilities of each possible change are given in this transition state diagram:



How would MarComm be able to determine the probabilities on the diagram?

Represent this information in a transition matrix, P, of the system.

If customers are all initially allocated to option A for the first year, then write down an initial state vector, $\vec{x}_0$, in the form $\vec{x}_0 = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

The probabilities for the next state can then be found by multiplying $\vec{x}_0$ by the transition matrix P.

Calculate the probabilities of being in each option after one year if the customer starts with option A.

Write down these probabilities in the form $\vec{x}_1 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

How might MarComm find this information useful?

Repeating this iteratively will give the probability of a customer being in any state in subsequent years.

Investigate the behaviour of the system over time.

Let π be the stationary distribution of the system, where $\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$.

You will now find the stationary distribution using three different methods.

Using technology

Calculate $\vec{x}_{19}$ and $\vec{x}_{20}$ (that is $P^{19}\vec{x}_0$ and $P^{20}\vec{x}_0$) using a calculator or computer. This will give an **indication** of the convergence to a stationary distribution (to a reasonable degree of accuracy).

Why would this **not** be considered a sophisticated approach to the problem?

Solving a system of equations

This method uses the fact that once the stationary distribution, π, is reached, multiplying by the transition matrix has no effect $\Rightarrow P\pi = \pi$.

So, by letting $\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$, you can solve the system of equations $[P]\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$.

Use this method with the information in the example to calculate π and to verify that the result is equivalent to the answer obtained using the previous method using technology.

Does this feel more sophisticated than the previous process using technology? Justify your answer.

Using eigenvalues and eigenvectors

If the stationary distribution π is such that $P\pi = \pi$, then as you have seen in this chapter the matrix P must have an eigenvalue of 1 with corresponding eigenvector π scaled so that the sum of its elements is 1.

Calculate the eigenvalues of P from the example and the corresponding eigenvectors (using technology).

How could you demonstrate that you understand fully what you are finding here and that you understand the process that is taking place beyond just getting the correct answers?

How will you organize the mathematical calculations so that the process from posing the problem to answering the problem is logical?

These methods give MarComm π, the probabilities of a customer being in each of the four options in the long term.

How could the company use this information?

Clearly all three methods produce the same approximate vector π correct to four significant figures.

However, which method do you prefer?

Which method is the most sophisticated? Justify your answer.

Extension

Does a system always reach a steady state?

How quickly will the process converge to the steady state?