10 Analyzing rates of change: differential calculus

Calculus is a mathematical tool for studying change. This chapter will focus on differentiation, which looks at how quickly one variable is changing **with respect to** another.

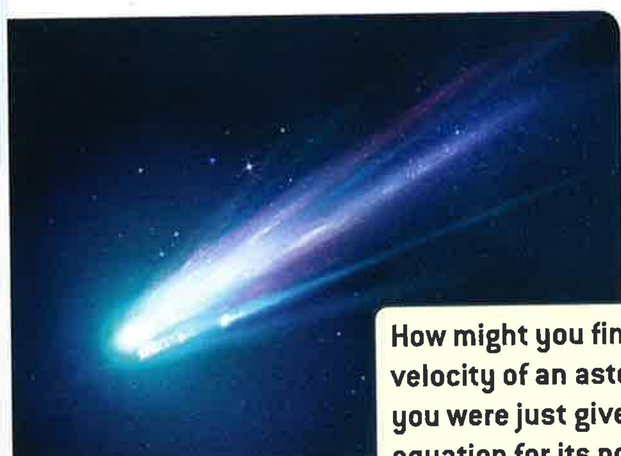
Change is all around us. Examples in physics, include change of position, velocity, density, current and power; in chemistry, we have rates of reaction, and in biology, the growth of bacteria; in economics, rates include marginal cost and profit; in sociology, we might want to measure the speed at which a rumour spreads or periodic changes in fashion. All of these can be applications of differentiation. But how do we go about measuring these changes?

Concepts

- Change
- Relationships

Microconcepts

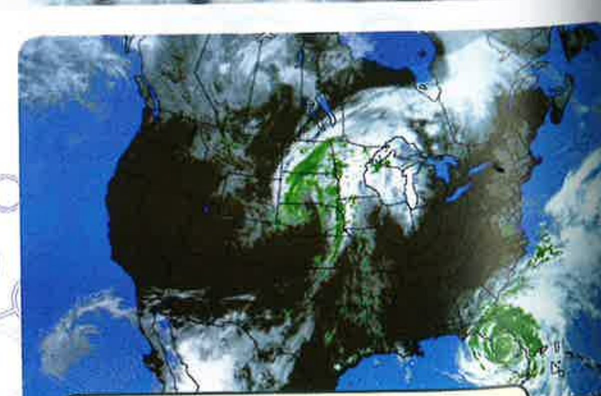
- Concept of a limit
- Derivatives of standard functions
- Increasing and decreasing functions
- Tangents and normals
- Optimization
- The chain rule
- The product rule
- The quotient rule
- The second derivative
- Kinematics



How might you find the velocity of an asteroid if you were just given the equation for its position?

If you are given a model for the number of people who catch a disease, how would you work out how quickly it is spreading at a particular time?

A company has a model that shows the profit it could make for different levels of output. How could it quickly find the output that would give maximum profit?



How do weather forecasters predict the weather when the changes seem so unpredictable?

A firm is about to launch a new product. They conduct a survey to find the optimum price at which it should be sold. They obtained the results shown in the table.

The firm decides that a **demand** equation is needed to model these results. Two models are suggested: a power model of the form $d = \frac{a}{x^b}$ and an exponential model of the form $d = ae^{-bx}$ or $d = a(c)^x$ where $c = e^{-b}$.

- Plot the data on your GDC.
- Explain why these two models might be suitable.
- Find best fitting equations for each of the models.
- The business would like to maximize their profit. What other information would they need?
- If this information were available, how might you work out the maximum value of a profit function?
- The gradient of the demand curve is called the "marginal demand". What might the marginal demand tell you? How could you work out its value from the curve?
- The marketing team has a model that links extra demand with the amount of money spent on advertising. How can this be incorporated into your model?

Price ($\$x$)	Percentage of the sample who would buy at this price (d)
5	68
10	55
15	43
20	33
25	22
30	14
35	8
40	4

Developing inquiry skills

What other information might you need to work out the profit?

What could the company do to persuade more people to buy the product?

How might this affect the model?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Find the equation of a line.

eg Find the equation of the line with gradient 5 and passing through the point (2, 1).

$1 = 5(2) + c$

$c = -9 \quad y = 5x - 9$

or use $y - 1 = 5(x - 2)$

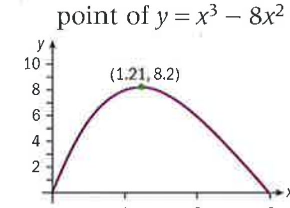
2 Use the laws of indices.

eg Write the following with a single exponent.

a $\frac{x^5}{x^3} = x^{5-3} = x^2$ **b** $\frac{1}{x^2} = x^{-2}$ **c** $\sqrt{x} = x^{\frac{1}{2}}$

3 Find maximum and minimum values of a function using your GDC.

eg Find the coordinates of the maximum point of $y = x^3 - 8x^2 + 15$, where $0 \le x \le 3$.



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Skills check

Click here for help with this skills check

1 Find the equation of the line perpendicular to the line $y = 2x - 3$ and passing through the point (3, 4).

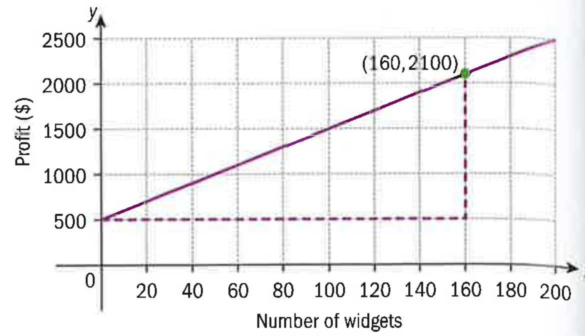
2 Write the following with negative exponents.

a $\frac{x^2}{x^7}$ **b** $\frac{1}{x}$ **c** $\frac{3}{\sqrt{x}}$

3 Find the minimum value of $y = x^4 - 5x^3 - 4x^2 + 20,\ 0 \le x \le 2$.

10.1 Limits and derivatives

The graph shows how the profits of a company increase with the number of widgets it sells.



From previous work on linear functions you will know that the rate at which the profit increases for each new widget made (profit per widget) is

$\frac{2100-500}{160-0}$ = \$10 per widget.

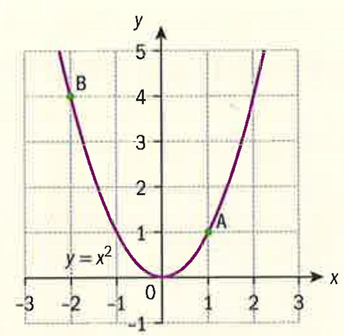
This is equivalent to the gradient of the curve. If the curve had a steeper gradient the profit per widget would be higher. If the gradient were less steep the profit per widget would be lower.

Hence, the rate of change of one variable with respect to another is equivalent to the gradient of the curve.

This is easy to calculate when the functions are linear. The following investigation will consider how the gradient at a point on a curve might be calculated.

Investigation 1

When a curve is increasing, we say that it has a positive gradient, and when it is decreasing we say that it has a negative gradient. Consider the curve $y=x^2$ shown in the diagram.



1 **a** Give the range of x values for which the gradient of the curve is

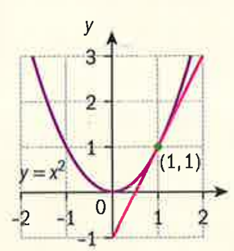
i negative

ii positive

iii equal to zero.

b At which point is the gradient of the curve greater, A or B?

The tangent to a curve at a point is the straight line that just touches but does not cross the curve at that point.



2 **a** On your GDC or other software, plot the curve $y=x^2$ and draw the tangent at the point (1, 1), as shown in the second diagram.

b Zoom in to the point (1, 1).

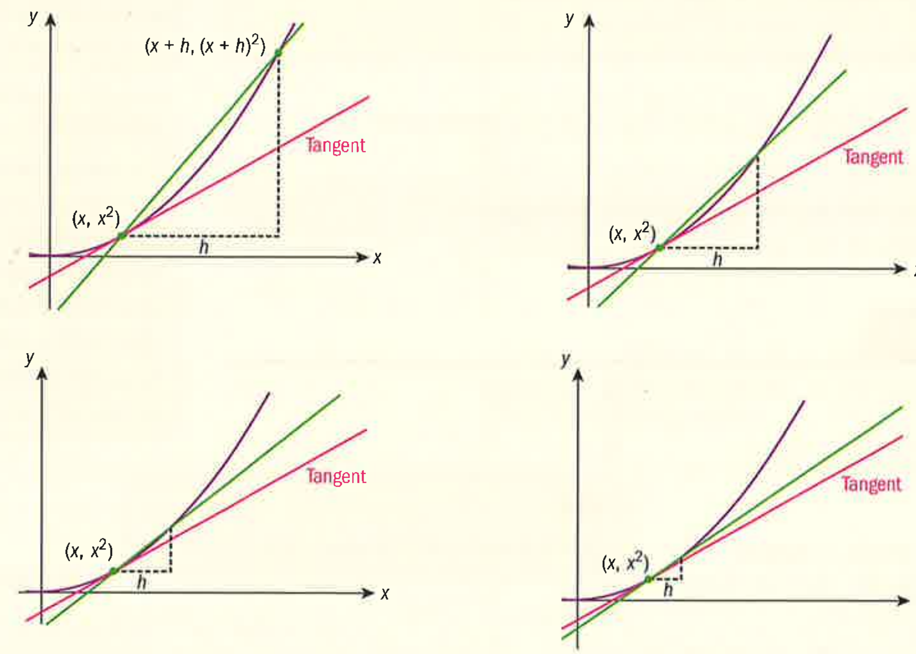
Factual What do you notice about the gradient of the tangent and the gradient of the curve at the point of contact?

c Use your GDC or online software to draw tangents to $y=x^2$ at the points listed below and in each case write down the gradient of the tangent.

x	−1	0	1	2	3
Gradient of tangent at x			2		

d Conjecture a function for the gradient of $y = x^2$ at the point on the curve with coordinates (x, x^2).

To justify the conjecture above we will consider the gradient of a chord from the point (x, x^2) to the point a distance h further along the x-axis. This is shown in the diagrams below for different values of h.



3 **a** What do you notice about the gradient of the chord as h decreases?

b What will happen as h approaches zero? (Note: we could write this as $h \to 0$.)

The answer to **3b** can be used to work out an expression for the gradient of the curve at the point (x, x^2).

4 **a** Explain why an expression for the gradient of the chord between the two points (x, x^2) and $(x+h, (x+h)^2)$ is $\frac{(x+h)^2-x^2}{h}$.

b What would happen if you let $h \to 0$?

c Expand and simplify your expression. What happens now if you let $h \to 0$?

d Compare your answer with your answer to question **2d**.

5 **a** Now consider the curve $y=x^3$. Draw tangents to the curve at the points listed below and in each case write down the gradient of the tangent.

x	−2	−1	0	1	2	3
Gradient of tangent at x						

b Conjecture a function for the gradient of $y=x^3$ at the point on the curve with coordinates (x, x^3).

c Write down an expression for the gradient of the chord between (x, x^3) and $(x+h, (x+h)^3)$.

d By first expanding and simplifying your answer to part **c** find an expression for the gradient of the curve at (x, x^3). Is it the same as the answer you conjectured in part **b**?

6 **Conceptual** How would you find the gradient of the curve at a point using the gradient of a chord from that point?

The function that gives the gradient of the graph $y=f(x)$ at the point x is written as $f'(x)$ where $f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$.

HINT
The notation $\lim_{h\to 0}(A)$ is read as "the limit of A as h tends to zero".

The gradient function, $f'(x)$, is referred to as the **derivative** of x.

Reflect What does the derivative of a function tell you?

EXAM HINT
You will not be asked questions that require you to use this notation. However, the ideas behind functions approaching limits will reappear at various points in the course.

Exercise 10A

Use the formula

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

to find the gradient function (the derivative) for the following functions.

1. $f(x)=4x$
2. $f(x)=-2$
3. $f(x)=3x-5$
4. $f(x)=c$
5. $f(x)=mx$
6. $f(x)=x^2-3x+7$
7. $f(x)=2x^2$
8. $f(x)=x^4$

HINT
$(x+h)^4=x^4+4x^3h+6x^2h^2+4xh^3+h^4$

9. $f(x)=\frac{1}{x}$
10. Conjecture an expression for $f'(x)$ given that $f(x)=ax^n$, $a, n\in\mathbb{R}$.

TOK
What value does the knowledge of limits have?

From the previous exercise, you may have conjectured the following rules:

- $f(x)=ax^n \Rightarrow f'(x)=anx^{(n-1)}, n\in\mathbb{R}$ (the power law)
- $f(x)=c \Rightarrow f'(x)=0$
- $f(x)=mx \Rightarrow f'(x)=m$

Notice that the last two follow the same rule as the first if you write c as cx^0 and mx as mx^1.

In addition, if $h(x)=f(x)+g(x)$ then $h'(x)=f'(x)+g'(x)$.

Reflect What is the power law for differentiation?

Example 1

Use the power law to find the derivative of each of:

a $f(x)=3x^2+2x+7$ **b** $f(x)=\frac{3}{x^2}$ **c** $f(x)=4\sqrt{x}+\frac{1}{\sqrt{x}}$

a $f'(x)=6x+2$	Use the rules above.
b $f(x)=3x^{-2}$	To use the power law you need to first write the expression using a negative index.
$f'(x)=-6x^{-3}=-\frac{6}{x^3}$	The answer can be left as a negative exponent, but sometimes writing it as a fraction makes subsequent work easier.
c $f(x)=4x^{\frac{1}{2}}+x^{-\frac{1}{2}}$	The first step again is to write the expression using fractional and negative powers.
$f'(x)=2x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$ $=\frac{2}{\sqrt{x}}-\frac{1}{\sqrt{x^3}}$	There are different ways to write the final answer; all are acceptable in an exam unless you are told otherwise.

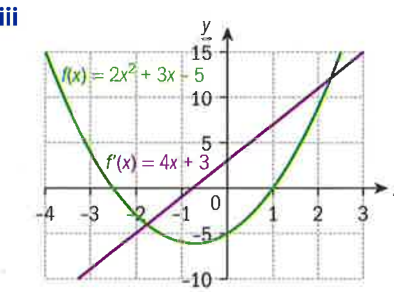
Example 2

For each of the functions below:

i. find $f'(x)$
ii. find the gradient of the curve at the point where $x=2$
iii. sketch the graph of the function and its derivative on the same axes
iv. write down the set of values of x for which the function is increasing.

a $f(x)=2x^2+3x-5$

b $f(x)=\frac{2}{x}+x, x\neq 0$

a **i** $f'(x)=4x+3$	Use the laws given above.
ii $x=2\Rightarrow f'(2)=11$	The derivative at a point gives the gradient at that point.
iii 	
iv $x>-0.75$	This can be found by either finding the minimum point of $f(x)=2x^2+3x-5$ or by finding where $f'(x)>0$.

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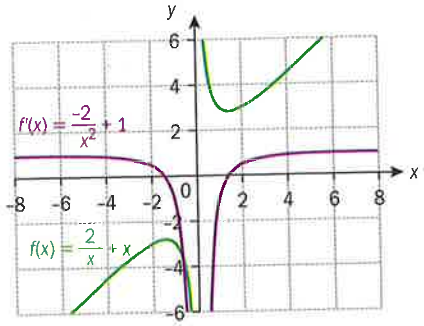
Calculus

b **i** $\frac{2}{x} = 2x^{-1} \Rightarrow f(x) = 2x^{-1} + x$

$f'(x) = -2x^{-2} + 1 = \frac{-2}{x^2} + 1$

ii $x = 2 \Rightarrow f'(2) = -0.5 + 1 = 0.5$

iii



iv $x > 1.41,\ x < -1.41$

Again the range in which the curve is increasing can be found by finding the maximum and minimum points using a GDC or by finding where $f'(x) > 0$.

Reflect What is the relationship between the gradient of a curve and the sign of its derivative?

Example 3

In economics, the marginal cost is the cost of producing one more unit. It can be approximated by the gradient of a cost curve.

A company produces motorbike helmets and the daily cost function can be modelled as

$C(x) = 600 + 7x - 0.0001x^3$ for $0 \le x \le 150$

where x is the number of helmets produced and C the cost in US dollars.

a Write down the daily cost to the company if no helmets are produced.

b Find an expression for the marginal cost, $C'(x)$, of producing x helmets.

c Find the marginal cost if **i** 20 helmets are produced **ii** 80 helmets are produced.

d State the units of the marginal cost.

a \$600	This is the value when $x = 0$.
b $C'(x) = 7 - 0.0003x^2$	C' represents the marginal cost.
c **i** $C'(20) = 6.88$	
ii $C'(80) = 5.08$	
d \$ per helmet	This is the extra cost in \$ for each new helmet made.

Alternative notation

There is an alternative notation for the derivative, which is very widely used.

For example, if $y = 2x^2 + 5$ then its derivative is $\frac{dy}{dx} = 4x$.

The notation comes from the fact that the gradient of a line is $\frac{\text{difference in } y}{\text{difference in } x}$.

One useful aspect to this notation is that it indicates clearly the variables involved, so if $s = 3t^3 + 4t$ then the derivative would be written as $\frac{ds}{dt} = 9t^2 + 4$.

International-mindedness

Maria Agnesi, an 18th century, Italian mathematician, published a text on calculus and also studied curves of the form $y = \frac{a^2}{x^2} + a$.

Usually the prime notation, $f'(x)$, is used when dealing with functions and the fractional notation, $\frac{dy}{dx}$, is used when dealing with equations relating two variables.

HINT

There are no set rules, however. $y' = 2x - 1$ and $\frac{df}{dx} = 3x$ are both correct, though less frequently seen.

Example 4

The tangent to the curve $y = 2x^2 + 3x - 4$ at the point A has a gradient of 11.

Find the coordinates of A.

$\frac{dy}{dx} = 4x + 3$	First find the derivative.
$4x + 3 = 11$ $4x = 8$ $x = 2$	You know that the derivative is the same as the gradient of the tangent – so, equate the derivative to 11 and solve for x.
When $x = 2$, $y = 2(2)^2 + 3(2) - 4 = 10$, So, the coordinates of A are (2, 10).	Now substitute 2 for x into the original equation.

The following exercise contains a mixture of both notations. Remember that they both indicate that the derivative of a function gives the gradient, or rate of change, of the variable.

Calculus

Exercise 10B

1 Find the derivative of each of the following functions with respect to x and find the gradient of the curve at the point where $x = 2$.

a $y = 6$
b $y = 4x$
c $f(x) = 3x^2$
d $f(x) = 5x^2 - 3x$
e $f(x) = 3x^4 + 7x - 3$
f $f(x) = 5x^4 - 3x^2 + 2x - 6$

2 Find the derivative of each of the following functions with respect to x and find the gradient of the curve at the point where $x = 1$.

a $f(x) = \sqrt[3]{x}$ **b** $y = 2\sqrt{x} + 3$
c $y = 2x^2 - \frac{3}{x}$ **d** $y = \frac{6}{x^3} + 4x - 3$
e $y = \frac{7}{x^3} + 8x^4 - 6x^2 + 2$

3 Find the derivative of each of the following functions, and find the gradient of the associated curve when $t = 16$.

a $s = 4t - \frac{8}{\sqrt{t}}$
b $v = 4t^{\frac{3}{4}} - \frac{16}{t^{\frac{1}{4}}}$

4 For each of the functions below:
i find the derivative
ii find the set of values of x for which the associated function is increasing.

a $y = (2x - 1)(3x + 4)$
b $f(x) = 2x(x^3 + 4x - 5)$
c $g(x) = x^3 + 3x^2 - 9x - 8$

5 The area, A, of a circle of radius r is given by the formula $A = \pi r^2$.

a Find $\frac{dA}{dr}$.
b Find the rate of change of the area with respect to the radius when $r = 2$.

6 The profit, $P, made from selling c cupcakes is modelled by the function $P = -0.056c^2 + 5.6c - 20$.

a Find $\frac{dP}{dc}$.
b Find the rate of change of the profit with respect to the number of cupcakes when $c = 20$ and $c = 60$.
c Comment on your answers for part **b**.

7 The distance of a bungee jumper below his starting point is modelled by the function $f(t) = 10t - 5t^2$, $0 \le t \le 2$, where t is the time in seconds from the moment he jumps.

a Find $f'(t)$.
b State the quantity represented by $f'(t)$.
c Find $f'(0.5)$ and $f'(1.5)$ and comment on the values obtained.
d Find $f(2)$ and comment on the validity of the model.

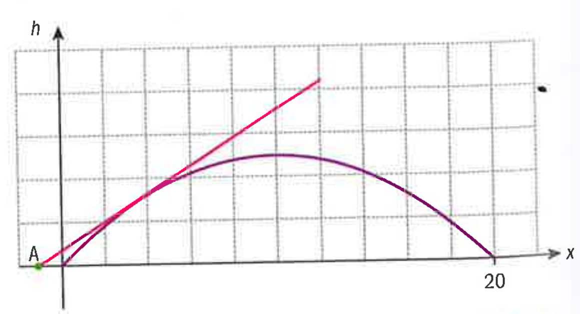
8 Points A and B lie on the curve $f(x) = x^3 + x^2 + 2x$ and the gradient of the curve at both A and B is equal to 3.

Find the coordinates of points A and B.

9 The outline of a building can be modelled by the equation $h = 2x - 0.1x^2$ where h is the height of the building and x the horizontal distance from one corner of the building, as shown in the diagram below.

An observer stands at A. The angle of elevation from his position to the highest point he can see on the building is 45°.

Find the height of that point above the ground.



Equations of tangents and normals

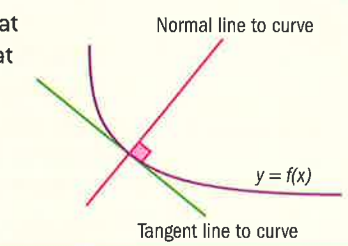
The equation for the tangent to a curve at a given point can easily be found using the gradient to the curve and the coordinates of the point.

Example 5

Find the equation of the tangent to the curve $y = 2x^2 + 4\sqrt{x}$ at the point where $x = 4$.

$y = 2x^2 + 4x^{\frac{1}{2}}$ $\frac{dy}{dx} = 4x + 2x^{-\frac{1}{2}}$	Write in exponent form so the power rule can be used.
$= 4x + \frac{2}{\sqrt{x}}$	If working without a GDC, it is easier to write the derivative as a fraction when working out its value.
When $x = 4$, $\frac{dy}{dx} = 17$. When $x = 4$, $y = 2 \times 4^2 + 4 \times 2 = 40$ so (4, 40) lies on the curve and on the tangent.	The gradient of the tangent is equal to the gradient of the curve so we know the tangent is of the form $y = 17x + c$. To find c, we need a point on the curve.
Equation of the tangent is $y - 40 = 17(x - 4)$ or $y = 17x - 28$.	The point–gradient form of the equation is often the easiest to use. Your answer can be left like this, though it is usual to write it in gradient–intercept form.

The **normal** to a curve at a point is the line that is perpendicular to the tangent to the curve at that point.



International-mindedness

The ancient Greeks used the idea of limits in their "Method of exhaustion".

Example 6

Find the equation of the normal to the curve for equation $f(x) = 2x^3 + 3x - 2$ at the point where $x = 1$.

$f(1) = 2(1)^3 + 3(1) - 2 = 3$ So the point is (1, 3).	Find the y-coordinate of the point.
$f'(x) = 6x^2 + 3$	Next find $f'(x)$.
$f'(1) = 6(1)^2 + 3 = 9$	Find $f'(1)$ to work out the gradient of the tangent line.

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HINT

If two lines are perpendicular then the product of their gradients is −1.

If the gradient of the tangent is $\frac{a}{b}$, then the gradient of the normal is $-\frac{b}{a}$.

Calculus

The gradient of the tangent line is 9.	
Gradient of the normal line $= -\frac{1}{9}$	Gradient of normal $= \frac{-1}{\text{gradient of tangent}}$
$3 = -\frac{1}{9}(1) + c$ $c = 3\frac{1}{9}$	Substitute the coordinates (1, 3) into the equation of the normal.
The equation of the normal is $y = -\frac{1}{9}x + 3\frac{1}{9}$.	In the general form, this is $x + 9y - 28 = 0$.

Example 7

The gradient of the normal to the graph of the function defined by $f(x) = kx^3 - 2x + 1$ at the point $(1, b)$ is $-\frac{1}{4}$. Find the values of k and b.

$f'(x) = 3kx^2 - 2$ If the gradient of the normal is $-\frac{1}{4}$, then the gradient of the tangent is 4. So, $f'(1) = 3k(1)^2 - 2 = 4$ $3k = 6$ $k = 2$	The derivative is the gradient of the tangent at any point. Here the point is where $x = 1$. So, you put $x = 1$ into the derivative and equate it to 4 (which is the gradient of the tangent at that point).
$f(1) = 2(1)^3 - 2(1) + 1 = 1$ So, $b = 1$.	To find b you need to substitute 1 for x into the original equation.

Exercise 10C

1 Find the equation of the tangent to the graph of the function defined by $f(x) = 2x^2 - 4$ at the point where $x = 3$.

2 Brian makes a seesaw for his children from part of a log and a plank of wood. The shape of the log can be modelled by the function $f(x) = -x^2 + 2x$ for $0 < x < 2$.

When the seesaw is level, the plank of wood is a tangent to the log at the point where $x = 1$.

By differentiating $f(x)$, find the equation of the line containing the plank.

3 Find the equation of the normal to the graph of the function defined by $f(x) = 3x^2 - 4x + 5$ at the point (1, 4).

4 Find the equations of the tangent and the normal to the functions defined by

a $y = x^4 - 6x + 3$ at the point where $x = 2$

b $y = 6\sqrt{x}$ at the point where $x = 9$.

5 The edge of a lake can be modelled by the function $f(x) = x^2$.

A fountain is to be placed in the lake at the point where the normals at $x = 2$ and $x = -2$ meet.

Find the equations of these two normals and the coordinates of the point where the fountain will be placed.

6 The gradient of the tangent to the graph of the function defined by $f(x) = ax^2 + 3x - 1$ at the point $(2, b)$ is 7. Find the values of a and b.

7 The gradient of the tangent to the graph of the function defined by $f(x) = x^2 + kx + 3$ at the point $(1, b)$ is 3. Find the values of k and b.

8 The gradient of the normal to the graph of the function defined by $y = ax^2 + bx + 1$ at the point $(1, -2)$ is 1. Find the values of a and b.

GDC techniques and local maximum and minimum points

There are many websites that will give the derivative of a function as an equation and indeed some calculators do this as well. Be aware these calculators are not allowed in exams.

It will be expected in exams that you can work out numerical values for the derivatives at given points and also draw the graph of the derivative. This widens the range of functions that you might need to find the gradient for.

EXAM HINT

Unless told otherwise, always use your GDC if it makes answering the question simpler.

TOK

Mathematics and the real world: the seemingly abstract concept of calculus allows us to create mathematical models that permit human feats, such as getting a man on the Moon. What does this tell us about the links between mathematical models and physical reality?

Example 8

Consider $y = \frac{x+2}{x-1}$, $x \neq 1$.

Find the gradient of the curve at the points where $x = 2$ and $x = 3$.

−3 and −0.75	The gradient is found using the numerical derivative function on your GDC.

Investigation 2

Consider the curve $s = 3t^3 + 3t^2 - 4t + 2$.

1 Find $\frac{ds}{dt}$.

2 On the same axes, sketch $s = 3t^3 + 3t^2 - 4t + 2$ and its derivative.

3 Solve the equation $\frac{ds}{dt} = 0$ and find the coordinates of the points at which this occurs.

4 What feature of $s = 3t^3 + 3t^2 - 4t + 2$ is indicated by these points.

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Calculus

5 What feature of the graph of $\frac{ds}{dt}$ allows you to say which of the points where $\frac{ds}{dt}=0$ is a maximum and which is a minimum?

6 If the domain of the function is restricted to $-2 \le x \le 2$, find the actual maximum and minimum values of the function.

7 **Conceptual** On a function with a restricted domain, where might the maximum and minimum points occur?

A local maximum or minimum point on a curve is a point at which the gradient moves from positive to negative or negative to positive, respectively.

For a differentiable function $f(x)$, $f'(x)=0$ at a local maximum or minimum point.

For a continuous curve, the maximum and minimum values will occur either at local maximum or minimum points or at the end points of the domain.

A differentiable function is a function whose derivative exists at every point in its domain.

Reflect How could you use the derivative to find a local maximum or minimum on a curve?

Example 9

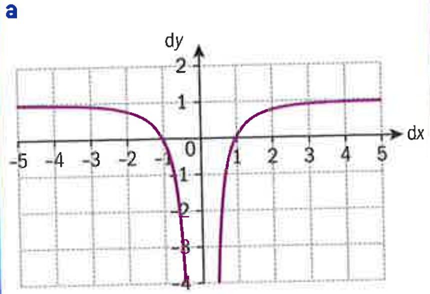
Consider the derivative function $\frac{dy}{dx}=1-\frac{1}{x^2}$, $x \ne 0$.

a Plot the curve on a GDC.

b Find the values of x at which $\frac{dy}{dx}=0$.

c State whether these points represent local maximum or minimum points on the curve for y, justifying your answer.

a



b $x = \pm 1$

These points can be found directly from the GDC.

c $x = 1$ will be a minimum point as the gradient changes from negative to positive.

$x = -1$ will be a maximum point as the gradient changes from positive to negative.

The graph of the derivative is sufficient justification of the change of sign of the gradient. It is not necessary in this case to find actual values.

EXAM HINT

A sketch of the curve is usually sufficient to justify whether or not a point is a maximum or minimum. However, you may be asked to determine maximum and minimum points from a graph of the derivative.

International-mindedness

German mathematician Gottfried Leibnitz represented the derivative by $\frac{dy}{dx}$ in the 17th century.

Exercise 10D

1 Use your GDC to find the gradient of the following curves at the point where $x = 3$.

a $y=\frac{x^2}{x-1}$ **b** $y = x\ln x$

2 Sketch the following curves and hence find the coordinates of any point where $\frac{dy}{dx}=0$.

In each case, state whether the point is a local maximum or a local minimum.

a $y = 3x^3 - 6x^2 - 15x + 7$

b $y = x+\frac{1}{x-1}$

c $y = xe^x$, $x > 0$

3 For each of the curves below, find the values of x at which $\frac{dy}{dx}=0$ and, by considering the sign of $\frac{dy}{dx}$, state whether the curve y has a local maximum or minimum point at these values.

a $\frac{dy}{dx}=2x-4$ **b** $\frac{dy}{dx}=3x^2-12$

c $\frac{dy}{dx}=8-\frac{2}{x^2}$

4 A business buys engine parts from a factory. The business is currently deciding which of three purchasing strategies to use for the next stage of their development. Their researchers produce models for each of the strategies. In these models, P is the expected profit in €10 000 and n is the number of parts they buy (in 1000s). The largest number of parts the factory can sell them is 5000 and there is no minimum.

a For each model find the maximum profit and the number of parts they need to buy to make this profit.

i $P=0.5n+1.5+\frac{4}{n+1}$

ii $P=\frac{n^3}{3}-\frac{5n^2}{2}+6n-4$

iii $P=\frac{n^3}{24}-\frac{5n^2}{8}+3n$

b State which strategy they should adopt.

5 John is a keen cyclist and is planning a route in the Alps. The profile of the route he would like to take can be modelled by the equation $y=-0.081x^4+0.89x^3-2.87x^2+3x$, $0 \le x \le 6$, where y ($\times$ 100 m) is the height of the point on the route which is a distance x ($\times$ 10 km) from his starting point.

Sketch the graph and find the total height John will climb on this route.

6 If $y=2^x$ then $\frac{dy}{dx}=k2^x$ where k is a constant. Find the value of k.

Optimization

If a function to be optimized has only one variable, then either a GDC or differentiation can be used directly to find the maximum or minimum point.

If a function to be optimized has more than one variable then a **constraint** must also be given. This constraint can then be written as an equation and substituted into the function to eliminate one of the variables.

International-mindedness

French mathematician Joseph Lagrange invented an alternative notation with a prime mark to denote a derivative as $f'(x)$.

Calculus

Example 10

A can of dog food contains 500 cm³ of food. The manufacturer, wanting to make sure that the company receives maximum profits, would like to make sure that the surface area of the can is as small as possible. Let the radius of the can be r cm and the height, h cm.

a Find an expression for the surface area S in terms of r.

b Find $\frac{dS}{dr}$.

c Hence, find the dimensions of the can that will result in the minimum surface area.

a Surface area, $S = 2\pi rh + 2\pi r^2$	Because the equation has two variables, you cannot find the minimum, so will need to eliminate h or r. The question is asking us to eliminate h.
$V = \pi r^2 h \Rightarrow \pi r^2 h = 500$	This is the constraint, written as an equation.
So, $h = \frac{500}{\pi r^2}$	By rearranging the expression we can substitute for h in the equation for surface area.
$S = 2\pi r\left(\frac{500}{\pi r^2}\right) + 2\pi r^2 = \frac{1000}{r} + 2\pi r^2$	
b $S = 1000r^{-1} + 2\pi r^2$ $\frac{dS}{dr} = -1000r^{-2} + 4\pi r$	If the only instructions in the question were to find the minimum value, this function could now be plotted and the minimum found directly from the GDC. The question though is guiding us towards solving the equation $\frac{dS}{dr} = 0$.
c $\frac{dS}{dr} = 0$ at a maximum or minimum point. So, $-1000r^{-2} + 4\pi r = 0$ $r = 4.3$.	Because the question says "hence", you need to provide some evidence that you know the local minimum will occur when the gradient is equal to 0.
So, the best dimensions for the can are $r = 4.3$ cm and $h = 8.6$ cm.	h is found using the equation from part **a**. If asked to justify this is a minimum, the graph of S or $\frac{dS}{dr}$ can be plotted or values of r either side of 4.3 can be substituted in to $\frac{dS}{dr}$ to show that the gradient changes from negative to positive across the point.

Exercise 10E

1 A vegetable garden is in the shape of a rectangle. The garden is surrounded by 100 metres of fencing.

a If the width of the garden is x metres, find an expression for the length.

b Show that the area of the garden is $A = x(50 - x)$ m².

c Find $\frac{dA}{dx}$.

d Hence find the maximum area of the garden and the corresponding dimensions that give this area.

2 An open cylinder has a volume of 400 cm³. The radius of the base is r cm and the height is h cm.

a Explain why $\pi r^2 h = 400$.

b Rearrange the equation in part **a** to make h the subject.

c Write down an expression for the surface area, A, of the open cylinder.

d Show that this can be written as $A = \pi r^2 + \frac{800}{r}$.

e Sketch the graph of $A = \pi r^2 + \frac{800}{r}$.

f Find the minimum area and the value of r when this occurs.

3 The total surface area of a closed cylinder is 5000 cm².

Find the dimensions of the cylinder that will maximize its volume and state this maximum volume.

4 A cone has radius r cm and height $(18 - r)$ cm.

a Find an expression for the volume V of the cone and hence show $\frac{dV}{dr} = \pi r(12 - r)$.

b Hence find the radius that will maximize the volume of the cone. Use the expression given in part **a** to justify the value found is a maximum.

5 A rectangular piece of card measures 20 cm by 24 cm. Equal squares of side x cm are cut out of each corner.

The rest of the card is folded up to make a box.

a Show that the volume, V, of the box can be expressed as $V = 4x^3 - 88x^2 + 480x$ and state the domain for this function.

b Find $\frac{dV}{dx}$.

c Find the value of x for which $\frac{dV}{dx} = 0$.

d Verify that the global maximum does not occur at the end points of the domain.

e Hence find the maximum volume of the box.

6 A school council does a survey to see how many students would buy charity cakes at different prices.

The survey revealed that if they sold each cake for $x, the demand ($d$) in the school could be expressed as the function.

$d = \frac{100}{x^2}$. The cost to produce the cakes is $0.75 per cake.

a Explain why the profit the council makes can be expressed by the equation $P = \frac{100}{x} - \frac{75}{x^2}$.

b Find $\frac{dP}{dx}$.

c Show that the solution to $\frac{dP}{dx} = 0$ is $x = 1.5$.

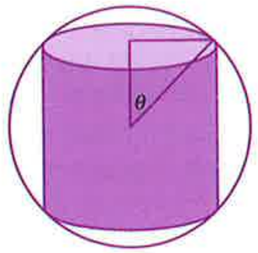
d By substituting $x = 1$ and $x = 2$ into the expression for $\frac{dP}{dx}$ justify that selling the cakes for $1.50 will bring in maximum profit.

7 A rectangle is inscribed inside a circle of radius 6. Let the width of the rectangle be $2x$ and the height be $2y$.

a Show that the area (A) of the rectangle is $A = 4x\sqrt{36 - x^2}$, $0 \le x \le 6$.

b Find the dimensions of the rectangle of largest area that can be inscribed in a circle with radius 6 cm and state the area of the rectangle.

8 A right circular cylinder is inscribed in a sphere of radius 8 cm.



If θ is the vertical angle made by the line segment from the centre of the sphere to the point at which the cylinder touches the sphere,

a show that the curved surface area of the cylinder is, $A = 256\pi \cos\theta \sin\theta$ cm^2, $0 \le \theta \le 90°$

b find the radius and height of the cylinder having the largest curved surface area and state this area.

Developing inquiry skills

In the opening problem for the chapter, a possible power model for the demand curve was found using the best fit function on a GDC to be $d = \frac{855}{x^{1.25}}$ where d is the percentage demand within the market for the new product at price $\$x$.

Given the cost ($\$C$) of producing n items is $C = 3n$ and the market size is estimated to be 10 000 people and assuming only sufficient items to match the demand are made, find an expression for the profit (P).

By differentiating this expression and solving $\frac{dP}{dx} = 0$, find the maximum profit and the price at which the product should be sold to achieve this.

10.2 Differentiation: further rules and techniques

The chain rule

Investigation 3

Consider the polynomial function $y = (2x^2 - 5)^2$, which is differentiable for all $x \in \mathbb{R}$.

1. Expand the brackets and simplify. Hence find $\frac{dy}{dx}$.
2. Note that y is a composite function.

   Let $u(x)$ be the first function in the composite, hence $u(x) = 2x^2 - 5$.

   a Write y as a function of u.

   b Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

   c Hence find the product $\frac{dy}{du} \times \frac{du}{dx}$ in terms of x.
3. Compare your answers from questions **1** and **2c**. What do you notice?
4. Consider the composite function $y = (x+1)^3$.

   a Expand the function and hence find $\frac{dy}{dx}$.

   Let $u(x)$ be the first function in the composite.

   b Write down an expression for $u(x)$.

   c Write y as a function of u.

   d Find $\frac{dy}{du} \times \frac{du}{dx}$.

   e What do you notice?
5. If $y(x)$ is a composite function, and $u(x)$ is the first function of the composite, conjecture an expression for $\frac{dy}{dx}$.
6. Test your conjecture by differentiating $y = (x + 2x^{-1})^2$

   a by first expanding the function

   b by using your conjectured expression.
7. **Conceptual** How do you find the derivative of a composite function?

TOK

Mathematics: invented or discovered?

If mathematics is created by people, why do we sometimes feel that mathematical truths are objective facts rather than something constructed by human beings?

The chain rule for composite functions can be written as $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

HINT

This relation is not the result of cancelling du, but the idea of treating the notation for a derivative as a fraction is often useful and provides a convenient way to remember the rule.

Example 11

Find the derivative of

a $y = \sqrt{3x^2 - 2}$ **b** $f(x) = \frac{1}{2x+3}$

a $u = 3x^2 - 5 \Rightarrow y = u^{\frac{1}{2}}$	Identify the first function and call it u.
$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ and $\frac{du}{dx} = 6x$	Find $\frac{dy}{du}$ and $\frac{du}{dx}$.
	Use the chain rule.
$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 6x = \frac{3x}{\sqrt{3x^2 - 5}}$	Substitute u and simplify.

Continued on next page

Calculus

b $f(x)=\dfrac{1}{2x+3}$	Again we need to identify the first function in the composite and call it u.
$u=2x+3,\ f(u)=\dfrac{1}{u}=u^{-1}$	Write the fraction as a negative power to make differentiating easier.
$\dfrac{du}{dx}=2,\ \dfrac{df}{du}=-u^{-2}=-\dfrac{1}{u^2}$	Though we often write $f'(u)$, $\dfrac{df}{du}$ is also correct notation.
$f'(x)=\dfrac{df}{du}\times\dfrac{du}{dx}$	Use the chain rule.
$f'(x)=-\dfrac{1}{u^2}\times 2=-\dfrac{2}{(2x+3)^2}$	Substitute u and simplify.

When you have practised a few of these you may find it is no longer necessary to explicitly write down an expression for u. Many people use the informal rule: the derivative of the inside function (u) multiplied by the derivative of the outside function.

For the first example above, this would be:

The derivative of $u = 3x - 5$, multiplied by the derivative of u to the power one-half.

Hence, $3\times\frac{1}{2}(3x-5)^{-\frac{1}{2}}$.

Exercise 10F

1 For each composite function below

  i identify the two functions that make up the composite function and write them as $u(x)$ and $y(u)$

  ii find $\dfrac{dy}{du}$.

a $y=(x^2+4)^3$ **b** $y=(5x-7)^2$

c $y=2(x^3-3x^2)^4$ **d** $y=\sqrt{4x-5}$

e $y=\dfrac{1}{(x^2+1)^2}$ **f** $y=\dfrac{2}{\sqrt{5x-2}}$

2 Find the derivative of each of the following functions.

a $f(x)=(x^2+1)^3$ **b** $g(x)=6(5x+2)^{\frac{1}{3}}$

c $h(x)=\left(\sqrt{x}-4\right)^3$ **d** $s(t)=3(t^2-2)^2$

e $v(t)=\dfrac{4}{5t-1}$ **f** $a(t)=\left(2-\dfrac{1}{t}\right)^3$

3 The table gives values for f, g, f' and g'.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	5	3	−2	1
3	1	5	5	2
5	5	3	2	−2

a If $F(x)=f\circ g(x)$, find $F'(1)$.

b If $H(x)=g\circ g(x)$, find $H'(3)$.

The product rule

When differentiating a product of two functions, it is not possible to differentiate the two functions separately and then multiply.

For example, if $y = 5x \times x^3$ then

$$y=5x^4 \Rightarrow \frac{dy}{dx}=20x^3$$

This is not the same as $\dfrac{d(5x)}{dx}\times\dfrac{d(x^3)}{dx}$, which is $5 \times 3x^2 = 15x^2$.

To differentiate a product of two functions, we need to use the product rule.

The product rule

If y, u and v are all differentiable functions of x and if $y = uv$ then

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

Proof of product rule

Let $h(x)=f(x)g(x) \Rightarrow h'(x)=\lim\limits_{h\to 0}\left(\dfrac{f(x+h)g(x+h)-f(x)g(x)}{h}\right)$.

This uses the limit definition of the derivative we met in the last section.

The next part looks complicated but it is just a trick. The term $f(x+h)g(x)$ is both added and subtracted in the denominator.

$$h'(x)=\lim_{h\to 0}\left(\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\right)$$

The rules of limits allow us to split this into two separate parts and then take out the factors in the next line.

$$h'(x)=\lim_{h\to 0}\left(\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}\right)+\lim_{h\to a}\left(\frac{f(x+h)g(x)-f(x)g(x)}{h}\right)$$

$$=\lim_{h\to 0}f(x+h)\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)+\lim_{h\to o}g(x)\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

The larger expressions are simply the limit formulae for the derivatives of $g(x)$ and $f(x)$, respectively, hence

$$h'(x)=f(x)g'(x)+g(x)f'(x).$$

EXAM HINT

You will not be required to reproduce this proof in an exam. The formula for the product rule is given in the formula book.

Example 12

Use the product rule to find $f'(x)$ if $f(x) = (x^2 + 2x)\left(\frac{1}{x} + 3\right)$.

Let $u(x) = x^2 + 2x$, then $u'(x) = 2x + 2$. Let $v(x) = \frac{1}{x} + 3$, then $v'(x) = -\frac{1}{x^2}$.	Define $u(x)$ and $v(x)$ and find their derivatives.
$f'(x) = (x^2 + 2x)\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + 3\right)(2x + 2)$ $f'(x) = 6x + 7$	The product rule can be thought of as, "Differentiate one term and leave the other the same, plus differentiate the other term and leave the first the same".

Example 13

Find the derivative of $y = (x^2 - 4)^2(x^3 - 1)^4$.

$u(x) = (x^2 - 4)^2$ and $v(x) = (x^3 - 1)^4$ $\frac{du}{dx} = 2(x^2 - 4)(2x) = 4x(x^2 - 4)$ $\frac{dv}{dx} = 4(x^3 - 1)^3(3x^2) = 12x^2(x^3 - 1)^3$	In this case, we need to use the chain rule to differentiate u and v first.
$\frac{dy}{dx} = (x^2 - 4)^2 12x^2(x^3 - 1)^3 + (x^3 - 1)^4 4x(x^2 - 4)$	Use the product rule for $\frac{dy}{dx}$.
$\Rightarrow \frac{dy}{dx} = 12x^2(x^2 - 4)^2(x^3 - 1)^3 + 4x(x^3 - 1)^4(x^2 - 4)$ $= 4x(x^2 - 4)(x^3 - 1)^3(3x(x^2 - 4) + x^3 - 1)$ $= 4x(x^2 - 4)(x^3 - 1)^3(4x^3 - 12x - 1)$	Often the product rule will generate factors that can be used to simplify the expression.

The quotient rule

Similar to the product of two functions, a quotient cannot be differentiated by differentiating the two parts separately, but has to be differentiated using the quotient rule.

The quotient rule
If u and v are differentiable functions of x and if $y = \frac{u}{v}$ then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ for $v(x) \neq 0$. The formula for the quotient rule is given in the formula book.

TOK

What is the difference between inductive and deductive reasoning?

The result is proved in Exercise 10G.

Example 14

If $y = \frac{x^2 + 3}{x + 1}$, $x \neq -1$, find

a $\frac{dy}{dx}$

b the equations of **i** the tangent and **ii** the normal at the point $(-2, -7)$.

a Let $u(x) = x^2 + 3$, then $u'(x) = 2x$. Let $v(x) = x + 1$, then $v'(x) = 1$.	Define $u(x)$ and $v(x)$ and find their derivatives.
$\frac{dy}{dx} = \frac{(x + 1)(2x) - (x^2 + 3)1}{(x + 1)^2}$ $= \frac{2x^2 + 2x - x^2 - 3}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2}$	Use the quotient rule.
b i Gradient of tangent $= m = \frac{-3}{1} = -3$	Substitute $x = -2$ into the derivative function.
$y = -3x + c$ $-7 = 6 + c$ $c = -13$	Point $(-2, -7)$ lies on the tangent.
Hence, the equation of the tangent is $y = -3x - 13$	An alternative method is to use the formula $y - y_1 = m(x - x_1)$.
ii Gradient of normal $= \frac{1}{3}$	If the gradient of the tangent is m then the gradient of the normal is $-\frac{1}{m}$.
Hence, the equation of the normal is $y + 7 = \frac{1}{3}(x + 2)$	Point $(-2, -7)$ lies on the normal as well.
$\Rightarrow y = \frac{1}{3}x - \frac{19}{3}$ or $x - 3y - 19 = 0$	As the question does not ask for a particular form, any of these answers would be acceptable.

HINT

It is important to use the correct technique in a particular question:

- Avoid rearranging a quotient into a product when using the product rule.

  For example, avoid writing $y = \frac{x^2 + 3}{x + 1}$ as $y = (x^2 + 3)(x + 1)^{-1}$.

  This will give the correct answer but the answer will not be set over a common denominator and so will be more difficult to manipulate in the later parts of a question.

Continued on next page

- An expression in which the numerator is a constant term should be differentiated using the chain rule and not the quotient rule.

For example, $y = \frac{2}{x^2+3}$, should be written as $y = 2(x^2+3)^{-1}$

$\Rightarrow \frac{dy}{dx} = -4x\left(x^2+3\right)^{-2}$

Exercise 10G

1 Use the product rule to find the derivative of the following equations. Write down your answers in factorized form where possible.

a $y = x^2(2x+1)$

b $y = (x+2)(x^2+3)$

c $y = (x^2+2x+1)(x^3-1)$

d $s = (2t^2+3)(4-t)^5$

e $f(x) = \frac{1}{x}(2x^3+x+4)$

f $g(t) = (2t+1)^4(t^3+1)^2$

2 Use the quotient rule to find the derivative of the following equations. Write down your answers as fractions, with any simple cancellations performed, where possible.

a $y = \frac{2x^3+x+4}{x}$ b $y = \frac{1-x}{x^2+1}$

c $y = \frac{x}{\sqrt{x+1}}$ d $s = \frac{4t}{2t+1}$

e $f(x) = \frac{x^3-1}{x+1}$ f $g(t) = \frac{t^{\frac{1}{3}}}{t^{\frac{1}{3}}-4}$

3 Find the derivative of each of the following.

a $y = x^2(x^2+1)^3$ b $y = \frac{4}{2x+3}$

c $y = \frac{x}{\sqrt{2x+1}}$ d $s = 4t\sqrt{2t-3}$

e $f(x) = \frac{x}{(3-2x)^2}$ f $g(t) = \frac{4t(t+1)^2}{2t-3}$

4 P is the point on the curve $y = x(2x-1)^2$ with coordinates $(1, 1)$.

a Find $\frac{dy}{dx}$.

b Hence find the equation of

i the tangent at P

ii the normal at P.

The tangent and normal at P meet the x-axis at Q and R, respectively.

c Find the area of the triangle ΔQPR.

5 Consider the curve $y = \frac{4x-2}{x+1}, x \neq -1$.

a Find the equation of the normal to the curve at the point $(2, 2)$.

b Find the coordinates of the second point at which the normal intersects the curve.

6 By writing $y = \frac{u}{v}$ as $y = u(v)^{-1}$, use the chain and product rules to prove $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Derivatives of trigonometric functions

Investigation 4

1 a With your GDC in radian mode, draw the graph of $y = \sin x$ for $0 \le x \le 2\pi$.

b From your graph, estimate the gradient of the curve at the points where $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π. Take into consideration the possible different scales on the two axes.

2 Use the numerical derivative function on your GDC to complete the following table. $(\sin x)'$ denotes the derivative of $\sin x$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin x$									
$(\sin x)'$									
$\cos x$									
$(\cos x)'$									

3 Conjecture an expression for the derivative of

a $\sin x$ b $\cos x$.

4 Put your GDC into degree mode. Use the numerical derivative function to calculate the gradient of

a $\sin x$ when $x = 0$ b $\cos x$ when $x = 90°$.

c Explain why these results are different from those obtained in question 2.

5 Use the definition of $\tan x = \frac{\sin x}{\cos x}$ and the quotient rule to show that the derivative of $\tan x$ is $\frac{1}{\cos^2 x}$.

6 **Factual** What units do you use when differentiating trigonometric functions?

7 From your answers to questions 3, 4 and 5, what can you say about the derivatives of $\sin x$ and $\cos x$?

TOK

Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work.

What does this suggest regarding intuition and imagination in mathematics?

- $f(x) = \sin x \Rightarrow f'(x) = \cos x$
- $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
- $f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$

HINT

All these derivatives are in the formula book.

Calculus

Example 15

Use the chain rule to find the derivative of

a $y = \sin 2x$ **b** $y = \cos 4t$.

a $\frac{dy}{dx} = \cos 2x \times 2$ $= 2\cos 2x$	Let $u = 2x$, $y = \sin u$ $\frac{du}{dx} = 2$, $\frac{dy}{du} = \cos u$
b $\frac{dy}{dt} = -\sin 4t \times 4$ $= -4\sin 4t$	Let $u = 4t$, $y = \cos u$ $\frac{du}{dt} = 4$, $\frac{dy}{du} = -\sin u$

HINT

Differentiating expressions of the form $y = \sin ax$ and $y = \cos ax$ is so common that it is useful to learn the derivatives: $y = \sin ax \Rightarrow \frac{dy}{dx} = a\cos ax$ and $y = \cos ax \Rightarrow \frac{dy}{dx} = -a\sin ax$

Example 16

Find the derivative of $y = \frac{\cos x}{x^2}$.

Let $u(x) = \cos x$ and $v(x) = x^2$.	We will use the quotient rule so we need to identify u and v.
$\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = 2x$	Find the derivatives of u and v.
$\frac{dy}{dx} = \frac{(x^2)(-\sin x) - (\cos x)(2x)}{(x^2)^2}$	Use the quotient rule.
$\frac{dy}{dx} = \frac{-x\sin x - 2\cos x}{x^3}$	Factorize and simplify.

Exercise 10H

1 Find the derivative of the following functions.

a $y = 4\sin x$

b $y = 2\cos x - x^2$

c $y = 5\tan x$

d $y = 2\cos 4t$

e $f(x) = \sin 5x + 4x^3$

f $g(t) = \sin 3t - 3\cos 2t - t^2 + 1$

2 Use the product and quotient rules to find the derivative of the following.

a $y = x\sin x$

b $y = \sin x\cos x$

c $y = 2x^2\tan x$

d $y = \frac{\sin t}{t}$

e $f(x) = \frac{\cos x}{\sin x}$

f $g(t) = \frac{\sin 2t}{\cos t}$

3 Find the derivative of the following.

a $y = \sin(x^2)$

b $y = 2\cos(3x)$

c $y = \tan(3x - 1)$

d $y = \sin^2 t$

e $f(x) = 3\cos^3 x$

f $g(t) = 2\sin 4t\cos 4t$

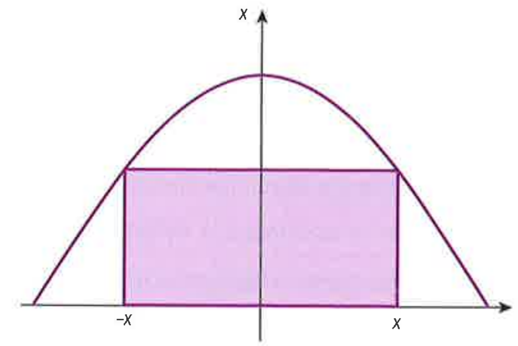
g $f(x) = \sin^2(3x)$

h $g(t) = \sin^2 t\cos t$

4 A performance hall is in the shape of a prism whose cross-sectional area can be modelled by the curve $y = 5.1\cos\left(\frac{\pi}{10}x\right)$, $-5 \le x \le 5$.

A temporary screen is to be fitted at one end of the hall. The cross-section of the screen will be in the shape of a rectangle.

A possible screen is shown in the diagram below.



The cross-sectional area of the screen should be as large as possible.

a Write down a model for the cross-sectional area (A) of the screen in terms of x, the x-coordinate of the right-hand vertical side.

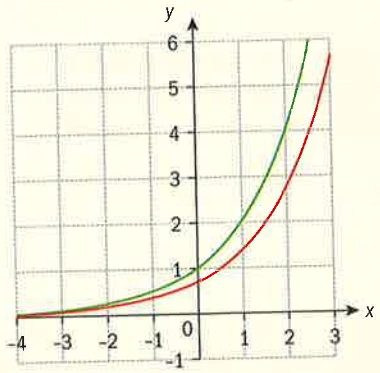
b Find $\frac{dA}{dx}$.

c Find the maximum value for the cross-sectional area of the screen.

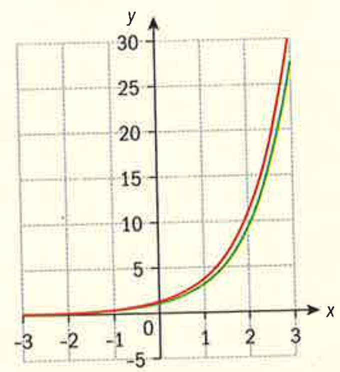
Derivatives of e^x and $\ln x$

Investigation 5

The diagrams below show the graphs of $y = 2^x$ and $y = 3^x$ in green and the graphs of their derivatives in red.



$y = 2^x$



$y = 3^x$

1 Comment on the any similarities or differences between the two graphs.

2 A function f is such that $f(x) = f'(x)$. Where might you look for such a function?

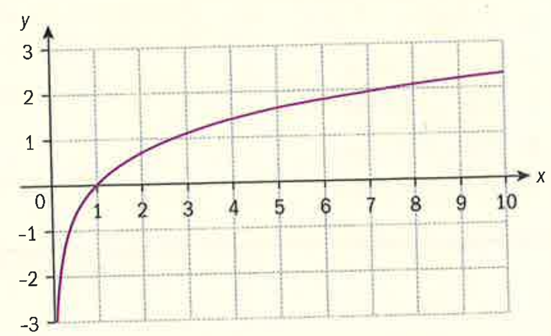
3 a Use the numerical derivative function on your GDC to complete the following table.

x	−1	0	1	2	3
e^x					
$(e^x)'$					

b Conjecture an expression for the derivative of e^x.

4 Consider the graph of $y = \ln x$ shown in this diagram.

Sketch a possible graph showing the gradients of points on the curve $y = \ln x$ and comment on its key features.



5 a Use the numerical derivative function on your GDC to find the value of the gradient of $y = \ln x$ at the following points.

x	1	2	3	4	5
$(\ln x)'$					

b i Conjecture an expression for the derivative of $\ln x$.

ii Does your sketch from question **4** support your answer?

iii Test your conjecture by finding the derivative when $x = 10$.

Differentiating exponentials and logarithmic functions in other bases gives a multiple of the results obtained here, so it is usual to use base e, as this is the simplest, and this is the only one required by the course.

6 **Factual** What base should be used when differentiating exponential or logarithmic functions?

7 **Conceptual** Why is the exponential function with base e so special?

- $f(x) = e^x \Rightarrow f'(x) = e^x$
- $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

HINT

These derivatives are in the formula book.

HINT

The results in Investigation 5 can be proved using the definition of e^x as

$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

which you met in Chapter 7.

Example 17

Find the derivative of **a** $y = x \ln x$ **b** $y = e^{4x}$.

a $\frac{dy}{dx} = \ln x + x \times \frac{1}{x}$ $= \ln x + 1$	Use the product rule.
b Let $u = 4x$ and $y = e^u$	Identify the two functions that make up the composite.
$\frac{du}{dx} = 4$, $\frac{dy}{du} = e^u$ $\frac{dy}{dx} = 4 \times e^u = e^{4x}$	Use the chain rule.

HINT

Differentiating expressions of the form $y = e^{ax}$ is so common that it is useful to learn the derivative:

$y = e^{ax} \Rightarrow \frac{dy}{dx} = ae^{ax}$.

Example 18

Find the derivative of $f(x) = \sin(\ln x)$.

Let $u(x) = \ln x$, then $f(u) = \sin(u)$	Identify the composite function.
$\frac{du}{dx} = \frac{1}{x}$ and $\frac{df}{du} = \cos(u)$	Find $\frac{du}{dx}$ and $\frac{df}{du}$.
$f'(x) = \cos(u) \times \frac{1}{x}$ $\Rightarrow f'(x) = \frac{1}{x}\cos(\ln x) = \frac{\cos(\ln x)}{x}$	Use the chain rule.

Calculus

Exercise 10I

1 Find the derivative of each of the following.

a $y = \ln(x^2 + 1)$
b $y = xe^x$
c $y = e^{2x^2}$
d $y = t^2 \ln t$
e $f(x) = (2x + 1)e^{2x}$
f $f(t) = \frac{\ln t}{t}$
g $s = 3t \ln(t^2 - 2)$
h $g(x) = \frac{2e^{4x}}{x}$
i $h(t) = te^{3t^2}$

2 Find $\frac{dy}{dx}$ using the log laws to simplify the expressions given.

a $y = \ln x^5$
b $y = \ln(2x + 3)^4$
c $y = \ln(x(x - 3)^2)$
d $y = \ln\left(\frac{2x+1}{x}\right)$
e $y = \ln(e^{x^2})$
f $y = \ln\left(\frac{x^2}{(2x+1)^3}\right)$

3 a By writing $\ln(ax)$ as $\ln x + \ln a$, deduce an expression for the derivative of $\ln ax$.
b Verify that the chain rule gives the same result.

4 i Find the derivative of each of the following functions.
ii Find the smallest positive value of x for which $f(x) = f'(x)$.

a $f(x) = 20 \sin(\ln x)$
b $f(x) = \ln(\cos x + 2)$
c $f(x) = e^{2\sin x}$

5 Consider the curve $y = \frac{e^{2x}}{x+3}$.

a Find the **exact** value of the coordinates of the point at which the tangent to the curve is parallel to the x-axis.
b Write down the equation of the normal to the curve at this point.

Developing inquiry skills

In the opening problem for the chapter, a possible exponential model for the demand curve was found, using the best fit function on a GDC, to be $d = 130e^{-0.0792x}$ where d is the percentage demand within the market for the new product at price \$$x$.

Given the cost (\$$C$) of producing n items is $C = 3n$ and the market size is estimated to be 10 000 people, and assuming only sufficient items to match the demand are made,

1 find an expression for the profit (P)
2 by differentiating this expression and solving $\frac{dP}{dx} = 0$, find the maximum profit and the price at which the product should be sold to achieve this.

TOK

Who do you think should be considered the discoverer of calculus?

10.3 Applications and higher derivatives

The second derivative

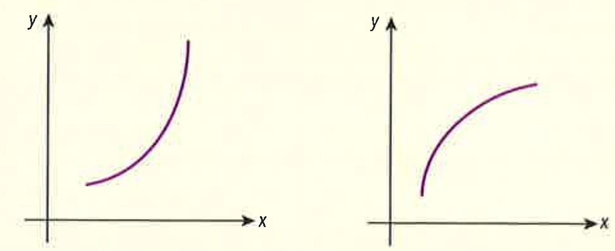
Investigation 6

In section 10.1 it was shown that for a function f

- if f is increasing at x then $f'(x) > 0$
- if f is decreasing at x then $f'(x) < 0$.

1 Consider the two curves shown below. Write down for which of the two curves

a the gradient of the curve is increasing
b the gradient of the curve is decreasing.



The first curve is an example of a curve that is **concave up** and the second an example of a curve that is **concave down**.

2 Consider the curve $y = (x - 1)^3$.

a Sketch the curve showing clearly the x-intercept.
b Find $\frac{dy}{dx}$.
c Explain from your answer to part **b** why the curve is always increasing.

The second derivative is obtained by differentiating the first derivative of a function. The notation is either $\frac{d^2y}{dx^2}$ (said as "d two y by dx squared") or $f''(x)$.

3 a Find the second derivative of $y = (x - 1)^3$.
b State the values of x for which

i $\frac{d^2y}{dx^2} > 0$
ii $\frac{d^2y}{dx^2} < 0$.

4 **Factual** What feature of a graph is indicated by the sign of the second derivative?

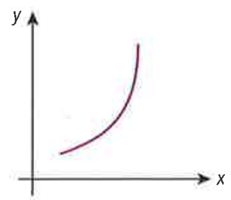
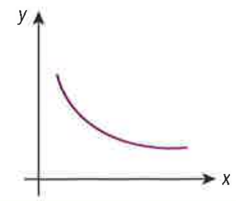
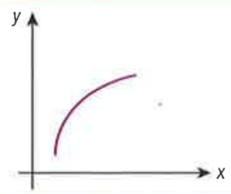
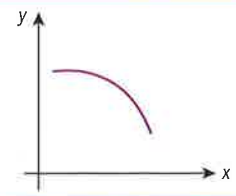
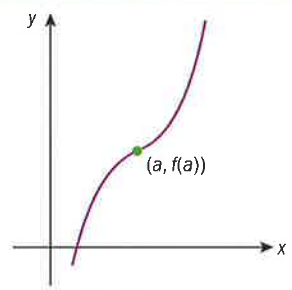
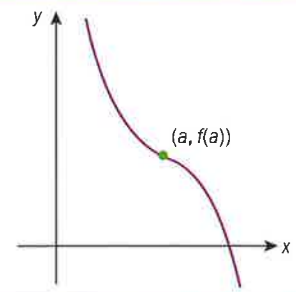
5 a Write down the value of x for which $\frac{d^2y}{dx^2} = 0$.
b What is significant about this point in terms of the concavity of the curve?

The point on a curve at which the concavity changes sign is called a **point of inflexion**.

6 **Conceptual** Given the equation of a curve, how would you find a point of inflexion if one exists?

Calculus

The first and second derivatives indicate the following features of a curve:

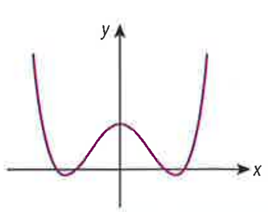
	$f'(x) > 0$	$f'(x) < 0$
$f''(x) > 0$		
$f''(x) < 0$		
$f''(a) = 0$		

Example 19

For the function $f(x) = x^4 - 3x^2 + 2$

a find the local maximum and minimum points, justifying the nature of each

b find the interval in which the curve is concave down.

a From the GDC, the local maximum point is (0, 0) and the minimum points are (1.225, −0.25) and (−1.225, −0.25). 	Often the easiest way to locate maxima and minima is by using the inbuilt functions on the GDC. The sketch is sufficient justification of the nature of the points.
b $\frac{dy}{dx} = 4x^3 - 6x$	It is difficult to find the point at which the concavity changes directly from a GDC and so this question is best done algebraically.
$\frac{d^2y}{dx^2} = 12x^2 - 6$	
$12x^2 - 6 < 0 \Rightarrow x^2 < 0.5$	When the graph is concave down, the second derivative is less than 0.
$-0.707 < x < 0.707$	The graph indicates that the range required is between the two boundaries of the inequality.

The second derivative provides an alternative way to distinguish whether or not the point $(a, f(a))$ at which $f'(a) = 0$ is a maximum, a minimum or a point of inflexion.

- If $f''(a) > 0$ the curve is concave up and so $(a, f(a))$ is a minimum point.
- If $f''(a) < 0$ the curve is concave down so $(a, f(a))$ is a maximum point.
- If $f''(a) = 0$ the result is inconclusive because, though it is likely to be a point of inflexion, there are a few curves for which it could be either a maximum or a minimum.

  For example, if $f(x) = x^4$ then $f''(x) = 0$ but the curve clearly has a minimum point at $x = 0$.

  In these cases it is important to check the sign of $f''(x)$ either side of the point to see if the concavity changes.

International-mindedness

The Greeks' mistrust of zero meant that Archimedes' work did not lead to calculus.

Reflect If $(a, f(a))$ is a local maximum, what is the sign of $f''(a)$? If $(a, f(a))$ is a local minimum, what is the sign of $f''(a)$?

EXAM HINT

In exams, a sketch of the curve is also sufficient to identify the nature of a point where the gradient is equal to zero.

Example 20

Consider the curve $y = 2x^3 - 9x^2 + 12x + 5$.

a Find $\frac{dy}{dx}$.

b Hence, find the coordinates of the points on the curve at which $\frac{dy}{dx} = 0$.

c Find $\frac{d^2y}{dx^2}$.

d Hence, determine whether the points are local maxima or local minima.

e Find the coordinates of the point of inflexion on the curve.

Continued on next page

Calculus

a $\frac{dy}{dx} = 6x^2 - 18x + 12$

b $6x^2 - 18x + 12 = 0$

$\Rightarrow x = 1, 2$

$(1, 10), (2, 9)$

It would be easy to find the maximum and minimum values by sketching the curve on your GDC to justify their nature, but because the question says "hence", you must use the previous result and show working to demonstrate that you have.

c $\frac{d^2y}{dx^2} = 12x - 18$

d When $x = 1$, $\frac{d^2y}{dx^2} = -6 < 0$; hence, it is a maximum point.

When $x = 2$, $\frac{d^2y}{dx^2} = 6 > 0$; hence, it is a minimum.

e $\frac{d^2y}{dx^2} = 0 \Rightarrow 12x - 18 = 0$

$\Rightarrow x = 1.5$

$(1.5, 9.5)$

Because the question says there is one point of inflexion and there is only one point at which the second derivative is equal to zero, there is no need to check for a change in concavity.

Exercise 10J

1 Find the second derivative for the following functions.

- **a** $y = x \sin x$
- **b** $f(x) = \frac{2}{(x-3)^2}$
- **c** $s = 2t \ln t$

2 Let $y = x^5 - 3x^2 + 5$.

- **a** Find any local maximum or minimum points on the curve, justifying whether they are maxima or minima.
- **b** Find a point of inflexion on the curve.
- **c** Hence, give the range of values of x for which the curve is concave up.

3 Let $y = \frac{e^x}{x}$, $x \neq 0$.

- **a** Find **i** $\frac{dy}{dx}$ **ii** $\frac{d^2y}{dx^2}$.
- **b** **Hence**, show that there is a minimum point at $x = 1$.
- **c** Prove that there are no points of inflexion on the curve.

4 If $f(x) = \frac{x}{x-3}$, $x \neq 3$,

- **a** show that f is decreasing for all x values in its domain
- **b** show that f has no points of inflexion.

5 For each function below

- **i** find any local maximum or minimum points
- **ii** find an expression for $\frac{d^2y}{dx^2}$
- **iii** find any points of inflexion
- **iv** determine the intervals where the function is concave up and where it is concave down.

- **a** $y = (x-2)^3(x+1)$
- **b** $h(x) = e^{-x}\cos x$, $0 \le x \le 2\pi$
- **c** $f(x) = x^3 e^{-2x}$

6 A population p at time t is given by the following logistic equation $p(t) = \frac{10}{1+4e^{-2t}}$.

- **a** **i** Find an expression for $p'(t)$.
  - **ii** Explain why $p'(t) > 0$ for all $t \ge 0$ and interpret your result in the context of population growth.
- **b** **i** Find an expression for $p''(t)$.
  - **ii** Hence, find the value of t at which $p''(t) = 0$.
  - **iii** Explain the significance of this point in the context of the rate of population growth.
- **c** **i** For the value of t found in part **b ii** find the value of $p(t)$.
  - **ii** Explain the significance of this point in the context of the carrying capacity of the population.

Kinematics

In Chapter 3 motion with constant velocity was introduced along with the idea of the displacement of an object being its position relative to an origin.

In this section, motion with a variable velocity in one dimension will be considered.

Though the focus will be on just one dimension, it is important to remember that velocity is still a vector quantity and hence has both magnitude and direction.

In the diagram below, A is moving along the x-axis and has a positive displacement. Because it is moving in a negative direction its velocity is -2.

Speed is the magnitude of the velocity and so is always positive. The speed of A is 2.

Let $s(t)$ be the displacement of an object at time t.

The average velocity between times t_1 and t_2 is

$$\frac{\text{change in displacement}}{\text{time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

And the velocity at time t will be the rate of change of the displacement at that time, $s'(t)$ or $\frac{ds}{dt}$.

In the same way as velocity is the rate of change of displacement (that is, a measure of how quickly displacement is changing), **acceleration** is the rate of change of velocity and hence is equal to $\frac{dv}{dt}$.

TOK

Does the fact that Leibniz and Newton came across calculus at similar times support the argument of Platonists over Constructivists?

Using the chain rule leads to another expression for acceleration.

$a = \frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds} = v\frac{dv}{ds}$, which is useful when velocity is given as a function of displacement rather than time.

> If an object has displacement $s(t)$, velocity $v(t)$ and acceleration $a(t)$ then
>
> $$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \text{ or } a = v\frac{dv}{ds}$$

Example 21

The height (m) of a rocket projected vertically into the air until it returns to the ground is represented by the function $h(t) = -0.11t^2 + 1.32t + 1.5$, $t \geq 0$, where t is the number of seconds after the rocket was launched.

a State the height at which the rocket was launched.

b Find the maximum height reached by the rocket.

c Calculate the velocity of the rocket at $t = 7.5$ s and state whether it is ascending or descending at this time.

d Find the other time at which the rocket is travelling at the same speed as when $t = 7.5$ s.

a 1.5 m	The initial height occurs when $t = 0$.
b $v(t) = -0.22t + 1.32 = 0$ $t = \frac{1.32}{0.22} = 6$ $h(6) = 5.46$ m	The maximum height will occur when $\frac{dh}{dt} = 0$, which is equivalent to $v(t) = 0$. Substitute the value for t into the equation for height.
c $v(7.5) = -0.22 \times 7.5 + 1.32 = -0.33$ m s^{-1} Descending	Because the velocity is negative, the height is decreasing and so the rocket is descending. Parts **a** to **c** could be done by sketching the curve on the GDC and using the maximum and numerical derivative functions. However, the simplest way to do part **d** is by using differentiation.
d $-0.22t + 1.32 = 0.33 \Rightarrow t = 4.5$ seconds	The rocket has the same speed, so its velocity is ± 0.33. We already know where it is -0.33 so we need to solve $v = 0.33$.

> The "dot" or Newton's notation is often used to signify a rate of change (derivative) with respect to time. If x is an object's displacement then $v = \dot{x}$ and $a = \ddot{x}$.
>
> This notation is particularly useful when considering motion in more than one direction.

HINT

If a derivative has time as its variable then Newton's dot notation is often used.

Hence, $v = \dot{s}$ and $a = \dot{v} = \ddot{s}$.

Exercise 10K

1 A ball is thrown vertically upwards.

The path of the ball can be modelled by the equation $h(t) = 12t - 4t^2$ where $h(t)$ is the height of the ball in metres after t seconds.

a Find the average velocity between 1 and 1.5 seconds.

b Find the instantaneous velocity at 1 second.

2 A particle moves in a straight line and its displacement from a fixed point is given as $s(t) = -t^3 + 2t^2 + 4t - 2$

where t is measured in seconds and s in metres.

a Find its average velocity in the first 3 seconds.

b Find its velocity and acceleration at $t = 3$.

c Determine if the speed of the particle is increasing or decreasing at $t = 3$.

d Find the value of t when the direction of the particle changes.

e **i** Find the value of t at which the acceleration of the particle changes direction.

ii Explain the geometrical significance of this point on the graph $s(t) = -t^3 + 2t^2 + 4t - 2$.

3 The velocity of a particle as a function of its displacement x is given as $v = 5\sin 3x$.

Use the expression $a = v\frac{dv}{ds}$ to find the particle's acceleration when $x = 2$.

4 A rocket is fired into the air from a platform which is 1 m above the ground.

The path of the rocket can be modelled by a quadratic function with equation $y(t) = -0.2t^2 + 2t + 1$, where $t \geq 0$ represents the time, in seconds, since the rocket took off, and $y(t)$ represents the height, in metres, of the rocket above the ground.

a Find an expression for the velocity of the rocket at time t.

b Hence, find

i the initial velocity of the rocket

ii the maximum height of the rocket

iii the speed at which the rocket hits the ground.

5 A marble is dropped from a certain height into a large tube containing a viscous liquid. The distance (s) fallen by the marble t seconds after the marble enters the liquid is given by the equation

$s = 0.4(2 + t - 2e^{-0.5t})$.

a **i** Find the distance fallen by the marble after 2 seconds.

ii Find the value of t at which the marble will have fallen 2 m.

b Find an expression for the velocity of the marble at time t.

c Write down the velocity of the marble as it enters the liquid.

The terminal velocity of a falling object is the velocity approached by the object as it falls.

d Write down the terminal velocity of the marble in the liquid.

e Find the time at which the marble is first moving within 1% of its terminal velocity.

6 A cyclist is cycling up a hill. The distance (x metres) cycled from the foot of the hill can be modelled by the equation $x = 3t + \ln(2t + 1)$ where t is the number of seconds after the cyclist begins the climb. The road up the hill is 200 m long.

a Find the value of t at which the cyclist reaches the top of the hill.

b Find an expression for the velocity of the cyclist at time t.

c Find

i the initial velocity of the cyclist

ii the velocity of the cyclist at the top of the hill.

d Show that the cyclist is always decelerating when climbing the hill.

7 The displacement from the equilibrium position (x cm) of a weight attached to a spring is given by the equation $x = 4e^{-0.4t}\sin(4t)$.

a Sketch the curve for $0 \le t \le 10$.

b Find the greatest value of t at which the weight is 1.2 cm from the equilibrium position.

c Find an expression for $\dot{x}$, the velocity of the weight at time t.

d Find

i the value of t at which the weight first returns to the equilibrium position

ii the velocity of the weight at this time.

Rates of change and related rates of change

The previous section considered the rates of change of displacement (velocity) and velocity (acceleration) but there are many other quantities that change with time.

For example, in a town, the temperature (T°C) t hours after midnight might be modelled by the equation $T = 4\sin\left(\frac{\pi(t-9)}{12}\right) + 13$.

In this case $\frac{dT}{dt}$ will represent the rate of change of temperature with time and the units will be °C per hour.

Related rates

Investigation 7

A spherical balloon is being inflated at a constant rate of 10 cm^3 s^{-1}.

This represents the rate of change of the volume of the balloon and so can be written as $\frac{dV}{dt} = 10$.

Suppose you now wish to find out how quickly the radius is increasing when $r = 1$ and $r = 3$.

1 At which of the two radii will the rate of increase of the radius be greater?

2 Write down the rate of increase of the radius as a derivative.

Clearly, the rates of increase of the radius and the volume are related. If the rate of increase of the volume changed, so would the rate of increase of the radius. This connection can be used to find one quantity if the other is known.

Recall that if y is a function of u and u is a function of x then by the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

3 Use the chain rule to find an expression for the rate of change of the volume in terms of the rate of change of the radius and one other quantity.

4 What do you still need to find?

5 Given the balloon remains spherical, find the relation between the two rates as a function of the radius.

6 Hence, find the rate of change of the radius at $r = 1$ and $r = 3$.

7 **Conceptual** How can you find an expression connecting two related rates?

TOK

How can you justify a tax rise for plastic containers, eg plastic bags and plastic bottles, using optimization?

If you are given a rate $\frac{dx}{dt}$ and wish to find the related rate $\frac{dy}{dt}$, the chain rule can be used to give $\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$.

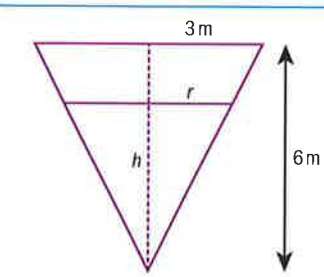
This can be found if the equation relating y and x is known and is differentiable.

HINT

If it is easier to calculate $\frac{dx}{dy}$ than $\frac{dy}{dx}$, then the chain rule may be written as $\frac{dy}{dt} = \frac{dx}{dt} \div \frac{dx}{dy}$.

Example 22

A water tank is in the shape of an inverted circular cone with base radius 3 m and a height of 6 m. If water is being poured into the tank at a rate of 5 m^3 min^{-1}, find the rate at which the water is rising when the water has a depth of 2 metres.



Draw a diagram.

Define:

t: time in minutes from the point when the water began to be added

h: depth of the water at time t

r: radius of the surface of the water at time t

V: volume at time t.

Define variables and write down the given information.

Continued on next page

Calculus

Rate given is $\frac{dV}{dt} = 5$	The process is always to write down the rate that you are given and the rate that you want and to use the chain rule to find the connection between the two.
Rate required is $\frac{dh}{dt}$	
Hence, $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$	
$V = \frac{1}{3}\pi r^2 h$	An equation connecting V and h is required. As the equation for the volume of a cone contains r, which is also a variable, it needs to be replaced with a function of h.
$\frac{h}{r} = \frac{6}{3} \Rightarrow r = \frac{h}{2}$	This result is obtained from the diagram above using similar triangles.
$\Rightarrow V = \frac{1}{12}\pi h^3$	
$\Rightarrow \frac{dV}{dh} = \frac{1}{4}\pi h^2$	
$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$	As it was easier to calculate $\frac{dV}{dh}$ than $\frac{dh}{dV}$, the alternative form of the chain rule is used.
$= \frac{20}{\pi h^2}$	
When $h = 2\,\text{m}$,	
$\frac{dh}{dt} = \frac{20}{4\pi} = 1.59\,\text{m min}^{-1}$.	

Exercise 10L

1 If V is the volume of a cylinder with radius r and height h, write down the rate of change of the volume when

  a the height is increasing at a rate of $\frac{dh}{dt}$ whilst r is kept constant

  b the radius is increasing at a rate of $\frac{dr}{dt}$ whilst h is kept constant.

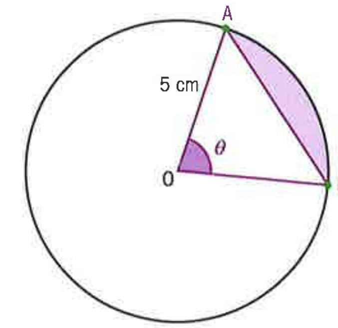
2 The volume of a cube is increasing at a rate of $2\,\text{m}^3\text{s}^{-1}$. Find the rate of increase of its surface area when the cube has a volume of $27\,\text{m}^3$.

3 A ladder 4 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at $1.5\,\text{m s}^{-1}$, find how fast the angle between this ladder and this wall will be changing when this angle is $\frac{\pi}{3}$ radians.

4 A 4 m long water trough is in the shape of a prism whose cross-section is an isosceles triangle with height 1.2 m and base length of 1.0 m. If the trough is being filled at a rate of $1200\,\text{cm}^3\,\text{min}^{-1}$, find how fast the water level will be rising when the water is 45 cm deep.

5 A and B are two points on a circle with centre O and radius 5 cm.

  Let the angle AOB be θ and be increasing at a rate of 0.2 radians s^{-1}.



Find the rate at which the area of the minor segment formed by the chord AB is increasing when $\theta = 1$.

6 a Show that if $f(x) = \frac{1}{\tan x}$ then $f'(x) = -\frac{1}{\sin^2 x}$.

  b A plane is flying at a constant height of 2000 m. An observer watching from the ground directly beneath the flight path of the plane measures the angle of elevation to the plane as 0.5 radians and the rate of change of the angle of elevation as 0.01 radians s^{-1}.

  Find how fast the plane is travelling.

Chapter summary

Differentiation rules

- $f(x) = c \Rightarrow f'(x) = 0$ where $c \in \mathbb{R}$
- $f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$ where $a, n \in \mathbb{R}$
- $f(x) = e^x \Rightarrow f'(x) = e^x$
- $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

If x is in radians, the derivatives of the trigonometric functions are

- $f(x) = \sin x \Rightarrow f'(x) = \cos x$
- $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
- $f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$

The chain rule for composite functions $y \times u(x)$

- $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

The product rule

- $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

The quotient rule

- $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

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Calculus

Increasing and decreasing functions

- $f'(x) > 0 \Rightarrow f(x)$ is increasing
- $f'(x) < 0 \Rightarrow f(x)$ is decreasing

Second derivative

- $f''(x) > 0 \Rightarrow y = f(x)$ is concave up
- $f''(x) < 0 \Rightarrow y = f(x)$ is concave down
- If the concavity of the curve $y = f(x)$ changes either side of $x = c$ then $(c, f(c))$ is a point of inflexion.
- If the curve is differentiable then $f''(c) = 0$.

Optimization

Optimization is the process of finding the maximum or minimum points of a function.

This can be done using a GDC or solving $f'(x) = 0$.

If the function is continuous and the domain is restricted, the global maximum or minimum may occur at the boundary of the domain or where $f'(x) = 0$.

To decide whether a point, $(c, f(c))$, at which $f'(c) = 0$ is a maximum, a minimum or a horizontal point of inflexion, either

- plot the curve on a GDC
- find the value of $f''(c)$ to see if the curve is concave up, down or neither, or
- find the sign of $f'(x)$ either side of c.

Related rates

- Related rates look at the effect that a change in a particular rate has on another rate.
- To solve a related rates problem, write down the rate you are given and the rate you want and connect the two using the chain rule.

Developing inquiry skills

The marginal profit is the extra profit made or lost when increasing production by one unit.

1 Explain why the derivative of the profit function with respect to the number of goods produced is often used as an approximation for marginal profit.

Use the equation for profit found previously, namely, $P = 13000e^{-0.0792x}(x-3)$ where x is the price of the goods and the original equation for demand $d = 13000e^{-0.0792x}$. Assume that the amount produced is equal to the demand.

2 Use related rates to find an expression for the marginal cost, $\frac{dP}{dd}$ in terms of d.

3 Find an expression for $\frac{d^2P}{dd^2}$ and hence show the marginal profit is always decreasing.

Generally, if the marginal cost is positive, it is worth increasing production.

4 Use this rule to find the maximum value of d at which the factory should produce goods.

Chapter review

Click here for a mixed review exercise

1 **a** Find the gradient of the curve $y = \frac{3}{2}x + 4x^2$ at the point $(4, 70)$.

b Find the point(s) on the curve $f(x) = 4x^{\frac{5}{4}} - 80x - 8$ where the gradient is zero.

c Given that $r = 6 + 10t - 7t^2$, find the rate of change of r with respect to t at the instant when $t = 5$.

2 Find the derivative of the following functions.

a $f(x) = 2\cos x + \sin 2x$ **b** $f(x) = \ln(\tan x)$

c $y = \frac{\cos x}{\sin x}$ **d** $y = xe^{2x}$

3 Find the first and second derivatives of each of the following.

a $f(x) = \frac{12}{x^2} + 3\sqrt{x}$ **b** $f(x) = (x-1)(3x-7)$

c $y = \frac{x-12}{x^2}$ **d** $y = \frac{x^2}{x-12}$

4 A function is defined by $f(x) = \frac{1}{e^{2x}+1}$.

Show that the function is increasing for all values of x.

5 Find the point of intersection of the normals to the curve $y = 3x^2 - 7x + 8$ at the points $(2, 6)$ and $(0, 8)$.

a Find the coordinates of the point at which $\frac{dy}{dx} = 0$ for the curve $y = 7x - 14\sqrt{x}$.

b Find an expression for $\frac{d^2y}{dx^2}$.

c **Hence**, state and justify whether the point found in part **a** is a maximum or minimum.

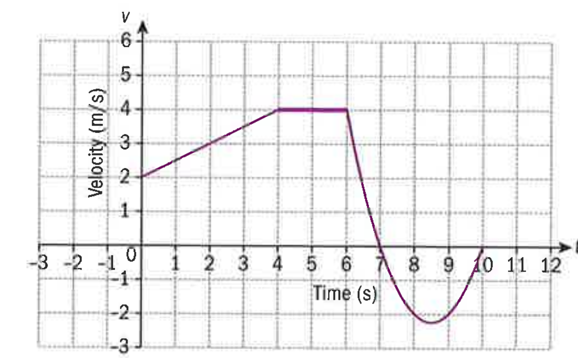
6 A rectangular enclosure is to be made in a field, using an existing wall as one side of the enclosure and 128 m of fencing to create the other three sides. Let x represent the length of the sides of the enclosure perpendicular to the wall.

a Find an expression for the area, A, of the enclosure in terms of x in its simplest form.

b Find the value of x for which A is a maximum.

c Find the maximum area of the enclosure.

7 The graph below represents the velocity v, in metres per second, of a particle moving along the x-axis over the time interval $t = 0$ to $t = 10$ seconds.



a At $t = 2$ s, determine whether the particle is moving to the right or left.

b Find the interval(s) in which the particle is moving **i** to the right **ii** to the left.

Explain your answer.

c At $t = 3$ s, determine whether the acceleration is positive or negative.

d Describe the motion of the particle in each of the intervals [0,4], [4,6], [6,7], [7, 8.5] and [8.5,10] in terms of direction of the motion, velocity and acceleration.

e Determine the time in the given interval when the particle is farthest to the right.

Calculus

8 Determine the value of x for which the gradient of the logistic function $f(x)=\frac{70}{1+e^{-2x+19}}$ is at its maximum.
Hence, find the coordinates of the point of inflexion of $f(x-5)$.

9 A particle is moving along the curve $y=\sqrt{x}+3$. As the particle passes through the point $(1, 4)$, its x-coordinate is increasing at a rate of $2\,\text{cm}\,\text{min}^{-1}$. Determine the rate of change of distance between the origin and the particle at this point.

Exam-style questions

10 P1: Find the equation of the normal to the curve $y=2-\frac{x^4}{2}$ at the point where $x=1$ (7 marks)

11 P1: A curve is given by the equation

$$y=\frac{2x^3}{3}-\frac{7x^2}{2}+2x+5.$$

a Determine the coordinates on the the curve where the gradient is -3. You must show all your working, and give your answers as exact fractions. (6 marks)

b Find the range of values of x for which the curve is decreasing. (2 marks)

12 P2: A rectangular piece of paper, measuring 40 cm by 30 cm, has a small square of side length x cm cut from each corner. The flaps are then folded up to form an open box in the shape of a cuboid.

a Show that the volume V of the cuboid may be expressed as $V=1200x-140x^2+4x^3$. (3 marks)

b Find an expression for $\frac{dV}{dx}$. (2 marks)

c Hence show that the cuboid will have a maximum volume when $x^2-\frac{70}{3}x+100=0$. (2 marks)

d Using technology, find the maximum possible volume of the cuboid. (4 marks)

13 P2: Consider the function defined by

$$f(x)=\frac{x^2}{2x^3-1},\ x\neq\sqrt[3]{\frac{1}{2}}$$

a Find an expression for $f'(x)$. (3 marks)

b Find the equation of the tangent to the curve at the point where $x=1$. (4 marks)

c Find the coordinates of the points on the curve where the gradient is zero. (4 marks)

d Determine the range of values of x for which $f(x)$ is an increasing function. (2 marks)

14 P1: A right circular cone with fixed height 50 cm is increasing in volume at a rate of $2\ \text{cm}^3\,\text{min}^{-1}$. Find the rate at which the radius of the base r is increasing when $r=0.4$ cm. Give your answer in exact form. (6 marks)

15 P1: A small object travels in a straight line so that its displacement, x metres, from a fixed point O after t seconds (where $0\le t<5$) is given by the equation

$$x=\frac{t}{t-5}+2\ln(1+t).$$

a Find the distance travelled by the object in the first two seconds. (2 marks)

b Find an expression for $\frac{dx}{dt}$ and hence find the value of t when the particle is stationary. (6 marks)

16 P2: A small metal bearing is attached to the end of a vertical spring inside a tube of dense liquid.

At time t seconds ($t\ge 0$), the displacement (x metres) of the bearing from its equilibrium position is given by the equation $x=e^{-2t}\cos\sqrt{12}t$.

a Show that $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+16x=0$. (6 marks)

b Sketch a graph of x against t for $0\le t\le 2$. (2 marks)

c Find the time when the bearing first comes to instantaneous rest and its distance from its equilibrium position at this time. (4 marks)

d Find the maximum speed of the particle during its first second of motion and its distance from its equilibrium position at this time. (5 marks)

Calculus

River crossing

Approaches to learning: Thinking skills: Evaluate, Critiquing, Applying

Exploration criteria: Personal engagement (C), Reflection (D), Use of mathematics (E)

IB topic: Differentiation, Optimisation

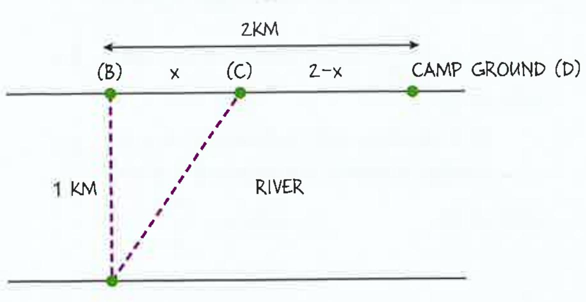
The problem

You are standing at the edge of a slow-moving river which is 1 kilometre wide. You want to return to your campground on the opposite side of the river. You can swim at 3 km/h and run at 8 km/h. You must first swim across the river to any point on the opposite bank. From there you must run to the campground, which is 2 km from the point directly across the river from where you start your swim.

What route will take the least amount of time?

Visualize the problem

Here is a diagram of this situation:



Discuss what each label in the diagram represents.

Solve the problem

What is the length of AC in terms of x?

Using the formula for time taken for travel at a constant rate of speed from this chapter, write down an expression in terms of x for:

1 the time taken to swim from A to C

2 the time taken to run from C to D.

Hence write down an expression for the total time taken, T, to travel from A to D in terms of x.

You want to minimize this expression (find the minimum time taken).

Find $\frac{dT}{dx}$.

Now solve $\frac{dT}{dx} = 0$ to determine the value of x that minimizes the time taken.

How do you know this is a valid value?

Use the second derivative test to show that the value you found is a minimum value.

For this value of x, find the minimum time possible and describe the route.

Assumptions made in the problem

The problem is perhaps more accurately stated as:

You are standing at the edge of a river. You want to return to your campground which you can see further down the river on the other side. You must first swim across the river to any point on the opposite bank. From there you must run to the campground.

What route will take the least amount of time?

Look back at the original problem.

What additional assumptions have been made in the original question?

What information in the question are you unlikely to know when you are standing at the edge of the river?

What additional information would you need to know to determine the shortest time possible?

The original problem is a simplified version of a real-life situation. Criticize the original problem, and the information given, as much as possible.

Extension

In an exploration it is important to reflect critically on any assumptions made and the subsequent significance and limitations of the results.

In this chapter you have been introduced to some classic optimisation problems in the examples and exercises. For example, there is the open-box problem in Q5 of Exercise 10E and Example 10 involving the surface area of a cylindrical can.

If you were writing an exploration and these problems were forming the basis or inspiration of that exploration, then:

- What assumptions have been made in the question?
- What information in the question are you unlikely to know in real-life?
- How could you find this missing information?
- What additional information would you need to know?
- Criticise the questions as much as possible!