# 11 Approximating irregular spaces: integration and differential equations

What do the graphs of functions that have the same derivative have in common? How do they differ?

This chapter explores integration, the reverse of differentiation. The area of an island, the surface area and volume of a building and the distance travelled by a moving object can all be represented by integrals. Integrals give you a way to estimate the values of areas and volumes that cannot be found using existing formulae.

**Concepts**

- Space
- Approximation

**Microconcepts**

- Lower limit
- Upper limit
- Antiderivative
- Definite integral
- Indefinite integral
- Numerical integration
- Reverse chain rule
- Area under the curve
- Volumes of revolution
- Exact solutions of differential equations
- Slope fields
- Euler's method


**How can you find the distance travelled when the equation velocity is given?**

**How can you estimate the area covered by oil spills out at sea?**

**How can you find the amount of glass in this building?**


**How can the volume of a building be found?**



San Cristóbal is the eastern most island of the Galapagos. Here is a map of the island.

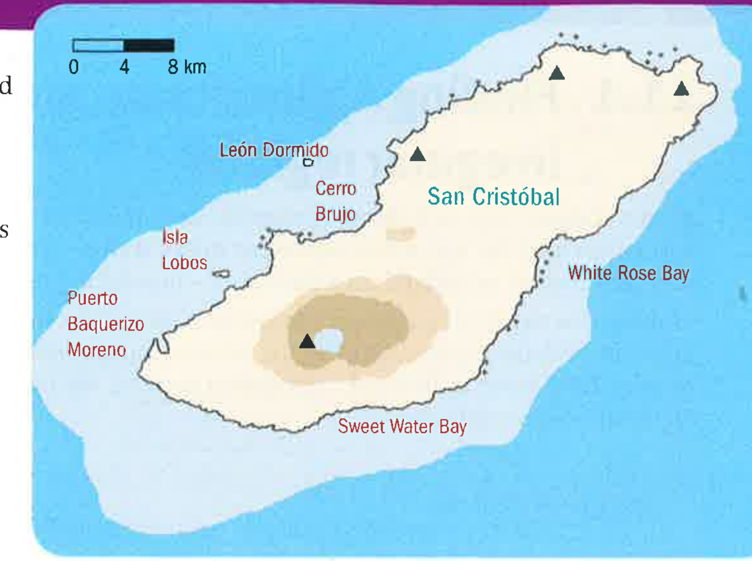
It is claimed that the total area of the island is 558 km². How can you test this value?

Use a rectangle to estimate the area of the island.

How did you use the map scale?

Does your result underestimate or overestimate the claimed area? Why?

What would you do to improve your estimate?




## Developing inquiry skills

Write down any similar inquiry questions you might ask to model the area of something different, for example the area of a national park, city or lake in your country.

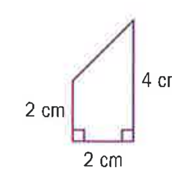
Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

## Before you start

**You should know how to:**

1 Find the area of a trapezium.

eg




$$\text{area} = \frac{1}{2}h(a+b) = \frac{1}{2}\times 2(2+4) = 6\text{ cm}^2$$

2 Differentiate functions, including with the chain rule.

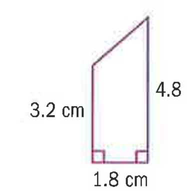
eg If $f(x) = \sin 2x - \cos x^2$ then

$f'(x) = 2\cos 2x + 2x\sin x^2$

**Skills check**

Click here for help with this skills check

1 Find the area of the trapezium.




2 Differentiate each function.

**a** $y = 3x^3 - 2\sqrt{x} + \frac{4}{x^2}$

**b** $f(x) = \cos 5x + \sin^2 x$

**c** $s = \ln 5t - 2e^{t^2}$

# 11.1 Finding approximate areas for irregular regions

In this chapter, you will discover that the area between the graph of a function and the $x$-axis can represent many things – distance, costs, total production or indeed an actual area – depending on the function.

Finding this value is clearly very important. For some functions the area can be found using calculus; for others, approximation methods or your GDC need to be used. In this first section, we use a GDC and approximation methods.

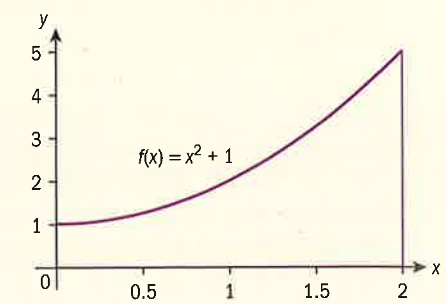
**International-mindedness**

Egyptian mathematician Ibn al-Haytham is credited with calculating the integral of a function in the 10th century.

## Investigation 1

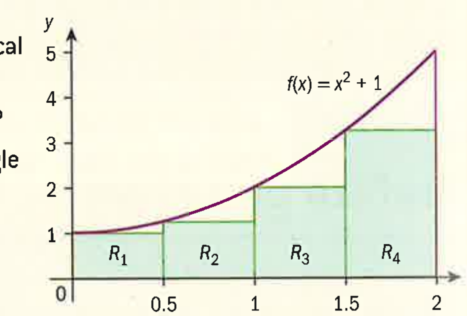
**1** Consider the area bounded by the graph of the function $f(x) = x^2 + 1$, the vertical lines $x = 0$, $x = 2$ and the $x$-axis.

Estimate the area of the region. Is your estimation an overestimation or an underestimation of the actual area? Discuss your method with a classmate.




**2** In this investigation, you will be using rectangles or vertical strips to estimate this area. At the end of the investigation you can check how close your estimation was to the actual area.

This graph shows four rectangles of the same width. The area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0$, $x = 2$ is also shown.




**a** What is the width of the rectangles? How did you calculate it?

**b** What is the relationship between the height of each rectangle and the graph of the function?

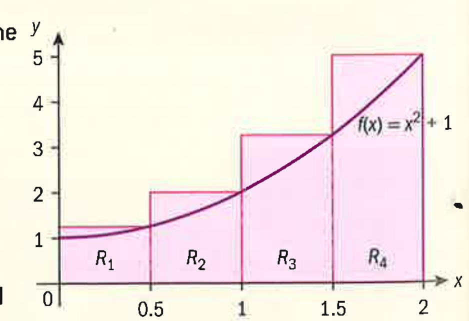
**c** Find the height of each of these rectangles.

**d** Find the area of each of these rectangles and then find **the sum** of the areas of these rectangles.

**e** Is this an underestimation or an overestimation of the actual area? Why?

The sum of these areas will be a **lower bound** of the area of under the curve. This will give an **underestimation** of the area.

**3** This second graph shows another set of four rectangles with the same width. The area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0$, $x = 2$ is also shown.




**a** What is the width of each of these rectangles?

**b** Find the height of each of these rectangles.

**c** Find the area of each of these rectangles and then find **the sum** of the areas of these rectangles.

**d** Is this an underestimation or an overestimation of the actual area? Why?

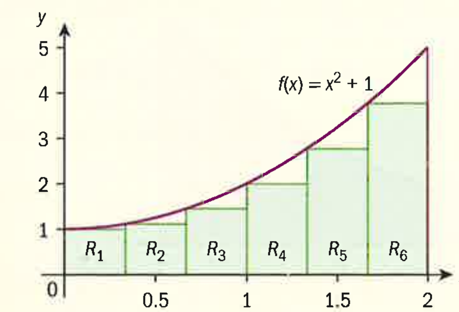
The sum of these areas will be an upper bound of the area under the curve. This will give an overestimate of the area.

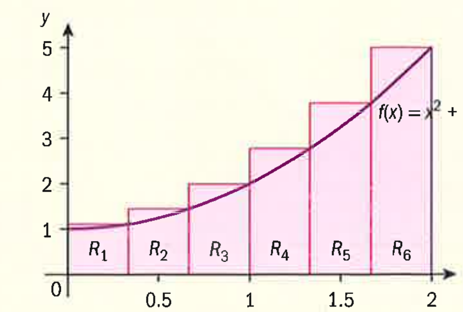
**e** If $L_S$ represents the lower bound, $A$ represents the actual area and $U_S$ represents the upper bound, write an inequality relating $L_S$, $U_S$ and $A$.





**4** In each of the following graphs there are six rectangles. The area under the graph of $f(x) = x^2 + 1$ between the vertical lines $x = 0$, $x = 2$ and the $x$-axis is also shaded.




The area under the curve will now be approximated by finding new upper and lower bounds.

**a** Why do you think that more rectangles are being used?

**b** Complete the following tables to organize the information. You can create a table with your GDC to calculate the height of the rectangles. How would you calculate the widths? Remember that they are all equal.

Lower bound with six rectangles:

| Rectangle | Width | Height | Area |
|---|---|---|---|
| $R_1$ | | | |
| $R_2$ | | | |
| $R_3$ | | | |
| $R_4$ | | | |
| $R_5$ | | | |
| $R_6$ | | | |
| | | | $L_S =$ |

Upper bound with six rectangles:

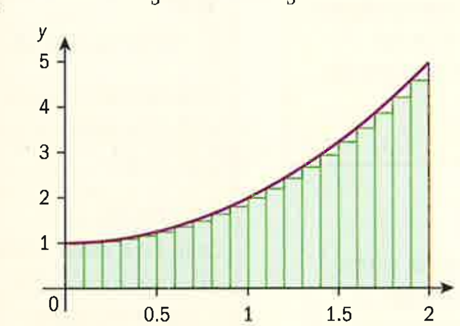
| Rectangle | Width | Height | Area |
|---|---|---|---|
| $R_1$ | | | |
| $R_2$ | | | |
| $R_3$ | | | |
| $R_4$ | | | |
| $R_5$ | | | |
| $R_6$ | | | |
| | | | $U_S =$ |

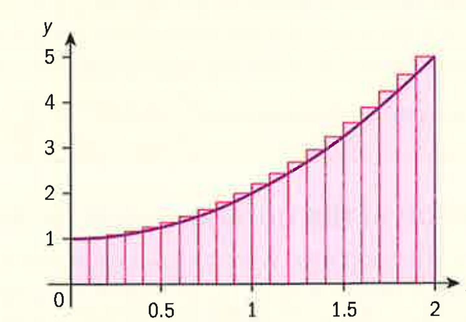
**c** Have the lower and upper bounds approached each other if you compare their values to those found with four rectangles? How do you think these estimations can be improved?

**d** Write a new inequality relating $L_S$, $U_S$ and $A$.

**5** Look at the graphs below. The number of rectangles, $n$, has been increased in each case. $L_S$ and $U_S$ are also given.

$n = 20$, $L_S = 4.47$, $U_S = 4.87$:

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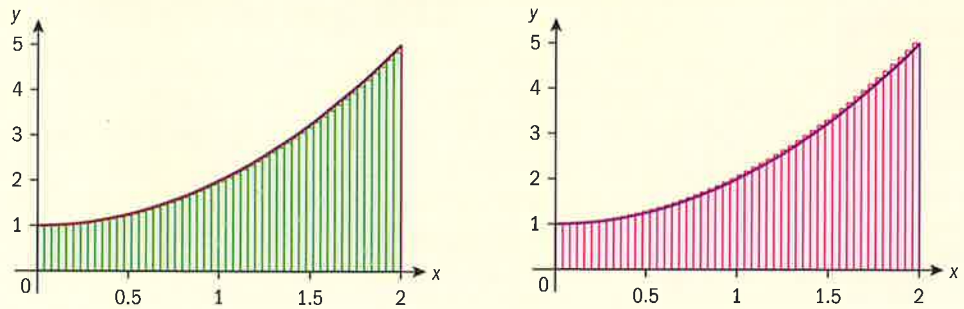
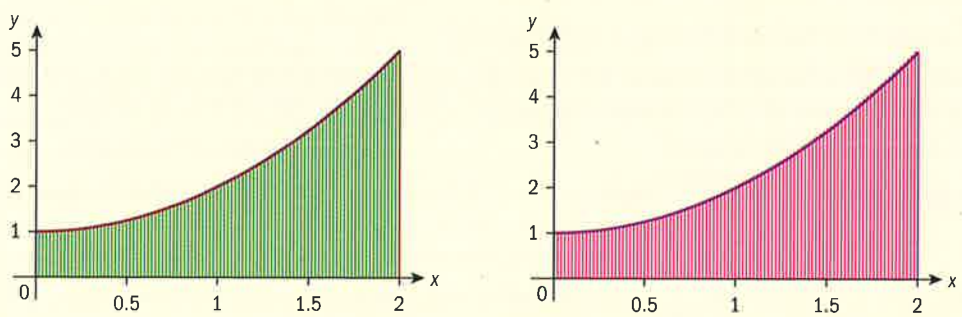
Calculus

$n = 50$, $L_S = 4.5872$, $U_S = 4.7472$:




$n = 100$, $L_S = 4.6268$, $U_S = 4.7068$:




**a** What can you say about the values of $L_S$ as $n$ increases?

**b** What can you say about the values of $U_S$ as $n$ increases?

The table shows more values for $n$, $L_S$ and $U_S$.

| $n$ | $L_S$ | $U_S$ |
|---|---|---|
| 500 | 4.65867 | 4.67467 |
| 1000 | 4.66267 | 4.67087 |
| 10 000 | 4.66627 | 4.66707 |

**c** What happens as $n$ tends to infinity (gets larger and larger)? What can you say about the value of $A$?

When $f$ is a non-negative function on the interval $a \le x \le b$, the area enclosed by the graph of $f$, the $x$-axis and the vertical lines $x = a$ and $x = b$ is the **unique** number between all lower and upper sums. This number is said to be the **definite integral** and is denoted as $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

**HINT**

$\int_a^b f(x)\,dx$ is read as "the definite integral of $f(x)$ between $x = a$ and $x = b$".

The number $a$ is called the **lower limit** of integration and $b$ is called the **upper limit** of integration.

## An explanation of the notation used for areas

If the width of the rectangle is $\delta x$ (the Greek letter $\delta$ (delta) is often used to indicate a small change, so $\delta x$ represents a small change in the value of $x$) then the area of the rectangles can be approximated as $A \approx \sum_{i=1}^{n} y_i \times \delta x$. The actual area will be $A = \lim_{\delta x \to 0}\left(\sum_{i=1}^{n} y_i \delta x\right)$. To indicate that the limit is being taken, the $\Sigma$ (Greek letter sigma) becomes an elongated S $(\int)$, the $\delta x$ becomes $dx$ and the bounds for $x$ are added, hence the notation $\int_a^b y\,dx$.





**TOK**

Where does mathematics come from?

Galileo said that the universe is a grand book written in the language of mathematics.

Does it start in our brains or is it part of the universe?

### Investigation 1 (continued)

**6** What is the lower limit of integration in the investigation? What is the upper limit of integration? Is the function positive between the lower and the upper limit? How would you represent $A$ using definite integral notation?

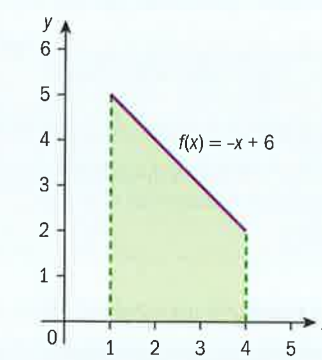
**7** **Conceptual** How do areas under curves within a given interval relate to the definite integral and to lower and upper rectangle sums on the same interval?

**Reflect** What is a definite integral?

### Example 1

**a** Write down a definite integral that gives the area of the shaded region.

**b** Calculate the definite integral by using the formula for a trapezoid.

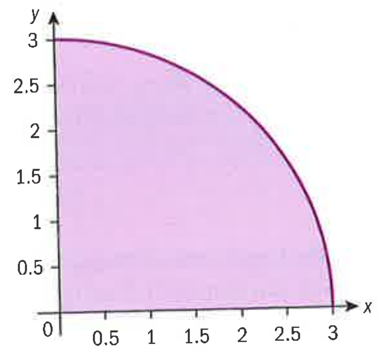



| | |
|---|---|
| **a** $\int_1^4 (-x+6)\,dx$ | The lower limit is $x = 1$.<br>The upper limit is $x = 4$.<br>The function is $f(x) = -x + 6$. |
| **b** $\int_1^4 (-x+6)\,dx = \frac{3}{2} \times (2+5) = 10.5$ | The shape is trapezoidal.<br>$a = f(1) = -1 + 6 = 5$<br>$b = f(4) = -4 + 6 = 2$<br>Height $= 4 - 1 = 3$ and substitute this into the trapezoid area formula. |

Calculus

## Exercise 11A

1 The equation for a circle centred at (0, 0) with radius 3 is $x^2 + y^2 = 9$. The sector of the circle in the first quadrant is shown below.




**a** **i** Find the equation of this sector in the form $y = f(x)$, carefully stating the domain.

**ii** Write an expression for the area of the sector in the form $\int_a^b f(x)\,dx$.

Parts **b** and **c** should be done using a spreadsheet or other technology.

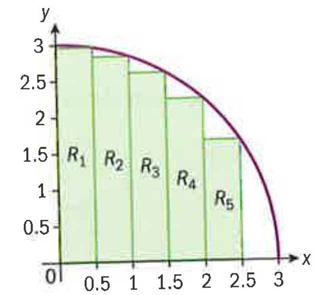
**b** Calculate the values of $y$ for the given values of $x$.

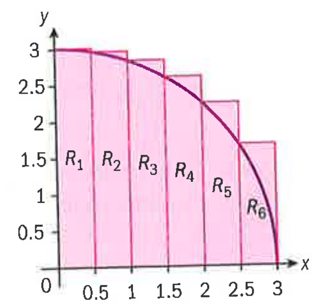
| $x$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**c** Use the table and the following diagrams to calculate the sum of the areas of the rectangles that form:

**i** a lower bound

**ii** an upper bound for the area of the sector.




**d** Find the mean of your upper bound and lower bound.

**e** Find the percentage error in using the answer to part **d** as an estimate for the area of the quarter circle.

2 A region $R$ is formed by the curve $y = \frac{1}{x}$, the $x$-axis and the lines $x = 1$ and $x = 6$.

**a** Sketch the graph of the curve and shade the region $R$.

**b** Write an expression for the area $R$ in the form $\int_a^b f(x)\,dx$.

**c** Use five rectangles to find

**i** a lower bound

**ii** an upper bound for the area of $R$.

**d** Write down an expression for the area of $R$ as an inequality in the form $p < R < q$.

## Using the GDC to evaluate areas

It is a requirement in the exam that your GDC can calculate areas between the curve and the $x$-axis (equivalent to finding the **definite integral** when the function is positive).

Use your GDC to find the value of $\int_0^2 (x^2 + 1)\,dx$ and compare your answer with the bounds obtained in Investigation 1.

**International-mindedness**

The definite integral notation was introduced by the German mathematician Gottfried Wilhelm von Leibniz towards the end of the 17th century.



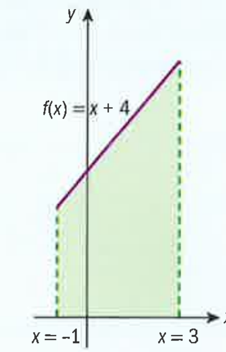


## Example 2

Find the area shown by

**a** using the formula for the area of a trapezium

**b** using the integral function on your calculator.



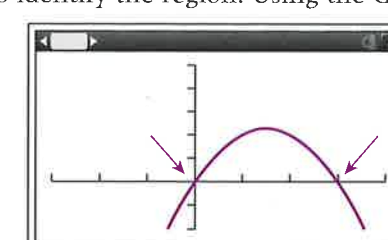

| | |
|---|---|
| **a** The function in this case is $f(x) = x + 4$. The lower limit is $x = -1$ and the upper limit is $x = 3$. | The shape is trapezoidal. The parallel sides of the trapezium are $a$ and $b$. |
| Hence the two parallel lines have lengths of 3 and 7. | $a = f(-1) = -1 + 4 = 3$ <br> $b = f(3) = 3 + 4 = 7$ |
| The area is therefore $\frac{1}{2} \times 4(3+7) = 20$. | Height $= 3 - (-1) = 4$ <br> Substitute into the trapezium area formula. |
| **b** $\int_{-1}^{3} (x+4)\,dx = 20$ | The area can be calculated directly from the GDC. |

## Example 3

Consider the area $A$ of the region enclosed between the curve $y = -x(x - 3)$ and the $x$-axis.

**a** Write down the definite integral that represents this area $A$.

**b** Find $A$.

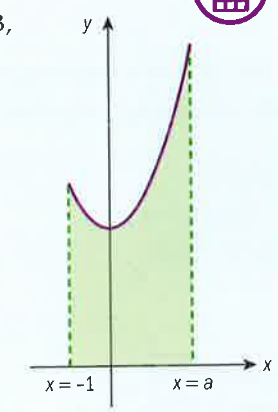
| | |
|---|---|
| **a** $\int_0^3 -x(x-3)\,dx$ | You first have to identify the region. Using the GDC:  From the graph it can be seen that the lower and upper bounds are the **roots** of the parabola. <br> $x = 0$ is one of the roots, the lower bound of the definite integral. <br> $x = 3$ is the other root, the upper bound of the definite integral. |
| **b** $A = 4.5$ | Once you have identified the region, write down the definite integral. |

Calculus

### Example 4

The region bounded by the graph of $f(x) = x^2 + 3$, the $x$-axis and the vertical lines $x = -1$ and $x = a$ with $a > -1$ has area equal to 12.

Find the value of $a$.




| | |
|---|---|
| $\int_{-1}^{a}(x^2+3)\,dx = 12$ | The unknown, $a$, is the upper bound of this area. First write down the definite integral. |
| $a = 2$ | Use your GDC to find the value of $a$. |

**International-mindedness**

A Riemann sum, named after 19th century German mathematician Bernhard Riemann, approximates the area of a region, by adding up the areas of multiple simplified slices of the region.

### Exercise 11B

1 For each of the following:

i write down the definite integral that represents the area of the enclosed region.

ii find the area.

a $y = x^2$, the $x$-axis and the lines $x = 2$ and $x = 4$

b $y = 2^x$, the $x$-axis and the lines $x = -1$ and $x = 1$

c $y = \frac{1}{1+x^2}$ and the $x$-axis in the interval $-1 \le x \le 1$

d $y = \frac{1}{x}$ and the $x$-axis in the interval $0.5 \le x \le 3$

e $f(x) = -(x-3)(x+2)$, the vertical axis and the vertical line $x = 1$

f $f(x) = -(x-3)(x+2)$, the vertical axis and the horizontal axis

g $f(x) = -(x-3)(x+2)$ and the horizontal axis

h $f(x) = -x^2 + 2x + 15$ and the vertical lines $x = -2$ and $x = 4.5$

i $f(x) = -x^2 + 2x + 15$ and the line $y = 0$

j $f(x) = 3 - e^x$, the vertical line $x = -1$ and the $x$-axis

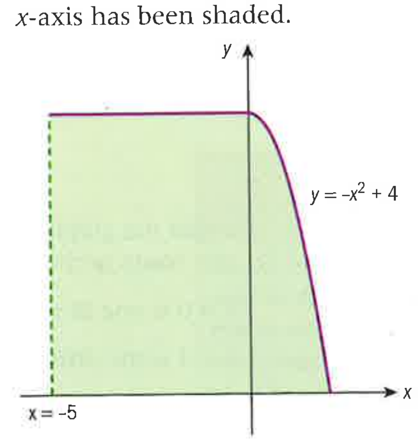
k $y = (x+2)^3 + 5$ and the coordinate axes

2 Consider the curve $y = -x^2 + 4$.

a Find the $x$-intercepts of this curve.

b Find the point where this curve cuts the $y$-axis.

The following graph shows a piecewise function $f$ made up of a horizontal line segment and part of the parabola $y = -x^2 + 4$. The area under the graph of $f$ and above the $x$-axis has been shaded.




c Find the area under the graph of $f$ in the interval $-5 \le x \le 0$.

d i Write down an expression for the area under the graph of $f$ and above the $x$-axis for $x > 0$.

ii Find the area.

e Find the shaded area.

3 The region bounded by the graph of $f(x) = -(x+1)(x-5)$, the $x$-axis, the $y$-axis and the line $x = a$ where $0 \le a \le 5$ has area equal to 24. Find the value of $a$.

4 The region bounded by the graph of $f(x) = 2^{-x}$ and the $x$-axis between $x = -3$ and $x = a$ where $a > -3$ has area equal to 9. Find the value of $a$. Give your answer correct to 4 significant figures.

## Negative integrals

### Investigation 2

1 Use your GDC to evaluate the following definite integrals.

a $A = \int_0^1 (x^2 - 1)\,dx$ b $B = \int_1^2 (x^2 - 1)\,dx$ c $C = \int_0^2 (x^2 - 1)\,dx$

2 Find an equation linking $A$, $B$ and $C$.

3 a Sketch the curve $y = x^2 - 1$ for $0 \le x \le 2$.

b Why is the **area** bounded by the curve, the $x$-axis and the lines $x = 0$ and $x = 2$ not equal to $\int_0^2 (x^2 - 1)\,dx$?

c **Factual** What do you notice about the definite integral when a function is below the $x$-axis?

4 a Sketch the graph of $y = |x^2 - 1|$.

b How might you use your GDC to find the **area** bounded by the curve, the $x$-axis and the lines $x = 1$ and $x = 2$?

5 **Conceptual** When evaluating areas bounded by curves, what must you consider when evaluating the definite integral?

If $f(x) < 0$ throughout an interval $a \le x \le b$ then $\int_a^b f(x)\,dx < 0$.

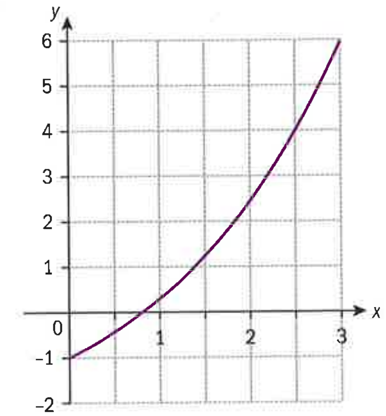
If the area bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is required and if $f(x) < 0$ for any values in this interval then the area should be calculated using $\int_a^b |f(x)|\,dx$.

## Example 5

**a** Sketch the curve $y = 2e^{0.5x} - 3$ for $0 \le x \le 3$.

**b** Find the area of the region bounded by the curve, the $x$-axis and the lines $x = 0$ and $x = 2$.

**a**




**b** $\int_0^2 \left|2e^{0.5x} - 3\right| dx \approx 1.74$

Because some of the curve lies below the $x$-axis, the modulus of the function is integrated.

## Exercise 11C

**1** **a** Sketch $y = \sin 2x$ for $0 \le x \le \pi$.

**b** Find the area of the region bounded by the curve, the $x$-axis and the following pairs of lines.

**i** $x = 0$ and $x = 0.5$

**ii** $x = 2$ and $x = 3$

**iii** $x = 0$ and $x = 2.5$

**2** **a** Sketch the curve $y = \frac{4}{x-2} - 2$.

**b** Explain why it is not possible to calculate $\int_0^3 \left(\frac{4}{x-2} - 2\right) dx$.

**c** Find the area of the region bounded by the curve, the $x$-axis and the lines given.

**i** $x = 3$ and $x = 4$ **ii** $x = 4$ and $x = 6$

**d** Hence or otherwise find the area of the region bounded by the curve, the $x$-axis and the lines $x = 3$ and $x = 6$.

## The trapezoidal (trapezium) rule

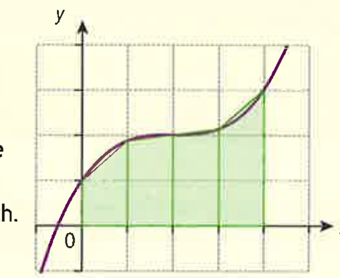
Sometimes it is not possible to use a calculator to work out areas, in particular in those cases where the equation of the function is not known. Though it would be possible to approximate the area using rectangles and finding the sum, a more accurate method is usually to divide the area between the curve and the $x$-axis into a series of trapezoids, as shown in the investigation below.





## Investigation 3

The area beneath the curve $y = f(x)$ is to be found by approximating it using four trapezoids.



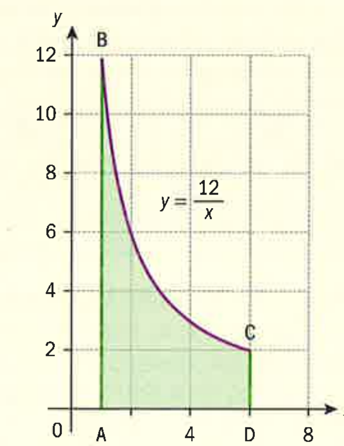

The formula for the area of a trapezoid is $A = \frac{1}{2}h(a+b)$ where $a$ and $b$ are the lengths of the two parallel sides and $h$ is the perpendicular distance between the two sides. $h$ is often referred to as the height of the trapezoid, though in the trapezoids shown it would be more natural to refer to the width.

Let the width of each trapezoid be $h$ and the lengths of the parallel sides be $y_0, y_1, y_2, y_3$ and $y_4$.

**1** Write down an expression for the area of each trapezoid and show that the sum can be written as $\frac{1}{2}h\left(y_0 + y_4 + 2(y_1 + y_2 + y_3)\right)$.

**2** Conjecture an expression for the approximate area between a curve and the $x$-axis found by using $n$ trapezoids with the first parallel line having length $y_0$ and the last $y_n$.

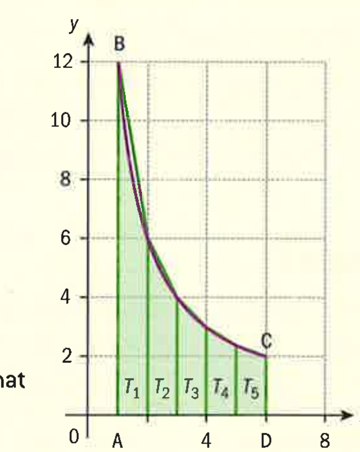
**3** Consider the curve $y = \frac{12}{x}$, where $1 \le x \le 6$.




The area of the region enclosed between the graph of $f(x) = \frac{12}{x}$ and the $x$-axis in the interval $1 \le x \le 6$ will be called $S$.

**a** Write down an expression for $S$ and find its value using your GDC. Give your answer correct to 2 decimal places.

This graph shows the shaded area subdivided into five trapezoids.




**b** Find the width ($h$) of each of the trapezoids.

**c** Use $f(x) = \frac{12}{x}$ to find the lengths of the vertical sides for each of the trapezoids.

**d** Hence, use the rule conjectured in question **2** to find an approximation for $S$.

**e** Find the percentage error for this approximation.

The area of $S$ will now be approximated using 10 trapezoids.

**f** Find the value of $h$.

**g** Find the lengths of the parallel sides of the trapezoids and hence, use the formula conjectured in question **2** to show that $S \approx 21.74$ correct to 2 decimal places.

**h** Find the percentage error for this approximation.

**i** What will be the limit of the sum of the areas of the trapezoids as the number of trapezoids tends to infinity?

**4** **Conceptual** How can we estimate the area underneath a curve?

**5** **Conceptual** How do we make this approximation more accurate?

For a positive continuous function, the area ($A$) between the graph of the function, the $x$-axis and the lines $x = a$ and $x = b$ can be approximated by the trapezoidal rule.

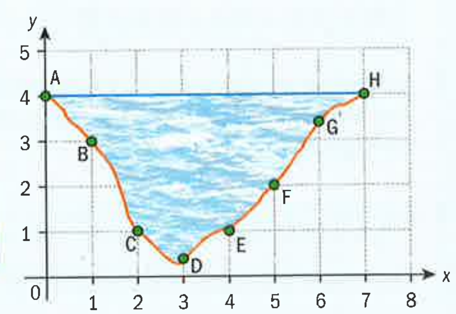
$$A \approx \frac{1}{2}h\left(y_0 + y_n + 2(y_1 + \ldots + y_{n-1})\right) \text{ where } h = \frac{b-a}{n}$$

It is particularly useful when the values on a curve are known, but not the function that defines the curve.

## Example 6

The cross-section of a river in which the water is flowing at $0.8\,\text{m s}^{-1}$ is shown in the diagram. Use the trapezoidal rule, with seven trapezoids, to find an approximation for the volume of water passing this point in one minute. All lengths are in metres.

| A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| (0, 4) | (1, 3) | (2, 1) | (3, 0.4) | (4, 1) | (5, 2) | (6, 3.4) | (7, 4) |




| | |
|---|---|
| $A \approx \frac{1}{2} \times 1\left(4 + 4 + 2(3 + 1 + 0.4 + 1 + 2 + 3.4)\right)$ | The lengths of the parallel lines are given by the $y$-coordinates in the table. |
| $= \frac{1}{2} \times 29.6 = 14.8\text{m}^2$ | The trapezoidal rule is applied to these values. |
| Cross-sectional area of river is $28 - 14.8 = 13.2\text{m}^2$ | |
| Volume of water per minute $= 13.2 \times 0.8 \times 60 \approx 634\text{m}^3$ | Volume of water is the amount that passes per second multiplied by 60. |

## Example 7

Find an approximate value of the integral $\int_1^4 \sin^2 x\,dx$ using the trapezoidal rule with four intervals.

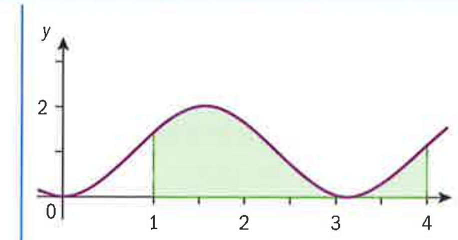
| | |
|---|---|
| $A \approx \int_1^4 (\sin^2 x)\text{d}x$ | The definite integral is the same as the area under the graph of the function when the function is always positive. You must always use radians for trigonometric functions unless told otherwise. |
| $h = \frac{4-1}{4} = 0.75$ | |





$$A = \frac{0.75}{2}\left[\begin{array}{l}\sin^2(1) + \sin^2(4) + 2\sin^2(1.75)\\ \quad + 2\sin^2(2.5) + 2\sin^2(3.25)\end{array}\right]$$

$A \approx 1.4839$




## Exercise 11D

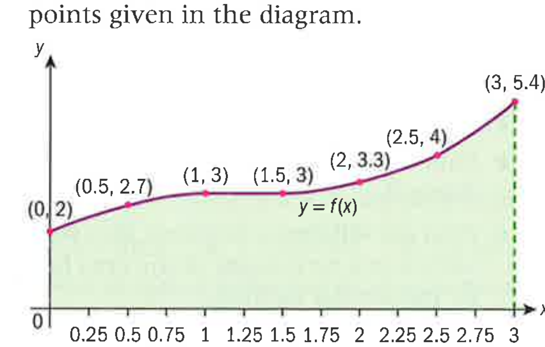
1 Use the trapezoidal rule to estimate the area under a curve over the interval $1 \le x \le 9$, with the $x$ and $y$ values given in the following table.

| $x$ | 1 | 3 | 5 | 7 | 9 |
|---|---|---|---|---|---|
| $y$ | 5 | 7 | 6 | 10 | 4 |

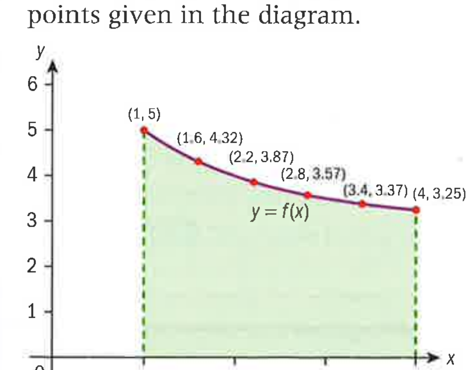
2 Use the trapezoidal rule to estimate the area under a curve over the interval $0 \le x \le 6$, with the $x$ and $y$ values given in the following table.

| $x$ | 0 | 1.5 | 3 | 4.5 | 6 |
|---|---|---|---|---|---|
| $y$ | 1 | 4 | 2 | 5.5 | 0 |

3 Use the trapezoidal rule to estimate the area under the graph of $y = f(x)$ using the data points given in the diagram.




4 Use the trapezoidal rule to estimate the area under the graph of $y = f(x)$ using the data points given in the diagram.




5 Use the trapezoidal rule to estimate the area between each curve and the $x$-axis over the given interval. Give your answers to 4 significant figures.

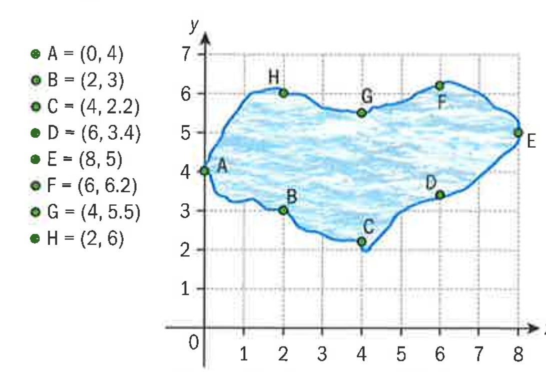
a $f(x) = \sqrt{x}$, interval $0 \le x \le 4$ with $n = 5$

b $f(x) = 2x$, interval $-1 \le x \le 4$ with $n = 4$

c $f(x) = \frac{10}{x} + 1$, interval $2 \le x \le 5$ with $n = 6$

d $y = -0.5x(x - 5)(x + 1)$, interval $0 \le x \le 5$ with $n = 5$

6 The diagram below shows the outline of a lake. The coordinates of seven points on the edge of the lake are given. All measurements are in kilometres.




a Use the trapezoidal rule with four intervals to find the area below the curve marking the upper boundary of the lake (through the points A, H, G, F and E) and the $x$-axis.

b Repeat for the area below the curve marking the lower boundary of the lake.

c Hence find an approximation for the area of the lake.

**7** Consider the region enclosed by the curve $y = -2(x-3)(x-6)$ and the $x$-axis.

**a** Sketch the curve and shade the region.

**b i** Write down a definite integral that represents the area of this region.

**ii** Find the area of this region.

**c** Use the trapezoidal rule with $n = 6$ trapezoids to estimate the area of this region.

**d** Find the percentage error made with the estimation made in part **c**.

**8** Consider the region enclosed by the graph of the function $f(x) = 1 + e^x$, the $x$-axis and the vertical lines $x = 0$ and $x = 2$.

**a** Sketch the function $f$ and shade the region.

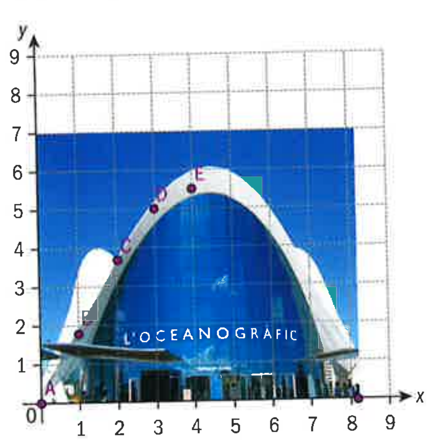
**b i** Write down a definite integral that represents the area of this region.

**ii** Find the area of this region. Give your answer correct to 4 significant figures.

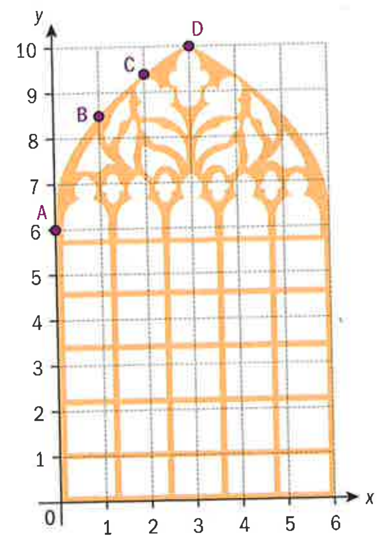
**c** Use the trapezoidal rule with $n = 5$ trapeziums to estimate the area of this region. Give your answer correct to 4 significant figures.

**d** Find the percentage error made with the estimation found in part **c**.

**9** The picture below shows the L'Oceanogràfic, a building in Valencia, Spain. Use the trapezoidal rule with four trapezoids to estimate the area of the window, given that the points A, B, C, D and E have coordinates (0,0), (1, 1.8), (2, 3.7), (3, 5), and (4, 5.5).




**10** A surveyor needs to work out the area of the old window shown in the diagram below. Based on a coordinate system with the origin in the bottom left-hand corner of the window, the points A, B, C and D have coordinates (0,6), (1, 8.5), (2, 9.4) and (3, 10)




**a** Use the trapezoidal rule to find an approximation for the area of the window.

A mathematician suggests to the surveyor that he might get a more accurate result if he fits a cubic curve through the four points.

**b** Find the area of the window using this method.

**c** Find the difference between the two values as a percentage of the area found by the second method.



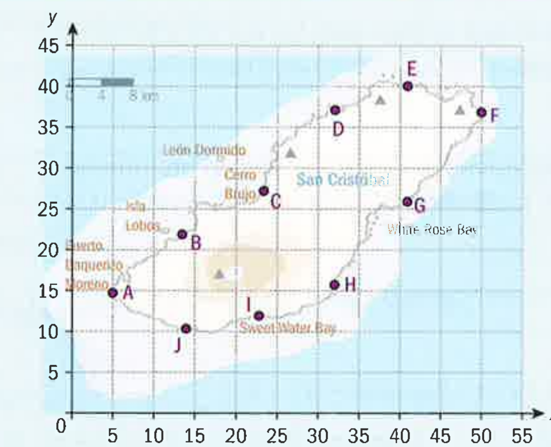


## Developing inquiry skills

In the opening problem for this chapter you looked at how to estimate the area of San Cristobal.

Use the trapezoidal rule to work out an estimate of the area using the points given. Compare your answer with the officially given figure of 558 km$^2$.

| A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|
| (5, 15) | (14, 22) | (23, 27.5) | (32, 37) | (41, 40) | (50, 37) | (41, 26) | (32, 16) | (23, 12.5) | (14, 11) |




## 11.2 Indefinite integrals and techniques of integration

### Investigation 4

With your GDC in radian mode, on the same set of axes, sketch the graphs of

**i** $y = \sin x$ **ii** $y = \sin x + 1$ **iii** $y = \sin x - 1$ **iv** $y = \sin x + 2$.

1. How can you describe their relative positions?
2. Write down the gradient of each of these curves when $x = \frac{\pi}{2}$.
3. Use your GDC to find the gradient of each of these curves when $x = 0$.
4. Differentiate each of these functions to find an expression for the gradient at $x$.
5. Write down another curve for which the gradient at $x$ is the same as the gradient of any of these curves.
6. **Factual** What is the formula of **any** curve whose gradient is the same as the gradient of the above curves?
7. **Conceptual** What does finding the indefinite integral lead to?

All these curves make up a **family of functions** with the same derivative.

Calculus

The family of curves whose derivative is $\cos x$ can be written using the following notation:

$\int \cos x \, dx = \sin x + c$

$\int \cos x \, dx$ is the **indefinite integral** of $\cos x$.

We would say, "the integral of $\cos x$ is $\sin x + c$".

**Reflect** What is an indefinite integral?

**TOK**

Is imagination more important than knowledge?

The reason for the similar notation to that used for the area between a curve and the $x$-axis will become apparent later in this section.

## Anti-derivatives

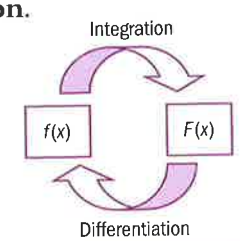
A function $F$ is called an antiderivative or integral of $f$ on an interval $[a, b]$ if $F'(x) = f(x)$ for all $x \in [a, b]$. That is,

$\int f(x)dx = F(x) + c$

The process of finding an antiderivative is called **integration**.

**HINT**

$[a, b]$ is equivalent to writing $a \leq x \leq b$.




For example,

- $F(x) = 3x$ is an integral of $f(x) = 3$ because $F'(x) = 3$.
- $F(x) = x^2 + 1$ is an integral of $f(x) = 2x$ because $F'(x) = 2x$.
- $F(x) = \cos x$ is an integral of $f(x) = -\sin x$ because $F'(x) = -\sin x$.

### Investigation 5

1 Write down the antiderivative of

a $f(x) = x$ b $f(x) = x^2$ c $f(x) = x^3$

2 Conjecture an expression for the antiderivative of $f(x) = x^n$ where $n \in \mathbb{R}$.

3 Verify your conjecture by first integrating then differentiating the following expressions for

a $f(x) = x^5$ b $f(x) = \sqrt{x}$ c $f(x) = \frac{1}{x^2} + x$

4 Conjecture an expression for the antiderivative of $f(x) = ax^n$ where $n, a \in \mathbb{R}$.

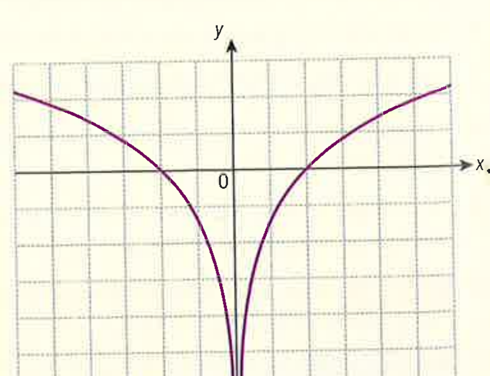
5 Write down the antiderivative of $f(x) = 12x^5 + 3x^2 + 4x + 7$, and verify your answer by differentiating.

6 a Use your conjectured formula to try to find the antiderivative of $f(x) = x^{-1}$.

b Explain why you cannot obtain a solution.

c From your knowledge of Chapter 10, write down the antiderivative of $f(x) = x^{-1}$, $x > 0$.

d The graph of $y = \ln|x|$ is shown here. Given your conjecture in part **c** explain why the antiderivative of $f(x) = x^{-1}$, $x \neq 0$ will be $\ln|x|$.




7 Write down the antiderivative of

a $f(x) = \sin x$ b $f(x) = \cos x$ c $f(x) = \frac{1}{\cos^2 x}$ d $f(x) = e^x$





8 a Use the chain rule to differentiate

i $f(x) = \sin 4x$ ii $f(x) = \cos 5x$ iii $f(x) = e^{7x}$

b Hence, find the antiderivatives of

i $f(x) = \cos 4x$ ii $f(x) = \sin 5x$ iii $f(x) = e^{7x}$

c Conjecture an expression for the antiderivatives of

i $f(x) = \cos ax$ ii $f(x) = \sin ax$ iii $f(x) = e^{ax}$

9 **Factual** What are the antiderivative rules?

10 **Conceptual** How can you evaluate antiderivatives given what you already know about derivatives?

$\int f(x)dx$ is an indefinite integral and the process of finding the **indefinite integral** is called **integration**.

If $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)dx = F(x) + c$ where $c$ is an arbitrary constant.

$F(x)$ is an antiderivative of $f(x)$.

Rules for integration can be derived from those for differentiation:

- The sum/difference of functions: $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
- Multiplying a function by a constant: $\int af(x)dx = a\int f(x)dx$
- Power rule: $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$
- $\int \sin x = -\cos x + c$
- $\int \cos x = \sin x + c$
- $\int \frac{1}{x} dx = \ln|x| + c$
- $\int e^x dx = e^x + c$
- $\int \frac{1}{\cos^2 x} dx = \tan x + c$

In addition:

- $\int \sin ax \, dx = -\frac{1}{a}\cos ax + c$
- $\int \cos ax \, dx = \frac{1}{a}\sin ax + c$
- $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

**International-mindedness**

The successful calculation of the volume of a pyramidal frustum was conducted by ancient Egyptians and seen in the Moscow mathematical papyrus.

Calculus

## Example 8

Find each of the following.

**a** $\int (x^2+3)\mathrm{d}x$  **b** $\int \frac{x^2+3}{x^2}\mathrm{d}x$  **c** $\int \cos 3t\,\mathrm{d}t$

| | |
|---|---|
| **a** $\int (x^2+3)\mathrm{d}x = \frac{1}{3}x^3+3x+c$ | Apply the power rule of integration. |
| **b** $\int \frac{x^2+3}{x^2}\mathrm{d}x = \int (1+3x^{-2})\mathrm{d}x$ | Separate the terms and write as a negative exponent. |
| $= x - \frac{3}{x} + c$ | Apply the power rule. |
| **c** $\int \cos 3t\,\mathrm{d}t = \frac{\sin 3t}{3}+c$ | As with differentiation, any letter can be used for the variable. |

## Example 9

The curve $y = f(x)$ passes through the point $(1, 3)$. The gradient function of the curve is given by $f'(x) = 2 - \frac{x}{3}$. Find the equation of the curve.

| | |
|---|---|
| $f(x) = \int \left(2-\frac{x}{3}\right)\mathrm{d}x = 2x - \frac{1}{3}\times\frac{x^2}{2}+c$ <br> $= 2x - \frac{x^2}{6}+c$ | Apply the power rule to find an antiderivative of $f'(x)$. |
| $f(1) = 2\times 1 - \frac{1^2}{6}+c$ <br> $3 = 2 - \frac{1}{6}+c$ <br> $c = \frac{7}{6}$ <br> Therefore, $f(x) = 2x - \frac{x^2}{6}+\frac{7}{6}$ | If the curve passes through the point $(1, 3)$ then $f(1) = 3$. |

### Exercise 11E

**1** Find the following indefinite integrals.

**a** $\int 10\,\mathrm{d}x$  **b** $\int 0.6x^2\mathrm{d}x$

**c** $\int x^5\mathrm{d}x$  **d** $\int (7-2x)\mathrm{d}x$

**e** $\int (1+2x)\mathrm{d}x$  **f** $\int \left(5+x-\frac{1}{3}x^2\right)\mathrm{d}x$

**g** $\int \left(-x+\frac{3x^2}{4}+0.5\right)\mathrm{d}x$  **h** $\int \left(1-x+\frac{x^3}{2}\right)\mathrm{d}x$

**i** $\int \left(x^2-\frac{1}{2}x+4\right)\mathrm{d}x$





**2** For $f(x) = x^2 - \frac{x}{3}+4$, find

**a** $f'(x)$  **b** $\int f(x)\mathrm{d}x$

**3** **a** Find $\int (t-3t^2)\mathrm{d}t$.

**b** Find $\int (4t^3-3t+1)\mathrm{d}t$.

**4** **a** Sketch on the same axes the curves $y = f(x)$ given that $f'(x) = 4x+8$ and:

**i** $f(0) = 1$  **ii** $f(0) = 4$

**b** Write down the vertex for each of the curves.

**5** Find

**a** $\int 2\sin x\,\mathrm{d}x$

**b** $\int 4\cos x\,\mathrm{d}x$

**c** $\int \frac{\sin x}{3}\mathrm{d}x$

**d** $\int \frac{3}{\cos^2 x}\mathrm{d}x$.

**6** Find

**a** $\int 2\mathrm{e}^x\mathrm{d}x$

**b** $\int \left(3\mathrm{e}^x+\frac{1}{x}\right)\mathrm{d}x$

**c** $\int \left(\frac{4}{x}-3\mathrm{e}^x\right)\mathrm{d}x$.

**7** Find

**a** $\int 2\mathrm{e}^{3x}\mathrm{d}x$

**b** $\int 2\sin 5t\,\mathrm{d}t$

**c** $\int (4-2\mathrm{e}^{-3x})\mathrm{d}x$

**d** $\int (\cos 2t - \sin 3t)\mathrm{d}t$.

**8** Find the following indefinite integrals.

**a** $\int \sqrt{5t}\,\mathrm{d}t$

**b** $\int \pi x^{-\frac{2}{3}}\mathrm{d}x$

**c** $\int (a^3-3a^2+2)\mathrm{d}a$

**d** $\int \sqrt{\frac{1}{x}}\mathrm{d}x$

**e** $\int \frac{x^2+2x+3}{x}\mathrm{d}x$

**f** $\int (\mathrm{e}^{2x}+3\sin 4x)\mathrm{d}x$

**g** $\int \frac{\sqrt{x}-\cos 4x}{3}\mathrm{d}x$

**9** **a** If $y = \frac{1}{\sin(x)}$

**i** find $\frac{\mathrm{d}y}{\mathrm{d}x}$

**ii** hence, find $\int \frac{\cos x}{\sin^2 x}\mathrm{d}x$.

**b** If $y = \sqrt{4x-1}$

**i** find $\frac{\mathrm{d}y}{\mathrm{d}x}$

**ii** hence, find $\int \frac{1}{\sqrt{4x-1}}\mathrm{d}x$.

**10** It is given that $\frac{\mathrm{d}y}{\mathrm{d}x} = x+\frac{x^2}{5}+2$ and that $y = 3$ when $x = 4$. Find an expression for $y$ in terms of $x$.

**11** It is given that $f'(x) = 3 - x$ and $f(3) = 2$. Find $f(x)$.

**12** Given $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x + 2\sin 2x$ and $y = 5$ when $x = \frac{\pi}{4}$, find an expression for $y$ in terms of $x$.

**13** Given $\frac{\mathrm{d}y}{\mathrm{d}x} = 8\mathrm{e}^{4x}+3x^2+1$ and $y = 4$ when $x = 0$, find an expression for $y$ in terms of $x$.

## Chain rule and reverse chain rule

The **chain rule** is a **rule** for differentiating compositions of functions.

If $y = f(g(x))$ then $\frac{dy}{dx} = f'(g(x)) \times g'(x)$.

We can reverse this rule to extend the range of functions we can integrate.

$\int g'(x) \times f'(g(x))dx = f(g(x)) + c$

To use this method, we need to look for a composite function multiplied by the derivative of the interior (first) function.

Consider $\int 2x(x^2+5)^4 dx$. We see that this is of the required form.

When you have had plenty of practice at these, they can be quickly solved but initially it might be helpful to use a similar method to when you learning the chain rule, and call the interior function $u$.

The integral becomes $\int 2x(u)^4 dx$, but $dx$ needs to become $du$ to match the variable we wish to use for the integration.

We know that $u = x^2 + 5$ so $\frac{du}{dx} = 2x$.

A trick is to say that this can be split to give the equation $du = 2x\,dx$ or $dx = \frac{du}{2x}$.

Although the derivative notation is not really a fraction, it is often used like one.

The integral now becomes $\int 2xu^4 \frac{du}{2x} = \int u^4 du$.

This can now be integrated in the usual way:

$\int u^4 du = \frac{1}{5}u^5 + c$

Finally, substitute back the expression for $u$:

$\int 2x(x^2+5)^4 dx = \frac{1}{5}(x^2+5)^5 + c$

The reverse chain rule methods can be used whenever the integral consists of the product of a composite function and a multiple of the derivative of the interior function. In these cases, this function will cancel when the substitution is made.

- The interior function is replaced by the variable $u$.
- $\frac{du}{dx}$ is evaluated and $dx$ is replaced by $\frac{du}{\left(\frac{du}{dx}\right)}$
- Any possible cancelling should then be carried out.
- The remaining function is integrated.
- The expression for $x$ is substituted back in for $u$.





### Example 10

Find the indefinite integral $\int x\sqrt{3-x^2}\,dx$.

| | |
|---|---|
| $u = 3 - x^2$ | Define $u$. |
| $\frac{du}{dx} = -2x$<br>$\Rightarrow dx = \frac{du}{-2x}$ | Evaluate $\frac{du}{dx}$. |
| $\int xu^{\frac{1}{2}} \frac{du}{-2x}$ | Rearrange and substitute into the integral. |
| $= -\frac{1}{2}\int u^{\frac{1}{2}} du$ | Cancel and move any constant factors outside the integral. |
| $= -\frac{1}{3}u^{\frac{3}{2}} + c$ | Integrate the remaining function. |
| $= -\frac{1}{3}(3-x^2)^{\frac{3}{2}} + c$ | Substitute back into the original integration. |

### Example 11

Find the equation of the curve passing through (2, 1) with $\frac{dy}{dx} = e^{2x-2}$.

| | |
|---|---|
| $y = \int(e^{2x-2})dx$ | $e^{2x-2}$ is a composite function. Although it does not look like a product, the derivative of the interior function is 2 rather than a function of $x$, hence the method can be used. |
| $u = 2x - 2$<br>$\frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2}du$ | Define $u$. |
| $\int e^u \frac{1}{2}du = \frac{1}{2}\int e^u du$<br>$= \frac{1}{2}e^u + c$<br>$\Rightarrow y = \frac{1}{2}e^{2x-2} + c$ | Substitute $u$ and $du$. |

Continued on next page

Calculus

| | |
|---|---|
| $1=\frac{1}{2}e^2+c$ | |
| $\Rightarrow c=1-\frac{1}{2}e^2\approx -2.69$ | Substitute (2, 1) and solve for $c$. |
| $\Rightarrow y=\frac{1}{2}e^{2x-2}-2.69$ | Write down the equation of the curve passing through (2, 1). |

## Exercise 11F

**1** Using the substitution $u = 4x - 1$, find the following indefinite integrals.

**a** $\int(4x-1)^2\,dx$ **b** $\int\frac{1}{(4x-1)^2}dx$

**c** $\int\frac{1}{\sqrt{4x-1}}dx$ **d** $\int\frac{3}{4x-1}dx$

**2** Find

**a** $\int 4(2x+3)dx$ **b** $\int 2xe^{1+x^2}dx$

**c** $\int 4\sec^2(4+2x)dx$ **d** $\int\frac{1}{2x+3}dx$

**e** $\int 4x\cos(x^2+2)dx$ **f** $\int\cos x\sin^2 x\,dx.$

**3** Given that the gradient function of a curve is $\frac{dy}{dx}=x+\frac{1}{x+2}$ and the curve passes through the point (0, 3), find the equation of the curve.

**4** Use a suitable substitution or the reverse chain rule to find the following integrals.

**a** $\int\cos(3x)e^{\sin 3x}dx$

**b** $\int(x-3)(x^2-6x+4)^5\,dx$

**5** **a** Find $\int\tan x\,dx$ using the identity $\tan x=\frac{\sin x}{\cos x}$.

**b** Hence, solve $\frac{d^2y}{dx^2}=\frac{1}{\cos^2 x}$ given that $\frac{dy}{dx}=3$ when $x=\frac{\pi}{4}$ and $y = 4$ when $x = 0$.

**6** **a** Find an expression in the form $\frac{y^2}{a}+\frac{x^2}{b}=1$ for the curve (an ellipse) with gradient function $\frac{dy}{dx}=-\frac{3}{2}\frac{x}{\sqrt{4-x^2}}$, given that when $x = 2$, $y = 0$.

**b** Find the $y$-intercepts and hence, sketch the curve.

**c** Find the area contained within the ellipse.

**d** If the area is equal to $c\pi$ where $c\in\mathbb{Z}$, find the value of $c$.

**7** In economics, marginal cost is the derivative of the cost function. A firm calculates that the marginal cost of producing its product is $\frac{dC}{dn}=\frac{4n}{n^2+1}$ where $C$ is the total cost of production (in €1000) and $n$ the number of items produced. Given that when no items are produced the cost to the firm is €1500, find:

**a** an expression for $C$

**b** the cost of producing seven items.

**c** Sketch the curve and comment on its shape as $n$ increases.

**8** The rate of spread of a rumour is modelled by the equation $R'(t)=\frac{18e^{-0.5t}}{5(1+12e^{-0.5t})^2}$

where $R(t)$ is the proportion of the population that knows the rumour at time $t$ hours.





**a** Given $R(0)=\frac{1}{5}$, find an expression for $R(t)$.

**b** Estimate how long it would take for 70% of the population to hear the rumour.

**c** Find the percentage of the population who will eventually hear the rumour.

**9** Water is flowing from a tank at a rate modelled by the function $R'(t)=5\sin\left(\frac{t}{120}\right)$.

Water flows into the tank at a rate of $S'(t)=\frac{10t}{1+2t^2}$.

Both $R'$ and $S'$ have units m³ per hour, and $t$ is measured in hours for $0\le t\le 5$.

**a** Find the intervals of time in $0\le t\le 5$ during which the amount of water in the tank is increasing.

At $t = 0$, there is 25 m³ of water in the tank.

**b** Find an expression for $T$, the amount of water in the tank at time $t$.

**c** Find the value of the maximum amount of water in the tank and the time, $t$, at which this occurs.

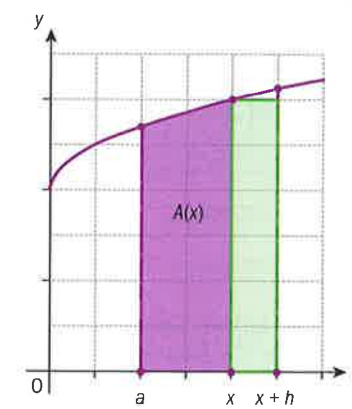
## Definite integrals

In this section you will explore the link between the definite integral used to find areas in Section 11.1 and the antiderivative (indefinite integral) from this section.

The following illustrates why the antiderivative gives the area under a curve.

Let $A(x)$ be the area under a curve $y=f(t)$ in the interval $[a,x]$.

For a small value of $h$, $A(x+h)$ will be approximately $A(x)$ plus the area of the rectangle below the curve between $t = x$ and $t = x + h$, as shown.




Therefore,

$A(x+h)\approx A(x)+hf(x)$.

As seen in Section 11.1, this result will become increasingly accurate as $h\to 0$.

The expression can be rearranged to give

$$f(x)\approx\frac{A(x+h)-A(x)}{h}$$

and as $h\to 0$, we can write

$$f(x)=\lim_{h\to 0}\left(\frac{A(x+h)-A(x)}{h}\right).$$

From Chapter 10, you will recall that this is the equation for the derivative. Hence, we obtain

**TOK**

We are trying to find a method to evaluate the area under a curve.

"The main reason knowledge is produced is to solve problems." To what extent do you agree with this statement?

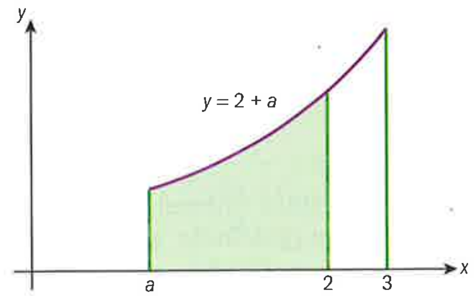
Calculus

$f(x)=\frac{\mathrm{d}A}{\mathrm{d}x} \Rightarrow A=\int f(x)\,\mathrm{d}x = F(x)+c$

The $+c$ term of the indefinite integral just reflects the arbitrary choice for the limits of the area. The reason it is not needed when calculating a definite integral is indicated below.

Consider the area under the curve $y = x^2 + 1$ between $x = 2$ and $x = 3$, which can be written as $\int_2^3 x^2+1\,\mathrm{d}x$.

We know the area from an arbitrary point $a$ to any value $x$ is given by $F(x)=\int x^2+1\,\mathrm{d}x=\frac{1}{3}x^3+x+c$




From the diagram, it can be seen that the shaded area is $F(2)$ and hence, the required area is $F(3)-F(2)=\left(\frac{1}{3}\times 27+3+c\right)-\left(\frac{1}{3}\times 8+2+c\right)$

$=12+c-\frac{14}{3}-c=\frac{22}{3}$.

Hence, $\int_2^3 x^2+1\,\mathrm{d}x=\frac{22}{3}$.

Thus, for a positive continuous function, the area between the curve $y=f(x)$, the $x$-axis and the lines $x=a$ and $x=b$ is given by

$\int_a^b f(x)\,dx = F(b)-F(a)$ where $F(x)$ is the antiderivative of $f(x)$.

This is **the fundamental theorem of calculus** and gives us a new way of calculating areas under a curve without using approximation methods or a GDC.

**International-mindedness**

The **fundamental theorem of calculus** shows the relationship between the derivative and the integral and was developed in the 17th century by Gottfried Wilhelm Leibniz and Isaac Newton.

## Example 12

Find the area of the region bounded by the curve $y=3x^2+\frac{2}{\sqrt{x}}$, the $x$-axis and the lines $x=1$ and $x=4$.

| | |
|---|---|
| $\int_1^4\left(3x^2+2x^{-\frac{1}{2}}\right)\mathrm{d}x$ | This is the definite integral required. Normally this would be found using your GDC but here it will be calculated using the antiderivative. |
| $=\left[x^3+4x^{\frac{1}{2}}+c\right]_1^4$ | Note the new notation. The square brackets with the limits are telling us to substitute the two values and subtract. |
| $=(64+8+c)-(1+4+c)$ | From the fundamental theorem of calculus, $\int_a^b f(x)\,\mathrm{d}x=F(b)-F(a)$ where $F(x)$ is the antiderivative of $f(x)$. |
| $=72+c-5-c=67$ | The $+c$ term will always cancel and so is omitted when working out a definite integral. |





## Example 13

Given $y=\cos 2x$:

**a** Find the area between the curve, the $x$-axis and the lines $x=0$ and $x=\frac{\pi}{4}$.

**b** Show that the area between the curve, the $x$-axis and the lines $x=0$ and $x=a$, where $\frac{\pi}{4}<a<\frac{3\pi}{4}$, is equal to $1-\frac{1}{2}\sin 2a$.

| | |
|---|---|
| **a** $\int_0^{\frac{\pi}{4}}\cos 2x\,\mathrm{d}x=\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$ | Because the curve is positive for the interval $x=0$ to $x=\frac{\pi}{4}$, the area is equal to $\int_0^{\frac{\pi}{4}}\cos 2x\,\mathrm{d}x$. |
| $=\frac{1}{2}\left(\sin\frac{\pi}{2}-\sin 0\right)$ | Note that this time the $+c$ terms are not included. |
| $=\frac{1}{2}$ | The can also be checked using your GDC. |
| **b** $\int_{\frac{\pi}{4}}^{a}\cos 2x\,\mathrm{d}x=\left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{a}=\frac{1}{2}\sin 2a-\frac{1}{2}\sin\frac{\pi}{2}$ | $y=\cos 2x$ is negative for $\frac{\pi}{4}<x<\frac{3\pi}{4}$. |
| Hence, area = $\frac{1}{2}+\left(-\frac{1}{2}\sin 2a+\frac{1}{2}\sin\frac{\pi}{2}\right)=1-\frac{1}{2}\sin 2a$ | Hence, the area between this part of the curve and the $x$-axis will need to be worked out separately and is equal to $-\int_{\frac{\pi}{4}}^{a}\cos 2x\,\mathrm{d}x$ |

Calculus

## Exercise 11G

**1** Find the **exact** area of the region bounded by the curve, the $x$-axis and the lines given.

**a** $y = e^{2x} + 1,\ x = 0,\ x = 1$

**b** $y = \frac{2}{x},\ x = 2,\ x = 4$

**c** $y = x\sqrt{x^2 - 5},\ x = 3,\ x = 4$

**2** Show that $\int_1^4 \frac{x}{x^2+2} = \frac{1}{2}\ln 6$.

**3** Given that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ show that

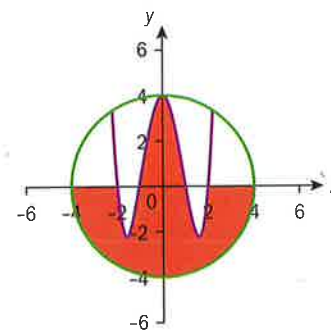
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x \, dx = \frac{2-\sqrt{3}}{4}.$$

**4** The area under a power against time curve, between the times $t_1$ and $t_2$, will give the energy used during this time.

Find an expression for the energy used between $t = 0$ and $t = t_1$ given the power at time $t$ is given by the equation

$$P = \frac{4t}{t^2+2}$$

**5** The diagram below shows a company logo created by the curve $y = x^4 - 5x^2 + 4$ inside a circle of radius 4 cm.



Find the area of the logo that is coloured red.

# 11.3 Applications of integration

In this section, we will look at some of the applications of definite integration, such as areas between curves, volumes of solids, displacement and total distance travelled.

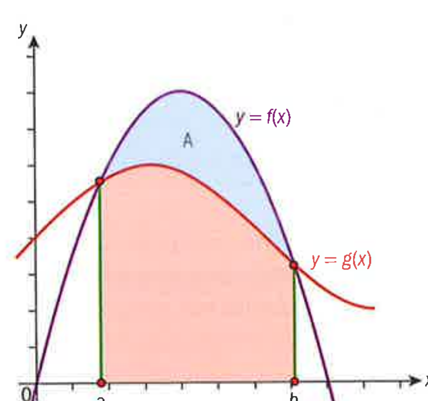
The O2 Arena in London is one of the largest buildings by volume in the world. How might you calculate the volume contained, given it has a diameter of 365 m and a height of 50 m?

## Areas between curves

From the diagram, it is clear that the area $A$ between the two curves can be calculated by $A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ where $a$ and $b$ are the two intersection points of the curves.

However, it is often simpler to find the area by combining the two integrals, hence

$$A = \int_a^b (f(x) - g(x))\,dx$$

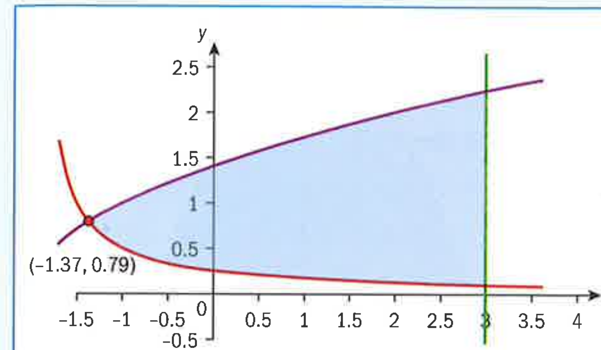







### Example 14

Sketch the region bounded by $y = \sqrt{x+2}$, $y = \frac{1}{2x+4}$ and $x = 3$. Find the area of the region.




$$A = \int_{-1.37}^{3}\left[(\sqrt{x+2}) - \left(\frac{1}{2x+4}\right)\right]dx$$

$A \approx 6.08$ (3 sf)

Sketch both graphs using your GDC or dynamic geometry software.

Find the intersection point of the two curves and identify the area to be calculated.

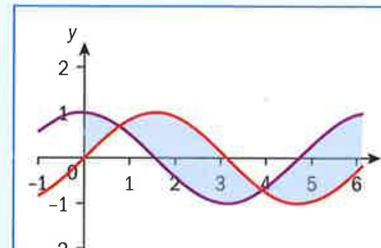
Remember that, even if you are using technology to find the value of the integral, you should always write down the definite integral representing the enclosed region.

The expression $A = \int_a^b (f(x) - g(x))\,dx$ is valid even when either of the curves is below the $x$-axis. However, if $f(x)$ is below $g(x)$, the integral (though not the area) will be negative.

In these cases, the expression $A = \int_a^b |f(x) - g(x)|\,dx$ should be used.

### Example 15

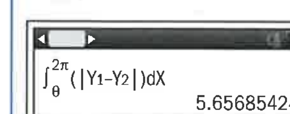
Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = 2\pi$.



$$A = \int_0^{2\pi} |\sin x - \cos x|\,dx$$

$A \approx 5.66$ (3 sf)

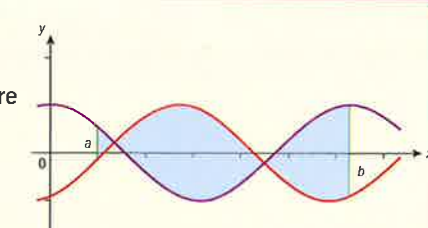
Sketch the graph and shade the enclosed area.




As $\cos x > \sin x$ in some intervals and $\sin x > \cos x$ in others, we need to use the absolute value.

The area $A$ of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$, where $f$ and $g$ are continuous, is given by

$$A = \int_a^b |f(x) - g(x)|\,dx$$



Calculus

## Finding the area between a curve and the $y$-axis

If the equation is given in the form $x = f(y)$ then the method is very similar to finding the area between a curve and the $x$-axis.

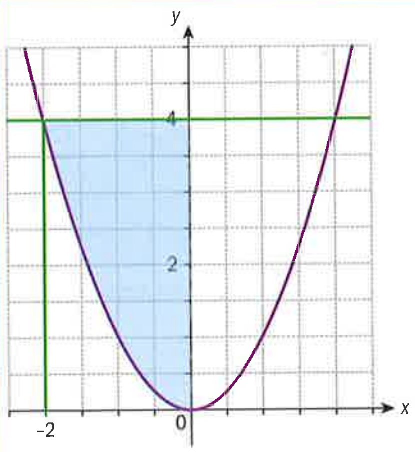
If the equation is given as $y = f(x)$ then it will need to be rearranged to make $y$ the subject.

> The area $A$ of the region bounded by the curve $x = h(y)$, and the lines $y = a$ and $y = b$, where $h$ is continuous for all $y$ in $[a, b]$, is given by
>
> $$A = \int_a^b |h(y)|\,dy$$

Note that, if we are trying the find the integral, we do not need the absolute value, but for areas we must use the absolute value to ensure we have positive values.

### Example 16

Find the area of the region bounded by the curve $y = x^2$ and the $y$-axis, $x \in [-2, 0]$.




Sketch the graph and shade the required area.

$x = 0 \Rightarrow y = 0$ and $x = -2 \Rightarrow y = 4$

Identify the lower and upper boundaries of $y$.

$y = x^2 \Rightarrow x = -\sqrt{y}$

Write $x$ as a function of $y$ for the given region.

$A = \int_0^4 \left|-\sqrt{y}\right| dy$

Write the area $A$ as a definite integral.

$A \approx 5.33$ (3 sf)

You can use your GDC or integration rules to find the value of the integral.

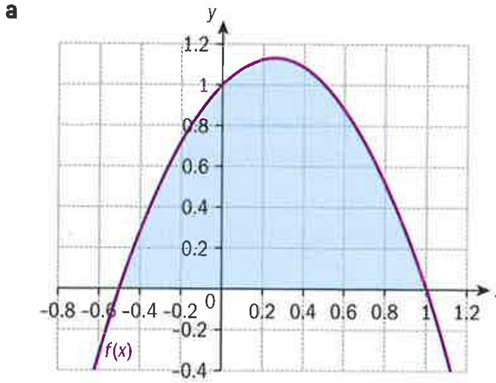





### Exercise 11H

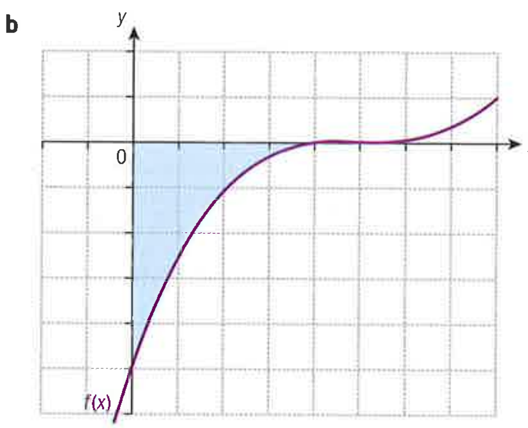
1 Find the area of the shaded regions.

**a**



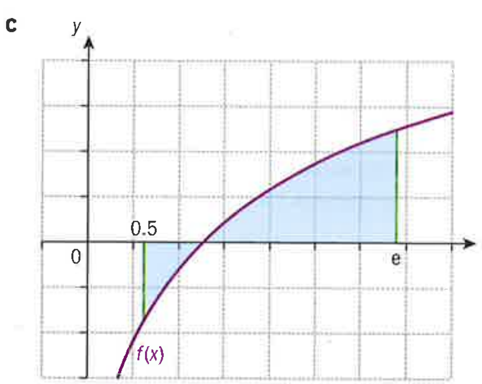

$f(x) = -2x^2 + x - 1$

**b**



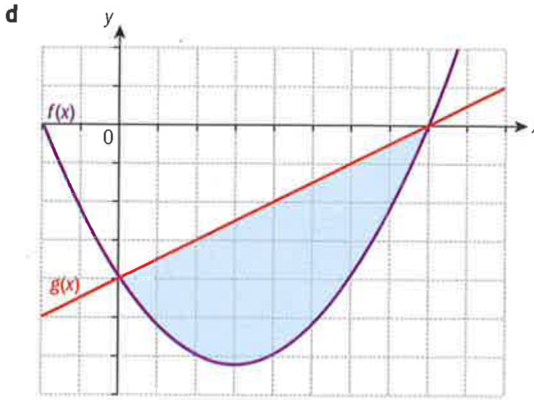

$f(x) = (x - 3)^3$

**c**




$f(x) = \ln x$

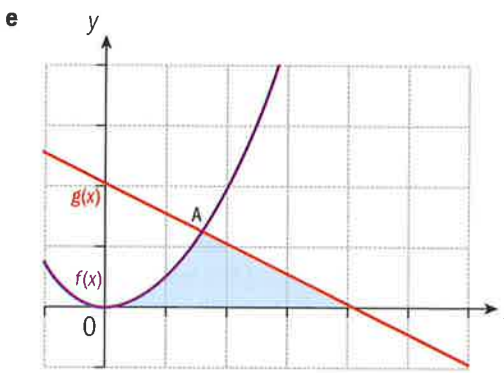
**d**




$f(x) = x^2 - 3x - 4$

$g(x) = x - 4$

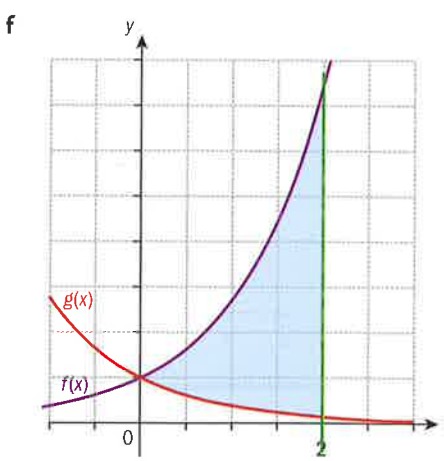
**e**




$f(x) = x^2$

$g(x) = 4 - x$

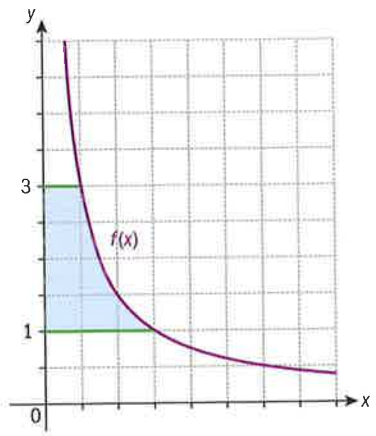
**f**




$f(x) = e^x$

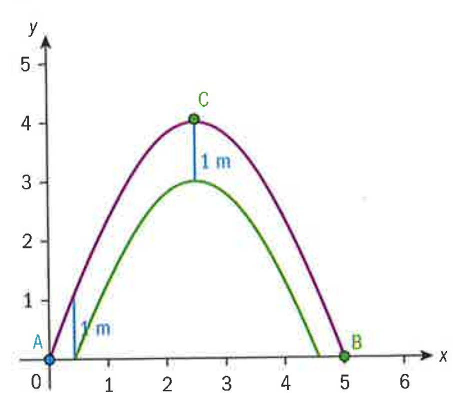
$g(x) = e^{-x}$

**2** Find the area of the shaded region bounded by $y = \frac{3}{x}$, the $y$-axis and the lines $y = 1$ and $y = 3$.




**3** Find the area of the region bounded by $x = \sin y$, the $y$-axis and the lines $y = 0$ and $y = 2\pi$.

**4** A building is in the shape of a prism with cross-section as shown in the diagram below. The building has a concrete roof that is 1 m thick when measured vertically. The outside of the cross-section of the roof forms a curve of the form $y = a\sin(bx)$ when placed on a coordinate system as shown. This curve touches the ground at the points A(0, 0) and C(5, 0) and the highest point of the roof is at $(2.5, 4)$.




**a** Find the values of $a$ and $b$.

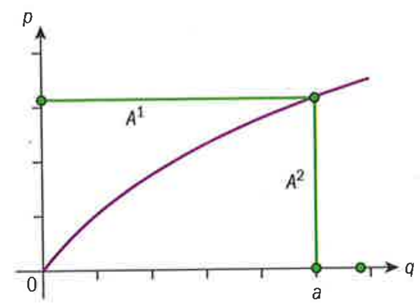
**b** Write down the equation for the inner surface of the roof.

The building is 15 m long.

**c** Find the volume of concrete in the roof.

**5** In economics, a supply function relates the quantity $(q)$ that a manufacturer is willing to supply for a particular price per unit $(p)$.

The diagram below shows the supply curve $p = \frac{1}{2}\ln(q + 1)$.




For a particular quantity, the area of the region above the curve $(A_1)$ gives the producer surplus for selling that quantity and the area below the curve $(A_2)$ gives the revenue needed by the producer. The area of the whole rectangle is the actual producer revenue. A quantity $q = a$ is produced.

**a** Find an expression in terms of $a$ for:

**i** the producer surplus

**ii** the revenue needed.

**b** If the surplus is at least 50% more than the revenue needed by the producer, find the maximum integer value of $a$ for this to occur.

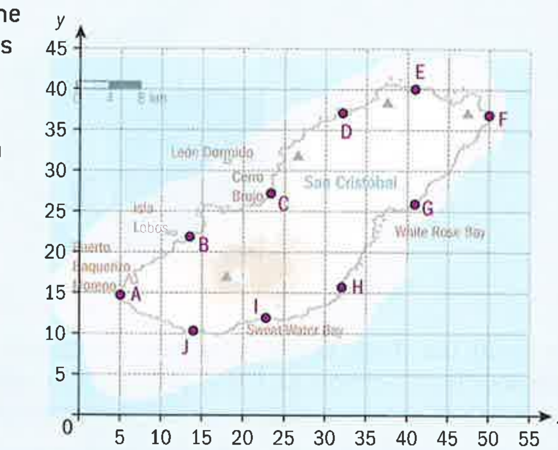




## Developing inquiry skills

Look again at the opening problem. To work out the area, we will now fit two curves through the points found previously: one curve through A, B, C, D, E and F and one through A, J, I, H, G and F.

Use cubic regression to find an equation for each curve and hence, an estimate for the area of San Cristobal.

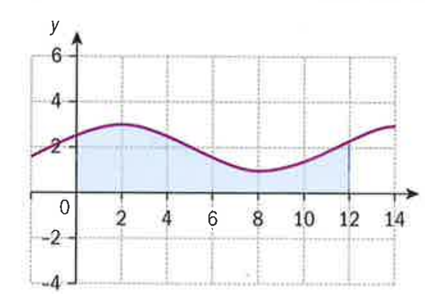



## Solids of revolution

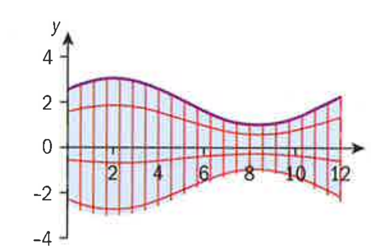
How do we calculate the volumes of irregular shapes?

We have formulae to calculate the volumes of regular shapes, for example a cylinder or a pyramid. We can extend our knowledge of volumes of regular shapes and integration to volumes of irregular shapes.

Consider $f(x) = \cos(0.5x - 1) + 2$ in the interval $[0, 12]$.




When you rotate this curve through $2\pi$ radians around the $x$-axis, you would get a 3D shape, like a vase.




**TOK**

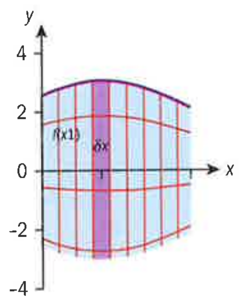
Consider $f(x) = \frac{1}{x}$, $1 \le x \le \infty$.

An infinite area sweeps out a finite volume.

How does this compare to our intuition?

Calculus

On a closer look at the shape, you can see that it has circular cross-sections when sliced perpendicular to the $x$-axis, centred on the $x$-axis, each with a radius of $y = f(x)$.




If we divide the shape into $n$ cylindrical discs, each of width $\delta x$ (remember from Section 11.1 that $\delta x$ is used to indicate a small change in the value of $x$), we can approximate the total volume.

The cross-sectional area of the disk with $x$-coordinate $x_i$ is $A_i = \pi[f(x_i)]^2$ using the formula for the area of a circle $A = \pi r^2$.

Using the formula for the volume of a cylinder, each disc has a volume of approximately

$V_i = \pi[f(x_i)]^2 \delta x$.

Thus, the volume formed by the revolution of the continuous function $y = f(x)$ through $2\pi$ radians around the $x$-axis is approximately

$$V \approx \sum_{i=1}^{n} \pi\left[f(x_i)\right]^2 \delta x.$$

As $\delta x \to 0$, the approximation will become more accurate.

Hence, $V = \lim\limits_{\delta x \to 0}\left(\sum_{i=1}^{n} \pi\left[f(x_i)\right]^2 \delta x\right)$

Recall from Section 11.1 that we can write the limit with the integral sign and that $\delta x$ becomes $dx$.

Hence, the formula for a volume formed by rotating $y = f(x)$ through $2\pi$ radians about the $x$-axis between $x = a$ and $x = b$ is

$V = \int_a^b \pi\left(f(x)\right)^2 dx$.

We can now use the formula to find the volume of the solid formed by rotating $f(x) = \cos(0.5x - 1) + 2$ through $2\pi$ radians about the $x$-axis between $x = 0$ and $x = 12$:

$$V = \int_0^{12} \pi\left[\cos(0.5x - 1) + 2\right]^2 dx \approx 167.27 \text{ unit}^3 \text{ (2 dp)}$$

To find the volume formed when rotating about the $y$-axis, we simply interchange $x$ and $y$.

**EXAM HINT**

This section has given an indication of the origin of the formula rather than a proof of the formula. Knowledge of the derivation is not required for the examinations.

**TOK**

You have been using radians to measure angles instead of degrees in recent chapters.

Why has this change been necessary? What are its advantages?

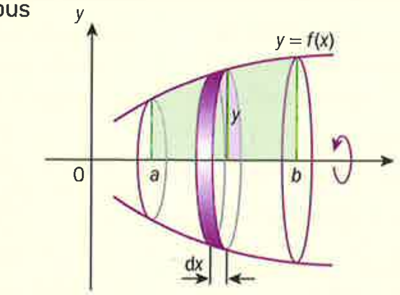




**Volumes of revolution**

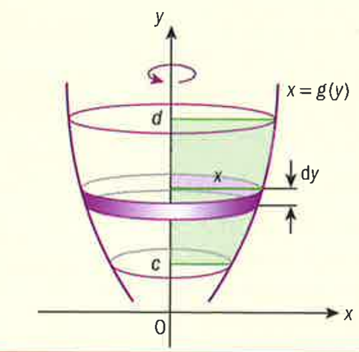
- The volume formed when a continuous function $y = f(x)$ is rotated through $2\pi$ radians about the $x$-axis is

  $V = \int_a^b \pi\left[f(x)\right]^2 dx$ or $V = \pi\int_a^b \left[y\right]^2 dx$




- The volume of the solid formed by revolution of $y = f(x)$ through $2\pi$ radians about the $y$-axis in the interval $y = c$ to $y = d$ is

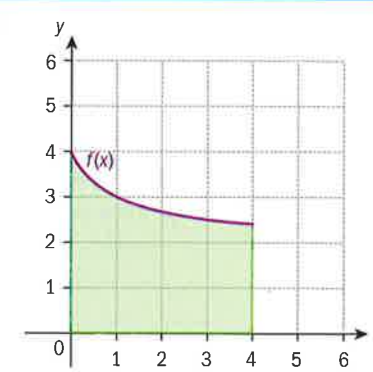
  $V = \int_c^d \pi x^2 dy$




## Example 17

Let $f(x) = \dfrac{2x+4}{x+1}$, $0 \le x \le 4$. Find the volume of revolution formed when the curve $f(x)$ is rotated through $2\pi$ radians about

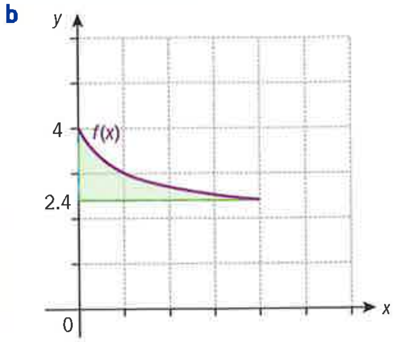
**a** the $x$-axis **b** the $y$-axis.

| | |
|---|---|
| **a**   | Sketch the graph of the curve. |
| $V = \int_0^4 \pi\left[\dfrac{2x+4}{x+1}\right]^2 dx$<br>$V \approx 101$ (3 sf) | Use the formula and evaluate the definite integral on the GDC. |

Continued on next page

Calculus

**b**




Sketch the graph of the curve.

$x = 0 \Rightarrow y = 4$ and $x = 4 \Rightarrow y = 2.4$

Find the boundaries for $y$.

$y = \dfrac{2x+4}{x+1} \Rightarrow x = \dfrac{-y+4}{y-2}$

The formula requires $x$ as a function of $y$, so we need to rearrange the formula.

$V = \int_{2.4}^{4} \pi \left[\dfrac{-y+4}{y-2}\right]^2 dy$

Use the formula and evaluate on a GDC.

$V \approx 9.93$ (3 sf)

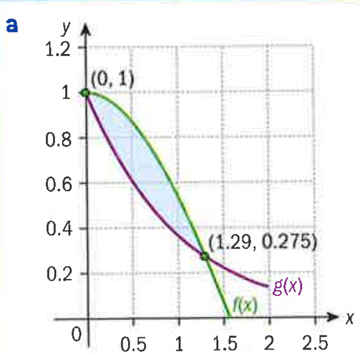
## Example 18

**a** Sketch $f(x) = \cos x$ and $g(x) = e^{-x}$ on the same axes for $0 \le x \le 1.5$.

**b** Find the intersection points of the two curves in the given interval.

**c** Find the volume of revolution formed when the region enclosed by the curves $f$ and $g$ is rotated through $2\pi$ radians about the $x$-axis.

**a**




Sketch the graphs within the given domain.

**b** Intersection points are $(0, 1)$ and $(1.29, 0.275)$.

Use your GDC to find the intersection points remembering to use at least 3 sf.

**c** $V = \pi \int_0^{1.29} [\cos x]^2 \, dx - \pi \int_0^{1.29} [e^{-x}]^2 \, dx$

If we subtract the volume formed by $y = e^{-x}$ from the volume formed by $y = \cos x$, we get the volume formed by the region between the two curves.

$V \approx 0.993$ (3 sf)

Use your GDC.





### Exercise 11I

**1** Each of the following functions are sketched on the domain $0 \le x \le 1$ and rotated about the $x$-axis by $2\pi$ radians. Find the volumes of the generated solids of revolution.

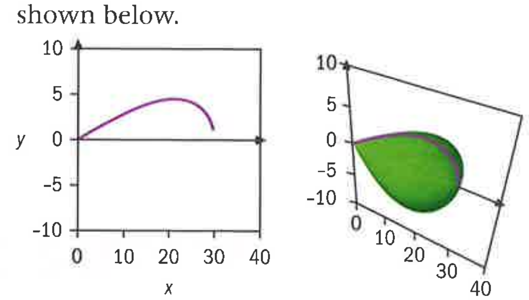
**a** $a(x) = x$ **b** $b(x) = x^2$

**c** $c(x) = x^3$ **d** $e(x) = e^x$

**2** The same parts of the functions in question **1** are rotated about the $y$-axis by $2\pi$ radians. Find the volumes of the generated solids of revolution.

**3** A teardrop-shaped earring is being designed. Its shape is based on the graph of

$y = \dfrac{x}{100}\sqrt{900 - x^2}$ for $0 \le x \le 30$ (where $x$ and $y$ are measured in mm), which is revolved by $2\pi$ radians around the $x$-axis as shown below.

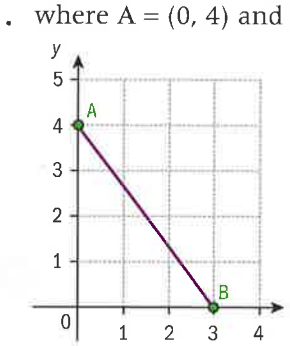



**a** Find $\int \left(\dfrac{x}{100}\sqrt{900 - x^2}\right)^2 dx$.

**b** Hence, find the volume of one earring.

**c** If 1 $cm^3$ of gold weighs 19 g, and 1 g of gold costs £30, find the cost of the gold used for the earring.

**4** The line segment [AB] is placed as shown, where A = (0, 4) and B = (3, 0).




**a** Write down integrals that represent the volume obtained by rotating [AB] by $2\pi$ radians:

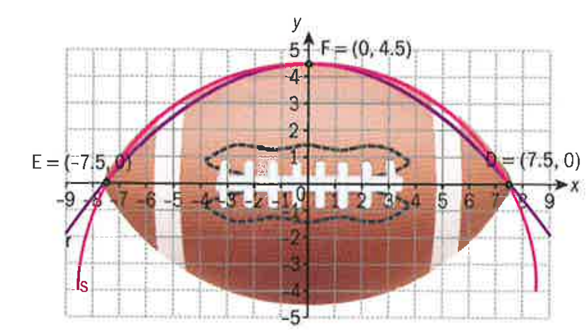
**i** about the $x$-axis

**ii** about the $y$-axis.

**b** Calculate the integrals and confirm your answers using the formula for the volume of a cone: $V = \dfrac{1}{3}\pi r^2 h$.

**5** Hany and Ahmed are trying to model an American football. Hany's model is $s(x) = -4 + \sqrt{72.25 - x^2}$ and Ahmed's is $f(x) = 4.5 - 0.079x^2$.

**a** Choose the best model and hence estimate the volume of the American football in cubic units using the information given in the diagram.




**b** Given that the length of the football is 11 inches, convert your answer to part **a** into

**i** cubic inches

**ii** cubic centimetres.

**6** The following regions are rotated about the $x$-axis by $2\pi$ radians. For each region:

**i** Sketch the information given.

**ii** Write down the integrals that quantify the volumes required.

**iii** Hence, calculate the volumes.

**a** The region bounded by $x = 0$, $y = x^2$ and $y = e^x$.

**b** The region bounded by $y = 0$, $y = \ln(x)$ and $y = 4 - x^2$.

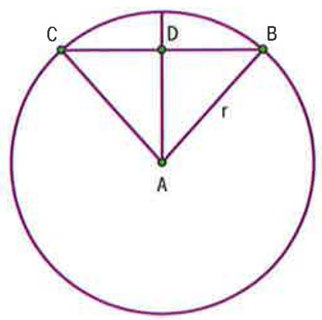
Calculus

**7** **a** Use technology to sketch the functions $y_2 = 40 + \sqrt{4 - x^2}$ and $y_1 = 40 - \sqrt{4 - x^2}$.

**b** Sketch the shape generated by rotating the region bounded by $y_2$ and $y_1$ about the $x$-axis by $2\pi$ radians.

**c** Write down an integral that quantifies the volume of this shape and hence, find its volume.

**8** In the diagram below, A is the centre of a circle with radius $r$ and the line segment [AD] passes through the middle of [BC].




**a** Prove that $\hat{ADB}$ is 90°.

The cross-section of the O2 Arena in London can be thought of as a segment of a circle with a radius $r$ m formed by a chord of length 365 m.

Given that the height of the O2 is 50 m and the equation of a circle of radius $r$ with centre at (0, 0) is $x^2 + y^2 = r^2$, find:

**a** the value of $r$

**b** the volume of the O2.

## Kinematics

The following results for an object with displacement $s$, velocity $v$ and acceleration $a$ at time $t$ were derived in Chapter 10: $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

We can now reverse the process to obtain $s = \int v\,dt$ and $v = \int a\,dt$.

**TOK**

Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage?

### Example 19

A stone is thrown vertically upwards with an initial velocity of 4.9 m s$^{-1}$ from a point 2 m above the ground.

Take the ground as the origin of the coordinate system with the upward direction as positive.

The acceleration due to gravity is $a = -9.8\,\text{m s}^{-2}$.

**a** Find the maximum height reached by the stone.

**b** Find the speed with which it hits the ground.





| | |
|---|---|
| **a** $a = -9.8$ | |
| $v = \int -9.8\,dt = -9.8t + c$ | Using $v = \int a\,dt$. |
| When $t = 0$, $v = 4.9 \Rightarrow c = 4.9$ | Using the boundary condition to find $c$. |
| $\Rightarrow v = 4.9 - 9.8t$ | |
| $s = \int (4.9 - 9.8t)\,dt$ $= 4.9t - 4.9t^2 + c$ | Using $s = \int v\,dt$. |
| When $t = 0$, $s = 2 \Rightarrow t = 2$ | |
| $\Rightarrow s = 4.9t - 4.9t^2 + 2$ | |
| When $v = 0$, $t = \frac{4.9}{9.8} = 0.5$ | The maximum height will occur when the velocity is equal to 0. |
| Hence, $s = 4.9 \times 0.5 - 4.9 \times 0.5^2 + 2 = 3.225$ | This value for time is put into the displacement equation to find the maximum height. |
| **b** When $s = 0$, $4.9t - 4.9t^2 + 2 = 0$ $\Rightarrow t = 1.31$ seconds | When the stone hits the ground, its displacement will be 0 as we are measuring displacement from ground level. |
| $v = 4.9 - 9.8 \times 1.31^2 = -7.95\,\text{m s}^{-1}$ | The velocity is negative as the stone is moving in the negative direction. |
| Speed = 7.95 m s$^{-1}$ | The question asks for the speed, which is the magnitude of the velocity. |

### Example 20

A particle moves so that its velocity at time $t$ is given by the equation $v = 4t\sin(t^2)$.

**a** Given that it is initially at the origin, find an expression for its displacement at time $t$.

**b** Find the first time at which it returns to its starting point.

| | |
|---|---|
| **a** $s = \int 4t\sin(t^2)\,dt$ $= -2\cos(t^2) + c$ | "Initially" implies when $t = 0$. |
| When $t = 0$, $s = 0$ | |
| $\Rightarrow 0 = -2\cos 0 + c$ $\Rightarrow c = 2$ $s = 2 - 2\cos(t^2)$ | It is important not to assume that $c$ will always be equal to 0 just because initially $s = 0$. |

Continued on next page

Calculus

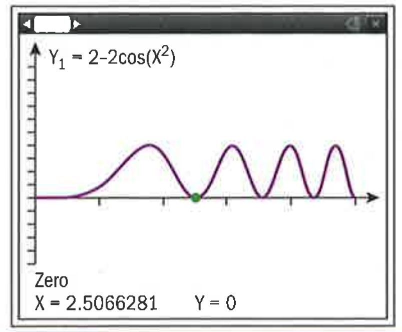
**b** $0 = 2 - 2\cos(t^2)$

$\Rightarrow \cos(t^2) = 1$

First positive solution is when

$t = 2.51$ seconds

The zeros can be found from rearranging and solving or by drawing a graph.




## Exercise 11J

**1** **a** Find expression for the velocity and displacement of a particle with acceleration $a(t) = \frac{3}{(t+1)^2}$, $t \geq 0$, if the particle is initially at rest at the origin.

**b** Determine the time it will take the particle to reach a distance of 50 m from the origin.

**2** A particle moves so that its velocity ($\text{m s}^{-1}$) at time $t$ seconds is given by the formula $v = e^{2\sin 3t}\cos 3t$.

**a** Find:

**i** an expression for the acceleration of the particle

**ii** the time $t$ at which the particle's acceleration is first equal to $-0.5\,\text{m s}^{-2}$.

**b** Find an expression for the displacement of the particle ($s$) from its initial position at time $t$.

**c** Write down the exact value of the maximum displacement of the particle.

**3** A particle moves so that its velocity at time $t$ is given by the equation $v(t) = \frac{t}{e^{kt^2}}$ for $k > 0$.

At time 0, it is 1 cm from the origin.

**a** Determine the maximum acceleration of the particle.

**b** Find the set of values of $k$ for which the particle will travel at least 5 cm from the origin.

**4** **a** Show that the vertical height of a projectile after $t$ seconds, if it is acted on only by gravity (acceleration $g = -9.8\,\text{m s}^{-2}$) is given by $s(t) = \frac{1}{2}gt^2 + v_0 t + y_0$, where $v_0$ is its initial vertical velocity and $y_0$ is its initial vertical position.

**b** Hence determine a formula for the maximum height reached by the particle.

**c** Determine, with reasons, whether a rocket fired with an initial vertical velocity of 310 m/s from 10 m above ground level is capable of reaching a target at a height of 4 km.

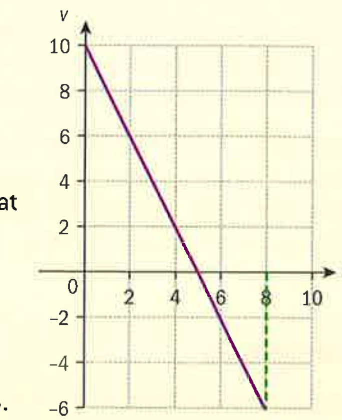




## Area under a velocity–time graph

### Investigation 6

The graph shows a **velocity–time graph** for the journey of an object. The horizontal axis shows the time taken from the start of the journey in seconds and the vertical axis shows the velocity of the object in $\text{m s}^{-1}$.




The equation for the velocity of the particle is $v = 10 - 2t$, $0 \leq t \leq 8$.

**1** Explain why the particle has its maximum displacement when $t = 5$.

**2** **a** Find an equation for the displacement ($s$) of the particle, given that $s = 0$ when $t = 0$.

**b** Sketch the displacement graph for $0 \leq t \leq 8$.

**3** **a** Find the maximum displacement of the particle.

**b** Find the displacement at $t = 8$.

**4** Hence, write down the total distance travelled by the particle for $0 \leq t \leq 8$.

**5** Find the area between the curve, the $t$-axis, $t = 0$ and $t = 8$.

**6** **Conceptual** How do you find the distance travelled by a particle?

**7** Explain why this is not the same as $\int_0^8 (10 - 2t)\,dt$.

## $\int_a^b f(t)\,dt$ and $\int_a^b |f(t)|\,dt$

The value of $\int_a^b v(t)\,dt = s(b) - s(a)$, which is the change in displacement of the particle between $t = a$ and $t = b$.

Consider an object moving along a line with a velocity at time $t$ given by $v(t) = \cos t$.

Using $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$, we obtain

$\int_a^b a(t)\,dt = v(b) - v(a)$, the net change in the velocity of the particle

$\int_a^b v(t)\,dt = s(b) - b(a)$, the net change in the displacement (or position) of the particle.

The **definite integral of a rate of change** in a given interval calculates the **net change**. This is a restatement of the fundamental theorem:

If $f$ is a **continuous** function on $a \leq x \leq b$ and $F$ is any antiderivative of $f$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If the rate of change of a function $F(x)$ is given by $f(x)$ then the total change in $F$ between $a$ and $b$ is $\int_a^b |f(x)|\,dx$.

Calculus

As $v$ is the rate of change of $s$ then the **total distance travelled** between $t = a$ and $t = b$ is $\int_a^b |v(t)| \mathrm{d}t$.

There are various applications of total and net change derived from equations for rates beyond kinematics. Whenever we are given an equation for a rate, the definite integral of the function will give us the net change between the two values and the definite integral of the modulus of the function will give us the total change.

For example, if we would like to know how much money is left in our bank account, we would use net change, but if we would like to calculate the total value of all transactions into and out of the account, we would use the total change.

**TOK**

Why do we study mathematics?

What's the point?

Can we do without it?

## Example 21

A hydro-electric power station generates electricity from water flowing through a pipe. During periods of low demand, water is pumped back up the pipe to the reservoir above.

Let the volume of water in the reservoir be $V$ and assume that the only water that enters or leaves the lake during this period is through the pipe. The rate at which water flows through the pipe during a 24-hour period is given by the following equation:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -8.2\sin\left(\frac{\pi}{12}t + \frac{15\pi}{12}\right) - 5$$

where $t$ is measured in hours after midnight and $V$ is measured in millions of litres.

**a** Sketch the curve for $\frac{\mathrm{d}V}{\mathrm{d}t}$ against time for a 24-hour period.

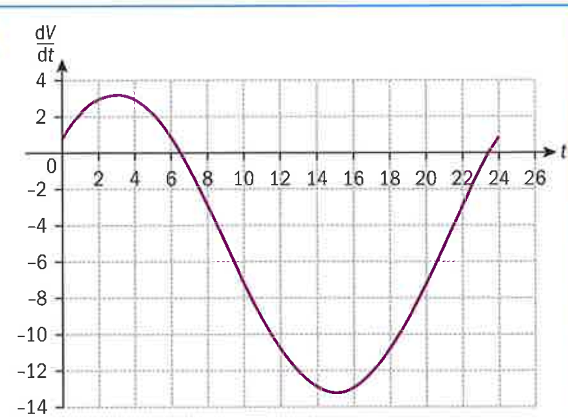
**b** By calculating an appropriate definite integral, find the net change in the volume of water in the reservoir over a 24-hour period.

**c** By calculating an appropriate definite integral, find the total amount of water that has passed through the pipe in a 24-hour period.

**d i** Determine a formula for the volume of water in the reservoir at time $t$, given that the volume is $V_0$ at midnight.

**ii** Find the value of $V$ when $t = 24$ and use this value to verify your answer to part **b**.

**a**




For most of the time the rate is negative, indicating that water is flowing out of the reservoir.

**b** $V = \int_0^{24}\left(-8.2\sin\left(\frac{\pi}{12}t + \frac{15\pi}{12}\right) - 5\right)\mathrm{d}t$

$= -120\,\mathrm{m}^3$

To find the net change, you take the definite integral between the two points.

**c** $V = \int_0^{24}\left|-8.2\sin\left(\frac{\pi}{12}t + \frac{15\pi}{12}\right) - 5\right|\mathrm{d}t$

$= 149.4\,\mathrm{m}^3$

To find the total change, you take the definite integral of the modulus function between the two points.

**d i** $V = \int\left(-8.2\sin\left(\frac{\pi}{12}t + \frac{15\pi}{12}\right) - 5\right)\mathrm{d}t$

$= \frac{98.4}{\pi}\cos\left(\frac{\pi}{12}t + \frac{15\pi}{12}\right) - 5t + c$

The reverse chain rule is needed to integrate the expression.

When $t = 0$, $V = V_0$

$\Rightarrow c = V_0 - \frac{98.4}{\pi}\cos\left(\frac{15\pi}{12}\right) = V_0 + 22.15$

$V = \frac{98.4}{\pi}\cos\left(\frac{\pi}{12}t + \frac{15\pi}{12}\right) - 5t + V_0 + 22.15$

**ii** When $t = 24$,

$V = \frac{98.4}{\pi}\cos\left(2\pi + \frac{15\pi}{12}\right) - 120 + V_0 + 22.15$

$= -120 + V_0$

So the difference is $-120 + V_0 - V_0 = -120$

The difference is independent of $V_0$ in the same way as you do not need to know the $+c$ term when working out definite integrals.

Both methods of working out a change are acceptable but generally, given the availability of technology, the method of part **b** is the more straightforward.

## Exercise 11K

**1** In an experiment, a liquid is cooled and then heated. The rate of change of the temperature of the liquid is given as $T'(t) = t^2 - 4t - 8$ where $t$ is the time in minutes and $T(t)$ the temperature in °C. When $t = 8$, find:

**a** how many degrees higher the liquid is than it was at $t = 0$

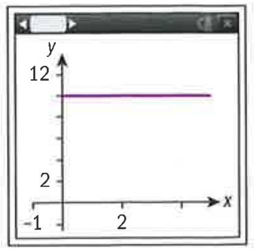
**b** the total change in temperature over that period.

**2** The graphs show the velocity–time graphs of four particles for a 5-second period of time. The $x$-axis is time (s) and the $y$-axis is velocity (m s$^{-1}$).

**a** For each graph, find the total distance travelled and the displacement after 5 seconds.

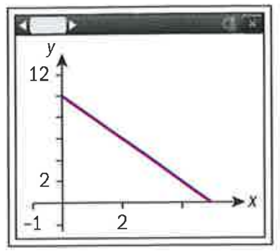
**i** Particle A

$v(t) = 10$



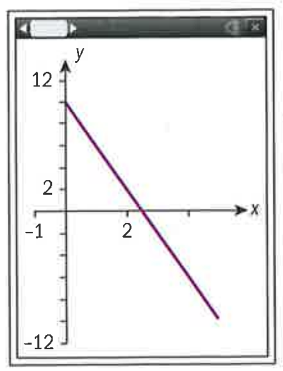
**ii** Particle B

$v(t) = 10 - 2t$



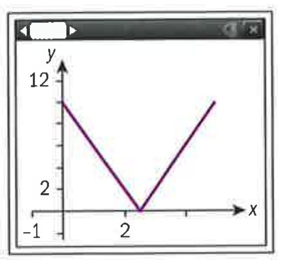
**iii** Particle C

$v(t) = 10 - 4t$



**iv** Particle D

$v(t) = \begin{cases} 10-4t, & 0 \le t \le 2.5 \\ 4t-10, & 2.5 < t \le 5 \end{cases}$



**b** Identify which of the particles changes direction.

**3** Use technology to help sketch the velocity–time graphs for each of the following moving bodies and hence, find the total distance travelled and the displacement on the time intervals given. Time is in hours and velocity in $\text{km h}^{-1}$.

**a** $v(t) = \cos t$, $\left[0, \frac{\pi}{2}\right]$

**b** $v(t) = 3.1\sin 2t - 1$, $\left[\frac{\pi}{6}, \frac{3\pi}{2}\right]$

**c** $v(t) = 17 - 2t^2$, $[1, 4.1]$

**d** $v(t) = 1 - e^t$, $[0, 2.39]$

**4** A particle moves with velocity function

$$v(t) = \begin{cases} 4, & 0 \le t \le 5 \\ 4 - \frac{4}{25}(t-5)^2, & 5 < t \le 10 \\ -1.6t + 16, & t > 10 \end{cases}$$

where time is measured in seconds and distance in metres.

**a** Find the total distance travelled by the particle after

**i** 5 seconds **ii** 10 seconds.

**b** Calculate the time taken for the particle to return to its original position.

**5** A suspension bridge is being designed. Immediately after the bridge sustains a heavy load, the vertical movement of a point on the bridge is modelled with a damped oscillation model, $v(t) = 5(0.65^t)(\cos 5t - 0.08616 \sin 5t)$, where time is measured in seconds and $v$ is measured in $\text{cm s}^{-1}$.

**a** Calculate the displacement of the point 5 seconds after sustaining a heavy load, and

**b** Find the total distance travelled by the point at this time.

**6** Two objects A and B are launched from a balloon at a height of 1000 m, with velocity functions $v_a(x) = 16 - 4e^{\frac{-t}{16}}\ \text{m s}^{-1}$ and $v_b(x) = 18 - 7e^{\frac{-t}{16}}\ \text{m s}^{-1}$. The downward direction is taken as positive.

**a** **i** Graph both velocity functions for $0 \le t \le 20$.

**ii** Calculate after how many seconds the objects are first travelling at the same velocity.

**b** Calculate how far above the ground A is at this time.

**7** The derivative of the energy, $E$ (mJ), of a particle is given as $\frac{dE}{dt} = \sin(0.5t) + a$. It is known that the change in the particle's energy between $t = 0$ and $t = \pi$ is 4 mJ. Find the value of $a$.

**8** The rate of change in the area ($A$) covered by mould on a petri dish after $t$ days is given as $\frac{dA}{dt} = 2e^{rt}$ where $r$ is the rate of growth. Between the times $t = 2$ and $t = 4$, the area covered by the mould increases by 7.2 $\text{cm}^2$. Find the value of $r$.





**9** The rate of change of the number of mosquitoes ($N$), measured in 10 000s, is modelled by the equation

$\frac{dN}{dt} = 4\sqrt{t}\cos\left(\frac{\pi}{6}t - \frac{1}{2}\right)$ where $t$ is measured in months from the beginning of a year (1 January).

**a** Find in which month each year the population of mosquitoes is a minimum.

**b** By how much does the population of mosquitoes increase:

**i** the first year

**ii** the second year.

The average increase in population is the actual increase divided by the duration.

**c** Find the average increase in population for the first two years.

# 11.4 Differential equations

A differential equation is an equation that contains a derivative.

For example, $\frac{dy}{dx} = 2x^2 + 5$.

You have already met several of these and solved them by integrating.

When no boundary condition is given then the solution to the equation will have an unknown constant (the $+c$ term). If a boundary condition is given, the particular solution can be found.

The equation above is written in the form $\frac{dy}{dx} = f(x)$, but often the context of the question will mean that the differential equation is of the form $\frac{dy}{dx} = f(x, y)$. These equations can often not be solved directly, so numerical solutions need to be found.

One situation in which they can be solved is when $\frac{dy}{dx} = f(x)g(y)$.

This form is called a **separable** differential equation.

It can be written as $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$ and both sides are then integrated with respect to $x$:

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx$$

It can be shown that $\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int \frac{1}{g(y)}dx$ as if we were cancelling the two $dx$'s. This is very similar to the process used when integrating using the reverse chain rule.

Both sides of the expression $\int \frac{1}{g(y)}dy = \int f(x)dx$ can now be integrated.

**International-mindedness**

French mathematician Jean d'Alembert's analysis of vibrating strings using differential equations plays an important role in modern theoretical physics.

Calculus

In practice, many people often go from $\frac{dy}{dx} = f(x)g(y)$ to

$\frac{1}{g(y)}dy = f(x)dx$ and then add the integral operators. Although this middle stage is meaningless, notationally it can be a convenient shortcut.

## Example 22

Find the solution of the differential equation $y\frac{dy}{dx} = x^2 + 1$ given that $y(0) = 4$.

| | |
|---|---|
| $\int y\,dy = \int (x^2+1)dx$ | The variables have been separated to give $y\,dy = (x^2+1)dx$. |
| $\frac{1}{2}y^2 = \frac{1}{3}x^3 + x + c$ | We do not need to have an integrating constant, $c$, for both sides of the equation.<br>If the question asks for the solution to be given in a particular form, we may need to rearrange the answer obtained from the integration. |
| $y = 4$ when $x = 0$<br>$\Rightarrow 8 = c$ | Substitute the given values to find $c$. |
| So $\frac{1}{2}y^2 = \frac{1}{3}x^3 + x + 8$ | |

## Example 23

Find a solution in the form $y = f(x)$ to the differential equation $\frac{dy}{dx} = 6x^2 y$ given that $y = 2$ when $x = 0$.

| | |
|---|---|
| $\int \frac{1}{y}dy = \int 6x^2 dx$<br>$\ln y = 2x^3 + c$ | Separate the variables and integrate. |
| $y = Ae^{2x^3}$ | Converting the $+c$ term into a coefficient when eliminating a natural log operator is so common that it is often done in a single step. The full process is<br>$y = e^{2x^3+c} = e^c e^{2x^3} = Ae^{2x^3}$ where $A = e^c$. |
| $y = 2$ when $x = 0$<br>$\Rightarrow 2 = A$ | Substitute to find $A$. |
| So $y = 2e^{2x^3}$ | |





Such an exponential model from a differential equation occurs whenever the growth or decay of a variable is proportional to the amount present.

## Example 24

The rate of growth of mould on a large petri dish is directly proportional to the amount of mould present. The area covered by the mould ($A$) is initially 2 cm² and after 2 days it is 9 cm².

**a** Find an expression for the area covered by the mould at time $t$ days.

**b** Find the value of $t$ at which the mould will cover 20 cm².

| | |
|---|---|
| **a** $\frac{dA}{dt} = kA$ | If two variables $a$ and $b$ are proportional to each other, then $a = kb$. |
| $\int \frac{1}{A}dA = \int k\,dt$<br>$\ln A = kt + c$<br>$A = Be^{kt}$ where $B = e^c$ | |
| $A = 2$ when $t = 0 \Rightarrow B = 2$<br>$A = 2e^{kt}$<br>$A = 9$ when $t = 2$ | There are two unknowns in the equation, so two sets of boundary conditions need to be used. |
| $\Rightarrow 9 = 2e^{2k}$<br>$2k = \ln\left(\frac{9}{2}\right) \Rightarrow k = 0.752$<br>So $A = 2e^{0.752t}$ | This equation and the one in part **b** can be solved using the log laws or directly on a GDC. |
| **b** $20 = 2e^{0.752t}$<br>$t = \frac{1}{0.752}\ln 10 = 3.06$ days | |

## Exercise 11L

1 Solve $4\frac{dy}{dx} = xy$, given that $y = 2$ when $x = 0$.

2 Solve $(x^2 - 1)\frac{dy}{dx} = 2xy$, given that $y(2) = 9$.

3 **a** The rate of decay of a radioactive substance is proportional to the amount $M$ g remaining. Form a differential equation in $M$ and show that this solves to give $M = Ae^{-kt}$ where $A$ and $k$ are positive constants.

**b** Given that the amount initially is 40 g and there is only 20 g remaining after 2 hours, find the value of $k$.

4 **a** Find the derivative of $y = \sqrt{4 - x^2}$.

**b** Verify that $y = \sqrt{4 - x^2}$ satisfies the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

**c** Solve $\frac{dy}{dx} = -\frac{x}{y}$ to find the general solution to the differential equation.

Calculus

**5** Solve the following differential equations.

**a** $\dfrac{dy}{dx} = e^{x+y}$  **b** $\dfrac{dy}{dx} = \dfrac{y + y\cos x}{y^2+1}$

**6** For each of the following differential equations, find the solution that satisfies the given initial condition.

**a** $\dfrac{dy}{dx} = \sqrt{\dfrac{x}{2y}}$, $y(1) = 2$

**b** $\dfrac{dy}{dx} = \dfrac{2x + \sin x}{y}$, $y(0) = 1$

**c** $y' \tan x = \tan y$, $y(2) = 2$

**7** Newton's law of cooling states that the rate of change of temperature, $\dfrac{dT}{dt}$, of an object is proportional to the difference between its own temperature, $T(t)$, and the ambient temperature, $T_a$ (ie the temperature of its surroundings), that is $\dfrac{dT}{dt} = -k(T - T_a)$.

Suppose you have just poured a cup of coffee with a temperature of 75°C in a room where the temperature is 24°C.

**a** State the time at which the coffee cools most quickly, justifying your answer.

**b** Write a differential equation using the initial conditions given.

**c** Solve the differential equation.

**d** Explain from the equation why the temperature will never go below 24°C.

**8** The number of bacteria in a liquid culture is observed to grow at a rate proportional to the number of cells present. At the beginning of the experiment there are $10^4$ cells and after 3 hours there are $3 \times 10^5$ cells.

**a** Calculate the number of bacteria after 1 day of growth if this rate of growth continues.

**b** Determine the **doubling time** of the bacteria.

**9** Indium-111 has a short half-life of 2.8 days, which makes it a useful tracer. It is used in a variety of diagnostic methods, including isotopic labelling of blood cell components, diagnosing rare cancers etc.

In the early 19th century, F. Soddy and E. Rutherford derived the radioactive decay formula as

$\dfrac{dN}{dt} = -kN$, where $N$ is the amount of a radioactive material and $k$ is a positive constant that depends on the radioactive substance, with $T$ being the half-life of the substance.

**a** Show that the formula for $k$ is, $k = \dfrac{\ln 2}{T}$.

**b** If $N(0) = 5$ g, find the mass remaining after 1 day.

**10** A cylindrical tank of radius $r$ is filled with water. A hole of area $A$ m² is made in the bottom of tank.

The velocity of water through the hole is $v$ m s$^{-1}$. The volume of water in the tank at time $t$ seconds after the hole is made is $V$ m³.

Let $h$ be the height of water remaining in the tank after $t$ seconds.

**a** Given that $v = \sqrt{2gh}$, show that

$\dfrac{dV}{dt} = -A\sqrt{2gh}$.

**b** **i** Hence, show that $\dfrac{dV}{dt} = -k\sqrt{V}$ where $k$ is a constant.

**ii** State the value of $k$ in terms of $A$, $r$ and $g$.

The volume of the cylinder when full is $V_0$.

**c** Find how long it will take the cylinder to empty in terms of $k$ and $V_0$.




## Investigation 7

In this investigation, you can assume that the number i can be treated the same as any other number when differentiating or integrating.

**1** Solve $\dfrac{dx}{d\theta} = ix$ given $x = 1$ when $\theta = 0$.

Let $x = \cos\theta + i\sin\theta$

**2** Verify that $x = 1$ when $\theta = 0$.

Show that $\dfrac{dx}{d\theta} = ix$.

**3** **Conceptual** What can you conclude about $e^{i\theta}$?

This is called Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

# 11.5 Slope fields and differential equations

It is not always possible to solve differential equations and find an explicit formula for the unknown function. **Slope fields,** or **direction fields,** are used to represent graphical solutions of differential equations. They are drawn using tangent lines at specific points. They can often be produced even when it is not possible to solve the differential equation and find an explicit formula for the unknown function.

**International-mindedness**

Differential equations first became solvable with the invention of calculus by Newton and Leibniz. In Chapter 2 of his 1736 work *Method of Fluxions*, Newton described three types of differential equations.

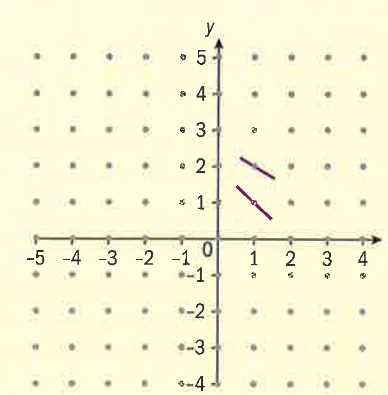
## Investigation 8

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y - 3x}{2}$.

The gradient of the tangent drawn to the curve $y$

- at $(1, 1)$ will be $-1$
- at $(1, 2)$ will be $-\dfrac{1}{2}$.

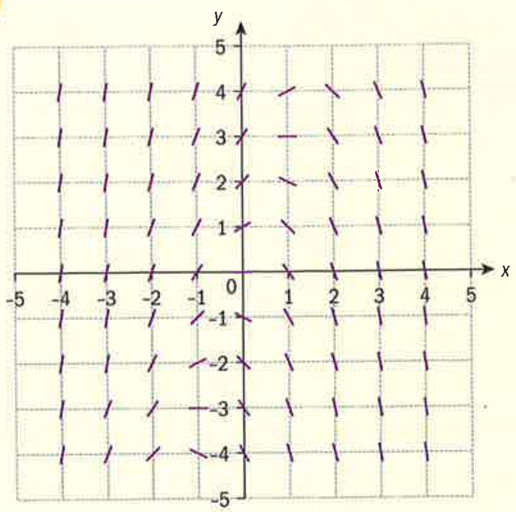
These points can be plotted on a grid and small line segments drawn with gradients of $-1$ and $-\dfrac{1}{2}$, respectively, as seen in the diagram.




The process can be continued and is shown below for all the integer values of $x$ and $y$ in the intervals $-4 \le x \le 4$ and $-4 \le y \le 4$.

Continued on next page

Calculus

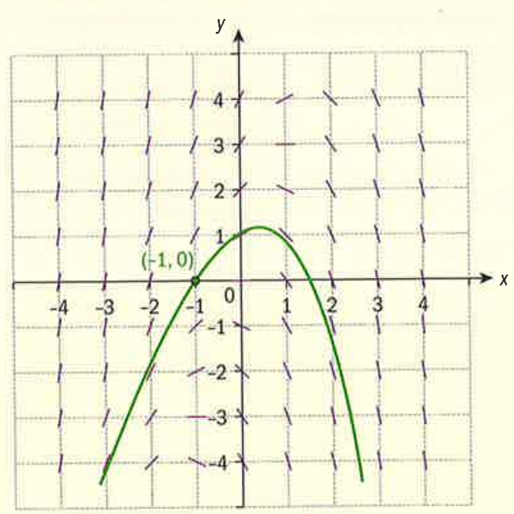


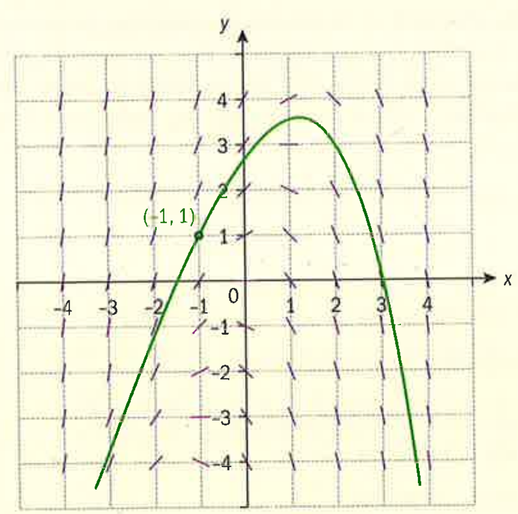

This type of diagram is called a **slope field.**

The slope field indicates the family of curves that satisfy the differential equation. At each point the direction of the curve is the same as the direction of the tangent. This means that given a starting point (a boundary condition), a solution curve can be drawn.

One way to think about this is that the slopes are like the current in a river and the curve follows the path a cork might take if released in the river at that point.

The diagrams below show the solution curves that pass through $(-1, 0)$ and $(-1, 1)$.




Certain features of the family of curves satisfying the differential equation can be worked out from the equation itself. For example, the maximum points will occur when $\frac{dy}{dx} = \frac{y-3x}{2} = 0$.

So, they will lie on the line $y = 3x$.





## Exercise 11M

**1** Consider the differential equation $y' = x^2 + y$.

**a** Find the missing values of the gradients of the tangents to the solution curves at the integer points for $-3 \le x \le 3$ and $-3 \le y \le 3$.

| $x$ \ $y$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| **−3** | 6 | 1 | −2 | −3 | −2 | 1 | 6 |
| **−2** | 7 | 2 | −1 | −2 | −1 | 2 | 7 |
| **−1** | 8 | 3 | 0 | −1 | 0 | 3 | 8 |
| **0** | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| **1** | 10 | 5 | 2 | 1 | | | |
| **2** | 11 | 6 | 3 | 2 | | | |
| **3** | 12 | 7 | 4 | 3 | | | |

**b** Sketch the slope field for the differential equation.

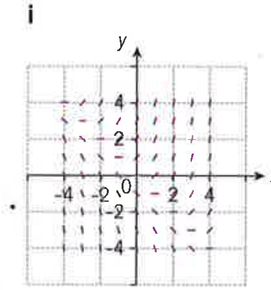
**c** Use the slope field to sketch the solution curve that passes through $(2, -2)$.

**2** By evaluating the gradient at various points, or otherwise, match each differential equation below with a slope field.

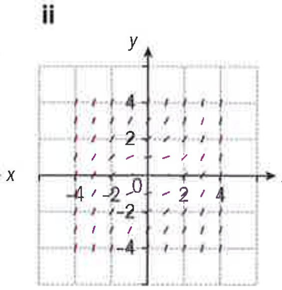
**a** $\frac{dy}{dx} = x - 2$ **b** $\frac{dy}{dx} = x + y$

**c** $\frac{dy}{dx} = y + 2$ **d** $y' = x^2 + y^2$

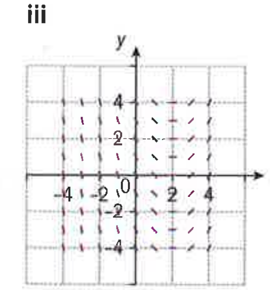
**i**



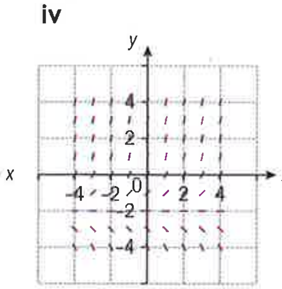
**ii**



**iii**

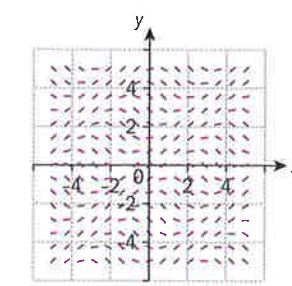


**iv**

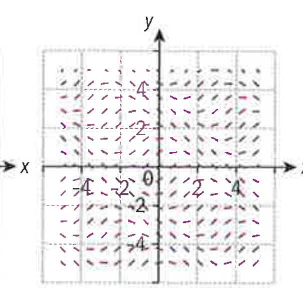


**3** One of the following slope fields is for the differential equation $y' = \cos(x + y)$, and the other is for the differential equation $y' = \cos(x - y)$.

**A**



**B**



**a** For **i** $y' = \cos(x + y)$ **ii** $y' = \cos(x - y)$ write down the equation of a line along which the slope field would have a gradient of zero.

**b** State which is the slope field for **i** $y' = \cos(x + y)$ **ii** $y' = \cos(x - y)$

**4** The Gompertz equation is a model that is used to describe the growth of certain populations. In the 1960s, AK Laird used the Gompertz curve to fit the data on the growth of tumours. A tumour is a cellular population growing in a confined space where the availability of nutrients is limited.

If $T(t)$ is the size of a tumour then under this model $\frac{dT}{dt} = -T\ln\left(\frac{T}{2}\right)$.

**a** Sketch a slope field for $T(t)$ over the first 6 months, $0 \le t \le 6$. Take $0 \le T \le 3$.

**b** On your slope field, sketch a curve that shows the growth of the tumour given that its volume is $0.5\,\text{cm}^3$ when $t = 0$.

**c** Write down the maximum possible size of a tumour according to this model.

**d** Explain why this model might be suitable for modelling the growth of a tumour.

## Euler's method for numerically solving differential equations

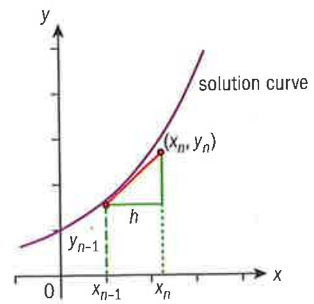
The idea used in slope fields can also be applied to obtain numerical approximations of differential equations.

Consider the differential equation $\frac{dy}{dx} = f(x, y)$.

Let $(x_{n-1}, y_{n-1})$ be a point on the curve and let $x_n = x_{n-1} + h$ $(h > 0)$. The method will use the gradient of the curve at $(x_{n-1}, y_{n-1})$ to evaluate $y_n$, an estimate of the $y$ coordinate of the point on the curve with $x$ coordinate $x_n$.

The gradient of the tangent at the point $(x_{n-1}, y_{n-1})$ is $f(x_{n-1}, y_{n-1})$. The diagram shows that it is also $\frac{y_n - y_{n-1}}{h}$. Hence,

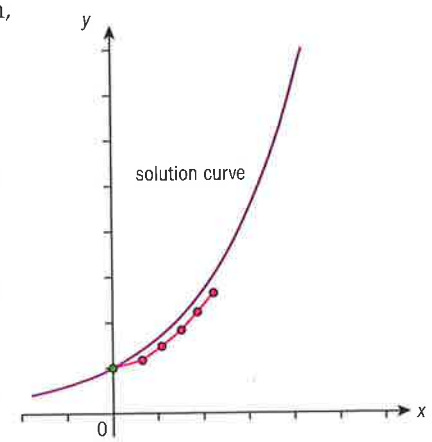
$f(x_n, y_n) = \frac{y_n - y_{n-1}}{h}$, which can be rearranged to give $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$.




Generally, we are given a boundary condition, which can be written as $(x_0, y_0)$. The formula is then used to find the coordinates $(x_1, y_1)$. $y_1$ is unlikely to be on the curve, but if $h$ is sufficiently small it should be quite close and so the coordinates can be used as the starting values to obtain $(x_2, y_2)$.

This will create a series of approximations that might look similar to the red lines in the diagram.

This iterative numerical approximation of differential equations is named after Euler as **Euler's method**.




**International-mindedness**

The Euler method is named after Swiss mathematician Leonhard Euler (pronounced "oiler"), who proposed it in his book *Institutionum Calculi Integralis* in 1768.

### Example 25

Use Euler's method with step size 0.1 to approximate the solution to the initial value problem $\frac{dy}{dx} = xy$ and $y(1) = 1$, and estimate the value of $y(2)$.

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$h = 0.1,\ f(x, y) = xy$

In examinations, it is important to write down the formula being used, as method marks can be awarded even if there are numerical errors in the calculations.





Hence,

$y_n = y_{n-1} + 0.1x_{n-1}y_{n-1}$

$\Rightarrow y_n = y_{n-1}(1 + 0.1x_{n-1})$

Using technology,

$y(2) \approx 3.860891894$

There is no need to simplify the formula, though sometimes this can make it easier to put into a GDC.

All the calculators recommended for this course have a simple way to calculate an iterative formula.

For this example, the full table of values is

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1.1 | 1.1 |
| 2 | 1.2 | 1.221 |
| 3 | 1.3 | 1.36752 |
| 4 | 1.4 | 1.5452976 |
| 5 | 1.5 | 1.761639264 |
| 6 | 1.6 | 2.025885154 |
| 7 | 1.7 | 2.350026778 |
| 8 | 1.8 | 2.74953133 |
| 9 | 1.9 | 3.24444697 |
| 10 | 2 | 3.860891894 |

### Exercise 11N

1 Consider the differential equation $\frac{dy}{dx} = \sqrt{\frac{x}{4y}}$ where $y = 4$ when $x = 4$.

Use Euler's method with step size 0.1 to find an approximate value of $y$ when $x = 4.5$.

2 Consider the differential equation $y' = 1 - y$ where $y = -1$ when $x = 0$.

Use Euler's method with step size 0.05 to find an approximate value of $y$ when $x = 0.15$.

3 Consider the differential equation $\frac{dy}{dx} = \frac{2x^2 + 1}{xe^y}$ where $y = 0$ when $x = 1$.

Use Euler's method with step size 0.025 to find an approximate value of $y$ when $x = 1.1$.

4 **a** Consider the differential equation $\frac{dy}{dx} = -xy$ where $y = 1$ when $x = 1$.

Use Euler's method with step size 0.1 to find an approximate value of $y$ when $x = 1.5$.

**b** Solve the differential equation and hence, find the error in your approximation.

5 **a** Consider the differential equation $\frac{dy}{dx} = \frac{2xy}{1 + x^2}$ where $y = 3$ when $x = 1$.

Use Euler's method with step size 0.1 to find an approximate value of $y$ when $x = 1.3$.

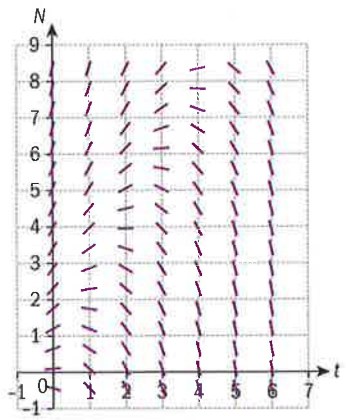
**b** Solve the differential equation and hence, find the absolute error in your approximation.

6 Let the population size of a colony of rare meerkats in units of 10 be $N$. At time $t$ years, the rate of change of the population can be modelled by the differential equation

$$\frac{dN}{dt} = 0.5N - t.$$

**a** Given that $N = a + bt$, $a, b \in \mathbb{R}$ is a solution to the differential equation for a particular initial population, find the values of $a$ and $b$.

The slope field for the differential equation is shown below.




**b** On a copy of the slope field diagram, show

**i** the line $N = a + bt$

**ii** the trajectory of the population if at $t = 0$, $N = 2$.

**c** Find the least number of meerkats in the colony at $t = 0$ that will ensure the population does not become extinct.

An environmentalist measuring the population calculates it will reach a maximum after three years and then will begin to decline.

**d** What is the population of meerkats at this time?

At $t = 3$, they decide to introduce more meerkats from another colony.

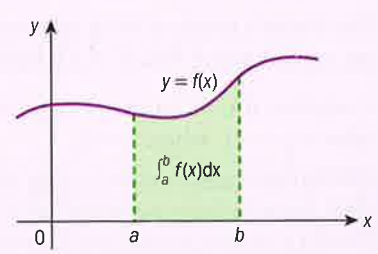
**e** If the model remains valid, find the least number of meerkats that would need to be introduced for the colony to increase in size continually?

After the introduction of extra meerkats, the size of the population is 120.

**f** Use Euler's method with a step size of 0.1 to estimate the population one year later.

## Chapter summary

- When $f$ is a non-negative function for $a \le x \le b$, $\int_a^b f(x)\,dx$ gives the area under the curve from $x = a$ to $x = b$.
- If $f(x) < 0$ in an interval $[a, b]$ then $\int_a^b f(x)\,dx < 0$.
- If the area bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is required and if $f(x) < 0$ for any values in this interval then the area should be calculated using $\int_a^b |f(x)|\,dx$.




- For a positive continuous function, the area $(A)$ between the graph of the function, the $x$-axis and the lines $x = a$ and $x = b$ can be approximated by the trapezoidal rule:

  $A \approx \frac{1}{2}h(y_0 + y_n + 2(y_1 + \ldots + y_{n-1}))$ where $h = \frac{b-a}{n}$.

- $\int f(x)\,dx$ is an indefinite integral and the process of finding the **indefinite integral** is called **integration**.

- If $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)\,dx = F(x) + c$ where $c$ is an arbitrary constant.
- $F(x)$ is the antiderivative of $f(x)$.
- Rules for integration can be derived from those for differentiation:
  - The sum/difference of functions: $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
  - Multiplying a function by a constant: $\int af(x)\,dx = a\int f(x)\,dx$
  - Power rule: $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$
  - $\int \sin x = -\cos x + c$
  - $\int \cos x = \sin x + c$
  - $\int \frac{1}{x}\,dx = \ln|x| + c$
  - $\int e^x\,dx = e^x + c$
  - $\int \frac{1}{\cos^2 x}\,dx = \tan x + c$
- In addition:
  - $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
  - $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
  - $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
- The reverse chain rule can be used whenever the integral consists of the product of a composite function and a multiple of the derivative of the interior function. In these cases, the derivative will cancel when the substitution is made.
- The area $A$ of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$, where $f$ and $g$ are continuous, is given by

  $A = \int_a^b |f(x) - g(x)|\,dx$

- The area $A$ of the region bounded by the curve $x = h(y)$, and the lines $y = a$ and $y = b$, where $h$ is continuous for all $y$ in $[a, b]$, is given by

  $A = \int_a^b |h(y)|\,dy$

- The **definite integral of a rate of change** in a given interval calculates the **net change** and the definite integral of the modulus of a function gives the total change.

  If $f$ is a **continuous** function on $[a, b]$ and $F$ is any antiderivative of $f$ then

  $\int_a^b f(x)\,dx = F(b) - F(a)$

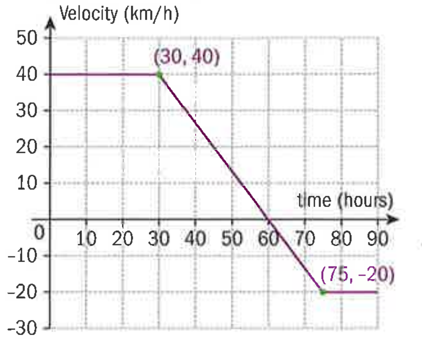
## Developing inquiry skills

Write down any further inquiry questions you could ask and investigate how you could find the areas of irregular shapes and curved shapes.

# Chapter review

Click here for a mixed review exercise

**1** The graph shows how the velocity of a prototype drone designed to monitor large areas of rainforest changed in a trial lasting 90 hours.




**a** Describe the motion of the drone during the entire trial.

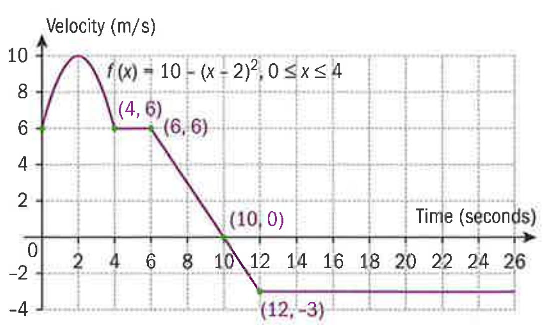
**b** Write down the interval during which the acceleration of the drone is negative.

**c** Hence, determine when the drone is slowing down.

**d** Find the total distance travelled in the trial.

**e** Calculate the position of the drone at the end of the trial.

**2** The graph gives information about the motion of a more advanced drone that is programmed to fly in a straight line and return to its starting point once it has collected enough photographic data to fill up its memory.




**a** Calculate the distance travelled in the first 6 seconds.

**b** Determine the time when the drone turns around to return to the starting point

**c** Hence, calculate how long the drone takes to complete its journey.

**3** A prototype solar-powered vehicle is tested. Sensors in the car record the following data every 5 seconds.

| Time (s) | Velocity (m/s) |
|---|---|
| 0 | 6.7 |
| 5 | 15.3 |
| 10 | 26 |
| 15 | 27.3 |
| 20 | 29.4 |
| 25 | 32.1 |
| 30 | 35.5 |

Apply the trapezoidal rule to estimate the distance the car has travelled by the time its velocity reaches $35.5\,\text{m s}^{-1}$.

**4** Identify which of the following integrals can be found analytically and which need to be found using technology.

**a** $\int_0^4 \sin(4x+3)\,dx$ **b** $\int_2^5 x\cos(x^2)\,dx$

**c** $\int_2^5 x^2\cos(x)\,dx$ **d** $\int_2^5 \cos(x^2)\,dx$

**e** $\int_0^4 (\sin(4x)+3)\,dx$ **f** $\int_5^{10} \frac{0.7x^4}{x^5-1}\,dx$

**g** $\int_5^{10} \frac{x^5-1}{0.7x^4}\,dx$ **h** $\int_5^{10} \frac{0.7x^4}{(x^5-1)^8}\,dx$

**i** $\int_5^{10} \frac{0.7e^x}{x^5-1}\,dx$

**5** A population of midges in an area of western Scotland is to be controlled through sustainable changes to the environment. The rate of decrease of the population $P$ is proportional to the number of midges at any time $t$. The population $P$ decreases from 600 000 to 500 000 over four years.

**a** Write down the relationship between $\frac{dP}{dt}$ and $P$.

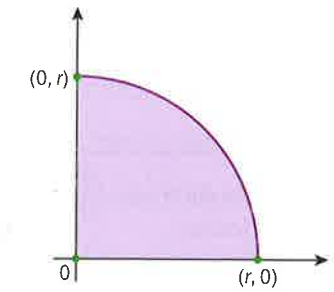
**b** Solve the differential equation for $P$.

**c** Hence, find the time taken for the population to decrease by 60%.





**6** The diagram shows the function $f(x)=\sqrt{r^2-x^2}$, $0 \le x \le r$, and the area beneath it.




**a** State the geometrical shape of the solid of revolution generated by rotating $f(x)$ around the $x$-axis by $2\pi$ radians.

**b** Write down an integral that represents the volume of this solid.

**c** Hence, show that the volume $V$ of a sphere of radius $r$ is $V=\frac{4}{3}\pi r^3$.

**7 a** Consider the differential equation $y'=2y(1+x)$ where $y=3$ when $x=0$. Use Euler's method with step size 0.1 to find approximate values of $y$ when $x=0.1$, 0.2 and 0.3.

**b** Solve the differential equation and hence, find the absolute errors for each of your approximations.

**8** A water tank in the shape of a cylinder of radius 40 cm and height 100 cm collects rainwater for recycling.

Water flows out of a tap at the bottom of the tank at a rate directly proportional to the square root of the depth of the water.

After heavy rainfall, the tank is full but the owner needs to empty the tank so it can be cleaned.

The tap is opened and after 10 minutes the level of the water has dropped to 64 cm.

**a** Using $V$ for the volume of water in the tank and $h$ for the depth of the water, write down a differential equation relating these variables.

**b** Apply the chain rule and the formula for the volume of a cylinder to show that $\frac{dV}{dt}=\frac{4\pi}{25}\frac{dh}{dt}$. Hence, write your answer to part **a** in terms of $h$ and $t$.

**c** Predict how long it will take for the tank to empty.

**9** Xavier is designing an art installation in a public space. It is a large water tank in the shape of a prism. He builds a scale model in his studio. The constant cross-sectional area of the model is bounded by the functions $f(x)=2^x+2^{-x}$ and $f(x)=\frac{9}{16}x^2+2$.

**a** Xavier sketches both functions on the domain $-2 \le x \le 2$. The $x$-axis is calibrated in metres. Calculate the area of the region bounded by the two graphs in square metres.

**b** Xavier uses his graph to build the model of the prism, which is 50 cm deep. Given that the model is 1:40 scale, calculate the volume of the art installation, correct to the nearest cubic metre.

# Exam-style questions

**10** **P1: a** Find the value of $\int_0^2 \frac{x}{x^2+4}$, giving your answer as a decimal correct to 3 significant figures. (2 marks)

**b** Find the integral $\int \frac{x}{x^2+4}\,dx$. (3 marks)

**c** Hence find the exact value of $\int_0^2 \frac{x}{x^2+4}\,dx$. (2 marks)

**11** **P1:** Find the following indefinite integrals.

**a** $\int x^2 + 3x + 1\,dx$ (2 marks)

**b** $\int \sin x + \cos(2x)\,dx$ (2 marks)

**c** $\int e^x + e^{-x}\,dx$ (2 marks)

**d** $\int (3x+1)^6\,dx$ (2 marks)

**e** $\int \frac{x^2+x}{x}\,dx$ (2 marks)

**12** **P1:** Farmer Davies owns the land enclosed between two roads. The roads can be modelled by curves with equations $y = x^2 - 8x + 23$ and $y = x + 9$ for $0 \le x \le 10$, where $x$ and $y$ each represent a unit of 100 m.

Find the area of Farmer Davies' land. (6 marks)

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**13** **P1:** Alun, a geographer, measures the depth of a river at 1 m intervals from the bank using a boat and a marked piece of line with a heavy weight at the bottom.

Alun's results are presented in the table below, where $x$ represents the perpendicular distance from the shore, and $d$ represents the depth of the river (both measured in metres).

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| $d$ | 0 | 6.71 | 10.4 | 12.15 | 12.8 | 12.95 | 12.96 | 12.95 | 12.8 | 12.15 | 10.4 | 6.71 | 0 |

**a** Use the trapezium rule to estimate the cross-sectional area of the river, giving your answer correct to 2 decimal places. (3 marks)

It is later discovered that the equation connecting $x$ and $d$ is $d = \frac{1296-(x-6)^4}{100}$.

**b** Calculate the true value of the cross-sectional area of the river. (3 marks)

**c** Find the percentage error in your estimation of the area you made in part **a**. (2 marks)

**d** Explain why, in this case, the trapezium rule underestimates the true value. (1 mark)

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**14** **P1:** The curve $y = e^{x^2}$ between $x = 0$ and $x = 1$ is rotated through $2\pi$ radians about the $x$-axis to form a solid of revolution. Find the volume of this solid. (4 marks)

**15** **P2:** The acceleration of a small rocket-propelled craft is given by $a = 4e^{-0.2t}\,\text{ms}^{-2}$.

**a** Find an expression for the velocity of the craft, $v\,\text{ms}^{-1}$, as a function of time $t$s, given that the craft started from rest. (4 marks)

**b** Find an expression for the displacement of the craft, $s$ m, as a function of time $t$s, given that the craft started from the origin. (4 marks)

**c** Write down the limiting velocity of the craft as $t$ gets very large. (1 mark)

**d** Find the craft's displacement one minute after it begins to move. (2 marks)

**16** **P2:** A differential equation is defined by $\frac{dy}{dx} = xy$, $x > 0$, $y > 0$ where $y(1) = 2$.

**a** Use Euler's method with a step size of 0.25 to find an approximation for $y(2)$. For each value of $x$, show the corresponding value of $y$ in your working. (5 marks)

**b** Solve the differential equation, giving the answer in the form $y = f(x)$ for some function $f$. (7 marks)

**c** Hence, find the exact value of $y(2)$. (2 marks)

**d** Find the absolute percentage error in the value of $y(2)$ given by Euler's method. Give your answer to 2 significant figures. (2 marks)





**17** **P1:** **a** Find the derivative of $x^2 \sin x$. (3 marks)

**b** Hence find $\int 4\cos x + 6x\sin x + 3x^2\cos x\,dx$. (2 marks)

**18** **P2:** In a slope field, an *isocline* is defined as a curve on which the gradient is constant.

In other words, an *isocline* is a curve where $\frac{dy}{dx} = k$, with $k$ a constant.

Consider the slope field of the following differential equation

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}, x \neq 0, y \neq 0.$

**a** State which of the following equations are isoclines of the above differential equation.

If an equation is an isocline, state the constant value of the gradient.

If an equation is not an isocline, justify why it is not.

**i** $x^2 + y^2 - 8xy = 0$

**ii** $y = 1$

**iii** $y = x$

**iv** $y = x + 1$

**v** $y = -x$ (10 marks)

**b** Determine whether the solution to the differential equation

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}, x \neq 0, y \neq 0$ has any turning points. Justify your answer. (3 marks)

Click here for further exam practice

Calculus

# In the footsteps of Archimedes

**Approaches to learning:** Research, Critical thinking

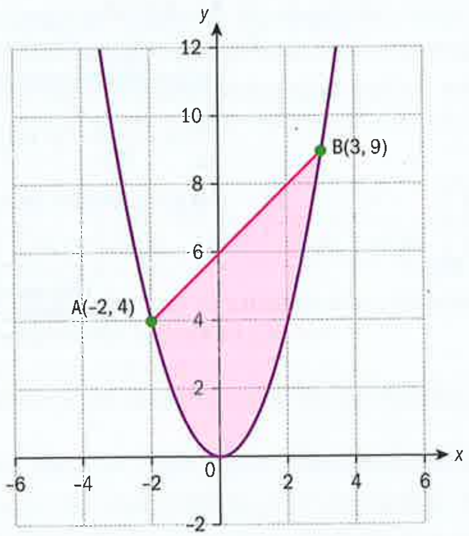
**Exploration criteria:** Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

**IB topic:** Integration, Proof, Coordinate geometry

## The area of a parabolic segment

A **parabolic segment** is a region bounded by a parabola and a line.

Consider this shaded region which is the area bounded by the line $y = x + 6$ and the curve $y = x^2$:



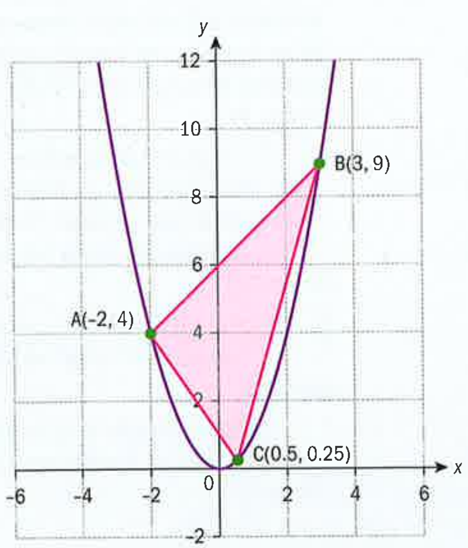

From this chapter you know that you can calculate the shaded area using integration.

On the diagram points A(−2, 4) and B(3, 9) are marked.

Point C is such that the $x$-value of C is halfway between the $x$-values of points A and B.

What are the coordinates of point C on the curve?

Triangle ABC is constructed as shown:




Archimedes showed that the area of the parabolic segment is $\frac{4}{3}$ of the area of triangle ABC.





Calculate the area of the triangle shown.

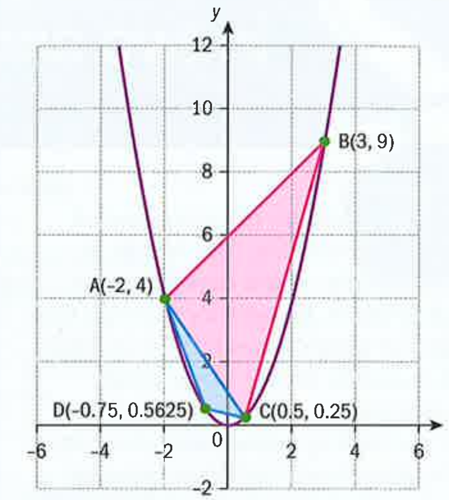
What methods are available to calculate the area of the triangle?

Use integration to calculate the area between the two curves.

Hence verify that Archimedes' result is correct for this parabolic segment.

You can show that this result is true for any parabola and for any starting points A and B on the parabola.

Consider another triangle by choosing point D on the parabola such that its $x$-value is halfway between the $x$-values of A and C, similar to before.




What are the coordinates of point D?

Calculate the area of triangle ACD.

Similarly, for line BC, find E such that its $x$-value is half-way between C and B.

What are the coordinates of point E?

Hence calculate the area of triangle BCE.

Calculate the ratio between the areas of the new triangles and original triangle ABC.

What do you notice?

You can see already that if you add the areas of triangles ABC, ACD and BCE, you have a reasonable approximation for the area of the parabolic segment.

You can improve this approximation by continuing the process and forming four more triangles from sides AD, CD, CE and BE.

If you add the areas of these seven triangles, you have an even better approximation.

How could the approximation be improved?

## Generalise the problem

Let the area of the first triangle be $X$.

What is the total area of the next two, four and eight triangles in terms of $X$?

If you continued adding the areas of an infinite number of such triangles, you would have the *exact* area for the parabolic segment.

By summing the areas of all the triangles, you can show that they form a geometric series.

- What is the common ratio?
- What is the first term?
- What is the sum of the series?
- What has this shown?

### Extension

This task demonstrates the part of the historical development of the topic of limits which has led to the development of the concept of calculus.

Look at another area of mathematics that you have studied on this course so far or one that interests you looking forward in the book.

- What is the history of this particular area of mathematics?
- How does it fit into the development of the whole of mathematics?
- How significant is it?
- Who are the main contributors to this branch of mathematics?