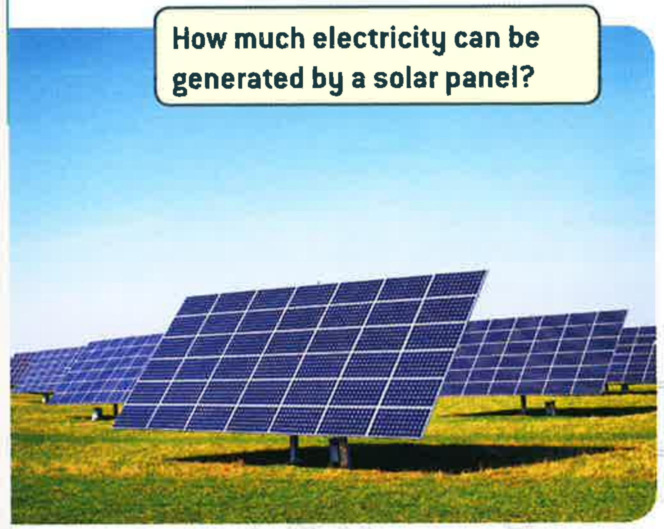
# 12 Modelling motion and change in two and three dimensions

**Concepts**
- Change
- Modelling

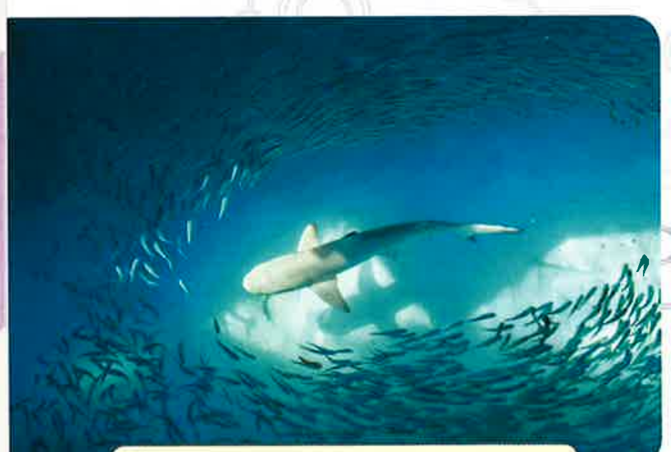
**Microconcepts**
- Vector quantities
- Component of a vector in a given direction evaluated using scalar or vector product
- Acceleration
- Two-dimensional motion with variable velocity
- Projectiles
- Phase portraits
- Euler method with three variables
- Coupled systems
- Eigenvalues and eigenvectors
- Asymptotic behaviour
- Second order differential equations

Change in a single variable is often connected to changes in another variable. The height of a projectile, for example, might depend on horizontal displacement, its velocity through the air and the time for which it has been travelling. Growth in a population, on the other hand, might depend not just on resources but on the size of another competing population. How can you develop the ideas from earlier chapters to predict change in complex systems such as these?

**How much electricity can be generated by a solar panel?**



**How much will a building move during an earthquake?**

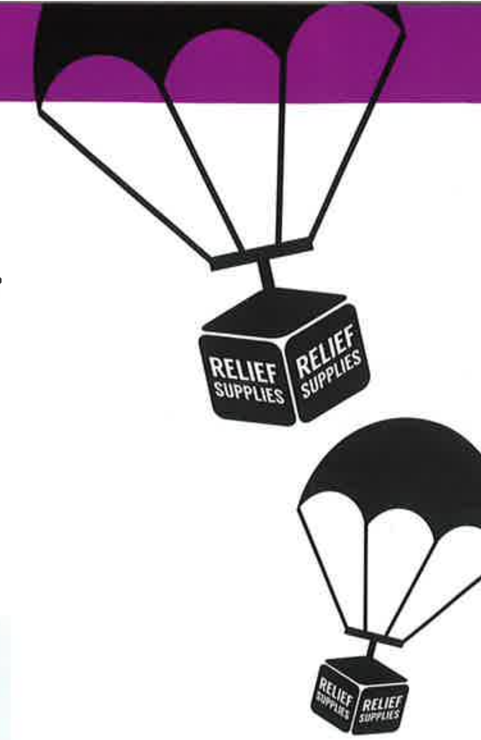


**How might the shark population be affected by changes in the fish population?**



An aircraft needs to deliver a supply package to a polar research station. The package will be dropped from the aircraft from a height of 150 m. If the aircraft is flying at 180 km h$^{-1}$ how far from the research station would the package need to be released if air resistance can be ignored?

- Make a guess at a possible answer to the question above.
- How would your answer change if the aircraft still had the same speed at the same height but was ascending at an angle of 30°?
- If air resistance cannot be ignored would the package need to be released further from or closer to the research station?




## Developing inquiry skills

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

## Before you start

Click here for help with this skills check

| You should know how to: | Skills check |
|---|---|
| **1** Integrate exponential and trigonometric functions.<br>eg Find $\int 3e^{2x} + 2\sin 5x\,dx$<br>$\int 3e^{2x} + 2\sin 5x\,dx = \frac{3}{2}e^{2x} + \frac{2}{5}\cos 5x + c$ | **1** Find:<br>**a** $\int 6e^{3x} - \frac{1}{e^{2x}}dx$<br>**b** $\int 3\cos 6x - 4\sin 2x\,dx$ |
| **2** Find and use the scalar and vector products of two vectors.<br>eg<br>**a** Find the angle between $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $2\boldsymbol{i} - 3\boldsymbol{k}$.<br>First take the scalar product<br>$1 \times 2 + 2 \times 0 + (-1) \times (-3) = 5$<br>If $\theta$ is the angle between the two vectors then<br>$\cos\theta = \frac{5}{\sqrt{6}\sqrt{13}} \Rightarrow \theta = 55.5°$ | **2 a** Find the angle between<br>$\begin{pmatrix}2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}-4\\2\\3\end{pmatrix}$ |

Continued on next page

**b** Find a vector perpendicular to

$\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\0\\-3\end{pmatrix}$

A vector perpendicular to two vectors will be parallel to the vector product of the two vectors.

$$\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}2\\0\\-3\end{pmatrix}=\begin{pmatrix}-6\\1\\-4\end{pmatrix}$$

**3** Find eigenvalues and eigenvectors.
eg Find the eigenvalues and eigenvectors for

$\begin{pmatrix}2&3\\1&4\end{pmatrix}$

First solve the equation

$$\begin{vmatrix}2-\lambda&3\\1&4-\lambda\end{vmatrix}=0$$

$(2-\lambda)(4-\lambda)-3=0$

$\lambda^2-6\lambda+5=0\Rightarrow\lambda=1,5$

For the eigenvalues above, solve

$\lambda=1, \begin{pmatrix}1&3\\1&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$x+3y=0\Rightarrow x=-3y$

Eigenvectors are parallel to $\begin{pmatrix}-3\\1\end{pmatrix}$

Similarly the second eigenvector is $\begin{pmatrix}1\\1\end{pmatrix}$

**b** Find a vector perpendicular to

$\begin{pmatrix}2\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-2\\2\\3\end{pmatrix}$

**3** Find the eigenvalues and eigenvectors for

$\begin{pmatrix}2&2\\5&-1\end{pmatrix}$





# 12.1 Vector quantities

In Chapter 3 you met the idea of using vectors to describe position and straight-line movement in two-dimensional and three-dimensional space.

In the 19th century, physicists extended this idea of vectors to many other quantities. They realized that physical quantities could be usefully divided into those that needed a direction, as well as size, to define them completely, and those that did not.

For example an object's mass does not need a direction but its velocity does. A velocity of 10 $ms^{-1}$ to the right is very different from 10 $ms^{-1}$ to the left.

Quantities which need a direction to define them fully were classed as **vectors** and those that do not were called **scalars**.

Vector quantities include:

- velocity
- acceleration
- force
- magnetic and electric fields
- momentum.

**EXAM HINT**

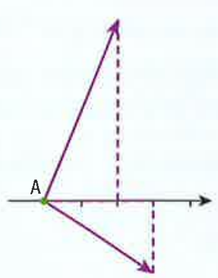
In exams you will not need to understand the concepts of the quantities being used except for those listed in the syllabus (displacement, velocity and acceleration). Other quantities might appear in questions but the context will make it clear how the question is to be answered.

The results obtained for displacement vectors in Chapter 3 apply equally well to any vector quantity.
In particular the single vector which will have the same effect as several vectors acting together is called the **resultant** and is equal to the sum of all the vectors.

## Example 1

A force is given by the vector $\begin{pmatrix}2\\5\end{pmatrix}$ Newtons (N) and a second force by $\begin{pmatrix}3\\-2\end{pmatrix}$ N relative to horizontal and vertical axes.

A particle (A) is subjected to both these forces as shown.




**1** What is the total horizontal force it experiences?
**2** What is the total vertical force it experiences?
**3** Hence write down as a column vector the resultant force experienced by the particle and show this force on a diagram.
**4** Find the magnitude of the resultant force experienced by the particle.

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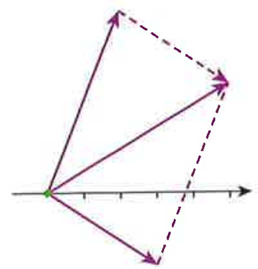
| | |
|---|---|
| 1 $2+3=5\text{N}$ | The total horizontal force is found by adding the two horizontal components. |
| 2 $5-2=3\text{N}$ | The total vertical force is found by adding the two vertical forces. |
| 3 $\begin{pmatrix}5\\3\end{pmatrix}\text{N}$  | The resultant force is the sum of the two vectors. It is equivalent to the two individual vectors so its direction is the direction the particle would move if acted on by no other forces. As can be seen from the diagram it can also be found geometrically by drawing the individual vectors following on from each other. |
| 4 $\sqrt{5^2+3^2}=5.83\text{ N}$ | The magnitude of the resultant is not the sum of the magnitudes of the individual vectors unless they are all acting in the same direction. If not, parts of the individual vectors act against each other and so cancel each other out. |

## Exercise 12A

1 Find the magnitude and direction of the following forces. Give the direction as an angle measured counter-clockwise from the positive $x$-axis.

a $(6\boldsymbol{i}+3\boldsymbol{j})$N  b $\begin{pmatrix}-3\\4\end{pmatrix}$N

2 Write the following velocities as column vectors:

a $7\text{ms}^{-1}$ on a bearing of 045°

b $12\text{ms}^{-1}$ on a bearing of 330°.

3 A particle experiences the following accelerations as a result of three different forces acting on it $\begin{pmatrix}2\\4\end{pmatrix}\text{ms}^{-2}$, $\begin{pmatrix}-1\\2\end{pmatrix}\text{ms}^{-2}$ and $\begin{pmatrix}-2\\0\end{pmatrix}\text{ms}^{-2}$

Find the magnitude and direction of the resultant acceleration experienced by the particle.

4 The torque ($\tau$) produced when turning an object with a force $\boldsymbol{F}$ acting at the end of a lever a displacement $\boldsymbol{r}$ from the object is given by the equation $\tau=\boldsymbol{F}\times\boldsymbol{r}$.

a State the direction of the torque in comparison to the other two vectors.

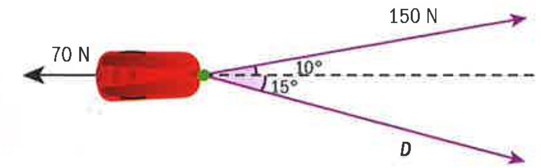
b Find the magnitude of the torque produced when a force of $\begin{pmatrix}3\\4\\0\end{pmatrix}$N is acting at a displacement $\begin{pmatrix}2\\1\\0\end{pmatrix}$ m from the object.





5 A sledge is being pulled by two dogs as shown. One of the dogs is pulling with a force of 150 N at an angle of 10° to the direction of travel and the second with a force of $D$ N at an angle of 15° to the direction of travel.

The sledge is also subject to a resistance force of 70 N as shown.




Take the components of the vectors to lie along the direction of travel and perpendicular to the direction of travel.

a Find expressions for the three forces as column vectors.

b Find the resultant force in terms of $D$.

c Find the value of $D$ if the sledge is heading in the intended direction.

Newton's second law states that the resultant force ($F$ N) acting on a body will be equal to the product of its mass (kg) and the acceleration ($\text{ms}^{-2}$) of the body ($F=ma$).

d Given that the mass of the sledge and occupant is 60 kg find the magnitude and direction of the sledge's acceleration.

## Components of vector quantities in given directions

It is often important to know how much of a vector quantity is acting in a given direction.

For example, if a book is lying on a table and is subject to a force of $3\boldsymbol{i}+4\boldsymbol{j}$ N then 3 N will be acting to move the book along the table and 4 N will be acting to lift the book off the table. Whether or not this will result in the book moving depends on the frictional force in the first case and the weight of the book in the second.

The situation is less straightforward when we are trying to find the component of the vector which is acting in a direction which does not lie along one of our base vectors.

### Investigation 1

Let $\theta$ be the angle between two non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Think of the direction of $\boldsymbol{b}$ as being along one of the axes.

1 Show that the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be written as $|\boldsymbol{a}|\cos\theta$ where $|\boldsymbol{a}|$ is the magnitude of $\boldsymbol{a}$.

2 Find a similar expression for the component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ within the plane formed by the two vectors.

3 Rewrite the expression for the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ in terms of the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$.

4 Rewrite the expression for the component of $\boldsymbol{a}$ perpendicular to the direction of $\boldsymbol{b}$ in terms of the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Continued on next page

**International-mindedness**

Greek philosopher and mathematician, Aristotle, calculated the combined effect of two or more forces called the Parallelogram law.

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5 Hence find the components of $\begin{pmatrix}1\\2\\1\end{pmatrix}$

a in the direction of $\begin{pmatrix}-3\\4\\0\end{pmatrix}$ b perpendicular to $\begin{pmatrix}-3\\4\\0\end{pmatrix}$.

6 **Conceptual** How do you find the component of a vector $\boldsymbol{a}$ acting

i parallel to a vector $\boldsymbol{b}$ ii perpendicular to a vector $\boldsymbol{b}$?

Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}$.

The component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$, within the plane defined by the two vectors, is $\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{b}|}$.

**International-mindedness**

René Descartes used $(x, y, z)$ to represent points in space in the 17th century.

In the 19th century, Arthur Cayley reasoned that we might go further than three values.

## Example 2

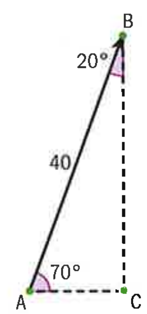
A sailboat is travelling on a bearing of 045°. The wind is blowing with a velocity of 40 km h$^{-1}$ **from** a direction of 200°.

**a** Write down the velocity of the wind as a column vector.

**b** Find the component of the velocity of the wind in the same direction as the path of the boat.

**a** $\begin{pmatrix}40\cos 70°\\40\sin 70°\end{pmatrix}=\begin{pmatrix}13.7\\37.6\end{pmatrix}$

A diagram is helpful to make sure that the direction of the vector is correct.




**b** Direction of sailboat $=\begin{pmatrix}1\\1\end{pmatrix}$

Required component

$=\frac{13.7\times1+37.6\times1}{\sqrt{1^2+1^2}}=36.3\,\text{km h}^{-1}$

Any vector in the required direction will give the same answer, as the scalar product is divided by the magnitude.





## Exercise 12B

1 **a** Find the magnitudes of the components of the vector $\begin{pmatrix}1\\4\end{pmatrix}$ parallel and perpendicular to the vector $\begin{pmatrix}-1\\2\end{pmatrix}$.

**b** Find the magnitudes of the components of the vector $\begin{pmatrix}2\\0\\5\end{pmatrix}$ parallel and perpendicular, in the plane defined by the two vectors, to the vector $\begin{pmatrix}2\\2\\-1\end{pmatrix}$.

2 Two forces are pulling a truck of mass 150 kg along a rail. The direction of the rail is $\begin{pmatrix}12\\5\\0\end{pmatrix}$ and the two forces are $\begin{pmatrix}16\\8\\2\end{pmatrix}$ N and $\begin{pmatrix}18\\10\\5\end{pmatrix}$ N.

**a** Find the component of the resultant force in the direction of the rail.

The acceleration of a body is equal to the resultant force divided by the mass of the body.

**b** Find the acceleration of the truck along the rail, assuming no other forces act in that direction.

3 In still water a boat can travel at 8 km h$^{-1}$ on full power. On a certain day the boat is travelling in the direction $\begin{pmatrix}1\\1\end{pmatrix}$ on full power. A current is flowing with speed 10 km h$^{-1}$ in the direction of $\begin{pmatrix}3\\4\end{pmatrix}$.

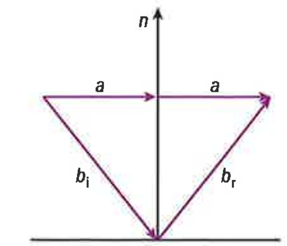
**a** Write the velocity of the current as a column vector.

**b** Find the component of the current acting in the direction the boat is travelling.

4 A computer graphics technician needs to calculate the direction of the ray reflected off a flat surface (known as ray tracing). He knows the direction of the incident ray and the direction of the normal to the surface.

Let the incident ray have direction $\boldsymbol{b}_i$, the reflected ray have direction $\boldsymbol{b}_r$ and the surface be perpendicular to the **unit** vector $\boldsymbol{n}$.

Let $\boldsymbol{a}$ be a vector perpendicular to $\boldsymbol{n}$ as shown in the diagram.




**a** Write down an expression for $\boldsymbol{b}_i$ in the form $\boldsymbol{b}_i=\boldsymbol{a}+k\boldsymbol{n}$, where $k$ is a function of $\boldsymbol{b}_i$ and $\boldsymbol{n}$.

**b** Find an expression for $\boldsymbol{b}_r$ in terms of $\boldsymbol{b}_i$ and $k\boldsymbol{n}$.

**c** A ray with direction $\begin{pmatrix}0\\-1\\4\end{pmatrix}$ strikes a surface with normal vector $\begin{pmatrix}2\\1\\-2\end{pmatrix}$. Find the direction of the reflected ray.

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# 12.2 Motion with variable velocity

A satellite is in a spiral descent back to the earth. What would the equation of its path look like? What features would it need to have?

In Chapter 3 you considered motion with constant velocity in two dimensions. In Chapters 10 and 11 you covered motion with variable velocity in one dimension. In this section the two concepts will be combined.

If the displacement of an object is given by $\boldsymbol{r}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ then

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}t} \\ \frac{\mathrm{d}f_2}{\mathrm{d}t} \end{pmatrix} \text{ and } \boldsymbol{a} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \begin{pmatrix} \frac{\mathrm{d}^2 f_1}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2 f_2}{\mathrm{d}t^2} \end{pmatrix}.$$

Integrating the acceleration vector (by integrating each component) gives the velocity vector, which can then be integrated to obtain the displacement vector. In the case of integration there will be two unknown constants which need to be found from the initial conditions.

### Example 3

A particle has velocity $\boldsymbol{v}(t)$ where $\boldsymbol{v} = (4t - 6)\boldsymbol{i} - (3t)\boldsymbol{j}$.

**a** Find the speed of the particle when $t = 3$.

**b** Find the time at which the direction of movement of the particle is parallel to $\boldsymbol{j}$.

**c** Show that the acceleration of the particle is constant.

**d** Given that the displacement of the particle when $t = 0$ is $4\boldsymbol{i}$ find an expression for its displacement at time $t$.

| | |
|---|---|
| **a** When $t = 3$, $\boldsymbol{v} = 6\boldsymbol{i} - 9\boldsymbol{j}$<br>Hence speed $= 10.8\ \mathrm{m\,s^{-1}}$ | |
| **b** $4t - 6 = 0 \Rightarrow t = 1.5$ | If parallel to $\boldsymbol{j}$ the component in the $\boldsymbol{i}$ direction must be equal to 0. |
| **c** $\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = 4\boldsymbol{i} - 3\boldsymbol{j}$ hence constant | There is no variable in the acceleration equation, so acceleration is constant.<br>Acceleration is a vector quantity and so should be left as a vector. If an answer of $\sqrt{13}\ \mathrm{m\,s^{-2}}$ were required, the question would need to ask for the magnitude of the acceleration. |
| **d** $\boldsymbol{r} = \left(2t^2 + 6t + c_1\right)\boldsymbol{i} - \left(\frac{3}{2}t^2 + c_2\right)\boldsymbol{j}$<br>At $t = 0$, $\boldsymbol{r} = 4\boldsymbol{i} \Rightarrow c_1 = 4$, $c_2 = 0$<br>$\boldsymbol{r} = \left(2t^2 - 6t + 4\right)\boldsymbol{i} - \left(\frac{3}{2}t^2\right)\boldsymbol{j}$ | Because there are two components to the vector two different constants of integration are required. |

**International-mindedness**

Different ways of representing vectors appear around the world such as row vectors being shown as <a,b>.

**TOK**

Do you think that one form of symbolic representation is preferable to another?

**International-mindedness**

Vectors developed quickly in the first two decades of the 19th century with Danish–Norwegian, Caspar Wessel, Swiss, Jean Robert Argand, and German, Carl Friedrich Gauss.

**TOK**

Why might it be argued that vector equations are superior to Cartesian equations?

## Exercise 12C

1 The position vector of a particle P at time $t$ is given as $\boldsymbol{r} = \begin{pmatrix} 4t + 2 \\ 3t^2 - t \end{pmatrix}$, where $\boldsymbol{r}$ is measured in metres and $t$ in seconds.

**a** Find the initial displacement of P.

**b** Find the initial velocity and speed of P.

**c** Show that the acceleration of P is constant and find its magnitude.

2 A particle moves with velocity $\boldsymbol{v}$ $(\mathrm{m\,s^{-1}})$ given by $\boldsymbol{v} = (4t - 1)\boldsymbol{i} + (5 - 2t)\boldsymbol{j}$.

Given the initial displacement of the particle is $2\boldsymbol{i} - \boldsymbol{j}$ from the origin O,

**a** find the displacement of the particle at time $t$

**b** find the distance of the particle from O when $t = 2$.

3 A particle P moves with acceleration $\boldsymbol{a}$ where $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\mathrm{m\,s^{-2}}$. The initial velocity of P is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}\mathrm{m\,s^{-1}}$.

**a** Find the velocity of P at time $t$.

The initial displacement of P is $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

**b** Find the displacement of P at time $t$.

4 The velocity of a charged particle in a magnetic field at time $t$ is given by $\boldsymbol{v} = (2t - 1)\boldsymbol{i} + 3t^2\boldsymbol{j}$.

**a** Find the time at which the particle is moving parallel to the vector $\boldsymbol{j}$.

**b** Find the velocity of the particle when $t = 2$ and hence the angle its direction makes with the vector $\boldsymbol{i}$.

The initial displacement of the particle is $2\boldsymbol{i} - 3\boldsymbol{j}$.

**c** Find an expression for the particle's displacement at time $t$.

**d** Find the time at which the distance of the particle from the origin is 16.

5 During the first few seconds after take-off an aircraft's acceleration $\boldsymbol{a}$ m s$^{-2}$ at time $t$ seconds ($t \geq 0$) is given by $\boldsymbol{a} = \begin{pmatrix} 6t \\ 2 \end{pmatrix}$, where the two components represent horizontal and vertical motion. When $t = 1$ the velocity of the particle is $\begin{pmatrix} 7 \\ 14 \end{pmatrix}$ m s$^{-1}$.

**a** Find an expression for the velocity at time $t$.

**b** Find the time at which the direction of motion of the aircraft is at 45° to the horizontal.

6 A particle P has velocity vector $\boldsymbol{v}$ given by $\boldsymbol{v} = \begin{pmatrix} 4t - 2.5 \\ 3t^2 \end{pmatrix}$. The initial displacement of P from an origin O is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

**a** Find an expression for the displacement of P at time $t$.

A second particle Q has a constant acceleration of $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$. Its initial velocity is 10 m s$^{-1}$ parallel to the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

**b** Find an expression for its velocity at time $t$.

**c** Given Q is initially at O show the two particles collide and state the time at which this occurs.

7 A boat has a displacement (km) given by $\boldsymbol{r} = (4t)\boldsymbol{i} + (t^2 - 3t)\boldsymbol{j}$ where $t$ is time in hours from the beginning of its motion.

Find the minimum speed of the boat and the value of $t$ at which it occurs.

## Projectiles

### Investigation 2

An object travelling under the force of gravity alone is called a projectile. For example, a baseball in flight would be a projectile.

The only acceleration experienced by a projectile is a downward acceleration of $g$ m s$^{-2}$ due to gravity. This acceleration is approximately equal to 9.81 m s$^{-2}$ at the surface of the Earth.

1 Write down, in terms of $g$, the acceleration vector for a projectile, taking the positive $y$-direction as "up".

At $t = 0$ a projectile is launched with a speed of $u$ m s$^{-1}$ at an angle of $\alpha$ to the horizontal.

2 Find the initial velocity of the projectile as a column vector in terms of $u$ and $\alpha$.

3 Use integration to find the velocity vector for the projectile at time $t$.

**International-mindedness**

Belgian/Dutch mathematician Simon Stevin used vectors in his theoretical work on falling bodies and his treatise "Principles of the art of weighing" in the 16th century.





At $t = 0$ the particle has displacement $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

4 Find an expression for the particle's displacement at time $t$.

A particle is projected from the point (0, 0) on level ground.

5 Use your answer to question **3** to find the time at which the particle reaches its maximum height.

6 Use your answer to question **4** to find an expression for this height.

7 If the initial speed of the particle is fixed, which value of $\alpha$ will result in the particle maximizing the height reached?

8 **a** Use your answer to question **4** to find the time at which the particle hits the ground.

**b** Compare your answer to part **a** with your answer to question **5**. What does this tell you about the motion of the particle?

The range of the particle is the total horizontal distance travelled until it returns to the same height as that at which it began.

9 Find an expression for the range of the particle.

10 If the initial speed of the particle is fixed which value of $\alpha$ will result in the particle maximizing its range?

A ball is projected from ground level with an initial speed of 15 m s$^{-1}$ at an angle of 30°. The acceleration due to gravity is 9.81 m s$^{-2}$.

11 Find the horizontal distance travelled by the ball before it hits the ground.

12 Find the maximum height reached by the ball.

13 In a more realistic model what additional force will need to be taken into account when predicting the flight of a ball?

14 **Factual** What does a column vector describe in projectile motion?

15 **Conceptual** How are vectors useful in describing projectile motion and how do we find the velocity equation?

### Exercise 12D

In the following questions take $g = 9.81$ m s$^{-2}$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are unit vectors in the horizontal and vertical directions respectively. All external forces, such as air resistance, may be ignored.

1 An object is projected with an initial velocity of $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ m s$^{-1}$ from a point 1.5 m above ground level.

**a** Write down the acceleration vector for the particle while in flight.

**b** Find an expression for

**i** the velocity

**ii** the displacement

of the object $t$ seconds after it has been projected.

**c** Find the time at which the particle strikes the ground and the horizontal distance it has travelled during its flight.

2 A stone is thrown from the top of a vertical cliff 50 m high with an initial velocity of $4\boldsymbol{i} + 2\boldsymbol{j}\ \text{m s}^{-1}$.

- **a** Write down the velocity vector for the stone at time $t$ seconds after the stone has been thrown.
- **b** Find the displacement vector for the stone, taking the point of projection as the origin of the coordinate system.
- **c** Find the time at which the stone hits the ground at the base of the cliff.
- **d** Find its distance from the base of the cliff when it hits the ground.

3 An particle is projected from ground level at an angle of 30° to the horizontal and with a speed of $14.7\ \text{m s}^{-1}$.

- **a** Write the initial velocity of the particle as a column vector.
- **b** Find an expression for the velocity of the particle at time $t$.
- **c** Find the time at which the particle attains its greatest height.
- **d** Find an expression for the displacement of the particle.
- **e** Find the greatest height reached by the particle.
- **f** Find the horizontal distance travelled by the particle while in flight.
- **g** Find the minimum speed attained by the particle during its flight. Fully justify your answer.

4 An object is projected at an angle of 30° to the horizontal from a point at ground level. It hits the ground again after 6.0 seconds.

Find the initial speed of the object.

5 A ball is thrown from a height of 1.5 m with a speed of $10\ \text{m s}^{-1}$ at an angle of 45° to the horizontal. It just clears a wall 10 m from the point of projection.

- **a** Find an expression for the displacement of the ball at time $t$.
- **b** Find the height of the wall.
- **c** Find the horizontal distance travelled by the ball before it hits the ground.

6 A particle is projected at a speed of $50\ \text{m s}^{-1}$ at an angle of 30° above the horizontal from a point 2 m above level ground.

- **a** Write down the initial velocity vector.
- **b** Find the displacement of the particle from the point of projection $t$ seconds after the particle is projected.
- **c** Find the time for which the particle's height above the ground is greater than 22 m.
- **d** Find the time at which the particle hits the ground and the horizontal distance travelled during the flight.

## Exponential and trigonometric motion

### Investigation 3

Consider a particle moving so that its position at time $t$ is given by the equation

$\boldsymbol{r} = \begin{pmatrix} r\cos\omega t \\ r\sin\omega t \end{pmatrix}$, where $r$ and $\omega$ are constant values.

1 By considering the distance of the particle from the origin at time $t$, show that the path followed by the particle is a circle centred on the origin and state its radius.





2 **a** Write down the initial position of the particle.
   **b** Give the value of $t$ when the particle first returns to its initial position.

3 **a** Find the velocity of the particle at time $t$.
   **b** Show that the speed of the particle is constant and state its value.
   **c** Show that the velocity is always perpendicular to the particle's displacement vector.
   **d** Explain what this means geometrically about the direction of the motion of the particle.

4 **a** Find the acceleration of the particle at time $t$.
   **b** Show that the acceleration is always perpendicular to the velocity vector.

Let $\boldsymbol{a}$ be the acceleration vector of the particle.

   **c** Show that $|\boldsymbol{a}| = \dfrac{|\boldsymbol{v}|^2}{r}$.
   **d** Show that $\boldsymbol{a} = k\boldsymbol{r}$ and state the value of $k$.

5 **Conceptual** How would you derive the velocity and acceleration equations for a particle moving in a circle from the general equation for its displacement?

**TOK**

How do we relate a theory to the author? Who developed vector analysis, Josiah Willard Gibbs or Oliver Heaviside?

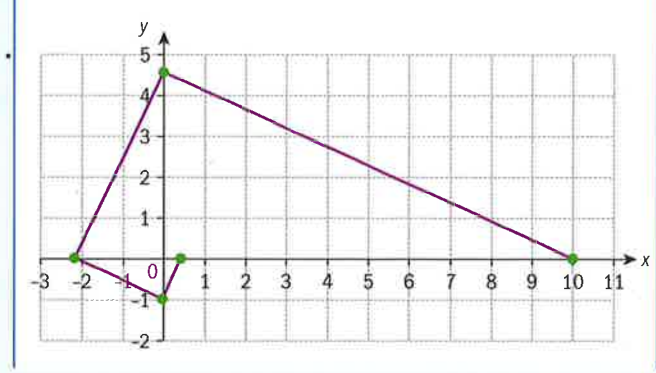
### Example 4

A particle has displacement $\boldsymbol{r}$ at time $t$ where $\boldsymbol{r} = \begin{pmatrix} 10e^{-0.5t}\cos t \\ 10e^{-0.5t}\sin t \end{pmatrix}$.

- **a** Find the position of the particle when $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $2\pi$ and show these points on a coordinate grid.
- **b** Describe the long-term behaviour of the particle.
- **c** Find the velocity of the particle at time $t$.
- **d** Find the magnitude and the direction of the velocity when $t = 0$.

**a** $\begin{pmatrix} 10 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4.56 \end{pmatrix}, \begin{pmatrix} -2.08 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.948 \end{pmatrix}, \begin{pmatrix} 0.432 \\ 0 \end{pmatrix}$

The position vectors are found by substituting the values of $t$ into the equation for the displacement.




Continued on next page

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**b** The particle will spiral towards the point $(0, 0)$.

**c** $v = \begin{pmatrix} -5e^{-0.5t}\cos t - 10e^{-0.5t}\sin t \\ -5e^{-0.5t}\sin t + 10e^{-0.5t}\cos t \end{pmatrix}$

$= \begin{pmatrix} -5e^{-0.5t}(\cos t + 2\sin t) \\ -5e^{-0.5t}(\sin t - 2\cos t) \end{pmatrix}$

The velocity is found by differentiating the displacement vector using the chain and product rules.

**d** $v = \begin{pmatrix} -5(1+0) \\ -5(0-2) \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$

Magnitude = 11.2, direction = 117° to the positive $x$-axis.

## Exercise 12E

**1** A particle moves such that its velocity at time $t$ is given by

$v = \begin{pmatrix} 2e^{2t} \\ e^{2t} + 2 \end{pmatrix}$. At $t = 0$ the particle is at $(0, 0)$.

**a** Find an expression for the particle's acceleration at time $t$.

**b** Find an expression for the particle's displacement at time $t$.

**2** A particle has acceleration at time $t$ given by

$a = \begin{pmatrix} 10e^{2t} - 1 \\ 4e^{2t} + 2 \end{pmatrix}$.

The particle is initially at rest at the origin. Find an expression for its displacement at time $t$.

**3** A particle moves so that its acceleration is given by the expression $a = \begin{pmatrix} 8\cos 2t \\ 8\sin 2t \end{pmatrix}$.

Given the particle has initial velocity of

$v = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and an initial displacement of $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$:

**a** Find an expression for the particle's displacement at time $t$.

**b** Find the particle's distance from the origin at time $t$.

**c** Hence describe the path of the particle.

**d** Given that $a = kr$ state the value of $k$.

**4** A particle moves such that its displacement at time $t$ is given by $r = \begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix}$.

**a** Find an expression for the velocity of the particle at time $t$.

**b** Find an expression for the acceleration of the particle at time $t$.

**c** Given that $a = kr$ state the value of $k$.

**5** A particle moves such that its displacement $t$ seconds after its motion begins is given by

$r = \begin{pmatrix} e^{-t}\cos 4t + 2 \\ e^{-t}\sin 4t + 5 \end{pmatrix}$.

**a** Find the particle's distance from $(2, 5)$ at time $t$.

**b** Hence describe the path of the particle.

**c** Show that the speed of the particle at time $t$ is $\sqrt{17}e^{-t}$.

**d** Find the value of $t$ at which the speed of the particle is reduced by half.





**6** A particle moves anti-clockwise at constant speed in a circle of radius 4 units centred on $(0, 0)$. Given that the time to complete one revolution is 6 seconds and the particle is initially at $(0, -4)$:

**a** Find an expression for the position of the particle at time $t$.

A second particle is moving anti-clockwise in a circle of radius 4 units centred on $(5, 4)$ with the same constant speed as the first particle. Initially it is at the point $(5, 0)$.

**b** Write down an expression for the displacement of this particle.

**c** Find the velocity vector at time $t$.

### Developing inquiry skills

Return to the opening problem for the chapter. The package will be dropped from the aircraft from a height of 150 m. The aircraft has a speed of 180 km h$^{-1}$. If air resistance can be ignored, find how far from the research station the package should be released:

**a** if the aircraft is flying horizontally

**b** if the aircraft is ascending at an angle of 30°.

## 12.3 Exact solutions of coupled differential equations

Two fungi, X and Y, are growing on a tree. X is the faster growing fungus but when it encounters Y, Y will dominate and benefit from the nutrients in X.

What will happen as the two fungi spread out?

This section will look at simple models where the growth of one variable is affected by the presence of another.

A coupled system is one in which either the equation for the derivative of $x$ contains a function of $y$ or the equation for the derivative of $y$ contains a function of $x$ or both of these.

## Investigation 4

Initially the investigation considers an uncoupled system.

1 Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x = 5$ when $t = 0$.

2 Solve the system of differential equations $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$, given that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ when $t = 0$.

The vector $\boldsymbol{x}$ is often used to stand for the general vector of the variables, so in the example above $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\boldsymbol{x}}$ will be the derivative of this vector with respect to time.

For any system of equations that can be written in the form $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$, and for which the eigenvalues of $\boldsymbol{M}$ are distinct, the solution is
$\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$

where $A, B \in \mathbb{R}$, $\lambda_1$, $\lambda_2$ are the eigenvalues of $\boldsymbol{M}$, and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are the corresponding eigenvectors.

3 a Verify the system of equations in question **2** can be written as $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$.

b Find the eigenvalues and eigenvectors of $\boldsymbol{M}$ and hence show that your solution can be written in the form $\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$, stating the values of $A$ and $B$.

4 Verify that $\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$ is a solution to $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$ more generally by differentiating the right-hand side and showing that it is equal to $\boldsymbol{Mx}$.

You will need to use the fact that from the definition of eigenvalues and eigenvectors, $\boldsymbol{Mp}_1 = \lambda_1\boldsymbol{p}_1$ and $\boldsymbol{Mp}_2 = \lambda_2\boldsymbol{p}_2$.

Two fungi, X and Y, are growing on a tree. X is the faster growing fungus but when it encounters Y, Y will dominate and benefit from the nutrients in X.

The area of tree covered by X is given as $x$ and that of Y by $y$. The growth of the two fungi in cm$^2$ can be approximately modelled by the following system of differential equations where $t$ is measured in weeks:

$\dot{x} = 0.4x - 0.2y$

$\dot{y} = 0.1x + 0.1y$

5 Explain why the signs of the different variables and the values of the coefficients indicate that these equations might be consistent with the information given.

6 Show that the right-hand side of this coupled system can be written in the form $\boldsymbol{Mx}$ where $\boldsymbol{M}$ is a $2 \times 2$ matrix and $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

7 Hence, find expressions for the areas covered by X and by Y at time $t$, given that X covers 10 cm$^2$ and Y covers 6 cm$^2$ when $t = 0$.

8 a If this model continues to be a good approximation to growth what will be the long-term ratio of the area covered by X to the area covered by Y?

b Comment on why this model is unlikely to be valid as $t$ increases.

9 **Conceptual** How are eigenvalues and eigenvectors useful when solving a system of equations?

The coupled linear system $\dot{\boldsymbol{x}} = \boldsymbol{Mx}$ has solution $\boldsymbol{x} = A\mathrm{e}^{\lambda_1 t}\boldsymbol{p}_1 + B\mathrm{e}^{\lambda_2 t}\boldsymbol{p}_2$, where $A, B \in \mathbb{R}$ and are dependent on the initial conditions.

## Example 5

**a** Find the solution to the following system of differential equations, given that $x = 4$ and $y = -2$ when $t = 0$.

$\dot{x} = 3x - 4y$

$\dot{y} = x - 2y$

**b** Determine the long-term ratio of $x$ to $y$.

| | |
|---|---|
| **a** $\begin{vmatrix} 3-\lambda & -4 \\ 1 & -2-\lambda \end{vmatrix} = 0$ <br> $\Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = 2, -1$ | The equations can be written as $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Hence the first stage is to find the eigenvalues and eigenvectors for $\begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}$. |
| Corresponding eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. | |
| Hence $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + B\mathrm{e}^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. | The form of the general solution is given in the formula book. |
| Putting in initial conditions <br> $\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 4A \\ A \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix}$ <br> $A = 2,\ B = -4$ <br> $\begin{pmatrix} x \\ y \end{pmatrix} = 2\mathrm{e}^{2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 4\mathrm{e}^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | |
| **b** The long-term ratio is 4:1. | As $t$ increases the second term tends to 0. |



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## Phase portrait

It can be useful to show on a diagram how the values of the two variables change over time. Such a diagram is called a phase portrait.

### Investigation 5

A system of linear equations has the following solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

**1** **a** Show that the equations of the lines through the origin which are in the direction of each of the eigenvectors are $y = 2x$ and $y = -x$.

**b** Choose an initial value for the system that lies on $y = 2x$, for example (2, 4), and find this particular solution to the system of equations.

**c** Choose an initial value for the system that lies on the line $y = -x$ and find this particular solution to the system of equations.

**d** What do you notice?

**2** A phase portrait indicates future **trajectories** for different initial values.

**a** Draw a set of coordinate axes and on them draw $y = 2x$ and $y = -x$.

**b** Indicate with arrows the direction of motion of any point initially on one of these lines.

**3** For any initial values not on these lines, the values of both $A$ and $B$ will be non-zero.

**a** Conjecture from your general solution the long-term trajectory for systems which are not initially on $y = 2x$ or $y = -x$.

**b** Conjecture the "history" of the trajectory by considering what happens as $t$ becomes increasingly negative.

**c** Find the particular solution for an initial value of (6, 6).

**d** Enter the top and bottom lines of your solution into a GDC and use the table function to look at the values of $x$ and $y$ as $t$ increases. Set the step interval on your table to 0.1. Consider the values of $x$ and $y$ as t increases in both the positive and negative directions. Does this support your conjectures in parts **a** and **b**?

**e** Show (6, 6) and the trajectory through that point on your phase portrait.

**f** Use technology to plot trajectories within each of the quadrants in the diagram drawn in question **3**.

**4** Now consider another system of coupled differential equations that has the following general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

**a** Write down the equations of the two lines through the origin that are parallel to the eigenvectors.

**b** By considering the values of $A$ and $B$ for a particular point on either of these two lines, describe the trajectory of any point initially on these lines.

**c** For those trajectories beginning away from the two lines given in part **a** conjecture the direction the trajectory approaches as $t \to \infty$. Verify your answer using your GDC or Geogebra.

**d** Which direction do the majority of trajectories tend towards as $t \to -\infty$?





**e** As they approach this point write down the approximate gradient for the trajectories that are not initially on either line through the origin parallel to the eigenvectors.

**f** Sketch the phase portrait for the solution given.

**g** How would your diagram change when drawing the phase portrait for the system with solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}?$$

**5** **Conceptual** Why are phase portraits a useful way of depicting the solutions to differential equations?

An equilibrium point (or equilibrium solution) is a point at which $\dot{x} = 0$ and $\dot{y} = 0$. An equilibrium point is classed as **stable** or **unstable**. If **all** points close to an equilibrium point will move towards the equilibrium point, it is stable, otherwise it is unstable.

In both the examples in Investigation 5, (0, 0) is an **equilibrium point**. At this point $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ so if a particle were initially at (0, 0) it would stay there.

In both cases (0, 0) is an **unstable** equilibrium point because particles that begin close to the origin will generally move away from it rather than towards the origin.

In the first example (0, 0) is a **saddle point**.

In the second example (0, 0) is a **source**.

When the matrix representing a coupled system of linear first order differential equations has two distinct real eigenvalues, the lines through the origin in the direction of the eigenvectors of the matrix give the gradients for the long-term behaviour of the system as $t \to \infty$ and as $t \to -\infty$.

When both eigenvalues are positive all trajectories move away from the origin as $t$ increases and towards the direction of the eigenvector associated with the larger eigenvalue. The origin is an unstable equilibrium point.

When both eigenvalues are negative all trajectories move towards the origin as $t$ increases, and towards the direction of the eigenvector associated with the more negative eigenvalue. The origin is a stable equilibrium point.

When one eigenvalue is positive and the other negative then the origin is a saddle point. As $t$ increases all trajectories not on the lines through the origin parallel to the eigenvectors move towards the direction of the eigenvector associated with the positive eigenvalue.

**TOK**

"There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world." – Nikolai Lobatchevsky

Where does the power of mathematics come from? Is it from its ability to communicate as a language, from the axiomatic proofs or from its abstract nature?

## Exercise 12F

**1 a** Find the solutions to the following systems of differential equations.

**i** $\dot{x} = 2x + 2y$
$\dot{y} = 5x - y \quad x = -6,\ y = 22$ when $t = 0$

**ii** $\dot{x} = x + 2y$
$\dot{y} = 3y \quad x = 3,\ y = 1$ when $t = 0$

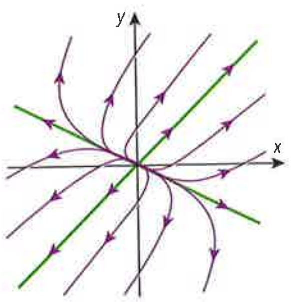
**iii** $\dot{x} = -2x + 2y$
$\dot{y} = -x - 5y \quad x = 2,\ y = 1$ when $t = 0$

**iv** $\dot{x} = x + 2y$
$\dot{y} = 3x - 4y \quad x = 1,\ y = 11$ when $t = 0$

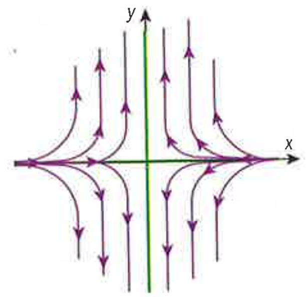
**b** Draw a phase portrait for the general solutions to the coupled differential equations obtained in part **a**. In each case give the Cartesian equations of the lines through the origin parallel to the eigenvectors.

**2** Match each of these phase portraits with the associated solution to a system of coupled differential equations.

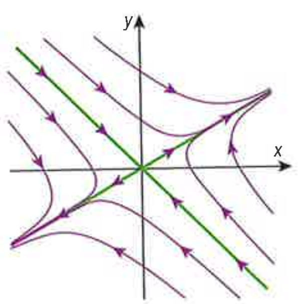
**a**



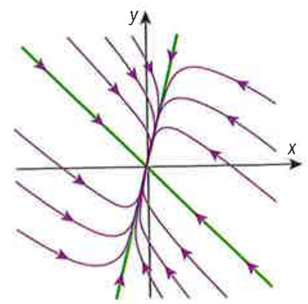

**b**




**c**




**d**




**A** $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

**B** $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**C** $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**D** $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{4t}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + Be^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

**3 a** Find the general solution of the following system of differential equations:
$\dot{x} = 2x + 3y$
$\dot{y} = x + 4y$

**b** Sketch the phase portrait for this system of equations, making clear any asymptotic behaviour.

**c** Find the equation of the particular solution given that $x = -9,\ y = -1$ when $t = 0$.





**4 a** Find the general solution of the following system of differential equations. Give all final solutions correct to 3 significant figures.
$\dot{x} = 3x + y$
$\dot{y} = 2x - y$

**b** Sketch the phase portrait for this system of equations, making clear any asymptotic behaviour.

**c** Find the equation of the particular solution given that $x = 1,\ y = 1$ when $t = 0$.

**5 a** By considering the system of equations below explain what would happen when the initial conditions are $x = 0$, $y = 0$.
$\dot{x} = ax + by$
$\dot{y} = cx + dy$

**b** By considering the phase portraits and solutions from part **a**, or otherwise, write down conditions on the eigenvalues for (0, 0) to be a stable equilibrium.

**c** Fred says that "Nearly all points close to the equilibrium point must tend towards it for the point to be stable?" Comment on Fred's statement.

**6** In Investigation 4 the area of tree covered by two fungi X and Y was given by the variables $x$ and $y$ respectively, where

$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{0.3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the values of $A$ and $B$ were dependent on the initial conditions.

**a** Draw a phase portrait for this general solution with $x, y \geq 0$.

The system of equations that led to this solution was
$\dot{x} = 0.4x - 0.2y$
$\dot{y} = 0.1x + 0.1y$

If the area covered by either of the fungi reaches 0 the fungus is extinct on that tree and will no longer grow.

**b** Write down the differential equations that will replace the system above if

**i** $x = 0$ **ii** $y = 0$

and give the general solution to these equations for times beyond this point.

Let the initial areas covered by the two fungi be $x_0$ and $y_0$ where $x_0, y_0 > 0$.

**c** Use your **phase portrait** to deduce the long-term proportions of $x:y$ if the initial conditions were such that:

**i** $0 < y_0 < \frac{1}{2}x$ **ii** $\frac{1}{2}x_0 < y_0 < x_0$

**iii** $y_0 > x_0$ **iv** $x_0 = y_0$

## Systems with imaginary or complex eigenvalues

Sometimes the system will have complex eigenvalues. You do not need to find the corresponding eigenvectors or the solution to the system in this case, but you do need to be able to draw the phase portrait and predict the long-term behaviour of the trajectories. Because complex eigenvalues will come from solutions to the characteristic equation they will always be a conjugate pair, for example 2 + 3i and 2 − 3i.

## Investigation 6

**1 a** Find the eigenvalues for the following system of equations:

$\dot{x} = 2y$

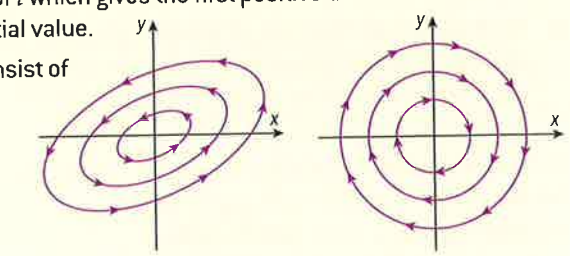
$\dot{y} = -2x$

In a similar way to the case of real eigenvalues, the general solution to a system of equations with imaginary eigenvalues $\pm bi$ can be written in the form: $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{bit}\boldsymbol{v}_1 + Be^{-bit}\boldsymbol{v}_2$, with $b \in \mathbb{R}$.

**b** State the value of $b$ for the system given in part **a** and write $e^{bit}$ in the form $\cos(bt) + i\sin(bt)$.

> **HINT**
>
> $A$, $B$ and the elements of the two eigenvectors can be complex numbers but values can be found in which both and $x$ and $y$ are real for all values of $t$. This is beyond this course but you need to note that the fact $e^{bit}$ can be written as $\cos(bt) + i\sin(bt)$ means the solution will be periodic with a period of $\frac{2\pi}{b}$.

**c** For the system given in part **a** find the value of $t$ which gives the first positive time when the coordinates $(x, y)$ are again equal to their initial value.




For imaginary eigenvalues the phase portrait will consist of concentric circles or ellipses centred on the origin.

**2 a** Find the eigenvalues for the systems below.

**A** $\dot{x} = -3x - 4y$ 
$\dot{y} = 2x + y$

**B** $\dot{x} = 2x - y$ 
$\dot{y} = x + 2y$

**b** Write down what you notice about

**i** the real part of the eigenvalue

and **ii** the imaginary part in both **A** and **B**.

**c** Explain why this is always the case.

**3** Write down a condition on $a$ for the trajectory to spiral towards the origin.

To decide whether the spiral is clockwise or counter-clockwise you can evaluate $\frac{dy}{dt}$ at a point on the $x$-axis or $\frac{dx}{dt}$ at a point on the $y$-axis.

**b** For each of the systems in question **2** find the values of $\frac{dy}{dt}$ at the point (1, 0) and deduce what this means about the direction of motion as $t$ increases.

**c** Use the chain rule to also evaluate the gradient at (0, 1).

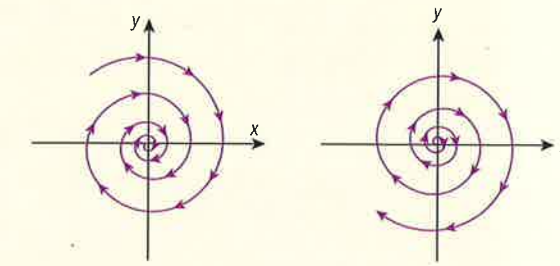
**d** Hence sketch the phase portrait for the two systems given in question **2**.

**4** **Conceptual** How is the motion of a particle different when the matrix for a coupled system has either imaginary or complex eigenvalues?





> The result from question **2** means that a solution can be written in the form $\begin{pmatrix} x \\ y \end{pmatrix} = e^{at}\left(Ae^{bit}\boldsymbol{v}_1 + Be^{-bit}\boldsymbol{v}_2\right)$ where $a, b \in \mathbb{R}$ and the periodic factor is the same as the one in question **1**.
>
> The solution will be either a clockwise or counter-clockwise spiral with the trajectories either moving towards or away from the origin as shown below.




> If the eigenvalues are $a \pm bi$ then the trajectories will:
>
> - form circles or ellipses with the origin as a centre if $a = 0$
> - spiral away from the origin if $a > 0$
> - spiral into the origin if $a < 0$.
>
> The direction of movement can be found by evaluating $\frac{dy}{dt}$ for a point on the $x$-axis or $\frac{dx}{dt}$ for a point on the $y$-axis.

**International-mindedness**

"As long as algebra and geometry have been separated, their progress has been slow and their uses limited, but when these two sciences have been united, they have lent each mutual force, and have marched together towards perfection." – 18th century French mathematician Joseph-Louis Lagrange

## Exercise 12G

**1** For each of the systems below:

**i** Find the eigenvalues.

**ii** Find $\frac{dy}{dt}$ and the gradient of the curve at the point (1, 0).

**iii** Sketch the phase portrait for the system.

**a** $\dot{x} = 2x - 5y$ 
$\dot{y} = x - 2y$

**b** $\dot{x} = -x - 3y$ 
$\dot{y} = 4x + 3y$

**c** $\dot{x} = -0.2x + y$ 
$\dot{y} = -x - 0.2y$

**d** $\dot{x} = x + 5y$ 
$\dot{y} = -5x + y$

**e** $\dot{x} = 3x - 9y$ 
$\dot{y} = 4x - 3y$

**2** A satellite is launched from the Earth. Its displacement at time $t$, measured with the Earth at the origin, is given as $(x, y)$ with the coordinates lying in the plane of the satellite's motion. The velocity of the satellite at time $t$ is given by the following system of equations:

$\frac{dx}{dt} = 2x + y$

$\frac{dy}{dt} = -x + 2y$

**a** Verify that the satellite's path will take it away from the Earth.

**b** Given that the satellite passes through the point (1, 1) sketch a possible future trajectory for the satellite.

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# 12.4 Approximate solutions to coupled linear equations

Many ecosystems contain species that are closely connected, often competing for limited resources. Is it possible to find population sizes where both species can exist together or will one always come to dominate over time? These kinds of equations are addressed through consideration of coupled non-linear differential equations.

Most systems of differential equations are not linear and cannot be solved exactly. Fortunately, there are many numerical methods that can be used to give approximate results, including the Euler method which you studied in Chapter 11.

For the equation $\frac{dy}{dx} = f(x, y)$ the Euler method equations are:

$x_{n+1} = x_n + h$, $y_{n+1} = y_n + hf(x_n, y_n)$, where $h$ is the step size.

> **HINT**
> Though easy to apply, the Euler method is not much used in practice due to its inaccuracies. Other methods include the improved Euler method and the Runge–Kutta method. But these are not in the Higher Level course.

> For the coupled case in which
> $\frac{dx}{dt} = f_1(x, y, t)$, $\frac{dy}{dt} = f_2(x, y, t)$ the Euler method equations are:
> $t_{n+1} = t_n + h$
> $x_{n+1} = x_n + hf_1(x_n, y_n, t_n)$
> $y_{n+1} = y_n + hf_2(x_n, y_n, t_n)$

> **HINT**
> This is provided in the formula book.

### Example 6

**a** Use the Euler method with a step size of 0.1 to find the approximate values of $x$ and $y$ when $t = 1$ and $t = 2$ for the following system of differential equations:

$\dot{x} = 3x - 4y$

$\dot{y} = x - 2y$

and $x = 4$ given $y = -2$ when $t = 0$.

**b** In Example 5 the exact solution to this system of differential equations was found to be

$\begin{pmatrix} x \\ y \end{pmatrix} = 2e^{2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 4e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the percentage error in your answers to part **a**.





| | |
|---|---|
| **a** The two equations are:<br>$x_{n+1} = x_n + 0.1(3x_n - 4y_n)$<br>$= 1.3x_n - 0.4y_n$<br>$y_{n+1} = y_n + 0.1(x_n - 2y_n)$<br>$= 0.1x_n + 0.8y_n$ | There is no need to simplify the equation. Only do so if you feel it will make the next stages more straightforward. |
| At $t = 1$, $x = 48.1$, $y = 11.0$<br>At $t = 2$, $x = 306$, $y = 76.2$ | There is no need to list intermediate solutions in an exam but the recurrence relation used should be given. |
| **b** The exact values from the equation are:<br>At $t = 1$, $x = 57.6$, $y = 13.3$<br>At $t = 2$, $x = 436$, $y = 109$ | Percentage errors are calculated using the formula $\frac{\text{actual error}}{\text{exact amount}} \times 100$. |
| Percentage errors are:<br>At $t = 1$, 16% and 17%<br>At $t = 2$, 30% and 30% | In this case, it is clear that the Euler method underestimates the actual values, and the errors increase as the iteration continues, though the asymptotic behaviour of the approximation is similar to the exact equation ($y$-value approximately four times $x$-value as expected from the exact equation). |

## Exercise 12H

**1** Solve each of the equations from question **1a** in Exercise 12F using the Euler method with a step size of 0.1 from $t = 0$ to $t = 1$. Use your results to sketch the trajectory from the initial point and compare your answer with the phase portrait drawn for question **1b**.

Comment on the accuracy of your answers, and in particular whether the ratio of $x$- to $y$-coordinates is tending towards the expected value.

**2** Use the Euler method with a step size of 0.1 to find approximations for the values of $x$ and $y$ when $t = 1$ for the following systems.

**a** $\frac{dx}{dt} = 2xy - x$

$\frac{dy}{dt} = 2y - xy$

$x = 1$, $y = 1$ when $t = 0$

**b** $\frac{dx}{dt} = 2x^2 - xy$

$\frac{dy}{dt} = 2xy - y^2$

$x = 1$, $y = 1$ when $t = 0$

**3** Use the Euler method with a step size of 0.1 to find approximations for the values of $x$ and $y$ when $t = 0.5$ for the following systems.

**a** $\frac{dx}{dt} = -2tx + 3y^2$

$\frac{dy}{dt} = -3x^2 + 3ty$

$x = -1$, $y = 2$ when $t = 0$

**b** $\frac{dx}{dt} = 2tx + y$

$\frac{dy}{dt} = -3x^2 + ty$

$x = 0$, $y = 1$ when $t = 0$

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## Predator–prey and other real-life models

### Investigation 7

When a population X of size $x$ increases at a steady rate $\alpha$, proportional to the size of the population, we can write the differential equation describing its rate of growth as $\frac{dx}{dt} = ax$. This equation leads to exponential growth.

Suppose now a different species Y, with population size $y$, is in competition with X.

The rate of increase of $x$ is now affected by the size of $y$, such that as $y$ increases the rate of increase of $x$ will diminish.

The simplest equation that will do this is $\frac{dx}{dt} = (a - by)x$.

As the population of Y ($y$) increases, the rate of growth of X decreases as it depends on $a - by$.

If Y is competing with X for the same resources then the equation for the growth in population of Y is likely to have a similar structure:

$$\frac{dy}{dt} = (c - dx)y$$

However, if species Y is a predator and species X is the prey then an equation of the form

$\frac{dy}{dt} = (cx - d)y$ might be more appropriate.

1 Explain why this equation might be more appropriate than the previous one for a predator–prey situation.

The populations, **in thousands**, of fish ($x$) and of sharks, **in hundreds** ($y$) at time $t$ (years) are given by the equations:

$$\frac{dx}{dt} = 3x - 3xy \text{ and } \frac{dy}{dt} = xy - 2y$$

2 Find the two equilibrium positions for the populations of fish and sharks.

Initially there are 3000 fish and 20 sharks.

3 Use the Euler method with a step size of 0.02 years to show that when $t = 0.02$ there are approximately 3144 fish and 20.4 sharks.

Use either a spreadsheet or a graphing calculator to answer the questions below.

4 a Plot the values of $x$ and $y$ on a set of axes for $0 \le t \le 3$. Plot values every 0.2 years.

b Join the points to create a trajectory.

What does this suggest about the non-zero equilibrium point?

5 Explain from a consideration of the differential equations why the population of sharks continues to grow once the population of fish has started to fall.

6 If using a spreadsheet draw a graph of $x$ and $y$ on the same axes, with $t$ on the horizontal axis. Comment on the significant features of the graph.

7 **Conceptual** How can we develop a predator–prey model and what can the model display?

**HINT**

These types of equations are often referred to as the **Lotka–Volterra equations.**





### Exercise 12I

1 In the predator–prey model given by the following equations let $P$ be the population of the predators and $Q$ the population of the prey. All population values are measured in 1000s of individuals and time is measured in years.

$$\frac{dP}{dt} = (Q - 2)P$$

$$\frac{dQ}{dt} = (1 - P)Q$$

a Find the two equilibrium points for the populations.

b Use the equations to explain what would happen to the population of the prey if the predators were absent ($P = 0$).

c Use the equations to explain what would happen to the population of the predators if the prey were absent ($Q = 0$).

d If the initial population of prey is 2000 and of the predators is 2000 use the Euler method with a step size of 0.1 year to find the population of each after 1 year.

2 Populations of rabbits and foxes live together in a large area of countryside.

The numbers in each population are linked by the following equations:

$$\frac{dx}{dt} = x - 2xy \text{ and } \frac{dy}{dt} = 1.8xy - 1.5y$$

where $x$ is the size of the population of rabbits (in 1000s) and $y$ is the population of foxes (in 100s).

a Find the non-zero equilibrium position for the populations of foxes and rabbits.

Initially there are 750 rabbits and 100 foxes ($x = 0.75$, $y = 1$).

b By putting the initial values into the differential equations state whether the populations of foxes and rabbits are initially increasing or decreasing.

c From the differential equations find:

i the size of the fox population when the rabbit population is a maximum

ii the size of the rabbit population when the fox population is a maximum.

d Using technology verify your answers to parts **b** and **c** and describe the change in the populations over one cycle.

**HINT**

Because of the approximations involved in the Euler method the trajectory becomes inaccurate and spirals out after the first cycle, rather than repeating the same cycle.

3 Two species of grazing animals, X and Y, are grazing on the same grass in competition with each other. The growth of their populations can be given by the following differential equations:

$$\frac{dx}{dt} = (2 - y)x$$

$$\frac{dy}{dt} = (3 - x)y$$

where $x$ is the population of X and $y$ is the population of Y, measured in hundreds.

a Use the Euler method with a step size of 0.1 to find the long-term outcome for the two populations if the initial population of X is 400 and of Y is 300. You may assume that when a population size is less than 1 ($x$ or $y < 0.01$) the population is extinct.

Use the points evaluated to plot a trajectory for the populations.

b i Write down the differential equation for the population growth of Y if X becomes extinct.

ii Hence write down a general equation for the population of Y if $t$ is the time from the extinction of X and $A$ is the population of Y at this time.

**c** Use the fact that $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ to explain why the axes are asymptotes.

**d** From the differential equation explain why:

**i** If the initial populations have $x > 3$, $y > 2$ the population of Y will decrease.

**ii** If the initial populations have $x < 3$, $y > 2$ the population of X will decrease.

**e** State the two equilibrium points in the system.

**f** Draw a phase portrait for the populations.

## Second order differential equations

A second order differential equation will contain a second derivative. We shall consider ones that can be written in the form:

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) \text{ or } \ddot{x} = f(x, \dot{x}, t).$$

These types of equations are a common occurrence in physics.

The questions can be approached by using the substitution $y = \frac{dx}{dt}$, and hence $\frac{d^2x}{dt^2} = \frac{dy}{dt}$, and solving the resulting system of equations:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = f(x, y, t)$$

**EXAM HINT**

An exam question is likely to set the differential equation in a context, but no knowledge of contexts outside the higher level Mathematics syllabus will be required to answer the question.

### Example 7

A company's pricing policy is given by the following differential equation:

$$\frac{d^2P}{dt^2} + 5\frac{dP}{dt} + 18P = 117$$

Given that at $t = 0$ the price is \$10 and the rate of change of price is \$1, use the Euler method to find the long-term stable price for the product.

| | |
|---|---|
| Let $\frac{dP}{dt} = y$. | |
| $\frac{dy}{dt} = -5y - 18P + 117$ | Because of the 117 this equation cannot be solved by considering the eigenvalues, so the Euler method must be used. |
| $y_{n+1} = y_n + 0.1(-5y_n - 18P_n + 117)$ | |
| $P_{n+1} = P_n + 0.1y_n$ | |
| $P_0 = 20$, $y_0 = 1$ | Repeated iterations of the Euler method lead to a stable price. |
| Stable price is \$6.50. | |





### Exercise 12J

**1** By writing each as a coupled first order differential equation solve the following equations to find $x$ in terms of $t$.

**a** $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$

$x = 2$, $\frac{dx}{dt} = 4$ when $t = 0$

**b** $\ddot{x} - 3\dot{x} - 4x = 0$, $x = 0$, $\dot{x} = 5$ when $t = 0$

**c** Draw a phase diagram for your solution and comment on the long-term relationship between $x$ and $\frac{dx}{dt}$.

**2** A particle moves such that its distance from the origin is given by the differential equation $\frac{d^2y}{dt^2} + 3\frac{dx}{dt} + 2x = 6t + 4$.

**a** Use the substitution $y = \frac{dx}{dt}$ to form a system of equations.

When $t = 0$, $x = 6$ and $\frac{dx}{dt} = 5$.

**b** Use the Euler method with a step size of 0.1 to find the value of $x$ when $t = 0.2$.

**c** Find the minimum value of $x$ as given by the Euler approximation and the value of $t$ at which it occurs.

**d** Find the limit of the velocity of the particle as time increases and hence write down an approximate solution for the particle's distance from the origin for large values of $t$.

**3** A weight on a spring undergoes forced oscillations about an equilibrium point. The distance $x$ cm from the point is given by the differential equation:

$$3\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 6x = 15\cos 5t$$

**a** Write this equation as a system of linear equations.

The weight is displaced by 2.5 cm and released $\left(\frac{dx}{dt} = 0\right)$.

**b** Use the Euler method with a step size of 0.1 to find the values of $x$ from $t = 0$ to $t = 6$. Show the values on a graph with time on the horizontal axis. Comment on the key features of the graph.

**HINT**

Part **b** is best done on a spreadsheet.

## Chapter summary

**Components of vectors**

- Acceleration and force are examples of vector quantities.
- The resultant vector has the same effect as all the individual vectors acting together.
- The component of vector $\boldsymbol{a}$ acting in the direction of vector $\boldsymbol{b}$ is $\frac{|\boldsymbol{a} \cdot \boldsymbol{b}|}{|\boldsymbol{b}|} = |\boldsymbol{a}|\cos\theta$.
- The component of a vector $\boldsymbol{a}$ acting perpendicular to vector $\boldsymbol{b}$, in the plane formed by the two vectors, is $\frac{|\boldsymbol{a} \times \boldsymbol{b}|}{|\boldsymbol{b}|} = |\boldsymbol{a}|\sin\theta$.

**Two-dimensional motion with variable velocity**

- The acceleration vector can be integrated to find the velocity vector, which can be integrated to find the displacement vector.

Continued on next page

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- Projectile motion occurs when the only acceleration is that due to gravity. The velocity and displacement vectors can be found by integrating $a = \begin{pmatrix} 0 \\ -g \end{pmatrix}$.
- The maximum height can be found when the vertical component of the velocity is equal to zero and the range found by setting the vertical component of the displacement to zero.
- Motion in a circle of radius $r$ and centre (0, 0) can be given as $r = \begin{pmatrix} r\cos\omega t \\ r\sin\omega t \end{pmatrix}$. Acceleration and velocity for motion in a circle can be found by differentiating the expression for the displacement.

**Coupled differential equations**

- Solutions can be found to a linear coupled system if the eigenvalues are distinct and real.
- If the system is written in the form $\dot{x} = Mx$ the solution is:
  $x = Ae^{\lambda_1 t}p_1 + Be^{\lambda_2 t}p_2$
  where $A, B \in \mathbb{R}$, $\lambda_1$, $\lambda_2$ are the eigenvalues of $M$, and $p_1$ and $p_2$ are the corresponding eigenvectors.
- Equilibrium points occur when all derivatives are equal to zero. If all points close to the equilibrium point move towards it, it is a stable equilibrium, if not it is unstable.
- Complex and imaginary eigenvalues lead to spiral or periodic motion. If the real part of the eigenvalue is positive the spiral will move away from the origin. If it is negative it will move towards the origin.
- Systems of non-linear coupled differential equations can be solved using the Euler method.
- For $\frac{dx}{dt} = f_1(x, y, t)$, $\frac{dy}{dt} = f_2(x, y, t)$ the Euler method equations are:
  $t_{n+1} = t_n + h$
  $x_{n+1} = x_n + hf_1(x_n, y_n, t_n)$
  $y_{n+1} = y_n + hf_2(x_n, y_n, t_n)$
- Second order differential equations can be written as a first order coupled system by letting $\dot{x} = y$.

## Developing inquiry skills

A plane needs to deliver a supply package to a polar research station. The package will be dropped from the plane from a height of 150 m. The plane has a speed of 180 km h$^{-1}$ and is flying horizontally. Air resistance acts on the package in such a way that the horizontal acceleration ($\ddot{x}$) and the vertical acceleration ($\ddot{y}$) are given by the following equations:

$\ddot{x} = -0.05\dot{x}$ $\qquad \ddot{y} = -9.8 + 0.02\dot{y}^2$

**a** **i** Use the substitution $u = \dot{x}$ to write $\ddot{x} = -0.05\dot{x}$ in terms of $u$.

**ii** Hence write down an equation for $u$ in terms of $t$, the time in seconds from when the package was dropped.

**iii** Find an equation for $x$, the horizontal distance travelled by the package.

**iv** If the package could be released from any height what would be the maximum horizontal distance from the research station that the package could be released.

**b** Use Euler's method with a step length of 0.1 seconds to find:

**i** the time at which the package will hit the ground to the nearest tenth of a second

**ii** the vertical speed at which the package will hit the ground.

**c** Hence find:

**i** the distance from the research station at which the package should be released

**ii** the speed (the magnitude of the resultant of horizontal and vertical velocity) at which the package hits the ground.

# Chapter review

Click here for a mixed review exercise

**1** **a** Find the general solution for the following system of linear differential equations:

$\dot{x} = y$

$\dot{y} = 6x - y$

**b** Sketch the phase portrait for the system giving the equations of any asymptotes.

**2** The displacement, $x$ cm, of an object from an equilibrium position is given by the equation

$\frac{d^2y}{dt^2} + 0.4x + e^{0.2t} = 0$ and $x = 3$, $\dot{x} = 0$ when $t = 0$ and $t$ is measured in seconds.

Use the Euler method with a step size of 0.1 to find:

**a** the time when the displacement is first 1.0, to 2 significant figures

**b** the speed of the object at this point.

**3** A particle has velocity given by the vector

$v = \begin{pmatrix} 2t^{\frac{1}{2}} + 1 \\ 2t \end{pmatrix}$.

**a** Given the particle is at the origin when $t = 0$ find the position of the particle at time $t$.

**b** Find the distance of the particle from the origin when $t = 1$.

**4** Populations of lions ($L$) and zebra ($Z$) inhabit a national park. The two populations can be modelled by the following differential equations, where the lion population is measured in 10s and the zebra population in 1000s:

$\frac{dL}{dt} = 2L + LZ - 2L^2$

$\frac{dZ}{dt} = 2Z - LZ$

**a** Find the three equilibrium points for this system.

**b** Use the Euler method with a step size of 0.1 years to find the population after two years for initial populations of:

**i** 30 lions and 4000 zebra

**ii** 50 lions and 1000 zebra.

**c** Using your findings, explain whether it is possible to have a stable population of lions and zebra in the national park.

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**5** The energy expended by a force in moving an object is called the "work done". If a force $\boldsymbol{F}$ moves an object from point A to point B, the work done ($W$) is given by the equation $W = \boldsymbol{F} \cdot \overrightarrow{AB}$.

**a** Find the work done by a force of $\begin{pmatrix}5\\6\end{pmatrix}$ newtons (N) in moving an object from (1, 3) to (6, 7) metres (the units of work done are the same as the units of energy, joules (J)).

**b** If the direction of the force is the same as the direction of the motion, explain why the equation for work done can be written as $W = |\boldsymbol{F}| \times |\overrightarrow{AB}|$.

**c** An aircraft takes off in a straight line from the point (0, 0, 0) to the point A(40, 55, 3). Given the force produced by the engines is 100 000 N, use the result from part **b** to find the energy expended by the engine during this part of the flight.

**6** The force produced by two engines in an aircraft during the first minute of its flight can be represented by the two vectors $\begin{pmatrix}62\\11\\21\end{pmatrix}$ and $\begin{pmatrix}62\\-11\\21\end{pmatrix}$, measured in kN (1 kN = 1000 N).

The force due to the air passing over the wings is $\begin{pmatrix}0\\0\\330\end{pmatrix}$ kN. The weight of the aircraft produces a force of $\begin{pmatrix}0\\0\\-294\end{pmatrix}$ kN and there is an air resistance of $\begin{pmatrix}17\\0\\0\end{pmatrix}$ kN.

**a** Find the resultant force acting on the aircraft.

The acceleration (m s$^{-2}$) of an object subject to a resultant force of $\boldsymbol{F}$ N is given by the formula $\boldsymbol{a} = \frac{\boldsymbol{F}}{m}$, where $m$ is the mass of the object in kilograms.

**b** Given that the mass of the aircraft is 30 000 kg, find the acceleration of the aircraft in m s$^{-2}$.

The aircraft is initially at (0, 0, 0) and takes off with an initial velocity of $\begin{pmatrix}55\\0\\0\end{pmatrix}$ m s$^{-1}$.

**c** Find:

**i** the displacement of the aircraft at time $t$

**ii** the displacement of the aircraft at $t = 60$.

**7** A child is spinning a ball attached to a string around her head. The motion of the ball is in the form of a horizontal circle. The string is 40 cm long and is being spun at an angle of 30° to the vertical. The ball does a complete revolution every second.

Taking the centre of the circle as the origin of a coordinate system with the position of the ball at time $t = 0$ on the positive $x$-axis, find:

**a** the radius of the circle

**b** the equation for the displacement of the ball in the form

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}r\cos bt\\r\sin bt\end{pmatrix}$$

**c** the velocity vector.

**d** The child releases the string. Find the initial speed of the ball as it is released, given that it is the same as its speed just before release.

**8** In a two-dimensional computer game the protagonists simultaneously fire beanbags into the air from points A and B, 17 units apart. The point A is at the origin of a coordinate system with B also on the horizontal axis. The velocity equations for the two beanbags are given by $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ where:

$$\boldsymbol{v}_A = \begin{pmatrix}5\\3-2t\end{pmatrix} \text{ and } \boldsymbol{v}_B = \begin{pmatrix}-4\\5-2t\end{pmatrix}.$$

**a** Write down vector equations for the displacements, $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$, of the two beanbags.

**b** Find an expression for the vector joining the centres of the two beanbags.

**c** Hence find the shortest distance between the two beanbags.

The player firing beanbags from point A adjusts $v_A$ to $\begin{pmatrix}a\\b\end{pmatrix}$.

**d** Find the values of $a$ and $b$ if the two beanbags collide when $t = 1$.

**e** Find the angle to the horizontal at which the beanbag is fired from A.

**9** A ball is thrown with an initial speed of 25 m s$^{-1}$. After 3.5 seconds it is descending at an angle of 45° to the horizontal.

Find the possible angles of projection of the ball.

## Exam-style questions

**10** **P2:** The displacement $\mathbf{s}(t)$ of an object at time $t$ is given by $\mathbf{s}(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}$, where $x$ and $y$ are functions of $t$. The object's acceleration is given by $\mathbf{a} = \begin{pmatrix}t\\4\end{pmatrix}$.

Initially, the object is at rest at the origin.

**a** In terms of $t$, find expressions for the object's

**i** velocity $\mathbf{v}(t)$

**ii** displacement $\mathbf{s}(t)$. (8 marks)

**b** When $t = 6$, find the object's

**i** velocity

**ii** displacement. (2 marks)

**11** **P1:** A force $\mathbf{F} = -12\mathbf{i} - 24\mathbf{j}$ Newtons is applied to the head of a nail with a hammer. The head of the nail is at point N($3\mathbf{i} + 4\mathbf{j}$) and the nail is embedded in a piece of wood at the origin.

The piece of wood lies along the $x$-axis.

**a** Find the scalar component of the force $\mathbf{F}$ in the direction of the vector $\overrightarrow{NO}$. (3 marks)

**b** Find the scalar component of the force $\mathbf{F}$ which is perpendicular to the vector $\overrightarrow{NO}$. (3 marks)

The force $\mathbf{F}$ can be resolved into two vector components; one acting in the direction of $\overrightarrow{NO}$ and the other acting perpendicular to $\overrightarrow{NO}$.

**c** Determine the component of the force $\mathbf{F}$ in the direction of the vector $\overrightarrow{NO}$ (from part **a** in vector form. Give your answer in terms of the unit vectors **i** and **j**. (3 marks)

**12** **P2: a** Consider the differential equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 10x = 0$.

An associated quadratic $\lambda^2 - 3\lambda - 10 = 0$ has roots of $\lambda = 5$ and $-2$.

**i** Verify that $x = Ae^{5t} + Be^{-2t}$ is a solution to this differential equation.

**ii** Find the solution given that $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{x=0} = 14$ (6 marks)

**b** Consider the differential equation $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = 0$.

An associated quadratic $\lambda^2 + 3\lambda = 0$ has roots of $\lambda = 0$ and $-3$.

**i** Verify that $x = A + Be^{-3t}$ is a solution to this differential equation.

**ii** Find the solution when $x(0) = 0$ and $\left.\frac{dx}{dt}\right|_{x=0} = 9$

**iii** Find the limit that $x$ tends towards as $t$ tends to infinity. (7 marks)







**13 P2:** During prehistory, Homosapiens and Neanderthal man were competing for resources. Let $x$ be the population of Homosapiens and $y$ be the population of Neanderthal man, both measured in tens of thousands. Time $t$ is measured in thousands of years.

This scenario can be modelled by the two coupled differential equations:

$$\begin{aligned} x'(t) &= 2x - 3y \\ y'(t) &= -y \end{aligned}\text{, where initially } x = 6 \text{ and } y = 4.$$

**a** Use the matrix-eigenvalue method to find the solution to these two differential equations. (10 marks)

**b** Find the two populations after 1000 years, giving your answers to 2 significant figures. (2 marks)

**c** As $t \to \infty$

**i** give a simpler approximation for the population of Homosapiens

**ii** state what happens to Neanderthal man. (2 marks)

**14 P2:** Geraint is training on a stationary bike. There is a logo on the tyre of his back wheel which is initially at the point $(0, 75)$. At time $t$ measured in seconds, it will be at position $(x, y)$ where the distances are measured in centimetres. The logo's velocity vector is given by

$$\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -105\pi\cos 3\pi t \\ -105\pi\sin 3\pi t \end{pmatrix}.$$

**a** Find its acceleration vector $\mathbf{a} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$. (3 marks)

**b** Find its displacement vector $\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix}$. (5 marks)

**c** When the vertical acceleration is zero, find the possible positions for the logo. (3 marks)

**d** State how many complete revolutions the wheel makes in one minute. (2 marks)

**15 P2:** For the following pairs of coupled differential equations, determine (with a worked reason) whether the solutions move away from the origin or move towards the origin.

**a** $x'(t) = 2x + y$, $y'(t) = x + 2y$

**b** $x'(t) = -2x + y$, $y'(t) = x - 2y$

**c** $x'(t) = 2x + y$, $y'(t) = -x + 2y$

**d** $x'(t) = -2x - y$, $y'(t) = x - 2y$. (12 marks)

**16 P2:** On a large island there is a population of lemmings and a population of snowy owls. The owls like to eat the lemmings. Let $x$ represent the number of lemmings (measured in thousands), and let $y$ represent the number of owls (measured in hundreds). Let $t$ represent time, in years.

Initially there are 1000 lemmings and 50 owls. The situation can be modelled by the coupled differential equations

$$x'(t) = x - xy$$
$$y'(t) = xy - 2y$$

**a** State the two equilibrium points for this model. (3 marks)

**b** Use Euler's method with a step size of 0.25 years to estimate the population of lemmings and the population of owls after 2 years (to the nearest integer). Show the intermediate values that are obtained in the working, in the format of a table. (7 marks)

**c** Suggest whether stating the number of lemmings to the nearest integer is a valid level of accuracy when using Euler's method in part **b**. (2 marks)





**17 P1:** Show that $\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)^2 + \left(\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}\right)^2 = |\mathbf{a}|^2$. (5 marks)

**18 P1:** Consider the differential equation $\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0$, where $b^2 > 4c$.

The quadratic equation $\lambda^2 + b\lambda + c = 0$ will have two distinct real roots, let these roots be $\lambda_1$ and $\lambda_2$.

Let $\dfrac{dx}{dt} = y$ and so $\dfrac{dy}{dt} = -cx - by$.

Solve these two coupled differential equations by the matrix-eigenvalue method to show that the solution to the original differential equation is $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$. (11 marks)

Geometry and trigonometry

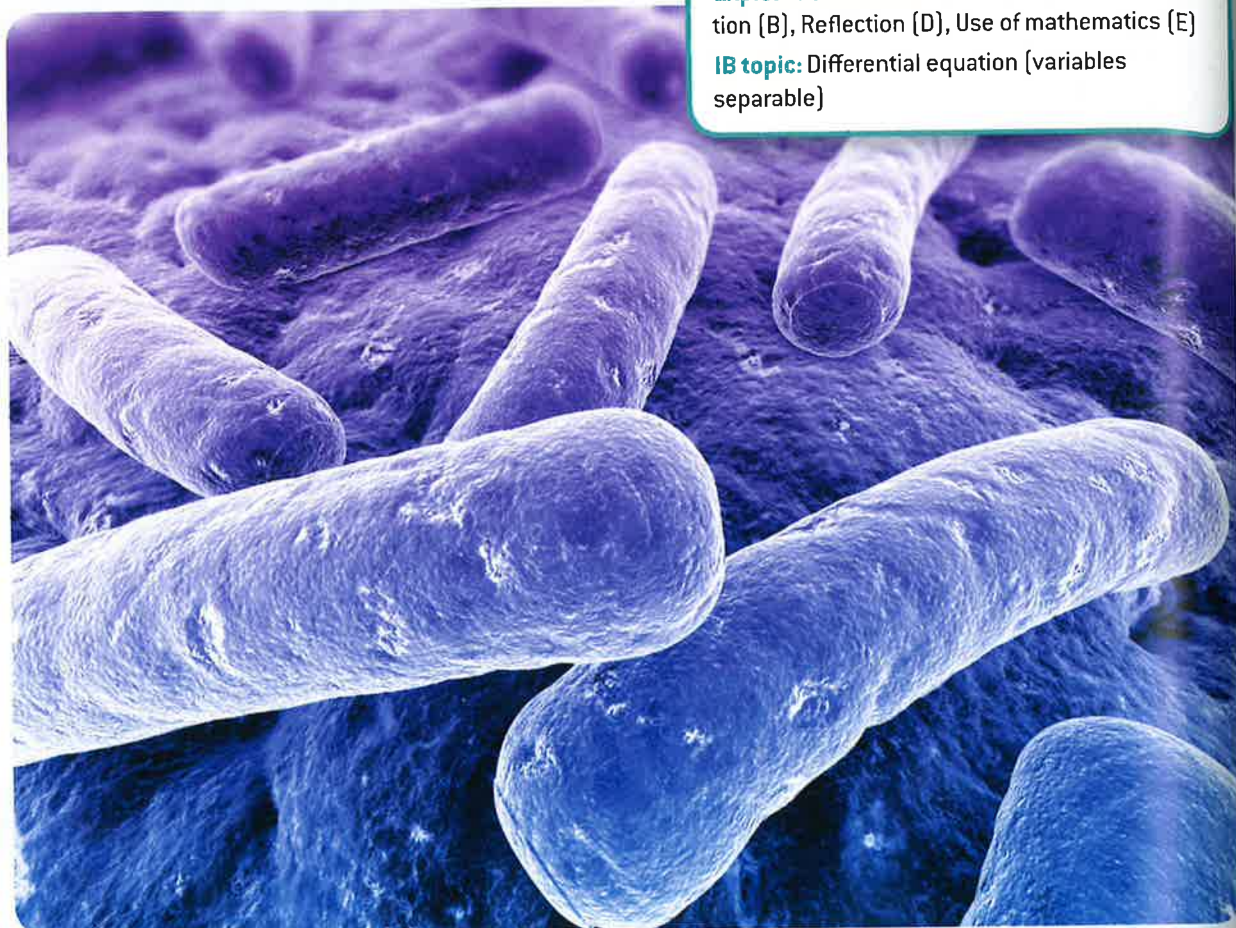
Calculus

# Disease modelling

**Approaches to learning:** Communication, Critical thinking

**Exploration criteria:** Mathematical communication (B), Reflection (D), Use of mathematics (E)

**IB topic:** Differential equation (variables separable)



### Hypothetical situation

Imagine a population threatened by an infectious disease.

Within the population there are two groups, those that are infected ($I$) and those that are healthy ($H$).

Assume that each year, the probability that a healthy person catches the disease is $c$, and the probability that an infected person recovers is $r$.

Let $x$ be the proportion of the population that are infected and $t$ be the number of years from the beginning of recording.

Show that a differential equation that will model the rate of change of the proportion of the population affected with respect to time is given by: $\frac{dx}{dt} = c(1-x) - rx$

Reflect on the assumptions made in this hypothetical situation and how they may differ in a real-life situation involving an infectious disease in a population.

By separating the variables find the general solution of this differential equation.

Using this general solution what would you expect to happen to the proportion of people affected by the disease over time? (ie what happens as $t$ tends towards infinity?)







## Simulation

You are now going to simulate a specific case of this situation.

Assume that $c = 0.3$ and $r = 0.2$.

Use these values of $c$ and $r$ to simulate the proportion of infected people over time

To do this, assume a population size of 10 of which, in year 0, seven are healthy ($H = 7$) and three are infected with a particular disease ($I = 3$).

You are going to simulate what happens to each of the 10 people over 10 years.

Consider the first person (who starts off healthy in year 0). Generate a series of 10 random numbers from a list of numbers from 1 to 10.

If the number is a 1, 2 or 3 and the person is healthy ($c = 0.3$) then they will catch the disease. Otherwise they will remain healthy.

If the number is a 1 or a 2 and the person is infected they will recover. Otherwise they will remain infected

Example:

| Year | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Random number | | 5 | 3 | 7 | 3 | 8 | 2 | 5 | 5 | 9 | 6 |
| Healthy or infected? | $H$ | $H$ | $I$ | $I$ | $H$ | $H$ | $I$ | $I$ | $I$ | $I$ | $I$ |

Conduct this simulation 10 times for each of the 10 people in your population.

For each year calculate the value of $x$, the proportion of people infected. Plot a graph of $x$ against time.

What does the graph suggest will happen as $x$ tends to infinity?

Now consider the differential equation again.

Rewrite the equation for $\frac{dx}{dt}$ but use the specific values of $c = 0.3$ and $r = 0.2$ from before.

Solve this differential equation by separating the variables and using the starting conditions of $t = 0$ and $x = 0.3$ to find the value of $b$ in the equation.

Sketch the graph of $x$ against $t$.

What happens as $t$ tends to infinity?

How well does this fit with your data in your simulation?

### Extension

What happens to the solution if you vary $c$ and $r$?

What happens to the solution if you vary the starting conditions of the proportion of infected people in the population?