iv Perform a χ^2 goodness of fit test at the 10% level to test the null hypothesis H_0: This new data fits the $B(40, 0.5)$ distribution.

State the number of degrees of freedom, the p-value and the conclusion of the test. [7 marks]

e You will now test if this data fits the normal distribution $N(20,10)$.

i If $Y \sim N(20, 10)$, calculate $P(21.5 < Y < 24.5)$.

ii Hence, Copy and complete the table below to show the expected frequencies when 100 values of a variable which fits the normal distribution $N(20, 10)$ are measured.

y	$y < 15.5$	$15.5 < y < 18.5$	$18.5 < y < 21.5$	$21.5 < y < 24.5$	$24.5 < y$
Expected frequency	7.73	24.1	36.5		

iii Perform a χ^2 goodness of fit test at the 10% level to test the null hypothesis H_0: This new data fits the $N(20,10)$ distribution.

State the number of degrees of freedom, the p-value and the conclusion of the test. [6 marks]

f Copy and complete the following conjecture by filling in the gaps.

For n ____ and p not close to 1 or 0, the binomial distribution can be approximated by the ____ distribution with the same ____and____, even though the binomial is a discrete distribution and the ____ is a ____ distribution. [1 mark]

[Total 30 marks]

Answers

Chapter 1

Skills check

1 **a** **i** 0.69 **ii** 28.71

iii 77.98

b **i** 0.694 **ii** 28.7

iii 78.0

2 **a** $2^{-3} = \frac{1}{8}$ **b** $27^{\frac{1}{3}} = 3$

3 **a** $x^2 = 9^2 + 13^2 = 250, x = 5\sqrt{10}$

b $7^2 = x^2 + 5^2, x^2 = 24, x = 2\sqrt{6}$

Exercise 1A

1 **i** 7.3 m (accuracy of the least accurate measurement)

ii 7.27 m

2 79 cm (to 2 s.f.)

Exercise 1B

1 **a** 23.5–24.5 mm

b 3.25–3.25 m

c 1.745–1.755 kg

d 1.395–1.405 g

2 **a** 0.09%

b 8847.5–8848.5 m

3 **a** 0.44 (2 s.f.)

b 2.7% (2 s.f.)

c Uncertainty much larger for the measurements done by group 1.

4 66×10^9 km (2 s.f.)

5 Max 0.35 min, 0.30

6 **a** Actual 10°C, approx 9°C

b 10%

7 **a** $7.20\text{ m} \le r < 7.21\text{ m}$ (3 s.f.)

b 0.005 m

Exercise 1C

1 **a** **i** 9×10^4 **ii** 9.936×10

b **i** 10^{11} **ii** 5.068×10^{11}

2 **a** 9.4×10^{-5}

b 8.35×10^3

c 5.24×10^{-19}

d 3.87×10^{-7}

3 **a** **i** $\frac{15}{x^{\frac{1}{2}}}$ **ii** $15 \cdot x^{-\frac{1}{2}}$

b **i** 7^1 **ii** $1 \cdot 7^1$

c **i** $\frac{1}{2^{3+3t}}$

ii 2^{-3-3t}

d **i** $\frac{25}{3^{2x}}$ **ii** $25 \cdot 3^{-2x}$

4 **a** 240, 339, 480

b Rate of growth is increasing. When $t = \frac{3}{2}$, we have the number of bacteria after one and a half hours.

5 **a** $\frac{1600}{2^{\frac{t}{8}}}$ **b** 1100

6 0.24 mm

7 18.1

8 2×10^{-3}

Exercise 1D

1 **a** $\theta = 28.8°$, $v = 8$ cm, $w = 4$ cm

b $\theta = 56°$, $y = 18.2$ cm, $x = 6.88$ cm

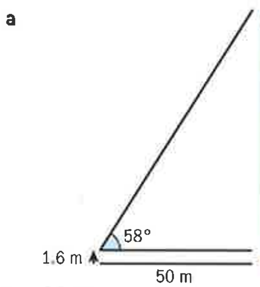
c $z = 5.7$ cm, $\alpha = 56°$, $\beta = 34°$

2 **i** 7.36 m

ii 4.25 m

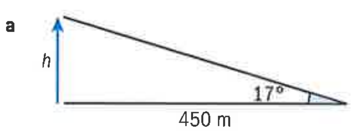
iii 6.96 m

3 **a**



b 81.6 m

4 **a**



b 138 m

5 1.62 m

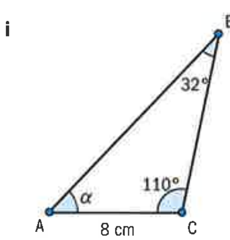
6 1300 m

7 45 km

8 **a** 1.20 m **b** 2.1%

Exercise 1E

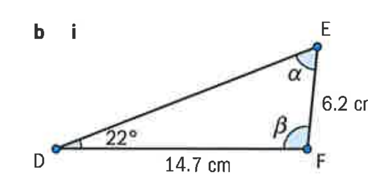
1 **a** **i**



ii 1 possibility as we have all 3 angles and 1 side

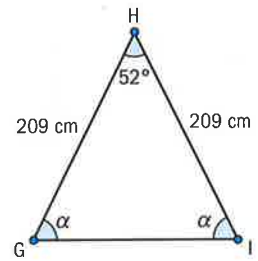
iii $\alpha = 47°$, $AB = 14.8$ cm, $BC = 10.7$ cm

b **i**



ii 2 possibilities, ambiguous case of sine rule

iii $\alpha = 62.6°$ $\beta = 95.4°$ $DE = 16.5$ cm or $\alpha = 153°$ $\beta = 5°$ $DE = 1.44$ cm c i



ii 1 possibility – properties of isosceles triangles

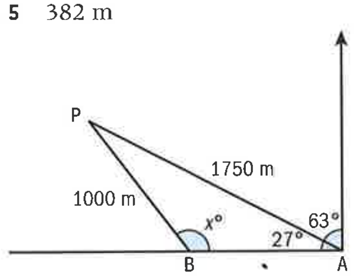
iii $\alpha = 64°$ $\beta = 52°$
$HI = 183$ cm

2 374 m

3 $PQ = 59$ m, $QT = 46$ m, $PT = 17$ m

4 381 km bearing 177°

5 382 m



Exercise 1F

1 a 41.0°, 87.0°, 106 mm
b 66°, 17.2 cm, 17.2 cm
c 11.11 cm, 39°, 88°
d $x = 20°$, 16 m, 22 m

2 Kristian's method is correct; cannot use the cosine rule in this case because the angle given is not between the two sides given. Solve $\sin\hat{C} = 40 \times \frac{\sin 35°}{25}$,
$\hat{C} = 67°$, $\hat{B} = 78°$, $AC = \frac{25}{\sin 35°} \times \sin 78° = 43$ cm

3 32°, 97°, 51°

4 a 82.8°, 6 cm b 5 cm²

Exercise 1G

1 a i 6 cm ii 15 cm²
b i 3 cm ii 6 cm²
c i 23.8 cm ii 125 cm²

2 $l = 62.7$ cm

3 a 209 m
b 13 min
c 24 min

4 a 0.96π
b 24 cm

5 a 18 cm²
b 41 cm²

6 1.1 cm²

7 59.9 m²

Exercise 1H

1 a 35 000 cm³
b 1500 cm³
c 3150 cm³ (3 s.f.)

2 a 121 m³ b 1.13 m

3 a i 1.4×10^{-29} m³
ii 1.0868×10^{21} m³
iii 8.661×10^{35} m³
b The Earth compared to an atom

4 a 704 000 m³
b 3.15×10^{9} kg

5 900 l

Exercise 1I

1 a 150 cm²
b 180 cm²
c 501 mm²
d 32 cm²

2 i 6.46 cm
ii 5.28 cm
iii 38 cm

3 a i $\frac{3}{r}$
ii 3.14×10^{-6} m
b i $\frac{6}{\left(\frac{4}{3}\pi\right)^{\frac{1}{3}} r}$
ii $\frac{42}{19} \times \frac{1}{\left(\frac{4}{19}\pi\right)^{\frac{1}{3}} r}$
iii Cube: 1.24, cylinder: 0.85

4 a 873 m²
b 27 c $432

5 19.60 cm²

6 a $V = 1400$ m³, $A = 750$ m² (2 s.f.)
b $V = 280$ cm³, $A = 240$ cm² (2 s.f.)

7 a $V = 396$ m³
b 37 l

8 $V = 213$ cm³, $A = 24$ cm²

Chapter review

1 a i 0.2 ii 0.154
b i 2.3 ii 2.30
c i 2.0 ii 1.99
d i 0.2 ii 0.248
e i 0.2 ii 0.248

2 a 5.6%
b Not enough time

3 $x = 1.15417$, 1.3×10^{-4}%

4 a Cosine rule gives the relation: $PR^2 - 2PQ \times PR \times \cos 31° + PQ^2 - QR^2 = 0$. Solve the quadratic equation to get $PR = 24.8$ cm.
b First, find the angle R: $\sin R = \frac{\sin 31°}{15} \times 13.4, R = 27°$, so Q = 122°. Apply the sine rule again to find $PR = \frac{15}{\sin 31°}\sin Q = 24.8$ cm.

5 Angle of elevation = 2.7°, angle of depression = 0.77°

6 a 52 670.25 m² $\le A <$ 53 130.25 m²
b $6.38 \le R < 8.30$

7 2.063×10^{9} km

8 a Volume of the cone $V_1 = \frac{1}{3}\pi r^2 h$, volume of the modified cylinder $V_2 = \pi r^2 h - \frac{\alpha}{2} r^2 h$. For volumes to be equal, $\alpha = \frac{4}{3}\pi$.
b Lateral area of the modified cylinder is $\left(2\pi - \frac{4}{3}\pi + 2\right)rh = \left(\frac{2}{3}\pi + 2\right)rh$.
Lateral area of the cone is $\frac{\pi r h}{\sin\theta}$. For them to be equal,
$\sin\theta = \frac{\pi}{\left(\frac{2}{3}\pi + 2\right)}, \theta = 50°$.

9 a $BM = \frac{1}{2}EF = 115.178\,28$ m,
$VM = 186.472\,85$ m,
$\widehat{M} = 51.853\,975°$,
$\hat{V} = 38.146025$
b $BF = \sqrt{2}\,BM = 162.886\,68$ m,
$VF = 219.176\,09$ m,
$\hat{F} = 41.997\,23°$,
$\hat{V} = 48.002\,77°$.
c $VE = VF = 219.176\,09$ m,
$\hat{E} = \hat{F} = 58.297\,71°$,
$\hat{V} = 63.40458°$,
$A = 21\,477.621$ m²
d $VB^2 = 21\,506.008$ m² so percentage error
$= \left|\frac{VB^2 - A}{VB^2}\right| \times 100\%$
$= 0.132\,365\,34\%$.

10 a $t = \frac{\sqrt{5^2 + (12-d)^2}}{70} + \frac{d}{110}$
b $t = \frac{5}{\cos\theta} \times \frac{1}{70} + \frac{12 - 5\tan\theta}{110}$

Exam-style questions

11 a $\frac{5.5}{0.25}$ ohms $\le R < \frac{6.5}{0.15}$ ohms,
22 ohms $\le R <$ 43 ohms
b 43%

12 a $12x^4$ b x^2 c $\frac{9y^2}{49x^6}$

13 a 1.12×10^{-27}
b 1 : 1840
c 9.8%

14 675

15 a 209 cm²
b 65.5 cm

16 22.5 cm

17 a 34° b 64°
c 112°
d 1400 cm³
e 840 cm²

Chapter 2

Skills check

1 Mean = 19.2, median = 19, mode = 22

Exercise 2A

1 Student answers.

2 Use quota sampling; take n students' marks at random, from each of the five year groups

a

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(x)$	10	8	16	13	21	9	10	9	5	4	4	3	3	2	1	2

b 24

c

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Sample size	4	3	7	5	9	4	4	4	2	2	2	1	1	1	0	1

3 As different sectors of society are likely to vote differently, it is important to have those sectors represented proportionately. Strata might be income, age, gender, geographical region.

Exercise 2B

1 a Mean = 6.068, median = 5.2, mode = 7.5, 17.8 and 25.0 may be outliers
b Mean = 3.533, median = 3.6, mode = 2.5
c Mean = 64.65, median = 62, mode = 62

2 a Mean = 1.8, median = 1.5, mode = 6
b SD = 1.89, variance = 3.56

3 a Mean = 8.9444, median = 10, mode = 12, most appropriate = mode
b Standard deviation = 4.0958, average distance from mean = 4
c Range = 14, IQR = 6, data is mainly concentrated within middle range as the IQR is a lot smaller than the range.

4 a Basketball: mean = 200.8, standard deviation = 10.4575; Men: mean = 172.27, standard deviation = 10.2541
b On average basketball players are taller by approx. 28 cm; however, they vary in height by the same amount.

5

	Mean	Median	s_n	Q_1	Q_3
Boys	76.5	81	12.9	65	84
Girls	69.4	69	7.88	61	77

6 Answers will vary.

7 Mean = 60, standard deviation = 6

8 a Mrs Ginger: mean = 84, standard deviation = 16; Mr Ginger: mean = 80, standard deviation = 20, Miss Ginger: mean = 76, standard deviation = 24

b Matty: Mrs Ginger: 44, Mr Ginger: 30, Miss Ginger: 16

Zoe: Mrs Ginger: 70, Mr Ginger: 62.5, Miss Ginger: 55

Ans: Mrs Ginger: 92, Mr Ginger: 90, Miss Ginger: 88

c For all students, Mrs Ginger's methodology gives them the highest mark. For students with low marks, their marks are always relatively similar, for middling students, their marks can vary a bit, but for low-scoring students, the mark can vary widely. If a student had a mark of 6 or lower, their new mark would be negative with Miss Ginger's methodology.

Exercise 2C

1 a i $150 \leq n < 180$

ii 111.625

iii 119.29

Modal class indicates most common number of cars was between 150 and 180. Median suggests that middle value of number of cars was 119 and the mean was 111. The mode is most appropriate here.

b i $50 \leq s < 55$

ii 54.4167

iii 53.913

Modal class indicates most common speed of cars was between 50 and 55. Median suggests that middle value of the number of cars was 53 and the mean was 54. All values here are similar; however, the most appropriate is the mean.

c i $7 \leq t < 8$ **ii** 5.8636

iii 5.9286

Mean and median very close, mode is higher (between 7 and 8). Median = best choice.

2 a $30 \leq c < 40$

b Mean = 34, median = £34.2857

c Standard deviation = £10.1158, the average amount of money spent more or less than the average amount was £10.11.

d Variance = 101.329, range = 50, IQR = 11.875; they assume the data is spread evenly within the classes.

3 a $180 \leq x < 190$

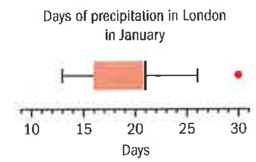
b Mean = 180.2, standard deviation = 10.9982, average distance from average height is approx. 11 cm

4 a Males: mean = 2546.3, standard deviation = 729.767
Females: mean = 2114.58, standard deviation = 635.257

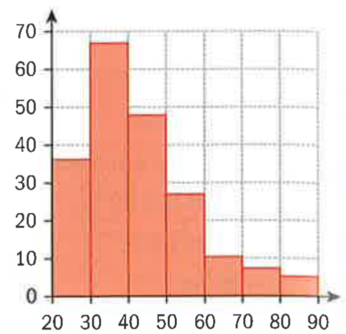
b On average, men earned more than women and male income varied more than female income.

Exercise 2D

1

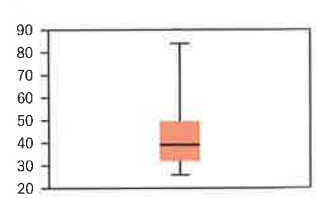


2 a

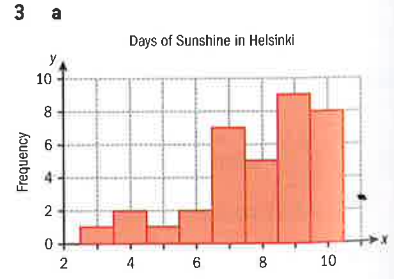


b Mean = 42.45, median = 39.5522, Q1 = 32.1875, Q3 = 49.7917, range = 70, at least 5 outliers, an estimated 7 outliers

c



3 a



b The data is negatively skewed.

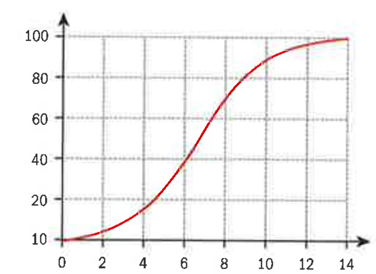
4 Students' own answers

Exercise 2E

1 a

Time, x (minutes)	Frequency	Cumulative frequency
$0 \leq x < 2$	5	5
$2 \leq x < 4$	11	16
$4 \leq x < 6$	23	39
$6 \leq x < 8$	31	70
$8 \leq x < 10$	19	89
$10 \leq x < 12$	8	97
$12 \leq x < 14$	3	100

b



c Median = 6.7, IQR = 3.5

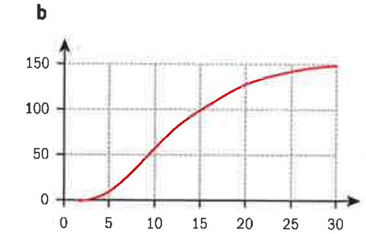
d 3.1

e 6

2 a

Number of words, x	Frequency	Cumulative frequency
$0 \leq x < 4$	5	5
$4 \leq x < 8$	32	37
$8 \leq x < 12$	41	78
$12 \leq x < 16$	28	106
$16 \leq x < 20$	22	128
$20 \leq x < 24$	12	140
$24 \leq x < 28$	7	147
$28 \leq x < 32$	3	150

b

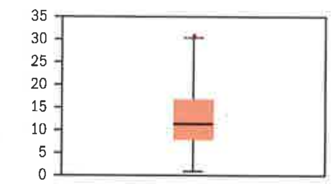


c Median = 11.5, IQR = 9

d There may or may not be outliers

e 22

f (assuming only one outlier):

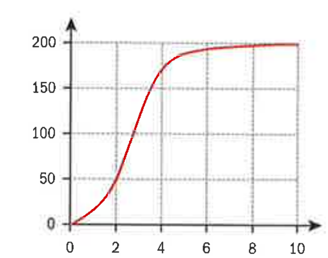


g Adults, because books more linguistically advanced than a children's book

3 a

Height, h (m)	Frequency	Cumulative frequency
$0 \leq h < 1$	17	17
$1 \leq h < 2$	35	52
$2 \leq h < 3$	69	121
$3 \leq h < 4$	51	172
$4 \leq h < 6$	22	194
$6 \leq h < 10$	6	200

b



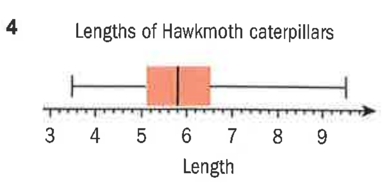
c 2.7 m

d IQR = 1.5

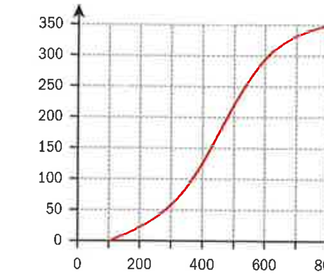
e 1.1 m

f 4.9 m

4



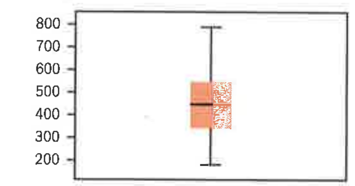
5 a



b Median = 450, IQR = 205

c No outliers

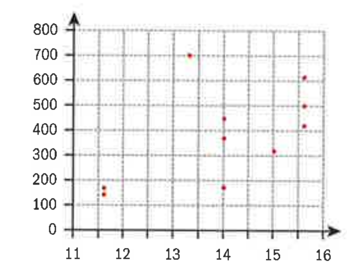
d



e 90

Exercise 2F

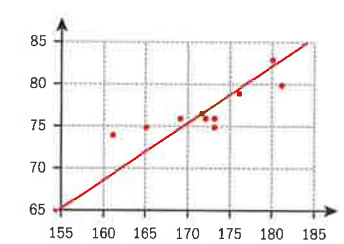
1 a



b Weak positive correlation

c A slight increase in price associated with an increase in screen size

2 a



b Red dot in graph above

c See graph

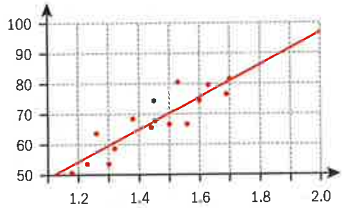
d 75 kg

e $r = 0.9150$

f Strong positive correlation, means the taller the player, the heavier they are on average

g Causation: a taller person has more body mass and so weighs more

3 a



b Red dot in graph above

c See graph above

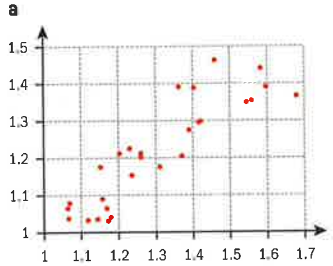
d 93%, not appropriate to use this as 1.9 is far outside the current data range

e $r = 0.884$

f Strong positive correlation

g Indicates taller people scored better on the vocabulary test – did not necessarily know better vocabulary

4 a



b $r = 0.8873$, strong positive correlation between price of unleaded and price of diesel

Chapter review

1 a Males: mean = 13.32, standard deviation = 3.374
Females: mean = 15.85, standard deviation = 4.228
On average, females are slower than males; however, time taken varies more for women than men.

b Mean = 14.33, standard deviation = 3.9397

c Answers will vary

d Mean + 15.425, standard deviation = 3.6324

e Take every 2.5th element of the data, i.e. take alternately every 2 and 3 points. Mean = 14.975, standard deviation = 4.1018

f Answers will vary. Random sample of 24 males and 16 females: mean = 14.3, standard deviation = 4.1845; random sample: mean is larger and the standard deviation is smaller than that of the whole population; systematic sample: mean and standard deviation are larger than that of the whole population; stratified sample: mean is the same and the standard deviation is larger than that of the whole population.

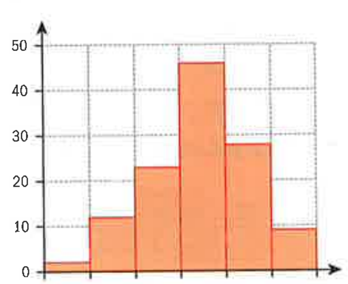
2 b Mean c Mode

d Mode

3 a $50 \le l < 60$

b Median = 55, mean = 54.4167, standard deviation = 11.2765

c



4 a $15 700

b $3249.62

c 8.33%

d i Misleading: makes it look like profit was multiple times higher each year

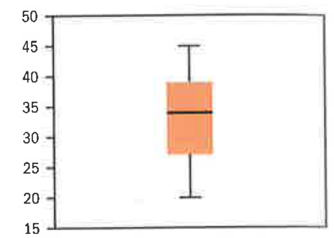
ii To make it seem profits are increasing more than they actually are

5 a Mean = 32.8, standard deviation = 7.5054

b Range = 25, IQR = 22

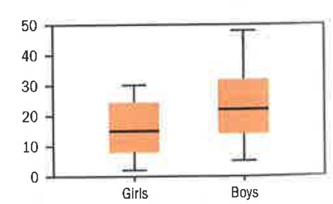
c 34, no outliers

d



6 a Girls: mean = 16.24, median = 15, $Q_1 = 8$, $Q_3 = 24$, range = 25, no outliers
Boys: mean = 24.48, median = 22, $Q_1 = 15$, $Q_3 = 31$, range = 16, no outliers

b

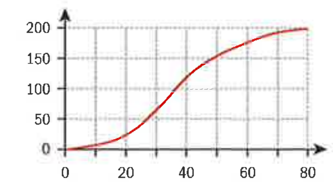


c Boys do more press ups on average and have a larger range, with a longer whisker in the top half of the boys, indicating that the data is more skewed.

7 a

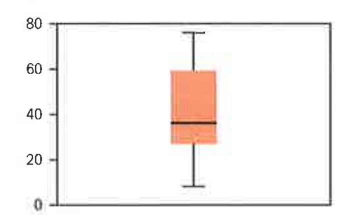
Number of hours, x	Frequency	Cumulative frequency
$0 \le x < 10$	8	8
$10 \le x < 20$	16	24
$20 \le x < 30$	41	65
$30 \le x < 40$	54	119
$40 \le x < 50$	36	155
$50 \le x < 60$	22	177
$60 \le x < 70$	17	194
$70 \le x < 80$	6	200

b



c Median = 36, IQR = 32

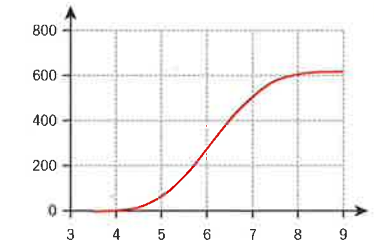
d



8 a Mean = 6.1326, standard deviation = 0.9213

b

Time (min), x	Frequency	Cumulative frequency
$3.5 \le x < 4$	6	6
$4 \le x < 4.5$	14	20
$4.5 \le x < 5$	48	68
$5 \le x < 5.5$	89	157
$5.5 \le x < 6$	121	278
$6 \le x < 6.5$	129	407
$6.5 \le x < 7$	103	510
$7 \le x < 7.5$	70	580
$7.5 \le x < 8$	30	610
$8 \le x < 8.5$	10	620
$8.5 \le x < 9$	2	622

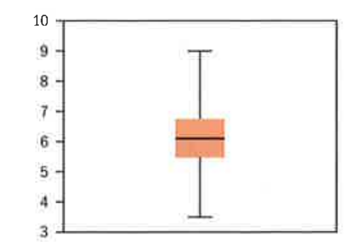


c i 6.1

ii $Q_1 = 5.5$, $Q_3 = 6.75$

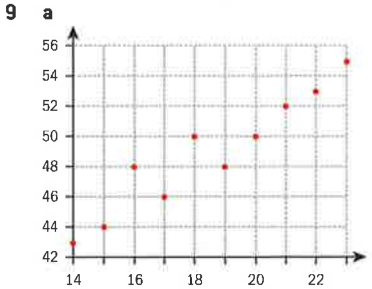
iii 1.25 iv 7.1

d



e There may be outliers at both ends of the data.

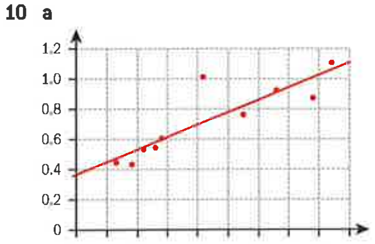
9 a



b Strong positive correlation

c It is likely that increasing the temperature increases the number of eggs as there is a strong positive correlation.

10 a



b (42, 1.02)

c (43.333, 0.698)

d See graph above

e $r = 0.9732$

f Very strong positive correlation

Exam-style questions

11 a 16 b 13.375

c 13.5 d 12

e 23.583

12 a Mean = 23.5833 °C

b Standard deviation = 3.3778 °C

c Mean = 22.8333 °C

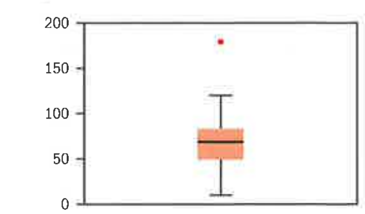
d Standard deviation = 5.5202 °C

e The mean temperature in Tenerife (23.58 °C) is higher than that in Malta (22.83 °C) and the temperature in Malta varies more than in Tenerife as the standard deviation is larger (5.5202 °C and 3.3778 °C)

13 a 69 b 50 c 81

d 170 e 180

f



14 a

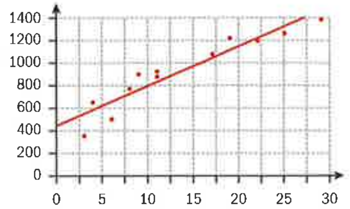
Height, (x cm)	Frequency
$20 \le x < 25$	3
$25 \le x < 30$	3
$30 \le x < 35$	4
$35 \le x < 40$	7
$40 \le x < 45$	4
$45 \le x < 50$	2
$50 \le x < 55$	1

b 35.8333 c 63.89

d 7.9931

e On average, the plants in Eve's neighbour's garden are 3.73 cm shorter than in Eve's garden, whilst the plants in both gardens vary by a similar amount.

15 a



b (13.667, 926.667), line drawn on graph above

c $r = 0.9441$, close to one, indicating a strong positive correlation

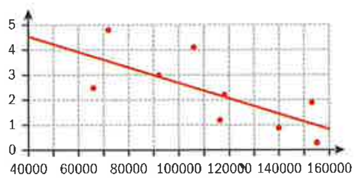
d When temperature is higher, sales of ice cream are higher.

16 a 6920 kg

b 430 kg

17 a −0.7659

b



c (106 600, 2.49), it is extrapolating much further than the current data provided, there is no guarantee that the decrease in salary with distance is linear

Chapter 3

Skills check

1 $\sqrt{4^2+2^2} = 4.47$

2 $\left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3, 3)$

3 a $x = 7.66$, $y = 6.43$

b 33.7°

4 a i 43.3 km **ii** 25 km

b 240°

Exercise 3A

1 a i 5.10 **ii** 2.45

b i (1.5, −1.5)

ii (1, 3.5, −2.5)

2 a i 36.1 **ii** 19.4

b i (9, 0.5)

ii (−9.5, 9.5, −6)

3 a 8.54 km

b Two distances are 33.9 km and 42.2 km, so the tracking station will only be able to detect the first aircraft.

4 a A(250, 0, 0), B(250, 400, 0), C(0, 400, 60), D(0, 0, 60)

b M(125, 400, 30)

c 840 m

5 a 52 900 m² to 3 s.f.

b 2 430 000 m³ to 3 s.f.

c (218.5, −31, 117)

d 213 m

Exercise 3B

1 a $x = 2$ **b** $y = 6$

c (2, 6)

2 a $y = 3x + 5$

b $y = -0.2x + 0.4$

c $y = 4.5x + 5$

d $y = 2x + 1$

3 a $y = -4x + 17$

b $y = 5x - 11$

c $y = \frac{1}{3}x + 3$

4 a $y = -2x + 20$

b i (0, 20)

ii (22, −24)

c 49.2 km

Exercise 3C

1 a i $y - 9 = 2(x - 3)$

ii $y = 2x + 3$

iii $y - 2x - 3 = 0$

b i $y - 5 = \frac{1}{2}(x - 6)$

ii $y = \frac{1}{2}x + 2$

iii $2y - x - 4 = 0$

c i $y + 7 = -\frac{1}{3}(x - 6)$

ii $y = -\frac{1}{3}x - 5$

iii $3y + x + 15 = 0$

2 a i $y - 5 = 2(x - 2)$

ii $y = 2x + 1$

iii $-2x + y - 1 = 0$

b i $y - 4 = -2x - 0$

ii $y = -2x + 4$

iii $2x + y - 4 = 0$

c i $y - 6 = 3(x - 2)$

ii $y = 3x$

iii $3x + y = 0$

d i $y + 6 = -\frac{2}{3}(x + 2)$

ii $y = -\frac{2}{3}x - \frac{22}{3}$

iii $2x + 3y + 22 = 0$

3 a $\frac{2}{3}$ **b** $-\frac{4}{7}$ **c** $-\frac{a}{b}$

4 a $y = -1.5x + 6.5$

b $2y + 3x - 13 = 0$

c $\frac{1}{2} \times 6.5 \times \frac{13}{3} \approx 14.1\text{m}^2$

Exercise 3D

1 a i $x = 7, y = 6$

ii $x = 0.5, y = 0.2$

iii $x = 3, y = 7$

iv $x = 1, y = -7$

b i $x = -8, y = 3$

ii $x = 2.6, y = -1.9$

2 a $\frac{1}{2} \times 50 - 100 = -75$

b (340, 70)

c i Distance of Alison from intersection is 481 m, distance of Bernard from intersection is 324 m, so Bernard arrives first.

ii 174 seconds

3 a 25%

b 15% (steeper going down)

c i $y = 0.1x$

ii $y = -0.15x + 0.3675$

d 0.147 km or 147 m

e 2.47 km

4 a Construct a right-angled triangle with two points on the line as the ends of the hypotenuse. The tangent of the angle is the opposite over the adjacent side in the triangle which is equivalent to the increase in the y values divided by the increase in the x values.

b $y = 0.069927x + 55.55$

c 55.55 m **d** 94.4 m

Exercise 3E

1 a $\frac{1}{2}$ **b** −3

c −2 **d** $\frac{7}{6}$

2 a Show $m_1m_2 = -1$

b $2x - y + 1 = 0$

c $2x + 3y - 27 = 0$

3 a Any point away from the perpendicular will form the hypotenuse of triangle with the perpendicular distance as one of the sides which will necessarily be shorter than the hypotenuse.

b $4x + 3y + 1 = 0$

c (−1, 1) **d** 10

4 a 5.83 **b** 4.43

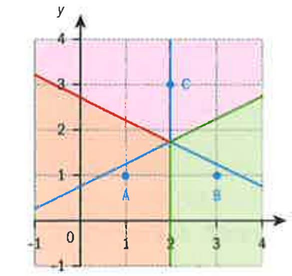
5 Intersection of $y = 2.5x - 19.25$ and $y = x + 6$, (16.8, 22.8)

6 a Station should be built at (11, 21)

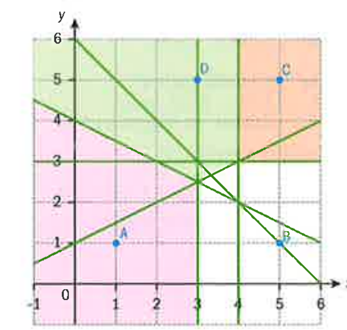
b (10, 43), 14.1 km

Exercise 3F

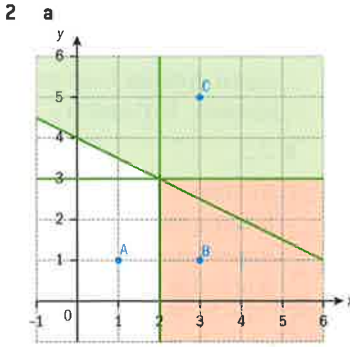
1 a Perpendicular bisectors are: $x = 2$, $x + 2y = 5.5$, $x - 2y = -1.5$



b

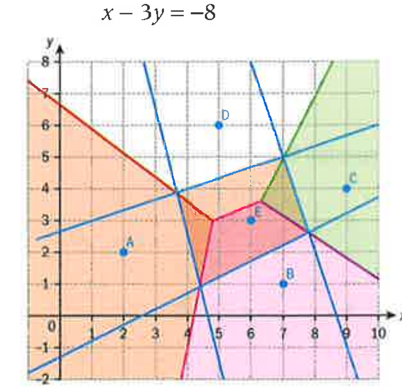


2 a



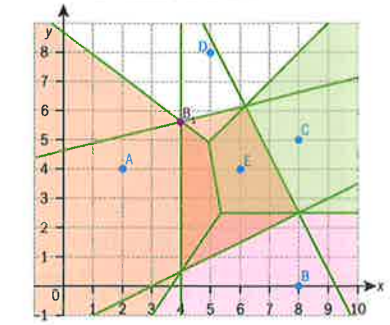
b 21°C

3 a Perpendicular bisector: $x - 3y = -8$



b Perpendicular bisector: $x = 4$, $x - 2y = 3$

i 19 mm **ii** 21 mm



c i 11 mm **ii** 12 mm

Exercise 3G

1 a The intersection of the perpendicular bisectors will be a vertex in the Voronoi diagram and hence the position of the solution to 'the toxic waste dump problem'.

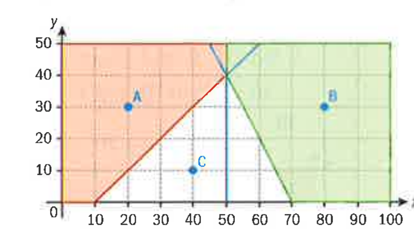
b $y = -5x + 21$ and $y = 2.5$

c (3.7, 2.5) **d** 2.75 km

2 a i $y = x - 10$

ii $y = -2x + 140$

b



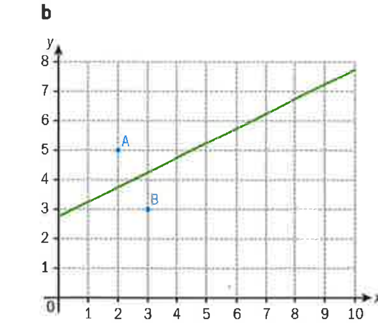
c i 0.24 **ii** 0.34

d i (50, 40) **ii** 31.6 m

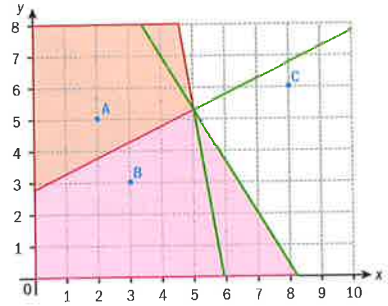
3 a $y - 4 = \frac{1}{2}(x - 2.5)$ or

$y = \frac{1}{2}x + 2.75$

b

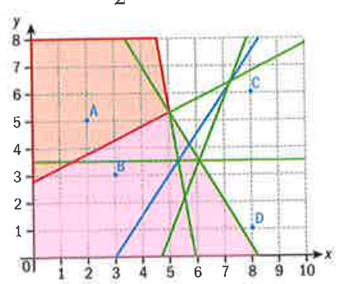


c



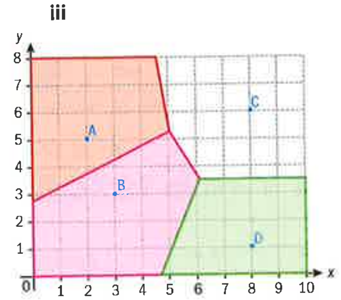
d **i** BD $y-2=\frac{5}{2}(x-5.5)$ or

$y=\frac{5}{2}x-11.75$ CD $y=3.5$



ii Because the perpendicular bisector of AD will always be outside the cell containing D (or, because either B or C are always closer than A to D). B and C are closer to D than they are to A.

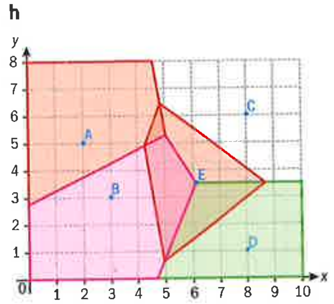
iii



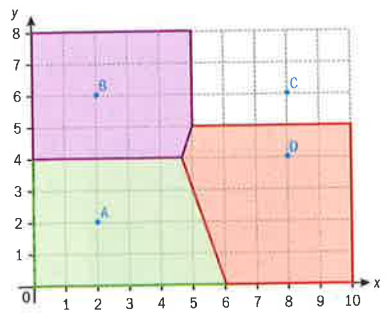
e (6.1, 3.5) and (5.04, 5.27)

f 28% **g** (6.1, 3.5)

h



4 **a**



b (4.67, 4)

c **i** 213 000 miles^2 (to 3 s.f.)

ii 198 000 miles^2

iii 150 000 miles^2

iv 239 000 miles^2

d So unable to support the other province

5 **a** **i** AB = 6, BC = 3, CD = 4, DA = 3.6055, total = 16.6 so 166 km

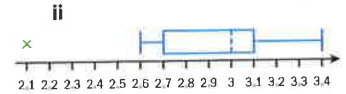
ii 0.289

b **i** Because (4, 3.5) is a vertex in the Voronoi diagram and so is an equal distance from three stations; B, C and D

ii 2.5

c **i** Median = 3.0, lower quartile = 2.7, upper quartile = 3.1, IQR = 0.4

ii



d The readings are all higher than the expected average except for one outlier so support claim.

Exercise 3H

1 **a** $\begin{pmatrix}0\\-3\end{pmatrix}$, $-3j$

b $\begin{pmatrix}-2\\-2\end{pmatrix}$, $-2i-2j$

c $\begin{pmatrix}2\\1\end{pmatrix}$, $2i+j$

d $\begin{pmatrix}1\\-3\end{pmatrix}$, $i-3j$

e $\begin{pmatrix}2\\0\end{pmatrix}$, $2i$

2 **a** $\begin{pmatrix}-3\\4\end{pmatrix}$ **b** $7i+4j$

c $\begin{pmatrix}-6\\2\end{pmatrix}$ **d** $5i-2j$

3 **a** $\begin{pmatrix}4\\6\end{pmatrix}$ **b** $\begin{pmatrix}6\\9\end{pmatrix}$

c $\begin{pmatrix}8\\12\end{pmatrix}$

Multiply each component in the vector by the scalar

d $3i+4j=3\begin{pmatrix}1\\0\end{pmatrix}+4\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix}+\begin{pmatrix}0\\4\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$

4 **a** **i** $\begin{pmatrix}1\\2\end{pmatrix}=\overrightarrow{AB}$

ii $\begin{pmatrix}-1\\-2\end{pmatrix}=-\overrightarrow{AB}$

iii $\begin{pmatrix}2\\4\end{pmatrix}=2\overrightarrow{AB}$

iv $\begin{pmatrix}-\frac{1}{2}\\-1\end{pmatrix}=-\frac{1}{2}\overrightarrow{AB}$

b Parallel vectors are scalar multiples of each other

5 a, c, e, g

6 **a** **i** $p=-1$, $q=2$

ii $p=2$, $q=\frac{1}{2}$

b **i** $p=\frac{1}{7}$

ii $q=-3$

Exercise 3I

1 **a** **i** $\begin{pmatrix}1\\4\end{pmatrix}$ **ii** $\begin{pmatrix}1\\1\end{pmatrix}$

iii $\begin{pmatrix}-2\\2\end{pmatrix}$ **iv** $\begin{pmatrix}2\\-2\end{pmatrix}$

b **i** $\overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AC}+\overrightarrow{CD}$

ii $\overrightarrow{BD}=-\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{CD}$

iii $\overrightarrow{BD}=\begin{pmatrix}1\\2\end{pmatrix}$ **iv** (3, 7)

2 **a** $\overrightarrow{AC}=\begin{pmatrix}0\\6\end{pmatrix}$ $\overrightarrow{CA}=\begin{pmatrix}0\\-6\end{pmatrix}$

b $\overrightarrow{DC}=\begin{pmatrix}-1\\4\end{pmatrix}$

3 **a** $\overrightarrow{AB}=\overrightarrow{DC}=\begin{pmatrix}1\\3\end{pmatrix}$

b Parallelogram as two sides are equal length and parallel

c $\overrightarrow{BC}$ and $\overrightarrow{AD}$

4 **a** $\begin{pmatrix}40\\-10\end{pmatrix}$

b $\overrightarrow{CA}=\begin{pmatrix}-40\\10\end{pmatrix}$ **c** 41.2 km

5 **a** $13i+11j$

b $-13i-11j$

Exercise 3J

1 **a** **i** 5.66 **ii** 45°

b **i** 2.24 **ii** 297°

c **i** 19.3 **ii** 21.3°

d **i** 8.06 **ii** 7.1°

2 **a** $\left|\begin{matrix}48\\20\end{matrix}\right|=\sqrt{48^2+20^2}$

$=\sqrt{2704}=\sqrt{16\times169}$

$=4\times\sqrt{169}=4\sqrt{12^2+5^2}$

$=4\left|\begin{matrix}12\\5\end{matrix}\right|$

b **i** 30 **ii** 50 **iii** 35

3 **a** $\frac{1}{5}\begin{pmatrix}24\\32\end{pmatrix}$ **b** $\begin{pmatrix}5\\7\end{pmatrix}$

c $\begin{pmatrix}7\\1\end{pmatrix}$

4 **a** $\begin{pmatrix}141.4\\141.4\end{pmatrix}$, $\begin{pmatrix}-175\\0\end{pmatrix}$

b 145 km

5 10.2 km, 260°

Exercise 3K

1 **a** 9 **b** 3

c 15

d 35

2 $\begin{pmatrix}-12\\-9\\-6\end{pmatrix}$

3 **a** To get from A to B, travel down the vector $\mathbf{AB}=b-a$ so to travel from A to the midpoint of [AB] travel down $\mathbf{AM}=\frac{1}{2}\mathbf{AB}=\frac{1}{2}(b-a)$.

Now $\mathbf{OM}=\mathbf{OA}+\mathbf{AM}$

$=a+\frac{1}{2}(b-a)$

$\mathbf{OM}=\frac{1}{2}(a+b)$.

b $\mathbf{PQ}=\begin{pmatrix}-2\\-3\\-1\end{pmatrix}$, $\mathbf{QR}=\begin{pmatrix}3\\4\\-6\end{pmatrix}$

c $\overrightarrow{PR}=\begin{pmatrix}1\\1\\-7\end{pmatrix}$ **d** (4, 7, 0)

e $\begin{pmatrix}0.5\\0.5\\-3.5\end{pmatrix}$ **f** $\begin{pmatrix}2.5\\3.5\\-2.5\end{pmatrix}$

g Diagonals of a parallelogram bisect each other

Exercise 3L

1 **a** 40.7° **b** 156°

c 25.1° **d** 40.8°

e 0° **f** 180°

2 **a** **i** $\overrightarrow{AB}$ and $\overrightarrow{AC}$

ii $\overrightarrow{BC}$ and $\overrightarrow{BA}$

b ∠A = 151.3° ∠B = 20.1° ∠C = 8.9°

c BC = 7.14

3 **a** $p=\frac{1}{4}$

b $p=-1, 4$

4 **a** $k=-1, 2$

b When $k=-1$,

$AC=\sqrt{1^2+3^2}=\sqrt{10}$.

$BC=\sqrt{(-1)^2+0^2}=1$.

Area of the triangle is $\frac{\sqrt{10}}{2}$.

When $k=2$,

$AC=\sqrt{1^2+(2-2)^2}=1$,

$BC=\sqrt{(-1)^2+(-1+2)^2}=\sqrt{2}$.

Area of the triangle $=\frac{\sqrt{2}}{2}$.

5 70.5°

Exercise 3M

1 **a** $\begin{pmatrix}5\\-1\\-3\end{pmatrix}$

b $\mathbf{a}\cdot\mathbf{c}=2\times5-1\times1-3\times3$
$=10-1-9=0$

$\mathbf{b}\cdot\mathbf{c}=1\times5-1\times-1+2\times-3$
$=5+1-6=0$

2 **a** $\begin{pmatrix}2\\8\\1\end{pmatrix}$ **b** $\begin{pmatrix}-3\\0\\6\end{pmatrix}$

c $2i-3j+k$

d $9i+4j+7k$

3 **a** $\begin{pmatrix}-4\\1\\-2\end{pmatrix}$

b $\frac{1}{2}\sqrt{5}=1.12$

4 $\frac{1}{2}\sqrt{42}=3.24$

5 **a** (4, 3, −3)

b $8\sqrt{10}=25.2$

c $\overrightarrow{\mathbf{AB}}\cdot\overrightarrow{\mathbf{AD}}=0\times3+0\times1+8\times0$
$=0$

6 21.0 m^2

7 27

Exercise 3N

1 a $r=\begin{pmatrix}2\\4\end{pmatrix}+t\begin{pmatrix}-1\\-3\end{pmatrix}$

b $r=\begin{pmatrix}2\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}$

2 a $a=0.5,\ b=2$

b $(0, -6, 3)$

3 a $s=3,\ p=4$

b i $t=-2$

ii $7\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}7\\14\\-21\end{pmatrix}$

4 a $4-t=2 \Rightarrow t=2$, $(4, -5, 1) + 2(-1, 3, 1) = (2, 1, 3)$

$4+2s=2 \Rightarrow s=-1$,

$(4, 2, 0) - (2, 1, -3) = (2, 1, 3)$

b 80.7°

5 a The direction vectors are multiples of each other.

b $3-t=1 \Rightarrow t=2$, $(3, 5, 2) + 2(-1, 2, 1) = (1, 9, 4)$

$3+2s=1 \Rightarrow t=-1$,

$(3, 5, 2) - (2, -4, -2) = (1, 9, 4)$

c The two lines are coincident (or the two equations are both for the same line).

6 a $3+s=1 \Rightarrow s=-2$, $(3, 1, 2) - 2(1, 3, -2) = (1, -5, 6)$

b $\mathbf{AB}=\begin{pmatrix}4\\2\\5\end{pmatrix}$, $\mathbf{AB}\cdot\begin{pmatrix}1\\3\\-2\end{pmatrix}$

$=1\times4+2\times3-2\times5=0$

7 a $(2+t, 3t, 1+2t)$

b $\begin{pmatrix}1+t\\3t-2\\3+2t\end{pmatrix}$ c $t=-\frac{1}{14}$

d $\left(\frac{27}{14}, -\frac{3}{14}, \frac{6}{7}\right)$

e 3.73

Exercise 3O

1 a i $i+2j$ ii $\sqrt{5}=2.24$

b i $\begin{pmatrix}-0.25\\0.5\end{pmatrix}$ ii 0.559

c i $-i+2j+k$ ii $\sqrt{6}=2.45$

d i $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ ii $\sqrt{2}=1.41$

2 a $\begin{pmatrix}-10\\0\end{pmatrix}$ b $\begin{pmatrix}-4.5\\6\end{pmatrix}$

c $\begin{pmatrix}-2\\-8\\16\end{pmatrix}$ d $\begin{pmatrix}-3.54\\-3.54\end{pmatrix}$

e $\begin{pmatrix}9.64\\11.5\end{pmatrix}$ f $\begin{pmatrix}10.4\\-6\end{pmatrix}$

3 a 36.1 m

b $p=\begin{pmatrix}20\\30\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}t$ c 1.71 m

4 a $r=\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}6\\-8\end{pmatrix}t$

b $\begin{pmatrix}27\\-31\end{pmatrix}$ c 41.1 m

d 40 m

5 a $\begin{pmatrix}-2\\9\\5\end{pmatrix}$

b i $r=\begin{pmatrix}-2\\9\\5\end{pmatrix}+\begin{pmatrix}1\\4\\0\end{pmatrix}(t-3)t,$

$t\geq3$

ii $r=\begin{pmatrix}-2\\9\\5\end{pmatrix}+\begin{pmatrix}1\\4\\0\end{pmatrix}t',\ t'\geq0$

c i $\begin{pmatrix}0\\17\\5\end{pmatrix}$ ii 17.7

6 a At 12:00 the ships are in the same position $(-1, 14)$

b $\begin{pmatrix}8-3t\\-6+3t\end{pmatrix}$

c 2 km at 12:40

d 1.41 km at 12:20

7 a i $\begin{pmatrix}3\\1\\1.5\end{pmatrix}$ ii 1.5 m s^{-1}

b $\begin{pmatrix}180\\60\\120\end{pmatrix}$ c $r=\begin{pmatrix}180\\60\\120\end{pmatrix}+\begin{pmatrix}3\\1\\-0.6\end{pmatrix}t'$

d 200 seconds e 822 m

8 a $\begin{pmatrix}0\\-9\end{pmatrix}$ b 22.4 km

c $\begin{pmatrix}20-9t\\10-21t\end{pmatrix}$ d $t=1$, $\begin{pmatrix}9\\17\end{pmatrix}$

e $\begin{pmatrix}20\\6\end{pmatrix}$ f 160.1°

Chapter review

1 a 1.74 km b B

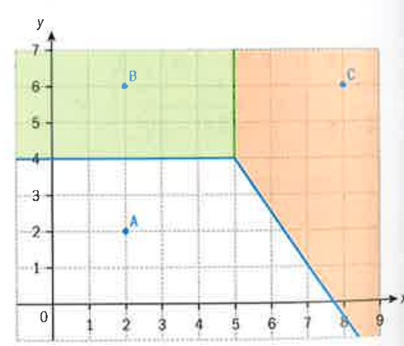
2 a $y=-0.4x+5.4$

b $(1, 5)$

3 a $y=5\ x=5$

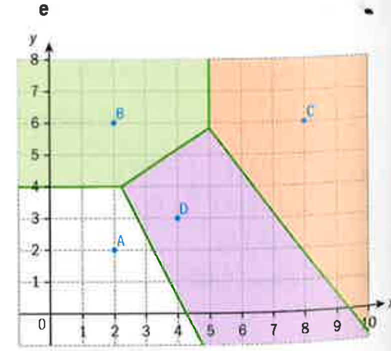
b $3x+y=20$

c



d $y=-2x+2.5$

e



4 a $y=8-x,\ y=\frac{1}{2}x+\frac{7}{4}$

b $(4.17, 3.83)$

c 2.17 km

5 a $\begin{pmatrix}5\\6\end{pmatrix}$ b $\begin{pmatrix}-3\\0\end{pmatrix}$

c 5 m min^{-1}

d 1.8 m when $t=1.52$ min

6 a $2q+p=0$

b $p=0.5,\ q=5$ c 40.2°

7 1.17

8 a $n=24$

b $p=82.5,\ q=-6.5$ and $u=82.5$

9 a

$r=\begin{pmatrix}0\\0\\8.2\end{pmatrix}+\frac{750\sqrt{2}}{2}t\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\2\end{pmatrix}$

b $r_B=\begin{pmatrix}0\\0\\13.1\end{pmatrix}+(t-0.5)\begin{pmatrix}400\\692.8\\-1\end{pmatrix}$

c 438 km

Exam-style questions

10 a $(1, 5.5)$

b -0.625

c i 1.6 ii $y=1.6x+3.9$

11 i Neither, gradients not the same and their product is not -1

ii Parallel, as both lines have a gradient of 3

iii Neither, gradients not the same and their product is not -1

iv Perpendicular, as product of gradients is -1

v Perpendicular, as product of gradients is -1

12 a 707 m b 1310 m

13 a $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ b $\frac{\sqrt{2}}{2}$

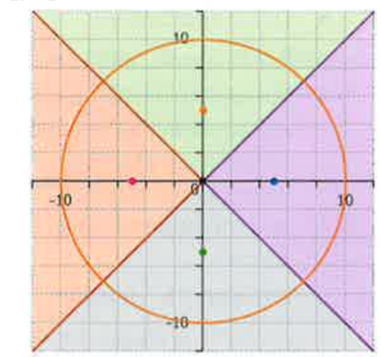
14 a -4.5 b $(-3.23, 1.85)$

15 a 126 b 8.37

c i $(1.5, 2.5, 3)$ ii 64.6°

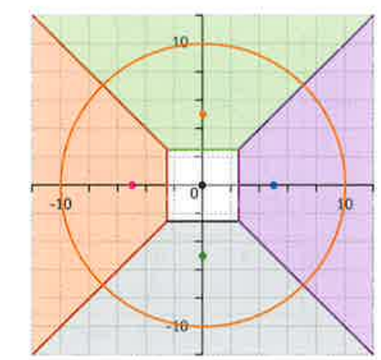
16 a $(150, 70)$ b 99.0 km

17 a



b 78.5 km

c



d 25 m^2

e 72.3 m^2

f 4

18 a $\begin{pmatrix}200\\240\\5\end{pmatrix}$

b The two flightpaths cross at $\begin{pmatrix}150\\180\\4\end{pmatrix}$

c The two aircraft do not collide as the first aircraft gets to the intersection point an hour before the second one does.

d 135 km

e 7.14 km, after 9 and a half hours

Chapter 4

Skills check

1 a $y=0.25x+1.75$

b $y=-3x-4$

2 a $y=23$

b $x=4$

Exercise 4A

1 a Not a function, because one x-value will have multiple y-values.

b y is a function of x, each month has exactly one number of people with a birthday in it.

2 a Function: every element from first set maps onto only one element from second set

b Not a function: input 3 has two outputs

c Not a function: fails the vertical line test

d Function: passes the vertical line test

e Function

f Not a function: fails the vertical line test

3 a $R: x \to \frac{1}{x}$ b Yes

c $\frac{1}{2}$ d $a=\frac{1}{3}$

4 a -1 b $x=-4$

c $B=\{-1, 0, 1, 8, 27, 64\}$

d It is a function.

Exercise 4B

1 a i 5 ii -3

b i 0 ii 3

c $x>3$

2 a i $f(2)=10-4(2)=2$

ii $f\left(-\frac{1}{2}\right)=10-4\left(-\frac{1}{2}\right)=12$

b $f(2.5)=10-4(2.5)=0$

c 4

3 a i 687 N ii 607 N

iii 88.4%. Jaime is 11.6% lighter on the space station.

b $h \approx 310$ km: gravity's force is 625 N at a height of 310 km above sea level.

c $F(653) = 170$

Exercise 4C

1 a On January 2nd last year the average temperature was 25°C.

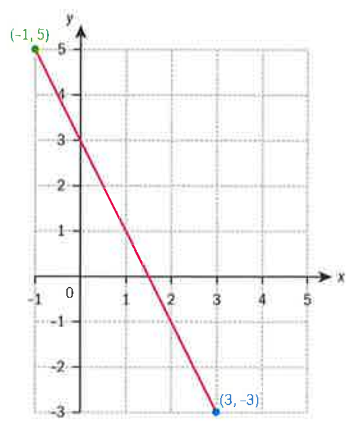
b $\{1, 2, 3, \ldots, 31\}$

c Estimates should be close to $\{T \mid 20 \le T \le 30\}$

2 a i 5 ii −3

b $x = \frac{1}{2}$

c



d $y \in [-3, 5]$ or $\{y \mid -3 \le y \le 5\}$

3 a Domain = $\{-5, -4, -3, -2, -1, 0, 1, 2\}$, range = $\{-2, 0, 2, 4, 6, 8\}$

b Domain = $\{x: -8 \le x \le 6\}$, range = $\{y: -4 \le y \le 3\}$

c Domain = $\{x: -7 < x \le 9\}$, range = $\{y: 0 < y \le 4\}$

d Domain = $\{x: -7 < x < 7\}$, range = $\{y: -4 \le y < 3\}$

4 a $\{p \mid p \ne -1\}$, $p \in \{3, 4, 5, 6, 7, 8\}$

b $w(3) = 12.75$ kg, $w(8) = 10.67$ kg

c $\{w \mid 10.67 \le w \le 12.75\}$

d 4–6

5 a $\{t \mid 0 \le t \le 5\}$

b $\{S \mid 1.2 \le t \le 3.9\}$

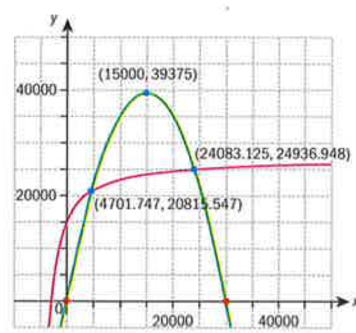
c \$2.09 d 2018

e The researchers would be extrapolating far beyond the data set; also the values of the function become negative after 6 years.

6 a €12 031 b €9844

c $q = 3571$, it costs €20 000 to produce 3571 bottles

d



Break-even when $q = 4702$ and $q = 24\,083$

e Company makes a profit when $\{q: 4702 < q < 24083\}$

7 a $a = 9625$, $b = 5230$

b 2017 c 2066

Exercise 4D

1 a Linear, gradient zero

b Linear, gradient −2

c not linear

d Linear, gradient 5

2 a i independent = time parked (in hours), dependent = cost of parking (in \$)

ii Linear, because there is a constant rate of change of \$12.50 per hour

b i independent = time (in years), dependent = population (in number of fish)

ii Not linear, as a decline of 7% of the current population will not be a constant number of fish per year

c i independent = amount of purchase (euros), dependent = amount of tax (euros)

ii Linear, because constant rate of change of 0.22 euro tax per euro spent

d i independent = daily high temperature (°C), dependent = number of daily passes sold

ii Linear, rate of change between every two points in the table is constant and equivalent to a decrease of 8 passes per degree Celsius.

3 Not linear. The rate of change from 1 to 3 is $\frac{4}{2} = 2$. The rate of change from 5 to 8 is $\frac{4}{3} \ne 2$.

4 a US \$30 b US \$180

c 2.8 kg

5 a $d(t) = 13 - 0.065t$

b 9.1 km

c $d(t) = 0$ when $t = 200$; Ewout will take 200 minutes or approximately 3 and half hours to reach home.

6 a $S(t) = 64 - 1.35t$ b 2028

c Foot Talker grows at 2.5% of total sales per year, whereas Sneakies declines by 1.35% of total sales.

d $F(t) = 13 + 2.5t$

e 2031

7 a 120 m s^{-1} b 8 sec

c −15 m s^{-1}

d $v(t) = -15t + 120$

8 a $50m + c = 20$

b $80m + c = 35$

c $L = 0.5W - 5$

d 40 cm

Exercise 4E

1 a $c = \frac{1}{.21}a$ or $c = 4.67a$

b one AUD = 4.67 CNY

c 51 AUD

2 a $F(x) = 23.4\,x$

b $x = 3.42$ m

c −35.1 N

Exercise 4F

1 a UK £1 = \$1.33

b UK £75.19

c i \$3.28 ii £75.19

iii UK £754.35

2 a Gradient: for each 1 euro that the price increases, the number of people willing to buy tickets deceases by 36. y-intercept: 5000 people will buy the tickets if they cost 0 euros.

b 2300

c $p = 138.89$; the price at which no one is willing to buy a ticket

d Domain $\{x \mid 0 \le x \le 139\}$, range $\{y \mid 0 \le y \le 5000\}$

e \$86

3 a i 90.2% ii 89.3°C

b There is a constant rate of change between the two variables: cooking time and boiling point temperature.

c $T(B) = 5 + \frac{2}{5}(100 - B)$

d 24 min

e Domain: $B \in [40, 100]$, range $T \in [15, 39]$

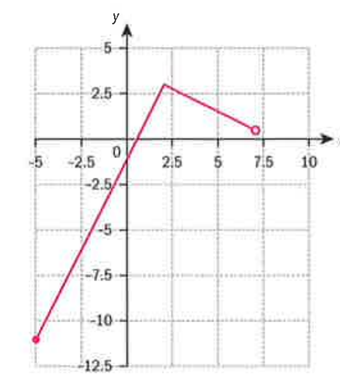
4 a x = number of regular dishes, y = number of premium dishes: $3.5x + 6.50y = 25$

b (8.33, 0) and (0, 3.85) so Alfie can buy 714 g of just Pepperoni or 385 g of just Parma Ham

c Alfie should buy 180 g of Parma Ham and 380 g of Pepperoni

Exercise 4G

1 a



b i 1.15 ii −7.4

c $x = 1.5$, $x = 4$

d The pieces connect at $x = 2$; $2(2) - 1 = 3$ and $4 - \frac{1}{2}(2) = 3$. Outputs match.

e Domain: $x \in [-5, 7)$, Range $y \in [-11, 3]$ (highest point is at (2, 3)).

2 a

$$f(x) = \begin{cases} 130x & 0 \le x \le 5 \\ 650 & 5 \le x < 7 \\ 130x - 260 & 7 \le x \le 10 \end{cases}$$

b At $x = 5$, $130(5) = 650$, and at $x = 7$, $130(7) - 260 = 650$, pieces connect so f is continuous

c 5.84 min

3 a $C(d) = \begin{cases} 35 & 0 \le d \le 1 \\ 35 + 60(d - 1) & d > 1 \end{cases}$

b i \$35

ii \$95

c When more than 1.4 GB of data are used

d i 1.78 GBP

ii Switch, she will save \$22.80

Exercise 4H

1 a $x = -\frac{1}{3}$ b $x = 8$

c $x = 2$ d $x = -6$

2 a Domain: $x \in (-6, 6]$, range: $y \in [2, 8)$

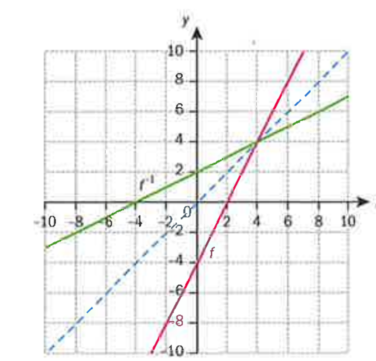
b $g(4) \approx -1$

3 1.27 (3 s.f.); a circle with circumference 8 units will have a radius of 1.27 units.

4 $c = 31$

Exercise 4I

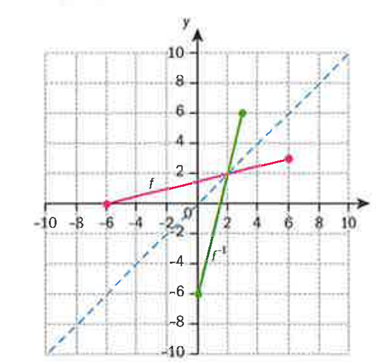
1 a i



ii Domain: $x \in \mathbb{R}$, Range: $y \in \mathbb{R}$

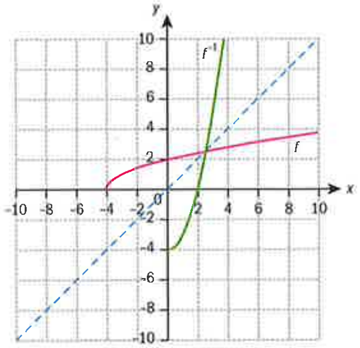
iii $x = 4$

b i



ii Domain: $\{x \mid 0 \le x \le 3\}$, range $\{y \mid -6 \le y \le 6\}$ $x = 2$

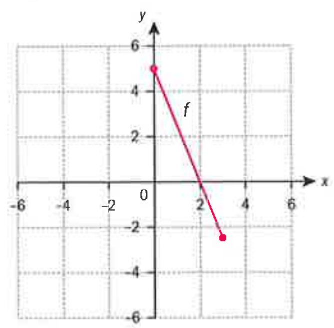
c **i**



ii Domain: $\{x \mid x \geq 0\}$, Range: $\{y \mid y \geq -4\}$

iii $x = 2.5$

2 **a**

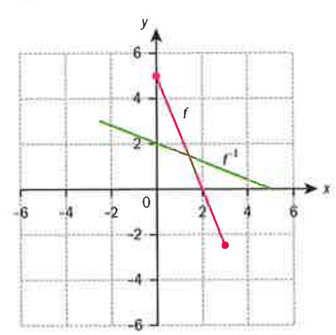


b $b = -2.5$

c

Function	Domain	Range
f	$0 \leq x \leq 3$	$-2.5 \leq y \leq 5$
f^{-1}	$-2.5 \leq x \leq 5$	$0 \leq y \leq 3$

d



e $\left(\frac{10}{7}, \frac{10}{7}\right)$

3 **a** $d(x) = 90x + 630$

b $d^{-1}(x) = \frac{x}{90} - 7$

c **i** 8:47 am

ii 10:47 am

iii 16:33 or 4:33 pm

d **ii** is most reasonable as it is closest to the middle of the day.

Exercise 4J

1 **a** -1 **b** 10.5

c $29 - 8x$ **d** $\frac{27}{8}x$

2 **a** $c(t(x)) = 1.10x + 2$ and $t(c(x)) = 1.10x + 2.2$. These are not equal because they are linear functions with different y-intercepts.

b $t \circ c(x)$ provides a 0.20 euro larger tip, because $2.20 - 2.00 = 0.20$ and the rest of the two functions are equal.

3 Answers can vary, e.g.

a $g(x) = \frac{7}{3}x$ and $f(x) = x - 5$

b $g(x) = x - 2$ and $f(x) = 4x$

c $h(x) = 12 - 4x$, so one possibility is $g(x) = 4x$ and $f(x) = 12 - x$

4 **a** The company loses \$750 if it operates for exactly 4 hours a day.

b $350t - 2150$

c At least 7 hours each day

5
$$C \circ F(x) = C\left(\frac{9}{5}x + 32\right)$$
$$= \frac{5}{9}\left(\frac{9}{5}x + 32\right) - 17.8$$
$$= x + 17.8 - 17.8$$
$$= x = i(x)$$

Similarly, $F \circ C(x) = x$

6 $x = 1, -5$

Exercise 4K

1 **a** **i** 41.7, 50.3, 58.9

ii Yes, $d = 8.6$

b **i** $-83, -105, -127$

ii Yes, $d = -22$

c **i** 151, 196, 250

ii No, difference is not constant, it increases by 9 each time

d **i** 1.25, 0.625, 0.3125

ii No, the numbers are divided by 2 each time instead of being added or subtracted by a constant amount.

2 **a** 9.5, 12, 14.5. This is arithmetic; $a_1 = 9.5$, $d = 2.5$

b 2650, 300, -2050. This is arithmetic; $b_1 = 2650$, $d = -2350$

c 3, 10, 21. Not arithmetic; $21 - 10 \neq 10 - 3$

3 **a** $a_n = 5 + 4(n - 1)$ or $a_n = 1 + 4n$

b Not a term of the sequence

4 **a** 95 in second year, 105 in third

b 175 **c** 21 years

5 **a** $a_n = 2.6 + 1.22(n - 1)$

b $a_{28} = 35.54$ m

c Between 2064 and 2065

Exercise 4L

1 **a** $d = \frac{11}{6}$ **b** $\frac{47}{3}$

2 **a** $d = 2.5$ **b** $u_3 = 5$

3 **a** $d = -25$. The frog hops 25 cm closer to the finish line with each hop.

b $a_{10} = 750$ cm. After 10 hops, the frog is 750 cm from the finish line

c 40

d $a_n = -225$ when $n = 49$. The frog hops 49 times.

4 **a** $12 = u_1 + 2d$ and $43.5 = u_1 + 9d$

b $u_1 = 3$, $d = 4.5$

c $u_{100} = 448.5$

5 **a** $3n + 19$ **b** 29

6 **a** $a_1 = 12$

b $a_3 = 12 + 2d$ and $a_9 = 12 + 8d$

c $2(12 + 2d)$

d $d = 3$

e $a_n = 100$ has solution $n = 30.3$, so Tyler should use an object that is at least 31 kg (!)

Exercise 4M

1 \$1350, \$10 593

2 UK £17 142.86

3 4.6%

4 16th year

Exercise 4N

1 -450

2 3720

3 **a** 42 **b** 10 794

4 **a** $1 + 5 + 9 + \ldots + 37$

b $\sum_{i=1}^{10} 4i - 3$

c $S_{10} = 190$; $S_n = n(2n - 1)$

d $n = 31.9$; it will take Janet 32 years to have a total of 2000 acres.

5 **a** $S_n = \frac{n}{2}(8 - 3(n - 1))$ or

$S_n = \frac{n}{2}(11 - 3n)$

b $S_{10} = -95$

c $n = 15$

6 **a** 8 km **b** 105 km

7 **a** $S_{29} = 1856$

b $d = 19$ will result in 6100 seats

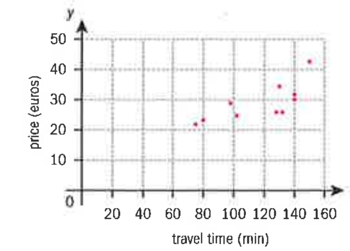
Exercise 4O

1 Approximately 547 years old

2 -4.17% of GDP

Exercise 4P

1 **a** A linear regression is appropriate because the data displays a roughly linear trend.

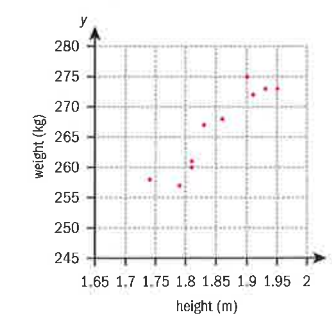


b $y = 0.16x + 9.84$

c $x = 120$, $y = 29.4$ euros

d 100 minutes by interpolation (within the data set). Predicting for 10 minutes would be extrapolation beyond the data set.

2 **a** A linear regression is appropriate because the data displays a roughly linear trend.



b $y = 93.7x + 92.8$

c $x = 1.8$, $y = 261$ kg

d A 1 cm increase in height corresponds to a 0.937 kg increase in weight.

3 **a** $f_1(x)$ is graph A because it has the lower y-intercept (8.11). $f_2(x)$ is Graph B.

b $SSR_1 = 0.576$, $SSR_2 = 0.614$. Since $f_1(x)$ has the smaller sum of square residuals, it is the least squares regression equation.

Exercise 4Q

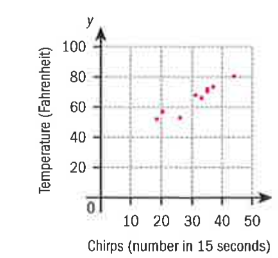
1 **a** The correlation coefficient is weak, so prediction from a linear regression will not be accurate.

b The data does not display a linear trend; it has a curved trend. A linear regression is not appropriate.

c A linear regression is appropriate here, but Kiernan is predicting the independent variable given as a value of the dependent variable. Only predictions of the dependent variable are valid.

d A linear regression is appropriate here, but Kiernan is extrapolating beyond the data set ($x = 80$ is outside the data values).

2 **a** The graph displays a linear trend.



b $r = 0.951$; this is a strong positive correlation

c Yes, it is a linear trend with a strong correlation.

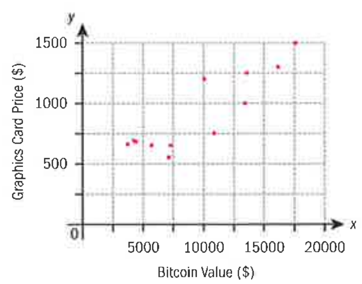
d **i** $T(c) = 1.14c + 30.3$

ii Invalid, can only use this regression to predict temperatures from known chirp numbers.

iii $T(40) = 76$ °F. Valid, because we are interpolating and have already checked that the linear regression is appropriate.

e Answers may vary. Rounding 14 seconds to 15, the Almanac's formula is $T_1(x) = x + 40$. The gradients of the two equations are similar, and the y-intercepts differ by 10 degrees. $T_1(40) = 80$ °F, which is 4 degrees difference from the regression prediction.

3 a The data follows a linear trend, as evidenced by the scatter plot:



b $r = 0.884$; strong positive correlation

c Yes, linear trend with a strong correlation

d i $G(b) = 0.06b + 342$

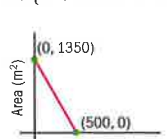
ii $G(12\,320) = \$1081.2$

e $m = 0.06$ means that for every dollar increase in Bitcoin value, graphics card prices increase by 6 cents.

Chapter review

1 a 500 hours

b D: $\{t \mid 0 \le t \le 500\}$, R: $\{A \mid 0 \le A \le 1350\}$

c


d $A(5) = 1336.5$; after 5 hours, 1336.5 m² of surface area remain to be cleaned.

2 a Function; not one-to-one

b Function; one-to-one (linear)

c Not a function: two cylinders with the same volume but different heights will have different radii.

3 a x = babies per woman; y = life expectancy; $y = -5.89x + 88.2$

b The scatter plot shows a linear relationship between the two variables; the correlation coefficient, $r = -0.726$, shows at least a moderately strong correlation.

c When $x = 2$, $y = 76.4$ years. Valid prediction because it is interpolation.

d $m = -5.89$; in the regression equation, an increase of 1 baby per woman corresponds to a decrease of 5.89 years of life expectancy.

4 a Graph 1, because it has a y-intercept of 0.

b Graph 1: gradient $\frac{1}{12}$; the plane uses 12 litres to travel 1 km Graph 2: gradient 0.08; the plane takes 8 minutes to travel 100 km

c $T(d(x)) = 20 + 0.08\left(\frac{1}{12}\right)x$
$= 20 + 0.0067x$;
$T(150\,000) = 1025$ min

5 a 61 months

b $k = \$111$

c 18 months

6 a $C(d) = 0.06d + 49$

b i Brussels: drive (costs €74.20); Hamburg: fly (drive costs €105.40); Paris: drive (costs €110.80)

ii Approximately €335

7 €40 000 at 1.6% interest

8 a CA $24 030.29

b $T(x)=\begin{cases} 0.15x & 0 \le x \le 46\,605 \\ 6990.85+0.205(x-46\,605) & 46\,605 < x \le 93\,208 \\ 16\,544.37+0.26(x-93\,208) & 93\,208 < x \le 144\,489 \end{cases}$

$29\,877.43+0.29(x-144\,489)$ $144\,489 < x \le 205\,842$

$47\,669.80+0.33(x-205\,842)$ $x > 205\,842$

c One payment: $T(162\,000) + T(122\,000) = 34\,955.62 + 24\,030.29$
$= 58\,985.91$

Two payments: $T(142\,000) \times 2 = (29\,230.29) \times 2$
$= 58\,460.58$ Ian should choose two payments; he will save $525.33.

9 a $f^{-1}(x)=\begin{cases} 15-3x, & 4 \le x \le 7 \\ -\frac{1}{2}x+5, & x < 4 \end{cases}$ domain: $\{x \mid x \le 7\}$, range: $\{y \mid y \ge -6\}$

b $f(x)$ is not one-to-one; for example, $f(0) = 5 = f(3.5)$. This can also be shown using the horizontal line test on the graph.

Exam-style questions

10 a i $m = \frac{4}{3}$, ii $c = 6\frac{2}{3}$

b Positive

c 126.67

d 86.67

11 a 10 days

b $\{C{:}186.90 \le C \le 245.35\}$

12 a i $r = 0.849$

ii Strong, positive correlation

iii $y = 0.24 + 0.94x$

b i $r = 0.26$

ii Weak, positive correlation

iii Data does not show a strong enough linear correlation for a linear regression line to be valid.

13 a $b = -\frac{5}{12}$, $a = 5$

b $f^{-1}(x) = 12 - \frac{12}{5}x$

c $3\frac{9}{17}$

d $h^{-1}(x)$ is a reflection of $h(x)$ on line $y = x$ so they intersect when $h(x) = x$

14 a $a = 3$, $b = 0.5$

b 27

15 a $\{f{:}\ f \in \mathbb{R}\}$

b $\{gf{:}\ gf \ge 18\}$

c $x = \pm\sqrt{3}$

d $g(x) = 2x^2 + 96x + 1170$

Chapter 5

Skills check

1 a $\frac{1}{3}$ b $\frac{7}{12}$ c $\frac{5}{12}$

2 a $\frac{57}{116}$ b $\frac{3}{29}$ c $\frac{49}{58}$

Exercise 5A

1 $\frac{1}{3}$

2 a $\frac{17}{20}$ b $\frac{3}{5}$ c 1

d $\frac{7}{20}$ e $\frac{2}{5}$ f $\frac{1}{5}$ g 0

3 a $\frac{73}{239}$ b $\frac{37}{136}$

4 a $\frac{1}{10\,000}$ b $\frac{99}{10\,000}$

c $\frac{99}{10\,000}$ d $\frac{9987}{10\,000}$

5 $\frac{1}{4}$

6 1917.26

7 2

Exercise 5B

1 a $\frac{7}{20}$ b $\frac{1}{5}$ c $\frac{3}{20}$

d $\frac{1}{3}$ e $\frac{4}{9}$ f $\frac{1}{2}$

2 a 5 b $\frac{41}{127}$

c 3228

3 a 9 b $\frac{3}{4}$

4 P(A wins) $= \frac{2}{3}$, P(B wins) $= \frac{1}{3}$,
P(C wins) $= \frac{2}{3}$, P(D wins) $= \frac{1}{3}$

5 P(A beats B) = P(C beats D) $= \frac{2}{3}$

6 $\frac{4}{9}$

Exercise 5C

1 a Independent (I)

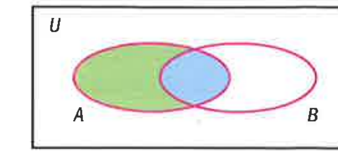
b Neither (N)

c N d I

e Mutually exclusive

f N g I

2 a Mutually exclusive as the areas shaded do not overlap.



b Using the Venn diagram from part **a**.

c Using the Venn diagram from part **a**.

d Independent as neither of the shaded sections for A or B′ encloses the other.

3 a i $\left(\frac{1}{6}\right)^5$ ii $\left(\frac{1}{6}\right)^5$ iii $\left(\frac{1}{6}\right)^5$

b $\frac{4651}{7776}$ c 1.286

d $\frac{1}{1296}$ e 3888 throws

4 a 0.9 b 0.72 c 0.9

5 a $\frac{113}{512}$ b Yes

6 $\frac{8}{13}$

Exercise 5D

1 $\frac{115}{351}$ 2 $\frac{116}{175}$

3 a 0.915 b 0.3115

c 170 d 0.6

4 0.4914 5 0.8

6 a 0.8831 b 23

Chapter review

1 0.6

2 a $\frac{5}{13}$ b No

3 a $1-\left(\frac{5}{6}\right)^n$ b 30

4 a 0.8115 b 0.5525

c 0.9025

5 a $\frac{5n+1}{2}$ b $\frac{n+1}{5n+1}$

6 9

7 a

		Chromosome inherited from mother	
		X	X
Chromosome inherited from father	X	XX	XX
	Y	XY	XY

b There are two outcomes from a sample set of 4 which cause female characteristics to develop.

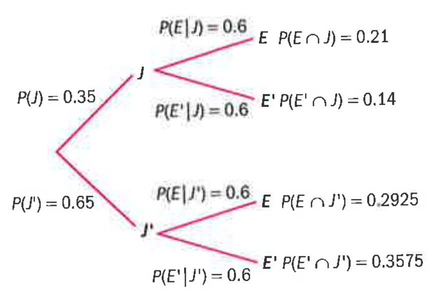
Exam-style questions

8 a $\frac{13}{40}$ b $\frac{7}{20}$

9 a 0.224 b 0.252

c 0.32 d 0.964

10 a Let J be the event that Jake solves it and E be the event that Elisa solves it.



b 0.6425 c 0.4179

11 a 0.4 b 0.6 c 0.75

12 a 9 b $\frac{23}{48}$

c Because

$$P(\text{Europe})\times P(\text{USA})=\frac{32}{48}\times\frac{25}{48}$$
$$=\frac{25}{75}\neq\frac{9}{48}$$
$$=P(\text{both})$$

13 a $\frac{28}{50}$ b $\frac{7}{25}$ c $\frac{5}{11}$

d 0 e $\frac{10}{17}$

14 a 0.15 b 0.65

c 0.15 d 0.5

15 a Events B and C are not independent because C is a subset of B so if C occurs then B must also have occurred by definition, so $P(B\cap C)=P(C)\neq P(B)\times P(C)$

b Events A and C are mutually exclusive as they do not overlap on the Venn diagram, meaning that it is not possible for them top both occur together.

c If A and B are independent, then $P(A)\times P(B)=P(A\cap B)$:
$P(A)\times P(B)=0.3\times 0.28$
$=0.084\neq 0.12=P(A\cap B)$
so not independent.

d Events A' and C' are not mutually exclusive as they overlap on the Venn diagram, meaning that it is possible for them top both occur together.

e 0.3

16 a $a=0.72$, $c=0.18$

b $b=0.08$, $d=0.02$

c 0.26 d 0.0889 e 0.1

f i They are not mutually exclusive as they overlap on the Venn diagram, i.e. there are people who play both squash and tennis.

ii Independent

Chapter 6

Skills check

1 $c=-3$

2 a $x=3, x=-2$

b $x=\frac{2}{3}, x=-1$

3 a $x=-4, x=2$

b $x=4, x=2$

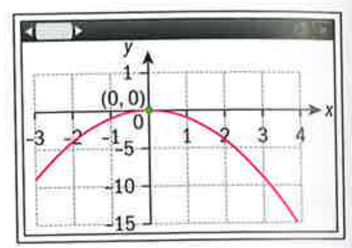
Exercise 6A

1 a $x=-\frac{1}{2}$ b $x=1$

c $x=-4$ d $x=3$

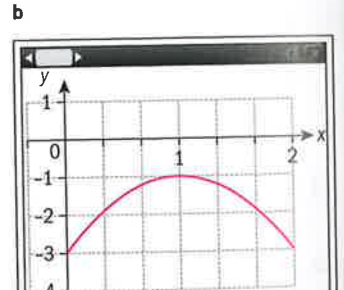
e $x=100$

2 a



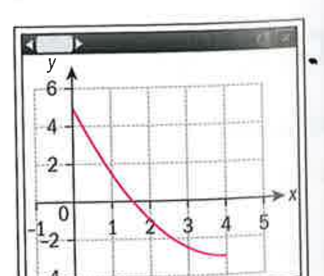
Range $-16\le y\le 0$

b



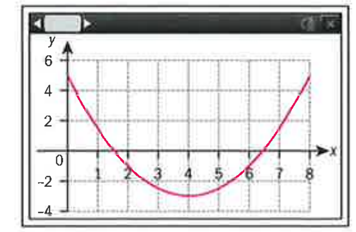
Range $-3\le y\le -1$

c



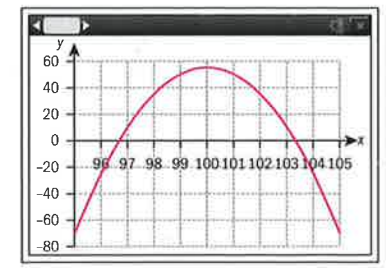
Range $-3\le y\le 5$

d



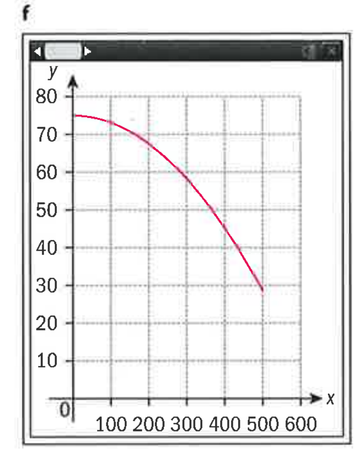
Range $-3\le y\le 5$

e



Range $-70\le y\le 55$

f



Range $28.5\le y\le 75$

3 a (–2.62, 0) and (7.62, 0)

b (0, –8) c $x=2.5$

d (1.255, –9.88)

4 a 5.05 m

b 31.75, ball travels 31.75 m horizontally

c (0, 1), Zander hits the ball when it is at a height of 1 m above the ground

5 a 4.31 m b $x=7.5$

c $x=16.79$, horizontal distance shot-put has travelled when it hits the floor

d $y=1.5$, height of the shot-put when it leaves Omar's hand.

6 a 6 m

b $x=10$

c $x=0$ and $x=20$; $x=0$ is position of the ball when Ziyue kicks it, and $x=20$ is the horizontal distance the ball has travelled when it hits the ground.

7 a 4.68 m

b (8.59, 0) and (29.8, 0), horizontal distance from the left hand side of the ramp as it passes through ground level.

8 a 8.01 m b 4 s

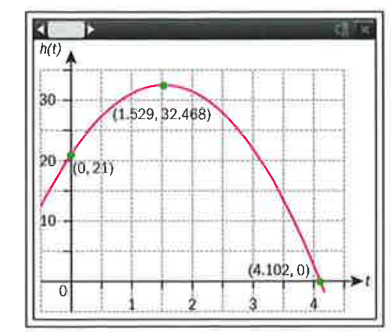
9 a 96 m b 164 m c 174 m

10 a, b

n	1	2	3	4	5	6	7	8
$S_n = 15n - 2n^2$	13	22	27	28	25	18	7	–8

c 28 d 7

11 a

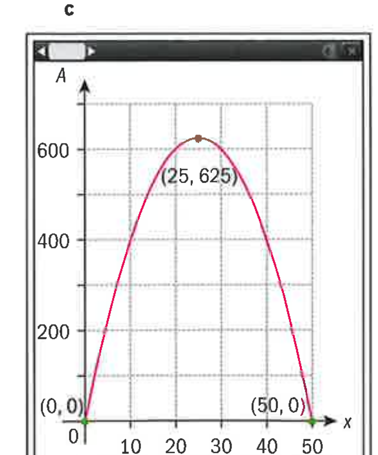


b 32.5 m c 4.10 s

Exercise 6B

1 a $y=50-x$ b $A=50x-x^2$

c



d (0, 0) and (50, 0)

e (0, 0) f $x=25$

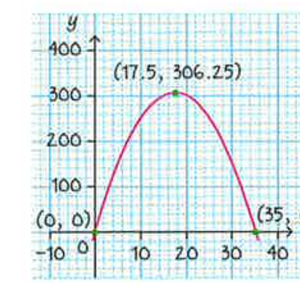
g (25, 625)

h 625 m², $x=25$ m and $y=25$ m

2 a $y=35-x$

b $A=35x-x^2$

c



d (0, 0) and (35, 0)

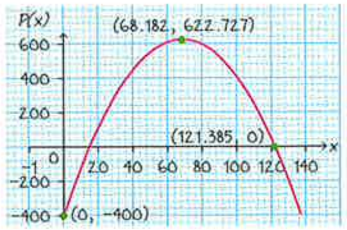
e These are the two values for the frame width which would give an area of zero. As such, these are the upper and lower limits for the width of the frame.

f $x=\frac{0+35}{2}=17.5$, passes through the maximum point of the graph, so gives value of x which gives maximum area of the frame.

3 a (Profit) = (income) – (cost)

$$P(x)=(-0.12x^2+30x)-(0.1x^2+400)$$
$$=(-0.12-0.1)x^2+30x-400$$
$$=-0.22x^2+30x-400$$

b



c The x-intercepts are the number of books which give a profit of €0.

d $x = 68.2$, since it is meaningless to produce 0.2 books, producing 68 books will maximise the profit.

4 a $a = 6$ and $d = 10 - 6 = 4$ so

$$S_n = \frac{n}{2}\left(2(6)+(n-1)(4)\right)$$
$$= \frac{n}{2}(12+4n-4)$$
$$= 4n + 2n^2$$

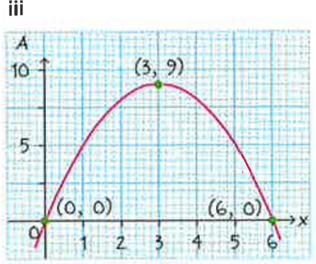
b $2n^2 + 4n - 880 = 0$

c $n > 20$

5 After 2.6 s

6 a i $y = 6 - x$ **ii** $A = 6x - x^2$

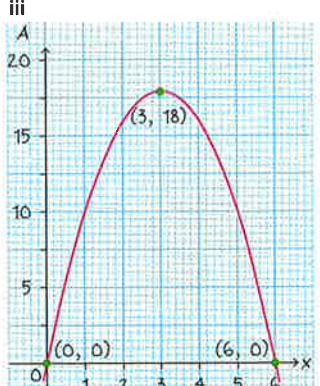
iii



iv $9\,\text{m}^2$

b i $y = 12 - 2x$ **ii** $A = 12x - 2x^2$

iii

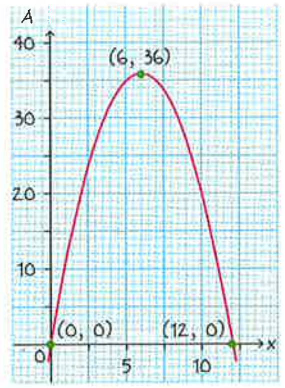


iv $18\,\text{m}^2$

c i $y = 12 - x$

ii $A = 12x - x^2$

iii



iv $36\,\text{m}^2$

v The design in part **c** will give the maximum area.

Exercise 6C

1 a $f^{-1}(x) = \sqrt{x-4}$, domain $x \geq 4$, range $y \geq 0$

b $f^{-1}(x) = 1 + \sqrt{x+2}$, domain $x \geq -2$, range $y \geq 1$

c $f^{-1}(x) = \sqrt{\frac{x}{2}} - 3$, domain $x \geq 0$, range $y \geq -3$

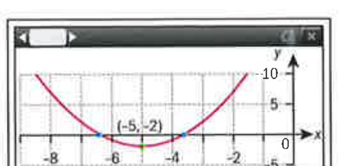
d $f^{-1}(x) = 2 + \sqrt{\frac{1-x}{3}}$, domain $x \geq 1$, range $y \geq 2$

2 a $R(p) = (175 - 3.5p)p$
$= 175p - 3.5p^2$

b $C(p) = 1750 - 17.5p$

c $P(p) = -3.5p^2 + 192.5p - 1750$

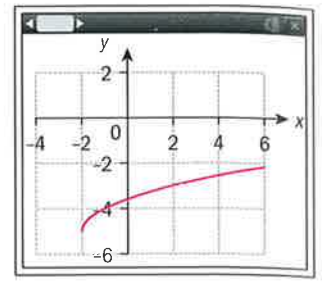
3 a



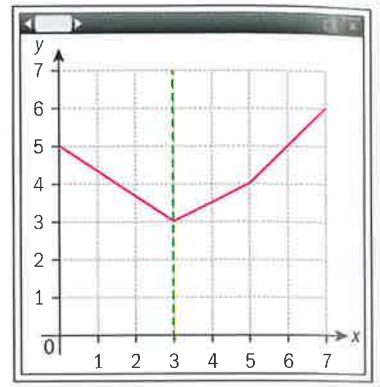
b $f(x)$ is not one-to-one if the domain is $x \in \mathbb{R}$. It fails the horizontal line test as it is symmetric about $x = -5$.

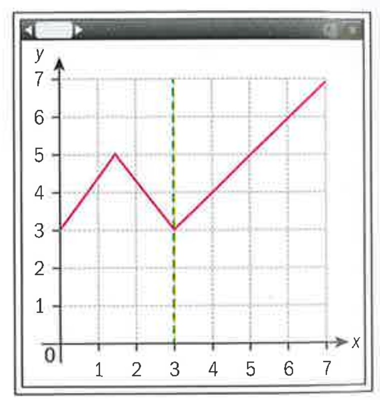
c $x \geq -5$

d



4





Exercise 6D

1 $y = x^2 + 3x - 4$

2 $y = -3x^2 + 3x + 6$

3 $y = 2x^2 - 7x - 4$

4 $y = 2x^2 - 12x + 22$

5 $y = -0.980x^2 - 9.80x - 12.5$

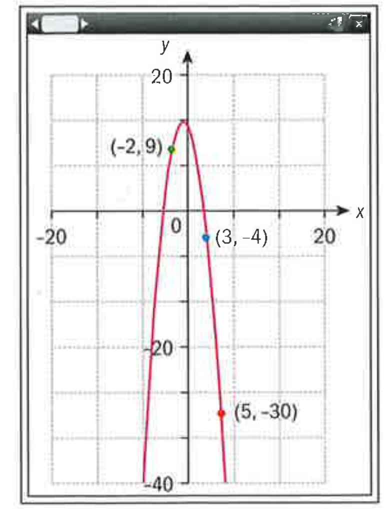
6 a Substituting $x = 3$ and $f(x) = -4$ into the function $f(x) = ax^2 + bx + c$ gives

$-4 = a \times 3^2 + b \times 3 + c$
$\Rightarrow 9a + 3b + c = -4$

b $9 = 4a - 2b + c$ and $-30 = 25a + 5b + c$

c $a = -\frac{52}{35}, b = -\frac{39}{35}, c = \frac{89}{7}$

d



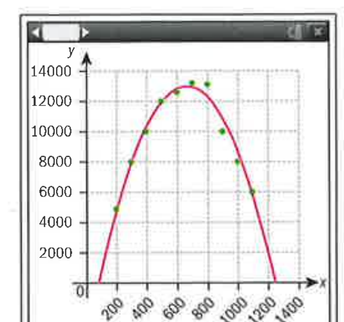
7 $y = -\frac{1}{48}x^2 + 192$

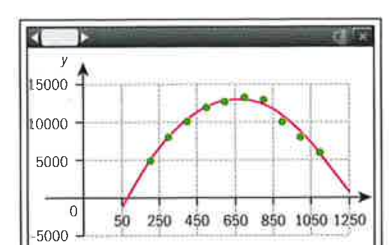
8 $y = -0.00915x^2 + 0.303x$

9 $f(x) = -0.12x^2 + 1.92x + 2$

Exercise 6E

1 a, b

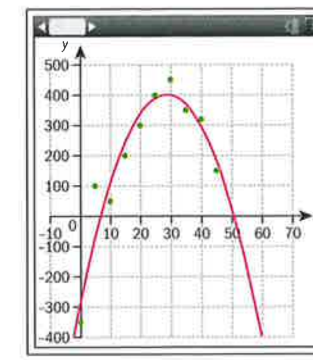


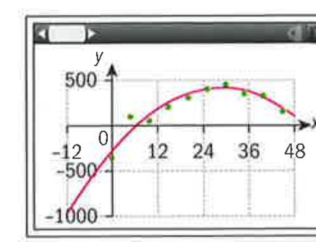


c $r = 0.988$ which is a very strong quadratic association, so equation $y = -0.0382x^2 + 50.5x - 3743$ is a good fit for this data.

d This equation only shows the association between the number of units sold and the profit. It is not linked to a particular period of the year, so could not be used to predict the company's profits at a particular time of year.

2 a, b

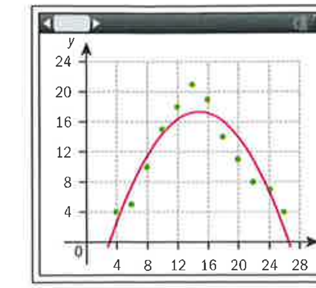




c $r = 0.954$ which is a very strong quadratic association, so equation $y = -0.835x^2 + 47.7x - 282$ is a good fit for this data.

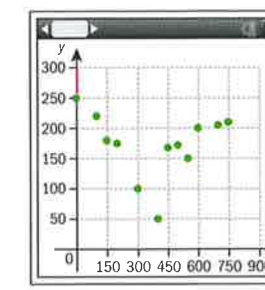
d Week 52 lies outside the range of data given. Using this data to make a prediction for week 52 would be extrapolation, and therefore unreliable.

3 a, b



c 16.6 °C, since $r = 0.924$ suggests a very strong quadratic association, and because 17:00 is within the data range so is interpolation, this is likely to be a reliable estimate.

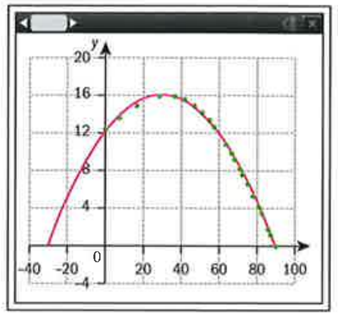
4 a, b

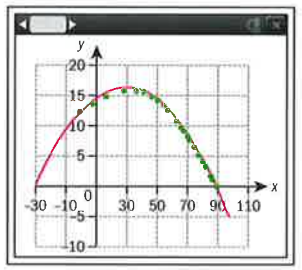


c A correlation coefficient of $r = 0.782$ suggests a strong quadratic association, and therefore Cindy can use this equation to accurately model the bridge.

d Since there is a strong correlation coefficient allowing Cindy to accurately use this equation to model the bridge, and also since 410 m is within the data range and is therefore interpolation, Cindy can use this equation to predict the height of the bridge at 410 m.

5 a





b 29.7°

c The coefficient of regression is 0.999, which indicates the data almost perfectly follows a quadratic model. Therefore, it is entirely appropriate to use the model function found in part **a**.

Exercise 6F

1 a $x^2 \to \left(\frac{x}{2}\right)^2$ = horizontal stretch, scale factor 2, $\left(\frac{x}{2}\right)^2 \to \left(\frac{x}{2}\right)^2 - 3$ = vertical translation of 3 units down, with vector $\begin{pmatrix}0\\-3\end{pmatrix}$

b $2(x-3)^2+4 \to 2(x-3)^2$ = vertical translation down 4 units, with vector $\begin{pmatrix}0\\-4\end{pmatrix}$,

$2(x-3)^2 \to \frac{1}{2}\left[2(x-3)^2\right]$

$=(x-3)^2$ = vertical stretch scale factor $\frac{1}{2}$,

$(x-3)^2 \to ((x-3)+3)^2 = x^2$ = horizontal translation 3 units to the left, vector $\begin{pmatrix}-3\\0\end{pmatrix}$

c $x^2 \to (x-1)^2$ is a horizontal translation 1 unit to the right, vector $\begin{pmatrix}1\\0\end{pmatrix}$

$(x-1)^2 \to 4(x-1)^2$ is a vertical stretch, scale factor 4

$4(x-1)^2 \to 4(x-1)^2+2$ is a vertical translation of 2 units up, vector $\begin{pmatrix}0\\2\end{pmatrix}$

d From part **b**, $2(x-3)^2+4 \to x^2$ = is a vertical translation of 4 units down, with vector $\begin{pmatrix}0\\-4\end{pmatrix}$, followed by a vertical stretch scale factor $\frac{1}{2}$, followed by a horizontal translation 3 units to the left, with vector $\begin{pmatrix}-3\\0\end{pmatrix}$, $(x)^2 \to (x-2)^2$ is a translation 2 units to the right, with vector $\begin{pmatrix}2\\0\end{pmatrix}$,

$(2-x)^2 \to -(2-x)^2$ is a reflection in the x-axis, $-(2-x)^2 \to 2-(2-x)^2$ = vertical translation of 2 units up, with vector $\begin{pmatrix}0\\2\end{pmatrix}$

2 a $-2(x+3)^2$ **b** $\left(\frac{1}{2}x+1\right)^2$

c $\left(\frac{1}{2}x+2\right)^2$ **d** $(2-x)^2+1$

3 a $(1, 5)$

b $y(3) = 2(3-1)^2+5$
$= 2\times4+5 = 13$ so $(3, 13)$ lies on the curve.

c i $\left(\frac{1}{2}, 10\right)$ **ii** $(0, 26)$

d $y = 2(-(x)-1)^2+5$

e Horizontal translation 2 units right, curve becomes

$y = 2\left(-\left(x-\frac{3}{2}\right)-1\right)^2+5$

$= 2\left(\frac{1}{2}-x\right)^2+5$

Vertical stretch s.f. 2, curve becomes

$y = 4\left(\frac{1}{2}-x\right)^2+10$

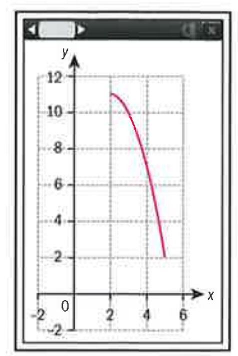
4 a $x^2 \to (x-2)^2$ horizontal translation $\begin{pmatrix}2\\0\end{pmatrix}$

$(x-2)^2 \to -(x-2)^2$ reflection in the x-axis

$-(x-2)^2 \to 5-(x-2)^2$ vertical translation $\begin{pmatrix}0\\5\end{pmatrix}$

b $h(x) = 13-2(x-1)^2$

5 a i



ii Range $2 \le y \le 11$

iii $x^2 \to (x-2)^2$ horizontal translation $\begin{pmatrix}2\\0\end{pmatrix}$

$(x-2)^2 \to -(x-2)^2$ reflection in the x-axis

$-(x-2)^2 \to 11-(x-2)^2$ vertical translation $\begin{pmatrix}0\\11\end{pmatrix}$

b $g(x) = 5-\frac{1}{2}(x-5)^2$

6 Need to reverse the transformations, in the reverse order in which they were applied.

Translation by $\begin{pmatrix}-2\\-5\end{pmatrix}$ to take $h(x) \to g(x)$

Translation $\begin{pmatrix}0\\15\end{pmatrix}$ maps $2(x+2)^2-15 \to 2(x+2)^2$

Stretch s.f. $\frac{1}{2}$ in y-direction maps $2(x+2)^2 \to (x+2)^2$

Translation $\begin{pmatrix}2\\0\end{pmatrix}$ maps $(x+2)^2 \to ((x-2)+2)^2 = x^2$

7 a $x^2 \to (x-1)^2$ translation $\begin{pmatrix}1\\0\end{pmatrix}$

$(x-1)^2 \to 3(x-1)^2$ vertical stretch s.f. 3

$3(x-1)^2 \to 3(x-1)^2+2$ translation $\begin{pmatrix}0\\2\end{pmatrix}$

b Translation $\begin{pmatrix}0\\-3\end{pmatrix}$ maps $3(x-1)^2+2 \to 3(x-1)^2-1$

Reflection in x-axis maps $3(x-1)^2-1 \to -3(x-1)^2+1$

c $h(x) = -3(x-1)^2+1$

8 a $(t \circ f \circ r)(x) = t(-x)^2$ (reflection in the y-axis)

$= (-x)^2+1$ (translation $\begin{pmatrix}0\\1\end{pmatrix}$)

b $(s \circ f \circ t)(x) = s(x+1)^2$ translation $\begin{pmatrix}-1\\0\end{pmatrix}$

$= 2(x+1)^2$ stretch s.f. 2 in y direction

c $(t \circ s \circ f)(x) = (t \circ s)x^2$

$= t(2x^2)$ stretch s.f. 2 in y direction

$= 2x^2+1$ translation $\begin{pmatrix}0\\1\end{pmatrix}$

d $(f \circ s \circ r)(x) = ((s \circ r)(x))^2$

$= (s(-x))^2$ reflection in y-axis

$= (-2x)^2$ stretch s.f. $\frac{1}{2}$ in x-direction

e Vertex of $f(x) = x^2$ is $(0, 0)$: part a: $(0, 0) \to (0, 0) \to (0, 1)$; part b: $(0, 0) \to (-1, 0) \to (-1, 0)$; part c: $(0, 0) \to (0, 0) \to (0, 1)$; part d: $(0, 0) \to (0, 0) \to (0, 0)$

9 a i $(4, 1)$

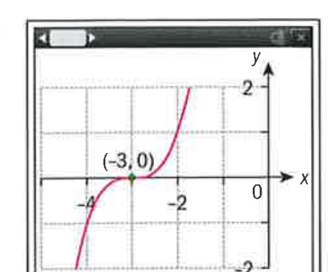
ii $(8, 1)$

b i $\left(\frac{1}{2}x-3\right)^2$

ii $\left(\frac{1}{2}x-3\right)^2$ These are the same equation.

Exercise 6G

1



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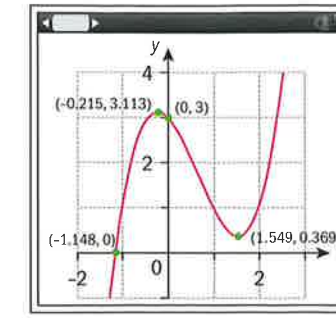
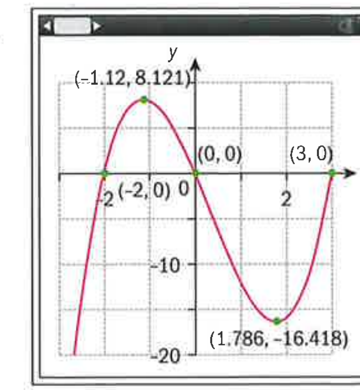
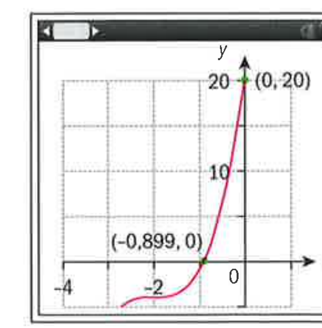


3



4



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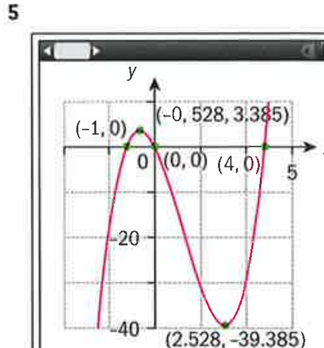
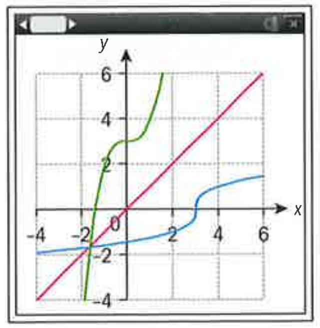
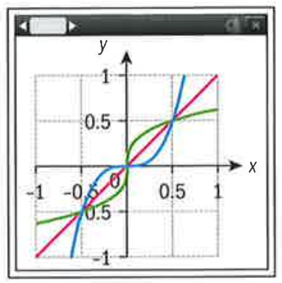
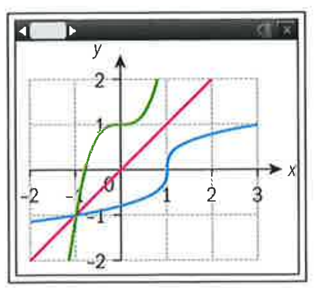


6 a 

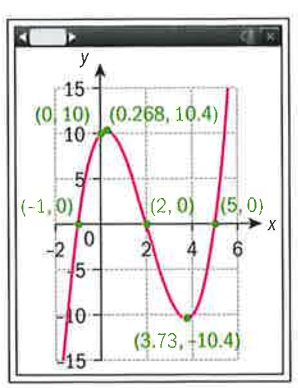
$f^{-1}(x) = x^{\frac{1}{3}} - 3$

b 

$f^{-1}(x) = \left(\frac{1}{4}x\right)^{\frac{1}{3}}$

c 

$f^{-1}(x) = \left(\frac{x-1}{2}\right)^{\frac{1}{3}}$

7 

a (−1, 0), (0, 10), (2, 0) and (5, 0)

b (0.268, 10.4) and (3.73, −10.4)

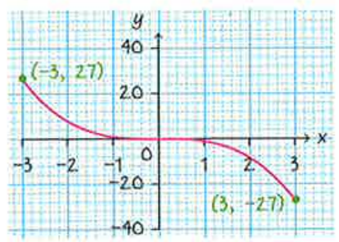
c $f(-x) = -x^3 - 6x^2 - 3x + 10$

8 a High = 24.7 °C, low = 19.2 °C

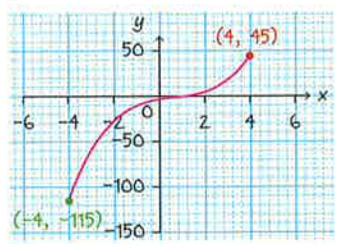
b 20.7 °C

c 11.7 hours

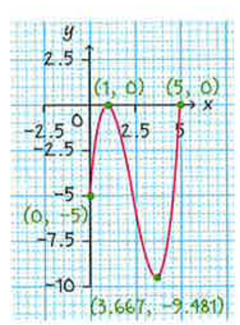
d The function descends rapidly after $t = 24$, so is not a realistic model for the temperature when $t < 24$. As such, the model would not be helpful to predict the temperature at 01.00 on Wednesday.

9 a 

$-27 \le f(x) \le 27$

b 

$-115 \le f(x) \le 45$

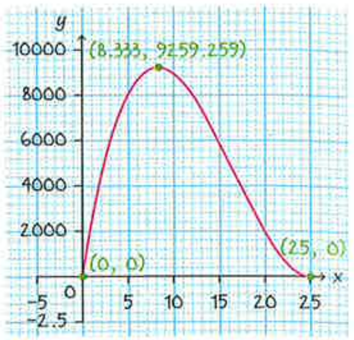
c 

$-9.48 \le f(x) \le 0$

10 a Length = width = $(50 - 2x)$ cm, height = x cm

b Volume = $x(50 - 2x)^2$

c $0 < x < 25$

d 

e Max volume = 9259 cm^3 when $x = 8.33$ cm

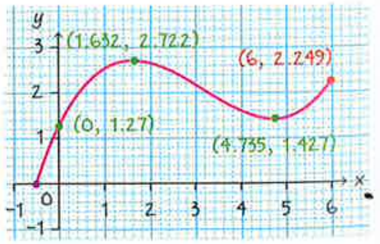
11 a 31.2 °C

b Lowest max in October, highest max in April

c 20.45

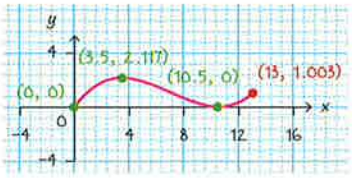
d July

e July is midway between April and October. The max temperature in July is roughly equal to the mean max temperature of April and October suggesting the temperature falls uniformly between April and October.

12 a 

b $1.427 \le f(x) \le 2.722$.

c Length = 6, height = 1.295

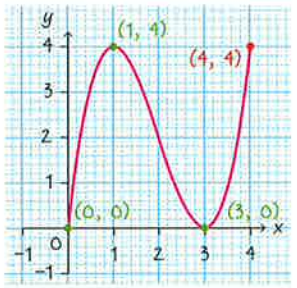
13 a 

b 13 m

c 2.117 m

d The model function needs to be translated 24 units upwards, in the y-direction. This gives the function

$$y = \frac{1}{81}x^3 - \frac{7}{27}x^2 + \frac{49}{36}x + 24.$$

14 a 

b (0, 0)

c Max at (1, 4) and min at (3, 0)

d $0 \le f(x) \le 4$

e $g(x) = -f(x)$,

$h(x) = \frac{1}{2}f(x)$,

$j(x) = f(x) - 4$

f $(j \circ h \circ g \circ f)(x)$

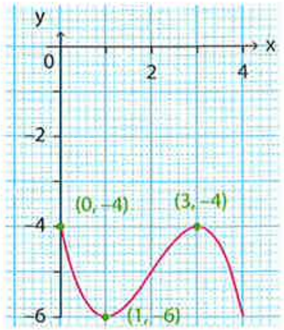
$= (j \circ h)(-f(x))$

$= j\left(-\frac{1}{2}f(x)\right)$

$= -\frac{1}{2}f(x) - 4$

g $f(x) = -\frac{1}{2}x^3 + 3x^2 - \frac{9}{2}x - 4,$

$0 \le x \le 4$

h 

i A positive coefficient of x^3 means that the graph increases to a maximum first, descends to a minimum, and then goes off to infinity as x becomes large.

A negative coefficient of x^3 means that the graph decreases to a minimum first, ascends to a maximum, and then goes off to minus infinity as x becomes large.

j The constant term determines the value of the y-intercept.

15 a Function is one-to-one along its entire domain, so invertible for $x \in \mathbb{R}$

b Function is one-to-one for $x \ge 1.618$, so invertible for $x \ge 1.618$

c Function is one-to-one along its entire domain, so invertible for $x \in \mathbb{R}$

d Function is one-to-one along its entire domain, so invertible for $x \in \mathbb{R}$

16 7.85 cm, 24.3 cm and 44.3 cm

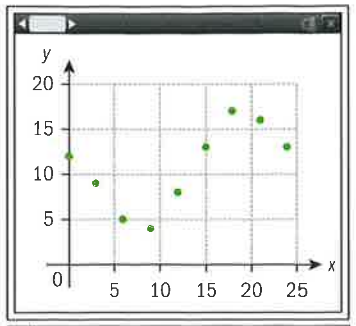
Exercise 6H

1 a $h(t) = 0.00758t^3 + 0.154t^2 - 1.662t + 6.227$

b 2.5 m

c $h(t) = 0.245t - 1.936t + 6.355$

d The cubic function is a better model, but only by a very small margin.

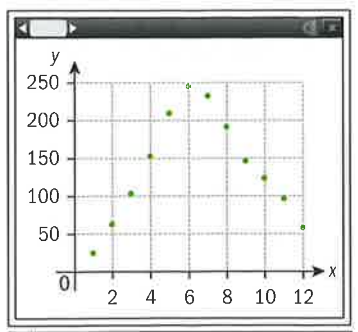
2 a 

x_1	0	3	6	9	12	15	18	21	24
y_1	12	9	5	4	8	13	17	16	13

$T(t) = -0.00848t^3 + 0.335t^2 - 3.162t + 13.283$

b The shape of the points in the scatter diagram follows the general shape of a cubic, as it has both a minimum and a maximum point. The coefficient of regression is $R^2 = 0.915$ which is very strong, and confirms the observation.

c Using the model to approximate the temperature within the times recorded is interpolation, and the model would be valid for these times. Using the model to approximate the temperature for the next day is extrapolation, and the model might not be valid for these times.

3 a 

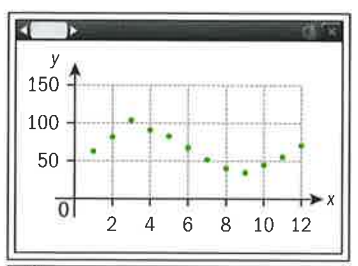
x_1	1	2	3	4	5	6	7	8	9	10	11	12
y_1	24	62	103	152	209	245	232	191	146	123	96	58

$y = 0.153x^3 - 9.15x^2 + 98.8x - 87.5$

b The scatterplot shows that the data appears to have only one vertex. This could mean that the data is in the shape of a parabola, in which case a best fit quadratic curve would be appropriate, or it could mean that a portion of a cubic curve might be best. The coefficient of regression for the quadratic curve is $R^2 = 0.909$, and for the cubic curve is $R^2 = 0.914$. This is marginally better, so use a best fit cubic curve.

c We do not know whether the data will continue to behave like this cubic function in the future, so the model is not useful in predicting the number of future cases.

4 a



x_1	1	2	3	4	5	6	7	8	9	10	11	12
y_1	63	82	104	91	83	68	52	41	35	45	56	71

$y = 0.504x^3 - 9.57x^2 + 46.4x + 27.2$

b By observation, the shape of the points in the scatter diagram in part **a** follows the general shape of a cubic, as it has both a minimum and a maximum point.

c The coefficient of regression is $R^2 = 0.947$ which is very strong, and confirms our observation from **b**. A cubic model is appropriate.

5 a i $y = -2.71x^2 + 18.7x - 7.40$

ii $R^2 = 0.902$

iii $y = 0.5x^3 - 7.96x^2 + 34.5x - 20$

iv $R^2 = 0.955$

b Cubic model

c A particle moving under gravity will always follow a parabolic path, so the quadratic function could be a better model.

6 a Max value is 0, at the two end points

b (0, 0)

7 a A quadratic model might be appropriate as the ball follows the general path of a parabola. However, it seems that the path is not quite symmetrical about the vertex (which a parabola is), so a quadratic model may not be appropriate.

b A cubic model does not need to be symmetric about a maximum point, so might be a better model.

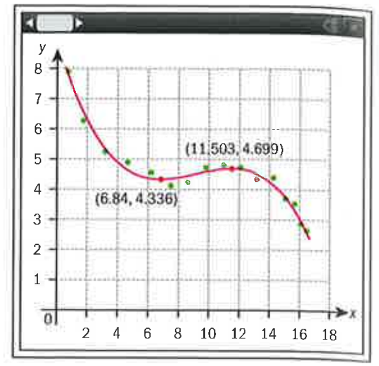
c The theory does not take into account air resistance. When air resistance is considered, quadratic functions are not wholly appropriate.

8 a By plotting the points on a scatter diagram, it appears that the F-hole approximately follows the shape of a cubic curve, so could be modelled by a cubic equation.

b $y = -0.00716x^3 + 0.197x^2 - 1.69x + 8.97,\ 0 \le x \le 16.5$

c The coefficient of determination is $R^2 = 0.991$ which is very strong, so this model is appropriate.

d

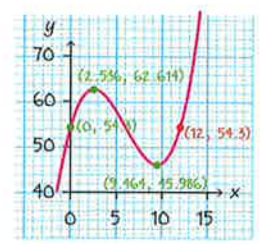


Best fit cubic curve gives minimum at (6.84, 4.34) and a maximum at (11.5, 4.70)

9 a 144 m **b** 5 m

c Roller coaster descends 139 m, so this is the greatest vertical descent

10 a



$0 \le x \le 12$

b 2.54 months

c $t = 9.46$ months

d Parts of August, September, October and November

Exercise 6I

1 a i $R = 2.1\,v,\ R = 1.05\,v^2$

b Linear: $R(3.2) = 2.1 \times 3.2 = 6.72,\ R(4.0) = 2.1 \times 4.0 = 8.4$

Quadratic: $R(3.2) = 1.05 \times 3.2^2 = 10.752,\ R(4.0) = 1.05 \times 4.0^2 = 16.8$

c Linear model much more likely as the sum of squares of residuals is much smaller than for the quadratic model.

d $R = 1.59v^{1.358}$

e Sum of squares of residuals is 0.809 which is much smaller than sum of squares of the linear model, so relationship between v and R is likely to be a power function.

2 a $d = kt^2,\ t = 2,$

$d = 9 \Rightarrow 9 = 4k \Rightarrow k = \frac{9}{4},$

$d = \frac{9}{4}t^2$

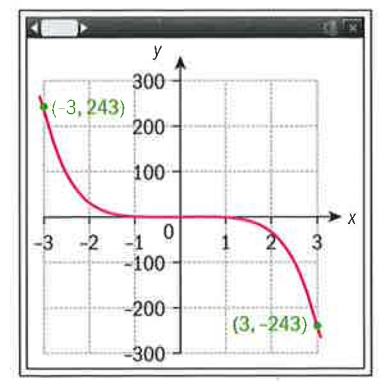
b $d(5) = \frac{9}{4} \times 5^2 = 56.25$ m

c $26.01 = \frac{9}{4}t^2 \Rightarrow t$

$= \sqrt{\frac{4}{9} \times 26.01} = 3.4$ s

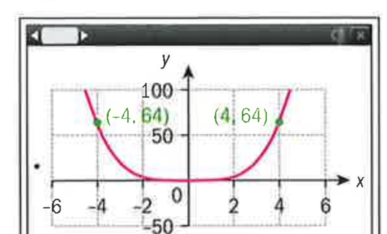
3 1570.83 g

4 a



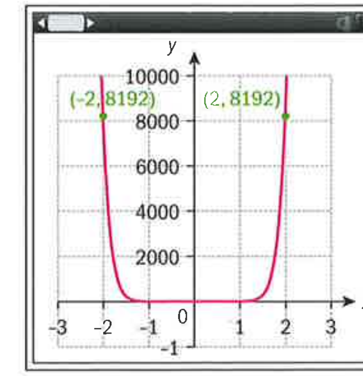
$-243 \le y \le 243$

b



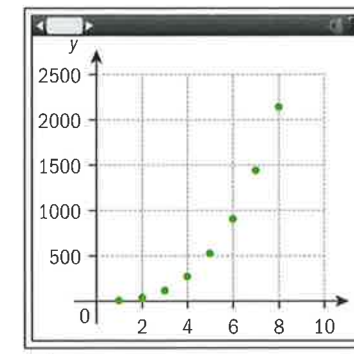
$0 \le y \le 64$

c



$0 \le y \le 8192$

5 a

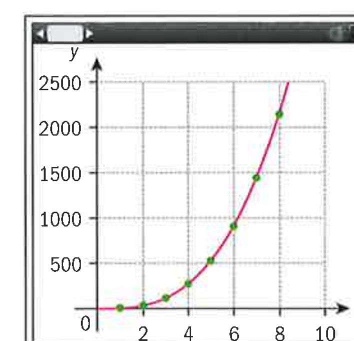


x_1	1	2	3	4	5	6	7	8
y_1	4.19	33.5	113	268	524	905	1440	2140

b $V = 4.19\,r^{3.00}$

c $R^2 = 1$ so the power model fits this data perfectly

d



x_1	1	2	3	4	5	6	7	8
y_1	4.19	33.5	113	268	524	905	1440	2140

Power function is a perfect fit.

e 4190

f 4188.79

6 a $W = \frac{6875}{132}p^4$

b 173 827 kg

c 7.87 m

7 a $f^{-1}(x) = \sqrt{x},\ x \ge 0$

b $f^{-1}(x) = \sqrt[3]{-x}$

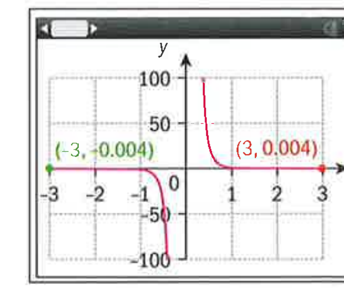
c $f^{-1}(x) = \sqrt[5]{\frac{x}{2}}$

Exercise 6J

1 $V = 4\,\text{Pa}$

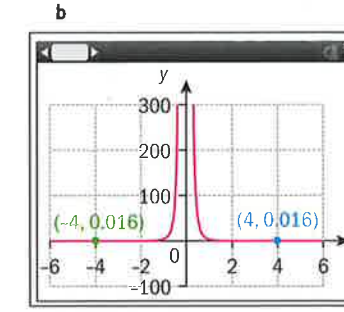
2 8

3 a



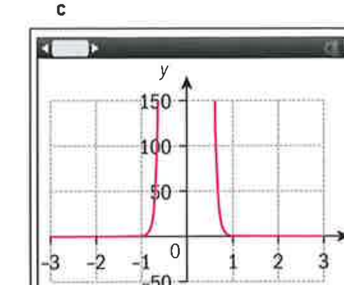
$-\infty < f(x) \le -0.004115$ and $0.004115 \le x < \infty$

b



$0.016 \le g(x) < \infty$

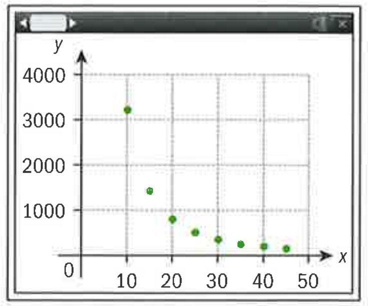
c



$\frac{1}{8192} \le f(x) < \infty$

4 a $\frac{1}{250}$ **b** 0.171 m

5 a



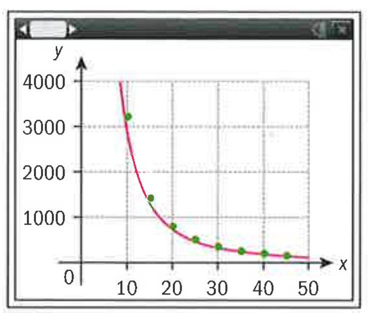
x_1	10	15	20	25	30	35	40	45
y_1	3232	1434	805	513	535	259	201	159

b Follows general shape of an inverse variation curve

c $I = \frac{261\,320}{x^{1.954}}$

d $R^2 = 0.975$ which is a very strong coefficient of determination, so the inverse variation function appears an appropriate model.

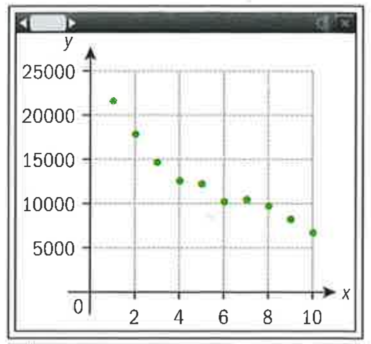
e



x_1	10	15	20	25	30	35	40	45
y_1	3232	1434	805	513	535	259	201	159

f 125 lux

6 a



x_1	1	2	3	4	5	6	7	8	9	10
y_1	21641	17890	14709	12613	12239	10223	10501	9744	8257	6772

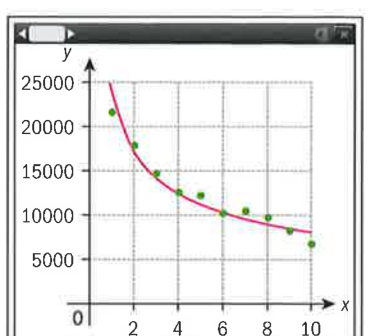
b The price of a car does not vary with age in a linear fashion. If it did, the value of the car would be worthless very quickly.

c The depreciation of the car is quite a lot in the early years, but levels off as the car gets older. An inverse variation function could model this.

d $P = 23688t^{-0.46253}$

e $R^2 = 0.939$ which is a very strong coefficient of determination, so the inverse variation function appears an appropriate model.

f



x_1	1	2	3	4	5	6	7	8	9	10
y_1	21641	17890	14709	12613	12239	10223	10501	9744	8257	6772

This model fits the general shape of the data well. There are a similar number of points above and below the curve.

g $4000 = 23\,688t^{-0.46253}$

Solving using GDC or logs $\Rightarrow t = 46.8$ years

7 a Invertible for $x \geq 0$

b Invertible for $x \in \mathbb{R}$

c Invertible for $x \geq 0$

8 a $f^{-1}(x) = -\frac{1}{x^{\frac{1}{4}}}$

b $f^{-1}(x) = \sqrt[3]{\frac{2}{x}}$

c $f^{-1}(x) = \frac{1}{\sqrt{x}}$

9 a $\frac{1}{x} \to \frac{1}{x-1}$, translation by vector $\begin{pmatrix}1\\0\end{pmatrix}$

$\frac{1}{x-1} \to \frac{2}{x-1}$, stretch in y-direction s.f. $= 2$

$\frac{2}{x-1} \to -\frac{2}{x-1}$, reflection in x-axis

$-\frac{2}{x-1} \to 3 - \frac{2}{x-1}$, translation by vector $\begin{pmatrix}0\\3\end{pmatrix}$

b $\frac{1}{x}$ has vertical asymptote at $x = 0$ as the function is not defined here

$\frac{1}{x}$ has horizontal asymptote at $y = 0$ as $\frac{1}{x} \to 0$ as $x \to \infty$

$3 - \frac{2}{x-1}$ has vertical asymptote at $x = 1$ as the function is not defined here

$3 - \frac{2}{x-1}$ has horizontal asymptote at $y = 3$ as $\frac{2}{x-1} \to 0$ as $x \to \infty$

c The same transformations which map f onto g also map the asymptotes of f onto the asymptotes of g.

10 $1 - \frac{2}{x-2} \to -\frac{2}{x-2}$

translation by vector $\begin{pmatrix}0\\-1\end{pmatrix}$

$-\frac{2}{x-2} \to \frac{2}{x-2}$ reflection in x-axis

$\frac{2}{x-2} \to \frac{1}{x-2}$ Stretch in y-direction s.f. $\frac{1}{2}$

$\frac{1}{x-2} \to \frac{1}{x}$ translation by vector $\begin{pmatrix}-2\\0\end{pmatrix}$

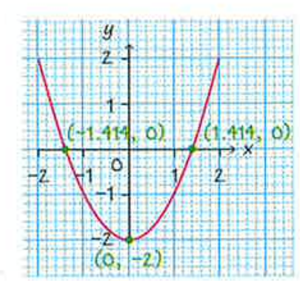
11 a i The shape of the demand curve resembles that of an inverse power law. As such, it must be true that $n < 0$.

ii As N gets large, this function suggests that a small percentage of the market might still buy the product, unlikely to be realistic when N is unreasonably large.

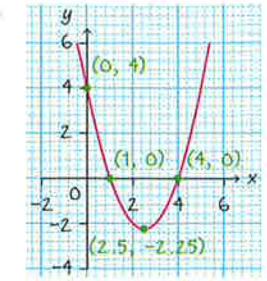
b $k = 57$, $n = -1.066$

Chapter review

1 a



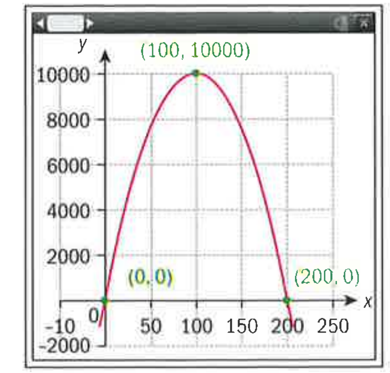
b



2 a $h = 200 - x$

b $A = 200x - x^2$

c



d (0, 0) and (200, 0), represent the limiting values of x for which the picture can be a rectangle.

3 a (0, −7)

b (−7, 0) and (1, 0)

c $x = -3$

d (−3, −16)

4 a (0, 1.9), stone is thrown from a height of 1.9 m

b 10.025 m

c 4 s

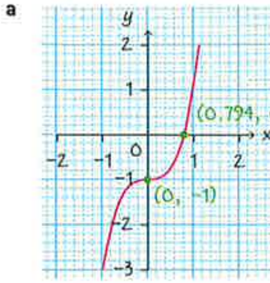
5 a i $x = 2$ and $x = 4$

ii $x = 3$ **iii** (3, −3)

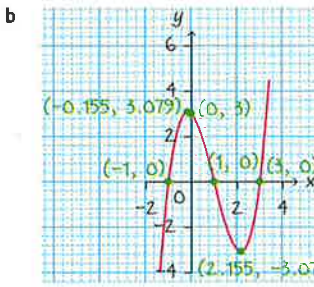
b i $x = -1$ and $x = 5$

ii $x = 2$ **iii** (2, −36)

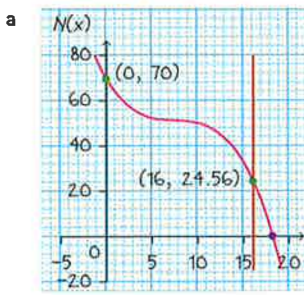
6 a



b



7 a



b 52.5 **c** 46.5

d 70, 2004 **e** 25, 2017

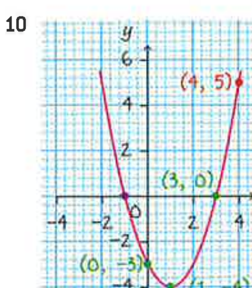
f 2001

8 a $m = 80t$ **b** 160 miles

c 3.75 hours

9 (−3.41, −0.204) and (1.91, 2.45)

10



Minimum occurs at vertex, where $y = -4$, maximum value occurs at $x = 4$, and is $f(x) = 5$

11 a $a = \frac{205.701}{t^2}$

b 20.3 s

12 a 3

b $f(x) = -\frac{8}{9}x^2 + \frac{16}{3}x$

13 a $n = -4$, $k = 512$

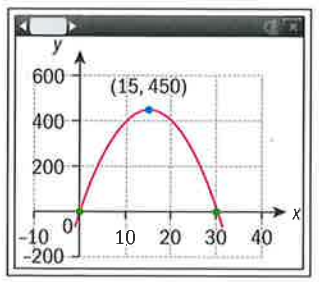
b Same results as part **a**

14 $0.438 \le x \le 4.562$

15 a $BC = 60 - 2x$

b $A = 60x - 2x^2$

c



Max value of the area is $A = 450$ when $x = 15$.

16 a 12 m

b 17 m

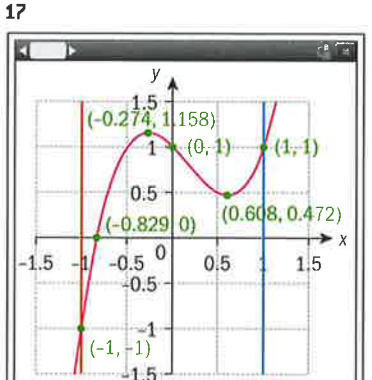
c i $17 = -t^2 + 6t + 12$

ii $t = 1$ or 5

d i $x = 3$

ii 21 m

17



$-1 \le y \le 1.158$.

18 a 126 mm (3 s.f.)

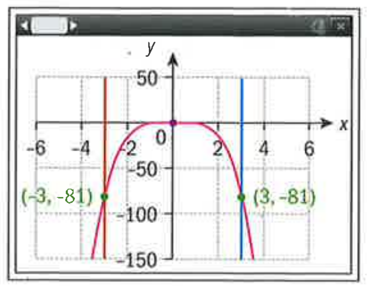
b Greatest in December, lowest in February

c 230.31 mm

d October

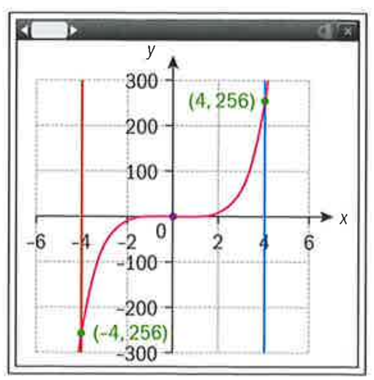
e The mean rainfall occurs much nearer to December than to February. This suggests that the majority of the rainfall occurs towards the end of the year, confirmed by the shape of the curve, which is concave-up.

19 a



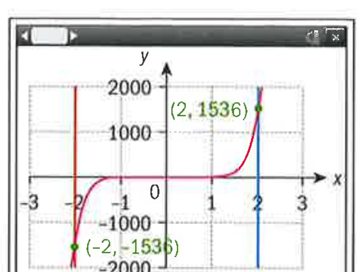
$-81 \le f(x) \le 0$

b



$-256 \le g(x) \le 256$

c



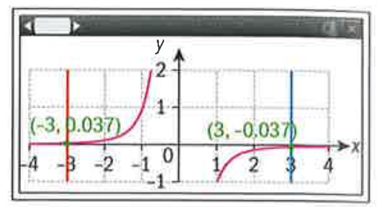
$-1536 \le h(x) \le 1536$

20 a $y = -0.00489x^3 + 29.6x^2 - 59\,561x + 40\,004\,542$

b Using this model to predict future gasoline prices would be extrapolation, so the model may not apply outside the given domain. However, this model does not suggest drastic changes over the 10 years after 2017, so it may well give a sensible estimate of gasoline prices, at least up to 2027.

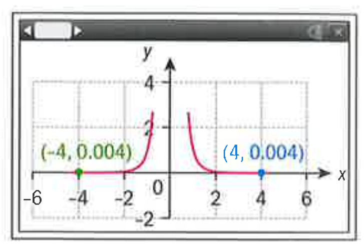
c 31.35 Euros

21 a



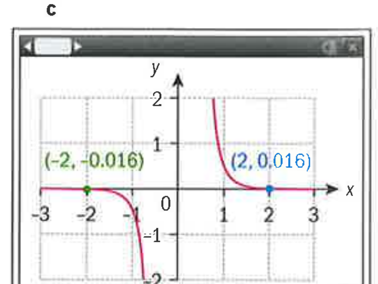
$-\infty < y \le -0.037$ and $0.037 \le y < \infty$

b



$0.004 \le y < \infty$

c



$-\infty < y \le -0.016$ and $0.016 \le y < \infty$

Exam-style questions

22 a $t = 10$s

b Reaches max height at $t = 5$s

c 126.7m

d The displacement-time graphs for Paul's bullet is a translation of Peter's graph by 0.1 m in the y-direction.

23 a $D = b^2 - 4ac = k^2 - 36$

i $k = 6$ or $k = -6$

ii $-6 < k < 6$

iii $k < -6$ or $k > 6$

b $D = b^2 - 4ac = k^2 + 36$, which is always positive, so there are always 2 roots.

24 a (3, 7)

b Domain $x \ge 3$; Range $y \ge 7$

c Domain $x \ge 7$; Range $y \ge 3$

25 a (50, 12.5) b (20, 26)

c $x > 73.4$ m

26 a 7.07 s (3 s.f.) b 12.8 m

c $d = 5$ m

27 1113 or 1311

28 a $k = -5$ b $l = 5$ or 1

c $r = 3$, $s = 4$

d $a < 0$, $b = 3$, $c = 4$

29 a $a = 9$, $b = -10$, $c = 11$

b The point does not lie on the quadratic.

c $p = 1$, $q = 2$, $r = 4$, $s = 3$

30 4 m

Chapter 7

Skills check

1 a x^6 b $\frac{b}{a}$

c $\sqrt{x^3}$ d $\sqrt[6]{x}$

e $\sqrt{x^5}$

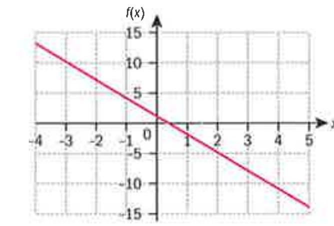
2 a 0.72 b 10.8

c 42

3 a $x = \{-1, 0, 1, 2\}$, $y = \{4, 1, -2, -5\}$

b $f(12) = 1 - 3 \times 12 = -11$

c



4 a 96 b 96 c 100

Exercise 7A

1 a $r = 2$, $u_7 = 320$, $u_n = 5 \times 2^{n-1}$

b $r = -5$, $u_8 = -1\,171\,875$, $u_n = (-3)(-5)^{n-1}$

c $r = \sqrt{3}$, $u_6 = 27\sqrt{2}$, $u_n = \sqrt{2} \times \left(\sqrt{3}\right)^{n-1}$

d $r = \frac{2}{3}$, $u_4 = \frac{4}{9}$, $u_n = \left(\frac{2}{3}\right)^{n-2}$

e $r = 10$, $u_{10} = 2 \times 10^9$, $u_n = 2 \times 10^{n-1}$

2 a $r = 2$; $u_n = 2 \times 2^{n-1}$

b $r = 1.5$, $u_n = 32 \times 1.5^{n-1}$

c $r = 3$, $u_n = 1 \times 3^{n-1}$

d $r = \frac{2}{3}$, $u_n = \frac{1}{2} \times \frac{2}{3}^{n-1}$

e $r = 3$ or $r = -3$; $u_n = -2 \times 3^{n-1}$

f $r = 3\sqrt{\frac{245}{9}}$; $u_n = 13.5 \times 3\sqrt{\frac{245}{9}}^{\,n-1}$

3 a $r = \frac{10}{2} = 5$, so $u_3 = 2 \times 5^2 = 50$, so, geometric sequence

b 31 250

c 2.6×10^{255}; not a reasonable answer as the total population of the world is less than 10^{10} people.

Exercise 7B

1 a 223 000

b 349 801

c No, too fast. In just 4 years the population almost doubled!

2 \$ 2.44

3 €33 079

4 a €10 222 b 8.59 years

5 a 1.125 b 0.27

c 0.89 d 1.001

6 a 1.69 m b 1.61 m

c 1.77 m

7 a 15.43 billion

b 4.73 billion

c 35.54 billion

d By 2029

e 0.36%

Exercise 7C

1 a 7.53 cm b 23.85 cm

2 a 12.3 million

b 63.28 million

c 350.89 million

d 27 hours

3 a 1.062 b 3.29 billion

c 193.5%

d At least 10 years

e It will soon reach scale of the population of the world, so it can no longer grow as fast.

4 a $r = 2$ b 768

c 9 d 20

5 a 976 562

b 479.275 c 3.55

6 a 0.998 b 30.1 s

c Yes

Exercise 7D

1 a Series diverges

b 2 000 000

c Series diverges d 1

2 a $r=\frac{6}{7}$ b 5

3 a $r=\frac{6}{7}$

b $u_1=4$

$u_2=4\cdot\frac{5}{6}=\frac{10}{3}$

$u_3=\frac{10}{3}\cdot\frac{5}{6}=\frac{25}{9}$

c $v_1=4^2=16$

$v_2=\left(\frac{10}{3}\right)^2=\frac{100}{9}$

$v_3=\left(\frac{25}{9}\right)^2=\frac{625}{81}$

d $r_{new}=\frac{\frac{100}{9}}{16}=\frac{25}{36}$

e $S_{\infty(new)}=\frac{v_1}{1-r_{new}}$

$=\frac{16}{1-\frac{25}{36}}=\frac{576}{11}$

f $\frac{\frac{576}{11}}{24^2}=\frac{1}{11}$

4 a 1.5 m b 1.125 m

c 0.834 75 m d 0.75

e No f 8 m

5 a 8π b 4π

c 12π d $\frac{\pi}{2}$

e 15.5π f 7

g 16π

6 a 0.792 m b 7

c 5.07 m d 9.09 m

Exercise 7E

1 Oswald

2 a 68 512 NIS

b 22.005 years

c 3.52%

3 a 5.1% b 8

4 6

5 €6000

6 a 6.14% b 6.17%

7 a 0.3% b $20 060

8 a 0.6% b $2012

Exercise 7F

1 a $111.02 b $5742.60

2 a AED 671.47

b AED 21 869.80

3 a TRY 951 026.40

b 542.5 months, or 45 years

4 a €178.12 b €2078.60

5 Yes

Exercise 7G

1 a i (0, 2) ii $y=1$

iii Increasing

b i (0, −3) ii $y=-3$

iii Decreasing

c i (0, 1) ii $y=0$

iii Increasing

d i (0, 3) ii $y=2$

iii Increasing

e i (0, −2) ii $y=-5$

iii Increasing

f i (0, 7) ii $y=3$

iii Decreasing

g i (0, 4) ii $y=-1$

iii Increasing

h i (0, 1) ii $y=-1$

iii Decreasing

2 a $2\times16^x+5$ b $7\times8^x+2$

3 a 22 b 6.6 hours

c $S_2(t)=14+10\times1.2^{-t}$

d $S_3(t)=12+10\times1.2^{-\frac{t}{2}}$

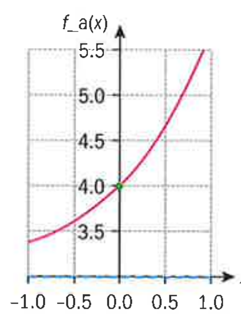
e $S_4(t)=24+20\times1.2^{-t}$

Exercise 7H

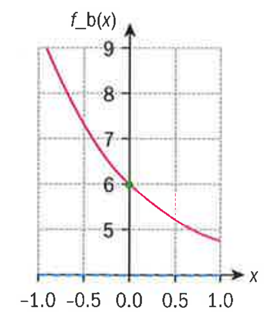
1

	y-intercept	Horizontal asymptote	Growth or decay	Range
$f(x)=e^x+3$	(0, 4)	$y=3$	Growth	[3, ∞]
$f(x)=2e^{-x}+4$	(0, 6)	$y=4$	Decay	[−∞, 4]
$f(x)=0.2e^{0.3x}-2$	(0, 0)	$y=-2$	Growth	[−2, ∞]
$f(x)=5-2e^{-3x}$	(0, 3)	$y=5$	Decay	[−∞, 5]

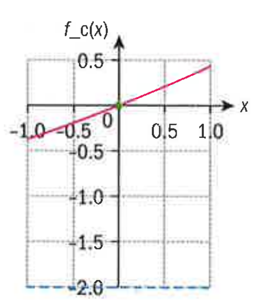
2 a



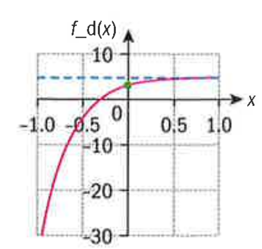
b



c



d



3 a $T(0)=80.4$

b It corresponds to the initial temperature at the moment it is removed from heat.

c Decay

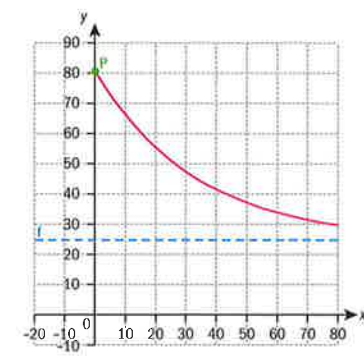
d A hot cup of water will cool down and thus its temperature will fall.

e 24.5

f The temperature of the water will tend towards the temperature of the room it is in.

g $24.5 < T \le 80.4$ The initial temperature of 80.4 falls towards the temperature of the room, which is 24.5

h



4 a 1.488

b $\frac{3}{2}$

Exercise 7I

1 a $k=0.15$

b 64 825

c 63

2 a 952 kg

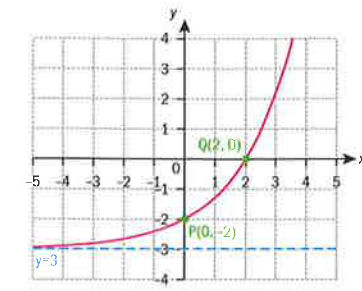
b 413.74 kg

c 2.079 min

d 6.908 min

3 a $y=-3$

b

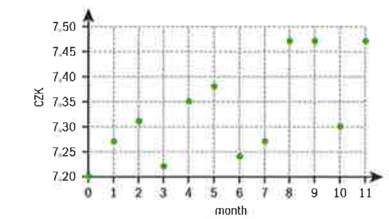


c $D_f: x\in, \mathbb{R}_f: y>-3$

4 a $a=5000$ b 0.1

c 40

5 a

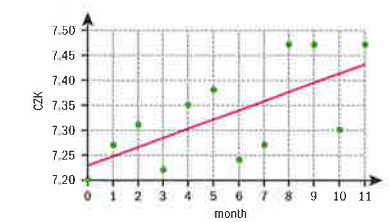


b It is an increasing set of points

c $CZK(t)=4.08e^{0.004t}+3.135$

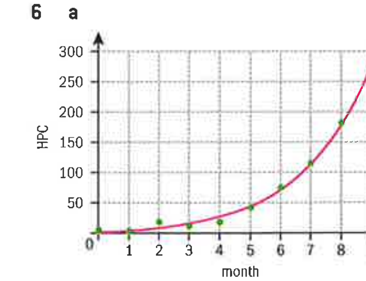
d 0.446

e



f 91 months, or 7.58 years

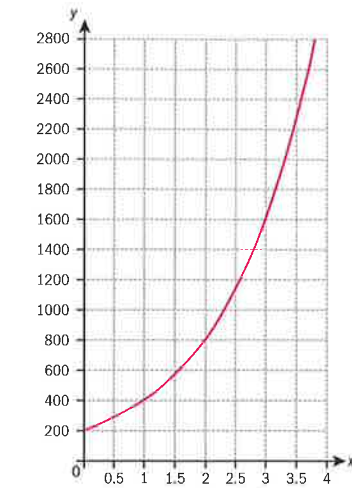
6 a



b Follows a geometric sequence with common ratio 2.

c $u_n=200\cdot2^n$ d 566

e



f The graph shows exponential growth. The domain is all the positive real numbers, whereas the range is $y\ge200$.

7 a $k=0.15$ b 64823

c 33

8 a $a=5000$ b $b=-0.1$

c $T=40$

9 a 1000 b $k=\frac{1}{2}$

c 16000 d $t=10$

Exercise 7J

1 a 2 b −1 c 1 d −1

e 2 f $\frac{1}{2}$ g $-\frac{1}{2}$

2 a 1 b 2 c log(38)

d 2 e ln(3)

f ln(0.3) g 0

3 a $\log(x^2)$ b $\log\sqrt[3]{x}$

c $\log(x^3y^2)$ d $\log\left(\frac{x}{y^3}\right)$

e $\ln(x^{-2})$ f $\log 100x$

g $\ln\left(\frac{x}{yz}\right)$

4 a $\ln\left(\frac{1}{2}\right)$ b $\frac{1}{2}e^4$

Exercise 7K

1 a i log(3)

ii log(75)

iii ln(5)

b i ln(5)

ii log(4)

iii $\ln\left(\frac{8}{3}\right)$

2 a $\log_a(c)$

b $\ln\left(\frac{2}{b}\right)$

c $\log\left(\frac{k}{2}\right)$

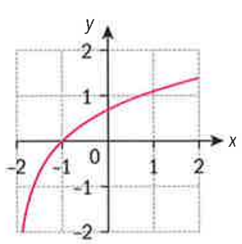
3 a x^3 b $\frac{x}{y}$

c $\frac{x^2}{y}$ d $\frac{1}{x^2}$

4 a $2x + 4$

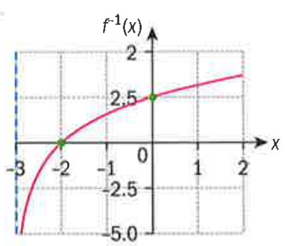
b Vertical asymptote at $x = -2$, $(-1, 0)$

c



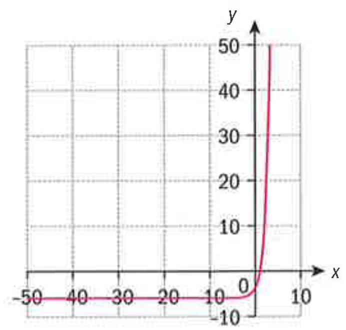
5 a $f^{-1}(x) = \log(x + 3)$

b Vertical asymptote: $x = -3$, $(-2, 0)$



c Domain: $x \in (-3, \infty)$, range: $y \in \mathbb{R}$.

6 a



Horizontal asymptote: $y = -6$, $(\ln(3), 0)$

b $f^{-1}(x) = \log\left(\frac{1}{2}(x+6)\right)$

c Domain: $x \in (-6, \infty)$, range: $y \in \mathbb{R}$

Exercise 7L

1 a $B = 1584.89$, $G = 3\,981\,071$, G is 2511.88 times greater than B

b From $A \to y = 10^{1.7} = 50.11$ to $J \to y = 10^{10.1} = 1.259 \times 10^{10}$

c Yes

2 a

	GNI–Log float64	LE–Log float64
1	4.62408	1.90687
2	4.59406	1.90417
3	4.14551	1.8704
4	4.28623	1.88705
5	4.38525	1.89098
6	4.62634	1.90255
7	4.31869	1.88252
8	4.58546	1.90472
9	4.55206	1.91169
10	4.60173	1.9058
11	4.41647	1.9058
12	4.30664	1.87506
13	4.52153	1.90634
14	4.51468	1.91381
15	4.25091	1.87274
16	4.29425	1.87099
17	4.80895	1.91116
18	4.63609	1.90795
19	4.31133	1.88366
20	4.38881	1.90417
21	4.18013	1.87216
22	4.34694	1.88138
23	4.43072	1.90255
24	4.50051	1.91381
25	4.62531	1.91222
26	4.55558	1.9058

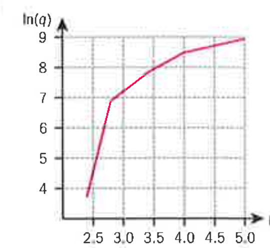
b $y = 0.079\log(x) + 1.54$

c $a = 34.86$, $b = 0.079$

d In the power model we get $a = 35.4$, $b = 0.077$; they are pretty close

3 a $\ln(q) \in \{1.6, 3, 3.4, 3.69, 3.91\}$

b



c $\ln(a) = 1.72$ $b = 0.44$

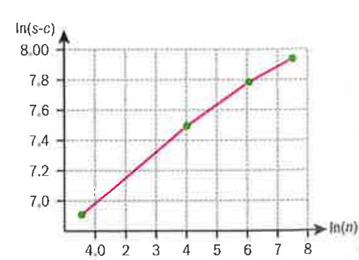
d $q(t) = 5.61e^{0.44t}$

e 5.38 min

4 a 1000

b $\ln(S - c) = b\ln(n) + \ln(a)$

c



d $\ln(a) = 3.99$, $b = 0.75$

e $\ln(S - 1000) = 54.05 \times n^{0.75} + 1000$

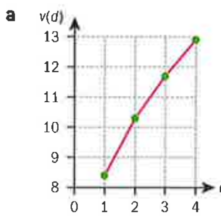
f $6715.61

g For many shirts the behaviour may be different, as in the cases analysed for geometric series.

h i $x = n$, $y = S - c$

ii $a = 55.2 \to \ln(a) = 4.01$, $b = 0.742$, similar to linear model

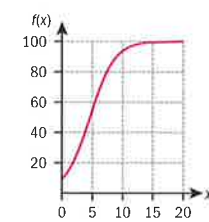
5 a



b $a = 4.5$ and $b = 12.9 - 2 \times 4.5 = 3.9$

Exercise 7M

1 a



b When $t = 0 \to f(0) = \frac{100}{11} = 9.09\%$, the initial recorded percentage of people with access to the Internet. When $t \to \infty \to f(t) = \frac{100}{1} = 100\%$ we are getting asymptotically to a state where everyone will have Internet, at least according to this model.

c 4.6 years d 99.9%

2 a 21 400

b 106 934

c 107 000

3 a $L = 120 \times 10^6$, $C = 11\,999$, $k = 0.346$

b 39 898

c 33.5 weeks

Chapter review

1 a 2.5

b 1220.70

c 2033.17

2 US $184.71

3 a 10%

b 272 727 euros

c 643 076 euros

4 a 11

b 177 146

5 a UK £9652

b UK £9787.13

6 a SGD 3453.80

b 21.76 years

7 2.35%

8 US$73.67

9 115.36 euros

10 a 42 296

b 6.116 years

11 a 95 °C

b 32.95 °C

c 7.46 minutes

d 21 °C

12 a $y = 16$

b (0, 20)

13 a 1101

b 10.13, so after 11 days

14 a 1.37 m

b 16.11, so 17 weeks

15 a $5 \times 2^n - 5$

b $5 \times 4^n - 5$

16 $9905.5

Exam-style questions

17 a 0.1kg

b 1.73 years (3 s.f.)

c 5.76 years (3 s.f.)

18 a $\log xy = \log x + \log y = p + q$

b $\log\frac{x}{y} = \log x - \log y = p - q$

c $\log\sqrt{x} = \frac{1}{2}\log x = \frac{1}{2}p$

d $\log x^2y^5 = 2\log x + 5\log y = 2p + 5q$

e $\log(x^y) = y\log x = 10^q p$

f $\log 0.01x^3 = \log\frac{x^3}{100} = \log x^3 - \log 10^2 = 3p - 2$

19 a $a = 6$, $b = 2$

b 2.53 m (3 s.f.)

c 19.5 years

20 a i 6000 ii 12000

b 40000

c $r = e^{-\left(\frac{\ln 2}{6000}\right)t}$

d 0.315 (3 s.f.)

21 a $r = 3$, $a = 2$

b i 4374 ii 6560

c $n = 13$

22 a i $u_n = ar^{n-1}$

ii $S_n = a\left(\frac{r^n - 1}{r - 1}\right)$

b v_n is an arithmetic progression

c $T_n = \frac{n}{2}(2\log a + (n-1)\log r)$

d $T_n \neq \log S_n$

23 a $r = \frac{2}{3}$

b $r = -\frac{1}{2}$

c Sum to infinity cannot be equal to $\frac{1}{3}$ times the first term.

24 a £97.09

b £74.41

c £84.69

25 a Scheme 1

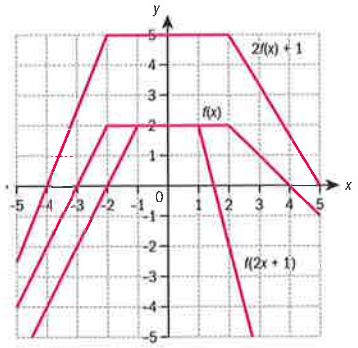
b 1083.26 euros

c No, scheme 1 is still better.

Chapter 8

Skills check

1 a, b



Exercise 8A

1 a $\frac{\pi}{6}$ b $\frac{11\pi}{12}$ c $\frac{3\pi}{2}$
d $\frac{5\pi}{3}$ e $\frac{7\pi}{6}$

2 a 60° b 240° c 108°
d 540° e 57.3°

3 a 14.66 m b 51.31 m²
c 21.2 m² d 30.1 m²

4 a 8.1 cm, 31.5 cm²
b 7 m, 30.1 m²
c 24 cm, 19.64 cm²

5 a 56.4° b 4.41 m²

6 a 11.08 cm b 6.92 cm²

7 7.36

Exercise 8B

1 a 0 b 1 c 0 d 0
e −1 f −1 g −1 h 0

2 a i $x = \{10°, 170°, 370°, 530°\}$
ii $x = \{17.5°, 162.5°, 376.5°, 522.5°\}$
iii $x = \{160°, 200°, 520°, 560°\}$

b i $x = \left\{\frac{2\pi}{5}, \frac{3\pi}{5}, \frac{12\pi}{5}, \frac{13\pi}{5}\right\}$
ii $x = \{0.3, 2.84, 6.58, 9.12\}$
iii $x = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}\right\}$

3 a i (0.51, 0.49)
ii (0.94, 0.81), (2.65, 0.47), (6.07, −0.21)

b Once the y-coordinate of points on the lines is greater than 1 or less than −1, the line will not intersect the curve again.

4 a On the unit circle the y-coordinate of one of the angles is the negative of the y-coordinate of the other angle.

b i 30° ii 150°

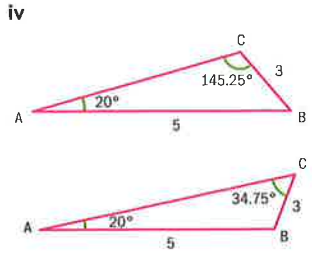
c i Using the sine rule:

$$\frac{\sin(20)}{3} = \frac{\sin(B\hat{C}A)}{5}$$

$$\rightarrow \sin(B\hat{C}A) = \frac{5\sin(B\hat{C}A)}{3}$$

ii 34.8° iii 145°

iv



5 4.72 cm and 21.18 cm

6 a i $\frac{\sqrt{3}}{2}$ ii $\frac{1}{2}$
iii $\frac{1}{2}$ iv $\frac{\sqrt{3}}{2}$

b i $\frac{1}{2}$ ii $\frac{1}{2}$
iii $\frac{\sqrt{3}}{2}$ iv $-\frac{\sqrt{3}}{2}$

7 a i $\frac{2\sqrt{2}}{3}$ ii $2\sqrt{2}$

b i $\frac{1}{3}$ ii $-\frac{2\sqrt{2}}{3}$

8 a $\frac{1}{2}$

b i $\frac{1}{\sqrt{5}}$ ii $\frac{2}{\sqrt{5}}$

c $\sin x \le 0$, $\cos x \ge 0$, cannot both be 0, hence $2\sin x \ne \cos x$

Exercise 8C

1 a i 3 ii $y = 3$
iii 2 iv $-\frac{3}{2}$

b i 4 ii $y = 0$
iii $\frac{2\pi}{3}$ iv 2

c i 1 ii $y = -1$
iii π iv −1.71

2 a $y = 2\sin\left(\frac{\pi}{2}x\right) + 1$

b $y = 2.5\sin\left(\frac{\pi}{3}x\right) - 1$

3 a i $y = 4\sin\left(\frac{\pi}{2}(x - 0.5)\right) - 1$

ii $y = 3\sin\left(\frac{\pi}{3}(x - 2)\right) + 2$

b i $y = 4\cos\left(\frac{\pi}{2}(x - 1.5)\right) - 1$

ii $y = 3\cos\left(\frac{\pi}{3}(x - 3.5)\right) + 2$

c $\frac{1\pi}{2b}$

4 a To have a scale, to represent date as a single ordered number

b Hour + $\frac{1}{60}$ second

c $f(t) = 1.5\sin(0.017t + 1.67) + 6.4$

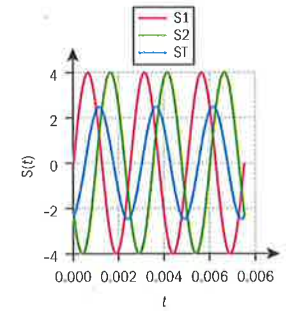
d Notice that 02-Feb-2019 is day 398 in our scale. Hence, $f(398) = 7.35 = 7{:}21$

e Good fit, it cannot reproduce anomalies

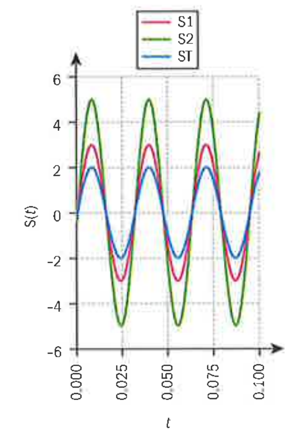
5 a $b = 2513$

b 0.001 s
$S = 4\sin(800\pi(t - 0.001))$

c Looks like a sinewave which is on counterphase with roughly half the amplitude



6 a



b max = 4.97 ≈ 5,
min = −4.97 ≈ −5

c i 5 ii 200

d $S_T(t) = 5\sin(200(t - 0.0004)) + 0$

7 a $a = 1.65$, $d = 2.45$

b take the mean between the time difference on the highest and lowest tides to get the period:

$P = \frac{1}{2}((15 - 15 - 3.033) + (21.32 - 8.9)) = 12.27.$

Hence, $b = \frac{2\pi}{P} = 0.512$

c −0.034

d $f(t) = 1.54\sin(0.518(t - 0.03)) + 2.5$, similar to model by inspection of the table

e 1.4 m

Exercise 8D

1 a 1.71 or 0.29

b $x = 5$ or $x = -1$

c $1 \pm \frac{1}{2}i$ d $= \pm\sqrt{10}\,i$

2 a $-5 + 8i$ b $8 - i$

c $\frac{4 + 7i}{13}$ d $5 - 12i$

e $-2 - 2i$ f $\frac{-69 + 58i}{13}$

3 $2 \pm i$, hence, $x^2 - 4x + 5 = (x - (2 + i))(x - (2 - i))$

4 $p = -6$, $q = 10$

Exercise 8E

1 a $r = 8$, $\theta = 90°$
b $r = 7$, $\theta = 180°$
c $r = 12$, $\theta = 0°$
d $r = 5$, $\theta = 270°$

2 a 3.61, −0.983
b 5.39, 0.38
c 3.16, −0.322
d 4.47, 0.464
e 5.39, −1.19
f 3.16, −1.25

3 a $4\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ b $4\text{cis}\left(\frac{\pi}{3}\right)$

c $6\text{cis}\left(-\frac{\pi}{6}\right)$ d $2\text{cis}\left(\frac{\pi}{6}\right)$

e $5\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ f $14\text{cis}\left(-\frac{\pi}{3}\right)$

4 a $1.5 + 2.59i$ b $-2 + 3.46i$
c $-1.72 - i$ d $4.61 + 1.95i$
e $-0.776 + 2.27i$
f $3.14 - 2.14i$

Exercise 8F

1 a $a = 2\text{cis}\left(\frac{\pi}{3}\right)$, $b = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$,

$c = 2\text{cis}\left(-\frac{\pi}{6}\right)$

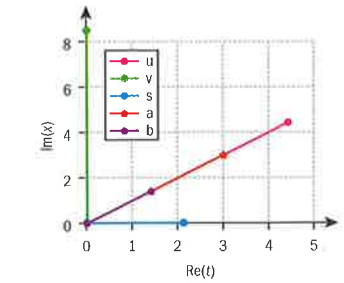
b i $2\sqrt{3} + 2i$ ii i iii −4
iv −4i v $1 - i$

2 $n = \{5, 15, 25, 35, \cdots\}$

3 a $2e^{\frac{i\pi}{4}}$

b (3, 3) in the Argand diagram

c



d u stays in the same line as a and b, v is a projection to the y-axis and s is a projection to the x-axis.

4 If $z = x + iy$, then $z^* = x - iy$.
Thus, $|z^*| = |z| = r$ and

$\arg(z^*) = \text{atan}\left(-\frac{y}{x}\right)$

$= -\text{atan}\left(\frac{y}{x}\right) = -\arg(z) = -\theta.$

Hence $z^* = re^{-i\theta}$.

5 $e^{i\pi} = \cos(\pi) + i\sin(\pi)$
$= -1 \rightarrow e^{i\pi} + 1 = 0$

6 a $\text{Re}(z) = \text{Im}(z)$

b i $t = 1$ ii $t = 2$

c Imaginary crossing: $z_I = 0 + i$, real crossing: $z_R = 2 + 0i$

d $z_1 = \left(1 + \frac{\theta}{\pi}\right)e^{i\theta}$,

$z_2 = \left(2 + \frac{\theta}{\pi}\right)e^{i\theta}$, $z_3 = \left(3 + \frac{\theta}{\pi}\right)e^{i\theta}$

e $|z_2| - |z_1| = 1$, $|z_3| - |z_2| = 1$, this relation will keep the same as the spiral is growing at a constant rate

Exercise 8G

1 Let the phase differences be 0° and 60°.

$110\,\mathrm{cis}\,0^\circ + 110^\circ\,\mathrm{cis}\,60^\circ$

$= 110 + 55 + 55\sqrt{3}i$

$= 165 + 55\sqrt{3}i$

$= 110\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 110\sqrt{3}\,\mathrm{cis}\,30^\circ$.

. So the voltage output will be $110\sqrt{3}$ V (with a phase shift of 30°).

2 a Let the phase differences be 0°, 120° and −120°.

$110\,\mathrm{cis}\,0^\circ + 110\,\mathrm{cis}\,120^\circ + 110\,\mathrm{cis}(-120^\circ) =$

$110 - 55 + 55\sqrt{3}i$

$-55 - 55\sqrt{3}i = 0$.

b Connecting in reverse is equivalent to reflecting in the *x*-axis as all positive becomes negative and vice versa. That is the same as shifting by 180°.

c Replacing −120° with 60°gives

$110\,\mathrm{cis}\,0^\circ + 110\,\mathrm{cis}\,120^\circ$

$+ 110\,\mathrm{cis}\,60^\circ = 110 - 55$

$+ 55\sqrt{3}i + 55 + 55\sqrt{3}i$

$= 110 + 110\sqrt{3}i = 220\,\mathrm{cis}\,60$

3 15.09 V

4 3.61, 88.9°

5 a $g(t) - f(t) = 2.19\sin(0.0165t - 1.23) + 18.0 - 2.14\sin(0.0165t + 1.81) - 5.97$
$= 2.19\sin(0.0165t - 1.23) - 2.14\sin(0.0165t + 1.81) + 12.03 = 2.19\sin(0.0165t - 1.23) + 2.14\sin(0.0165t + 1.81 + \pi) + 12.03 =$
$2.19\sin(0.0165t - 1.23) + 2.14\sin(0.0165t + 4.95) + 12.03$

b Longest day is 12.03 + 4.32 = 16.35 hours (or 16 hours 21 minutes) and the shortest day is 12.03 − 4.32 = 7.71 hours (or 7 hours 43 minutes)

c The longest day occurs on day 172 = 21 June. The shortest day occurs on day 363 which is 29 December.

Chapter review

1 R^2

2 a 1.14 **b** 7

3 Perimeter = 20 cm, area = 25 cm²

4 $\frac{5\pi}{3}$

5 $z_1 = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right)$, $z_2 = 2\,\mathrm{cis}\left(-\frac{\pi}{6}\right)$

a $2\sqrt{2}\,\mathrm{cis}\left(\frac{5\pi}{12}\right)$

b $\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{12}\right)$

c $16\sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$

6 $C = B + v_\perp = 1 - 2i$,
$D = A + v_\perp = 3 + 2i$

7 a $|w| = \sqrt{8}$ and $\arg(w) = \frac{\pi}{4}$

b 64

8 $|z_2| = \sqrt{2}, |z_1| = 2\sqrt{2}$,

$z_1 = 2\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{6}\right)$ and

$z_2 = \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

9 $\frac{\sqrt{7}}{2}$

10 17.32 A

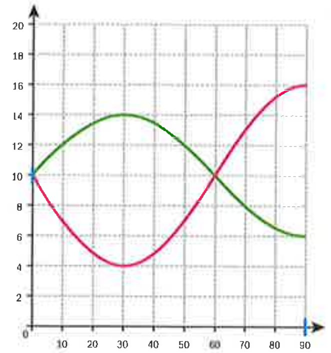
Exam-style questions

11 a $p = 3.5, q = 2, r = 3$

b $17.2^\circ < x < 62.6^\circ$ and $137.2^\circ < x \leq 180^\circ$

12 $32\sqrt{2} - 8$ cm

13 a



b $p = 10, q = 6, r = 3$

14 a 206 V

b 105.9°

15 a 12i

b $\frac{64}{27}i$

c $-8 + 8\sqrt{3}i$

16 0.872

Chapter 9

Skills check

1 $u_n = 18\left(-\frac{2}{3}\right)^{n-1}$

2 $S_{15} = 10.8$ (to 3 s.f.)

3 $S_{12} = 12$ (to 3 s.f.)

Exercise 9A

1 $a_{1,2} = -3$; $a_{2,3} = 6$; $b_{1,3} = 5$; $c_{2,2} = 0$; and $c_{3,1} = 1$

2 a 3 × 4

b $q_{2,4} = 100$; Manufacturer 4 produces 100 units of product 2.

c $T = \begin{bmatrix} 960 \\ 1200 \\ 990 \\ 335 \end{bmatrix}$

3 a $3W = \begin{bmatrix} 3 & -6 & 9 \\ 9 & 6 & -3 \end{bmatrix}$

a $R - W$ is undefined since dimensions of R are not equal to the dimensions of W.

b $4U + S = \begin{bmatrix} 35 & -14 & -7 \\ 28 & 29 & 1 \\ -9 & 10 & -2 \end{bmatrix}$

c $\frac{1}{3}T - \frac{1}{2}V = \begin{bmatrix} 0 & -3 \\ 4 & 8 \end{bmatrix}$

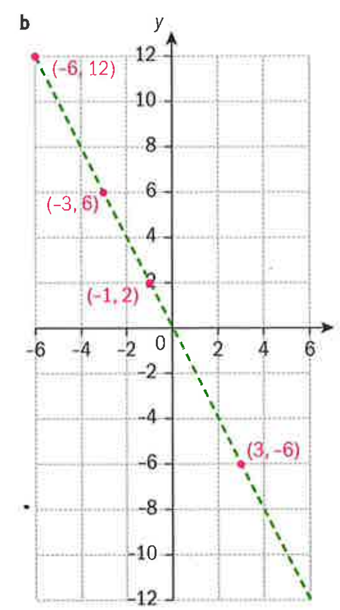
4 a $k = 0.075$

$\mathbf{T} = \begin{bmatrix} 2433.75 \\ 1404.38 \\ 1813.12 \\ 1443.75 \end{bmatrix}$

b $C = 0.075P + P = 1.075P$

5 a i $\begin{bmatrix} -6 \\ 12 \end{bmatrix}$ **ii** $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **iii** $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$

b



c The points described by kX lie along the line described by $y = -2x$

6 $\mu\begin{bmatrix} -9 & 12 \\ -6 & 3 \end{bmatrix} + \lambda\begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} -8 & 12 \\ -10 & 4 \end{bmatrix} \Leftrightarrow \begin{cases} -9\mu + \lambda = -8 \\ 12\mu - 2\lambda = 12 \\ -6\mu + 3\lambda = -10 \\ 3\mu - \lambda = 4 \end{cases}$

$\therefore \mu = \frac{2}{3}; \lambda = -2$

Exercise 9B

1 a $\begin{pmatrix} -4 & 1 \\ 13 & 16 \\ 5 & -3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ 18 \\ -7 \end{pmatrix}$

c $\begin{pmatrix} 18 & -23 \\ -13 & 17 \end{pmatrix}$

d Impossible **e** $\begin{pmatrix} -1 & -7 \\ 1 & 3 \end{pmatrix}$

2 $m = 2, n = 3$

3 $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4 a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c Matrix A is not a square matrix, so its left and right multiplicative identities are not equal, i.e. a multiplicative identity does not exist.

5 a $R = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.6 \end{pmatrix}$,

$N = AR = \begin{pmatrix} 45.0 \\ 36.8 \\ 94.8 \end{pmatrix}$

b $AC = \begin{pmatrix} 410 & 365 & 470 & 430 \\ 348 & 308 & 400 & 360 \\ 890 & 792 & 1020 & 928 \end{pmatrix}$,

entry $(AC)_{i,j}$ gives the total daily cost of removing all pollutants while producing the product *i* at the plant *j*.

Exercise 9C

1 a Not possible because the number of the columns of matrix A is not equal to the number of the rows of matrix B.

b $BA = \begin{pmatrix} -9 & 4 & 12 \\ 4 & -1 & -4 \end{pmatrix}$

c Not possible because A is not a square matrix.

d $B^2 = \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}$

2 a $\begin{pmatrix} -11 & -13 \\ 18 & -9 \\ 5 & 50 \end{pmatrix}$

b Not possible because number of columns of matrix T is not equal to number of rows of matrix R.

c $\begin{pmatrix} -14 & -5 \\ 5 & 11 \end{pmatrix}$

d $\begin{pmatrix} -20 & 3 & 22 \\ 2 & 9 & -4 \end{pmatrix}$

e Not possible because the number of the columns of matrix $S - U$ is not equal to the number of the rows of matrix W.

3 $UR + R = \begin{pmatrix} 1 & -33 \\ -22 & -25 \\ 8 & 3 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 3 & 7 \\ -5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & -34 \\ -19 & -18 \\ 3 & 7 \end{pmatrix}$

$(U + I)R = \begin{pmatrix} 5 & -3 & -2 \\ 5 & -3 & 2 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 3 & 7 \\ -5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & -34 \\ -19 & -18 \\ 3 & 7 \end{pmatrix}$

4 $(T+X)(T-X)=\begin{pmatrix}-48 & 32\\ 77 & -35\end{pmatrix}$,

$(T+X)^2=\begin{pmatrix}64 & 0\\ -105 & 49\end{pmatrix}$

$(T+X)(T-X)=T^2-TX+XT-X^2$ and $(T+X)^2=(T+X)(T+X)$ $=T^2+TX+XT+X^2$ because matrix multiplication is not commutative and $TX\neq XT$.

5 $(SU)^2=\begin{pmatrix}2 & 0 & -11\\ 21 & -19 & -3\\ -3 & 3 & 22\end{pmatrix}^2=\begin{pmatrix}37 & -33 & -264\\ -348 & 352 & -240\\ -9 & 9 & 508\end{pmatrix}$

$S^2U^2=\begin{pmatrix}-6 & -3 & 11\\ 37 & -22 & -5\\ -15 & 29 & -18\end{pmatrix}\times\begin{pmatrix}1 & -2 & -12\\ 0 & 3 & -20\\ 5 & -5 & 3\end{pmatrix}=\begin{pmatrix}49 & -52 & 165\\ 12 & -115 & -19\\ -105 & 207 & -454\end{pmatrix}$

$(SU)^2=(SU)\times(SU)=SUSU\neq SSUU=S^2U^2$ because matrix multiplication is not commutative.

6 $2(RT)=2\begin{pmatrix}1 & -3\\ -4 & 27\\ -9 & 2\end{pmatrix}=\begin{pmatrix}2 & -6\\ -8 & 54\\ -18 & 4\end{pmatrix}$, $(2R)T=\begin{pmatrix}0 & -2\\ 6 & 14\\ -10 & 8\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}=\begin{pmatrix}2 & -6\\ -8 & 54\\ -18 & 4\end{pmatrix}$,

$R(2T)=\begin{pmatrix}0 & -1\\ 3 & 7\\ -5 & 4\end{pmatrix}\begin{pmatrix}2 & 4\\ -2 & 6\end{pmatrix}=\begin{pmatrix}2 & -6\\ -8 & 54\\ -18 & 4\end{pmatrix}$.

Scalar multiplication is commutative with matrices.

7 **a** $AB=BA=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ **b** $(2AB)^{10}=2^{10}I_2=\begin{pmatrix}1024 & 0\\ 0 & 1024\end{pmatrix}$.

8 **a** The conjecture is not true. Consider, for example,

matrices $A=\begin{pmatrix}-3 & 5\\ 1 & -2\end{pmatrix}$, $T=\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$ and $n=2$. Then,

$(AT)^2=\begin{pmatrix}-8 & 9\\ 3 & -4\end{pmatrix}^2=\begin{pmatrix}91 & -108\\ -36 & 43\end{pmatrix}$,

but $A^2T^2=\begin{pmatrix}14 & -25\\ -5 & 9\end{pmatrix}\begin{pmatrix}-1 & 8\\ -4 & 7\end{pmatrix}=\begin{pmatrix}86 & -63\\ -21 & 23\end{pmatrix}$.

b The conjecture is true because of the scalar commutativity: $(kA)^n=k^nA^n$

Exercise 9D

1 **a** $\begin{pmatrix}-2.5 & 2.0\\ -1.5 & 1.0\end{pmatrix}$

b $\begin{pmatrix}0 & -1\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$

c $\begin{pmatrix}2.4 & -1.6\\ -1.6 & 2.4\end{pmatrix}$

d $\begin{pmatrix}-\frac{4}{11} & \frac{16}{11}\\ \frac{15}{11} & -\frac{5}{11}\end{pmatrix}$

2 **a** $\begin{pmatrix}-2 & 1 & -4\\ 1 & 0 & 2\\ 1.5 & -0.5 & 2.5\end{pmatrix}$

b $\begin{pmatrix}\frac{-20}{79} & -\frac{5}{158} & \frac{70}{79}\\ \frac{-52}{79} & \frac{33}{79} & \frac{24}{79}\\ -\frac{9}{79} & \frac{57}{158} & \frac{-8}{79}\end{pmatrix}$

c $\begin{pmatrix}\frac{24}{175} & \frac{54}{175} & \frac{6}{25}\\ -\frac{36}{25} & -\frac{6}{25} & \frac{12}{25}\\ -\frac{12}{35} & -\frac{27}{35} & \frac{2}{5}\end{pmatrix}$

3 **a** $\begin{pmatrix}-12\\ 30\\ 4\end{pmatrix}$ **b** $\begin{pmatrix}\frac{20}{11}\\ \frac{16}{11}\end{pmatrix}$

c $\begin{pmatrix}-\frac{91}{254}\\ -\frac{172}{127}\\ -\frac{456}{127}\\ -\frac{447}{508}\end{pmatrix}\approx\begin{pmatrix}-0.36\\ -1.35\\ -3.59\\ -0.88\end{pmatrix}$

4 $W=\begin{pmatrix}2 & 1 & 1 & 3\\ 1 & 3 & 2 & 4\\ 2 & 1 & 2 & 2\\ 3 & 4 & 1 & 2\end{pmatrix}$, $X=\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}$,

$Z=\begin{pmatrix}240\\ 380\\ 280\\ 400\end{pmatrix}$, $X=\begin{pmatrix}36\\ 48\\ 60\\ 20\end{pmatrix}$

5

Machine	Product A	Product B	Product C	Product D
I	120	240	240	80
II	380	126	190	95
III	140	280	140	140
IV	133	100	400	200
Total	773	746	970	515

6 **a** Use the formula for the inverse of 2 × 2 square matrices:

$(AB)^{-1}=\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}$,

$A^{-1}B^{-1}=\begin{pmatrix}-3 & 5\\ -1 & 2\end{pmatrix}$

b

i $B^{-1}A^{-1}=\begin{pmatrix}1 & 1\\ -1 & -2\end{pmatrix}=(AB)^{-1}$

ii e.g. $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, $B=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$,

$B^{-1}A^{-1}=\begin{pmatrix}-2 & 1\\ \frac{3}{4} & -\frac{1}{4}\end{pmatrix}=(AB)^{-1}$

iii $(AB)(AB)^{-1}=I_2$,
$A^{-1}AB(AB)^{-1}=A^{-1}$,
$B(AB)^{-1}=AA^{-1}$,
$B^{-1}B(AB)^{-1}=B^{-1}A^{-1}$,
$(AB)^{-1}=B^{-1}A^{-1}$

7 Mathematics is the music of reason.

Exercise 9E

1 **a** $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ **b** $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

c $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ **d** $\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

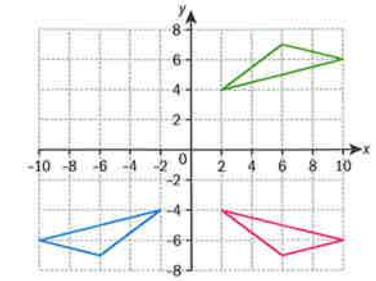
e $\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$ **f** $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

g $\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$

2 **a** $\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}$ **b** $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

c $\begin{pmatrix}-4 & 0\\ 0 & -4\end{pmatrix}$, which is equal to the enlargement matrix with scale factor −4

3 **a**



$T=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$,

$TP=\begin{pmatrix}2 & 6 & 10\\ -4 & -7 & -6\end{pmatrix}$

b $T'=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$,

$T'TP=\begin{pmatrix}-2 & -6 & -10\\ -4 & -7 & -6\end{pmatrix}$

c Since $TT=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, reflecting the triangle ABC in x axis twice will result in the same triangle ABC.

d $T'T=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

= anticlockwise rotation by 180° as seen in the graph above.

4 **a** $\begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{pmatrix}$

b $T^2=\begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{pmatrix}^2$

$=\begin{pmatrix}\cos^2(2\alpha)+\sin^2(2\alpha) & 0\\ 0 & \cos^2(2\alpha)+\sin^2(2\alpha)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

5 **a** $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

b $R^8=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, corresponds to rotation by 2π, i.e. the object stays the same

6 **a** 4 **b** **i** 72 **ii** 64

7 **a** If $T=\begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{pmatrix}$, $\det T=-\cos^2(2\alpha)-\sin^2(2\alpha)=-1$.

New area gets multiplied by $|\det T|=1$, i.e. does not change.

b If $T=\begin{pmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{pmatrix}$, $\det T=\cos^2(2\alpha)+\sin^2(2\alpha)=1$.

New area gets multiplied by $|\det T|=1$, i.e. does not change.

8 **a** $\begin{pmatrix}102\\ 76\end{pmatrix}$ **b** $\begin{pmatrix}-54\\ 30\end{pmatrix}$

9 **a** $\begin{pmatrix}-12\\12\end{pmatrix}$ **b** $(-12, 12)$

d $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ = reflection in y-axis

10 $\begin{pmatrix}1&-1\\3&-2\end{pmatrix}$

e $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ = reflection in x-axis

11 **a** $\boldsymbol{R}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \boldsymbol{T}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$

b $TR=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

c Triangle with vertices $(1, -1)$, $(3, -1)$, $(3, -3)$

12 **a** $\boldsymbol{E}=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix}$

b $E^n=\begin{pmatrix}0.5^n&0\\0&0.5^n\end{pmatrix}$

13 **a** $\begin{pmatrix}-5.33\\0.77\end{pmatrix}$

b $\begin{pmatrix}\cos 20°\cos 40° - \sin 20°\sin 40° & -\cos 40°\sin 20° - \sin 40°\cos 20°\\ \sin 40°\cos 20° + \cos 40°\sin 20° & -\sin 40°\sin 20° + \cos 40°\cos 20°\end{pmatrix}$

c Rotation by $\alpha+\theta$ is given by the following matrix:

$$\begin{pmatrix}\cos(\alpha+\theta) & -\sin(\alpha+\theta)\\ \sin(\alpha+\theta) & \cos(\alpha+\theta)\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$$=\begin{pmatrix}\cos\theta\cos\alpha-\sin\theta\sin\alpha & -\cos\alpha\sin\theta-\sin\alpha\cos\theta\\ \sin\alpha\cos\theta+\cos\alpha\sin\theta & -\sin\alpha\sin\theta+\cos\alpha\cos\theta\end{pmatrix}$$

Hence
$\cos(\alpha+\theta)=\cos\theta\cos\alpha-\sin\theta\sin\alpha$, $\sin(\alpha+\theta)=\sin\alpha\cos\theta+\cos\alpha\sin\theta$.

Exercise 9F

1 **a** $A'(18, 21)$, $B'(-2, 12)$, $C'(6, -18)$, $D'(-14, -29)$

b $(3, 1)$

2 **a** $\begin{pmatrix}12\\21\end{pmatrix}$ **b** $\begin{pmatrix}0\\3\end{pmatrix}$

c $\begin{pmatrix}-a\\a\end{pmatrix}$ **d** $\begin{pmatrix}-\frac{a}{3}\\\frac{a}{3}\end{pmatrix}$

3 **a** $T=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}3&0\\0&3\end{pmatrix}=\begin{pmatrix}0&-3\\3&0\end{pmatrix}$,

$B=\begin{pmatrix}2\\3\end{pmatrix}$

b $\begin{pmatrix}x''\\y''\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}$

$=\begin{pmatrix}0&3\\-3&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}-\begin{pmatrix}2\\3\end{pmatrix}$,

i.e. $A=\begin{pmatrix}0&3\\-3&0\end{pmatrix}, c=-\begin{pmatrix}2\\3\end{pmatrix}$

4 **a** **i** $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

ii $\begin{pmatrix}\cos 135° & \sin 135°\\ -\sin 135° & \cos 135°\end{pmatrix}$

iii $\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$ **iv** $\begin{pmatrix}2&0\\0&4\end{pmatrix}$

b $\begin{pmatrix}1.9&0.5\\-1.0&3.9\end{pmatrix}$

c $\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-2-\sqrt{3}\\2\sqrt{3}-1\end{pmatrix}$

5 **a** 7.75

b 7.75

c **i** $\frac{1}{4}$ **ii** $1.73\left(\sqrt{3}\right)$

iii $\frac{\sqrt{3}\left(1-0.25^6\right)}{1-0.25}=3.463$

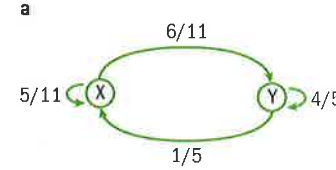
d $\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$

e $\begin{pmatrix}\frac{1}{4} & \frac{\sqrt{3}}{4}\\ -\frac{\sqrt{3}}{4} & \frac{1}{4}\end{pmatrix}$

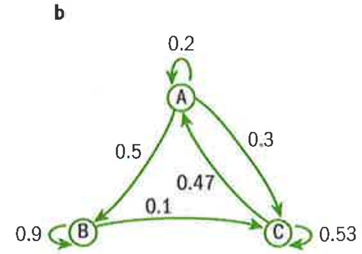
f $\begin{pmatrix}0.004&0.008&0\\0.007&0&0\end{pmatrix}$

Exercise 9G

1 **a**



b



2 **a** $T=\begin{pmatrix}0.75&0.2\\0.25&0.8\end{pmatrix}$

b $T^3=\begin{pmatrix}0.54&0.37\\0.46&0.63\end{pmatrix}$

3 **a** $T=\begin{pmatrix}0.65&0.26&0.05\\0.25&0.7&0.05\\0.10&0.04&0.90\end{pmatrix}$

b 0.18

4 **a** $\begin{pmatrix}0&\frac{1}{4}&0\\1&\frac{1}{2}&1\\0&\frac{1}{4}&0\end{pmatrix}$

b $\begin{pmatrix}0&\frac{1}{9}&0&0\\1&\frac{4}{9}&\frac{4}{9}&0\\0&\frac{4}{9}&\frac{4}{9}&1\\0&0&\frac{1}{9}&0\end{pmatrix}$

c Consider the general case nWm B with $m<n$ without loss of generality. Then, the transition matrix is of size $(m+1)\times(m+1)$. The element $T_{2,1}=1$ and all the other entries in the first column are 0. The element $T_{m,m+1}=1$ and all the other entries in the last column are 0. Then, to find the elements of the inner columns (elements $T_{j,j}$, $T_{j-1,j}$, $T_{j+1,j}$, all other elements in an inner column j are 0) consider having $j-1$ blue coins in the box with n coins and $j-1$ white coins in the box with m coins.

Then, the probability of moving to the state with $j-2$ blue coins in the box with n coins is $\frac{(j-1)^2}{nm}$ (this is element $T_{j-1,j}$). Similarly, the probability of moving to the state with j blue coins in the box with n coins is

$\frac{(n-j+1)(m-j+1)}{nm}$,
(this is element $T_{j+1,j}$).
Finally, staying in the same state has probability

$1-\frac{(j-1)^2}{nm}-\frac{(n-j+1)(m-j+1)}{nm}$

(this is element $T_{j,j}$). These formulae are valid for cases with $n=m$, too.

5 **a** $X=T^nB, T=\begin{pmatrix}0.959&0.032\\0.041&0.968\end{pmatrix}$,

$B=\begin{pmatrix}45520\\38745\end{pmatrix}, X=\begin{pmatrix}a\\b\end{pmatrix}$,

where a, b is the number of people voting for candidates A and B respectively.

c Candidate A would win by 1360 votes, 1.62% of all votes

Exercise 9H

1 **a** $\begin{pmatrix}0.45&0.45\\0.55&0.55\end{pmatrix}$

b $\begin{pmatrix}0.14&0.14&0.14\\0.51&0.51&0.51\\0.35&0.35&0.35\end{pmatrix}$

c $\begin{pmatrix}0.35&0.35&0.35&0.35\\0.10&0.10&0.10&0.10\\0.25&0.25&0.25&0.25\\0.29&0.29&0.29&0.29\end{pmatrix}$

2 **a** $\begin{pmatrix}0.32\\0.27\\0.41\end{pmatrix}$

b $\begin{pmatrix}0.32&0.32&0.32\\0.27&0.27&0.27\\0.41&0.41&0.41\end{pmatrix}$

3 **a** $\begin{pmatrix}A\\B\\C\end{pmatrix}=T^2\begin{pmatrix}45\\25\\30\end{pmatrix}=\begin{pmatrix}34.6\\36.5\\28.9\end{pmatrix}\%$

b $\begin{pmatrix}0.2085&0.2085&0.2085\\0.5787&0.5787&0.5787\\0.2128&0.2128&0.2128\end{pmatrix}$

c Convergence to the long-term matrix is slow even when the transition probabilities remain constant. Realistically, once the electricity providers notice the trend, they will respond to these changes quickly by changing the price or quality of their services in order to prevent customers from switching in which case the transition probabilities are unlikely to stay constant.

4 **a** Zeros signify that the cars that need minor or major repair will not be repaired. Fully broken-down cars will not be repaired to be functioning but still needing a repair, either.

b Check that all columns sum to 1

c $\begin{pmatrix}388\\70\\28\\39\end{pmatrix}$

d The company should repair functioning cars which need a minor or major repair before they are fully broken down to reduce the number of broken-down cars.

Exercise 9I

1 **a** $\lambda^2+3\lambda+2=0$

$\lambda=\{-2,-1\}$

b $\lambda^2-25=0$

$\lambda=\{\pm 5\}$

c $\lambda^2-3=0$

$\lambda=\{\pm\sqrt{3}\}$

2 No since $\lambda=-1$ is not a solution to the characteristic equation $\lambda^2-11\lambda+18=0$

3 **a** $\lambda_1 = \lambda_2 = 1;\ X_1 = X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b $(\lambda_1 = 7$ and $\lambda_2 = -2$

$X_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\lambda_{1,2} = 1$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

d $\lambda_1 = -3$, $\lambda_2 = 3$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

e $\lambda_1 = 1$, $\lambda_2 = -0.3$, $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4 **a** Find the eigenvalues from $\lambda^2 + (b - a - 1)\lambda + (a - b) = 0$, $\lambda_1 = 1$, $\lambda_2 = a - b$

b Then, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

gives $y = \dfrac{1-a}{b}x$ and

$T\begin{pmatrix} x \\ y \end{pmatrix} = (a-b)\begin{pmatrix} x \\ y \end{pmatrix}$ gives

$y = -x$, so possible

eigenvectors are $\begin{pmatrix} b \\ 1-a \end{pmatrix}$

and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Exercise 9J

1 $R = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

2 **a** $\lambda_1 = 2$ and $X_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$;

$\lambda_2 = 1$ and $X_2 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

A is diagonalizable since the eigenvalues are distinct real values.

b $A = \begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 3 & -7 \end{pmatrix}$

c $A^n = \begin{pmatrix} 15 \times 2^n - 14 & 35(1 - 2^n) \\ 6(2^n - 1) & 15 - 14 \times 2^n \end{pmatrix}$

d $A^4 = \begin{pmatrix} 226 & -525 \\ 90 & -209 \end{pmatrix}$

3 **a** $\lambda_1 = 1$ and $X_1 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$;

$\lambda_2 = -0.35$ and $X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

and therefore

$\begin{pmatrix} 5 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -0.35 \end{pmatrix}\begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{4}{9} & -\frac{5}{9} \end{pmatrix}$

b $\begin{pmatrix} 0.562 & 0.547 \\ 0.438 & 0.453 \end{pmatrix}$ (to 3sf)

c $T^n \to \begin{pmatrix} 0.56 & 0.56 \\ 0.44 & 0.44 \end{pmatrix}$ as $n \to \infty$

4 **a** $\begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{pmatrix}$

b $\begin{pmatrix} \mathbf{3620} \\ \mathbf{8080} \end{pmatrix}$ so more customers choose company S

c $\begin{pmatrix} 4 & 1 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{12} \end{pmatrix}\begin{pmatrix} \frac{1}{13} & \frac{1}{13} \\ \frac{9}{13} & -\frac{4}{13} \end{pmatrix}$

d $T^n = \begin{pmatrix} 4 & 1 \\ 9 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{12}\right)^n \end{pmatrix}\begin{pmatrix} \frac{1}{13} & \frac{1}{13} \\ \frac{9}{13} & -\frac{4}{13} \end{pmatrix}$

$= \dfrac{1}{13}\begin{pmatrix} 4 + 9p^n & 4 - 4p^n \\ 9 - 9p^n & 4p^n + 9 \end{pmatrix}$,

$p = -\dfrac{1}{12}$

e $\begin{pmatrix} 2900p^n + 3600 \\ 8100 - 2900p^n \end{pmatrix}$

f $\begin{pmatrix} \frac{2900}{12^2} + 3600 \\ 8100 - \frac{2900}{12^2} \end{pmatrix} = \begin{pmatrix} 3620 \\ 8080 \end{pmatrix}$

g 3600

Chapter review

1 **a** $\begin{pmatrix} -11 & 9 \\ -4 & -7 \end{pmatrix}$ **b** DNE

c $\begin{pmatrix} -4 & -2 & 7 \\ 5 & -20 & 10 \end{pmatrix}$

d $\begin{pmatrix} 4 & 7 & -11 \\ -1 & 4 & -9 \end{pmatrix}$

e $\begin{pmatrix} -6 & -1 & 3 \\ 0 & -3 & -2 \\ -7 & -11 & 13 \end{pmatrix}$

f $\begin{pmatrix} 8 & -20 \\ 10 & 17 \end{pmatrix}$ **g** $\begin{pmatrix} 0 \\ -4 \\ 8 \end{pmatrix}$

h $\begin{pmatrix} 2 & -1.5 \\ 1 & -0.5 \end{pmatrix}$ **i** $\begin{pmatrix} 47 \\ 74 \end{pmatrix}$

j $\begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix}$

2 $a = 4;\ b = -\dfrac{1}{2}$

3 **a** $\begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -2 & -\frac{2}{3} \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{13}{5} & \frac{34}{5} \\ -\frac{4}{5} & -\frac{8}{5} \end{pmatrix}$

c $\begin{pmatrix} 10 & 7 \\ 8 & 5.5 \end{pmatrix}$

4 **a** $\begin{pmatrix} \frac{3}{13} \\ -\frac{20}{13} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{21}{19} \\ \frac{3}{19} \\ -\frac{21}{19} \end{pmatrix}$

c $\begin{pmatrix} -3 \\ -\frac{5}{2} \end{pmatrix}$

5 **a** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}X + \begin{pmatrix} -\frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}$

b P′(−3.5, −1.5), Q′(−2, −6), R′(2.5, −4.5) T′(1, 0)

c $k = \dfrac{1}{4}$

6 **a** $\begin{pmatrix} 13 \\ 25 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

c $\begin{pmatrix} a & -1 \\ 7a & +2 \end{pmatrix}$

7 **a** **i** $\begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}$

ii $\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

b **i** $\begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix}$

ii $C_2 = \begin{pmatrix} 0 & -\frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$,

$C_3 = \begin{pmatrix} -\frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \\ \frac{1}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} \end{pmatrix}$,

$C_4 = \begin{pmatrix} -0.625 & 0 \\ 0 & -0.625 \end{pmatrix}$

iii 26.7

Exam-style questions

8 **a** $\begin{pmatrix} 8 & 0 \\ 2 & -4 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix}$

c $\begin{pmatrix} 10 & 4 \\ 5 & 6 \end{pmatrix}$ **d** $\begin{pmatrix} 43 & -4 \\ 6 & 19 \end{pmatrix}$

9 Find the determinant:
$x(3 - x) - (-1) \times 4 = 0$,
$-x^2 + 3x + 4 = 0$, $x = -1$ $x = 4$.

10 **a** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **b** 6

11 **a** $A^2 - 10A + 21I$

$= \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix} - \begin{pmatrix} 40 & 10 \\ 30 & 60 \end{pmatrix}$

$+ \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix}$

$= 0$

b $A^3 = 79A - 210I$

c $A^4 = 1000A - 1659I$

12 **a**

$A^{-1} = \begin{pmatrix} 0.125 & -0.875 & 0.625 \\ 0.25 & 1.25 & -0.75 \\ -0.125 & -0.125 & 0.375 \end{pmatrix}$

b $\begin{pmatrix} 0.75 & -0.875 & 0.75 \\ -0.5 & 1.25 & -0.5 \\ 0.25 & -0.125 & 0.25 \end{pmatrix}$

13 **a** $A = \begin{pmatrix} 5 & 0 & 3 \\ 1 & -2 & 5 \\ 0 & 3 & 7 \end{pmatrix}$,

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 23 \\ 23 \\ 122 \end{pmatrix}$

b $\begin{pmatrix} \frac{29}{136} & -\frac{9}{136} & -\frac{3}{68} \\ \frac{7}{136} & -\frac{25}{136} & \frac{11}{68} \\ -\frac{3}{136} & \frac{15}{136} & \frac{5}{68} \end{pmatrix}$

c $X = A^{-1}B = \begin{pmatrix} -2 \\ 15 \\ 11 \end{pmatrix}$

14 180

15 **a** $\lambda_1 = 2$, $\lambda_2 = 6$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b $D = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}, P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

c $A^n = \dfrac{1}{2}\begin{pmatrix} 2^n + 6^n & 2^n - 6^n \\ 2^n - 6^n & 2^n + 6^n \end{pmatrix}$

Chapter 10

Skills check

1 $y = -\dfrac{1}{2}x + \dfrac{11}{2}5$

2 **a** x^{-5} **b** x^{-1} **c** $3x^{-\frac{1}{2}}$

3 −20 (1, 12)

Exercise 10A

1 $f'(x) = 4$

2 $f'(x) = 0$

3 $f'(x) = 3$

4 $f'(x) = 0$

5 $f'(x) = m$

6 $f'(x) = 2x - 3$

7 $f'(x) = 4x$

8 $f'(x) = 4x^3$

9 $f'(x) = -\dfrac{1}{x^2}$

10 $f'(x) = anx^{n-1}$

Exercise 10B

1 **a** 0 **b** 4 **c** 12
d 17 **e** 103 **f** 150

2 **a** $\dfrac{1}{3}$ **b** 1 **c** 10
d −14 **e** −1

3 **a** $\dfrac{ds}{dt} = 4 + \dfrac{4}{t^{\frac{3}{2}}}$, $4\dfrac{1}{16}$

b $\dfrac{dv}{dt} = 3t^{-\frac{1}{4}} + 4t^{-\frac{5}{4}}$, $\dfrac{52}{32}$

4 a i $\frac{dy}{dx} = 12x + 5$

ii $12x > -5, x > -\frac{5}{12}$

b i $f'(x) = 8x^3 + 16x - 10$

ii $x > 0.544$

c i $g'(x) = 3x^2 + 6x - 9$

ii $x < -3$ and $x > 1$

5 a $2\pi r$ b 4π

6 a $-0.102c + 5.6$

b When $c = 20$, $\frac{dP}{dc} = 3.56$, when $c = 60$, $\frac{dP}{dc} = -0.52$

c When derivative is positive, increasing the number of cupcakes sold increases profit, but when derivative is negative, increasing the number of cupcakes sold decreases the profit.

7 a $10 - 10t$ b Speed

c $f'(0.5) = 5, f'(1.5) = 1 - 5$, positive value represents the bungee jumper going downwards while negative value represents the bungee jumper going upwards.

d $f(2) = 0$, but $f'(2) = -10$, model suggests correctly that bungee jumper ends at starting point, but it also predicts a large upwards speed at the end of the jump.

8 $A = -1, B = \frac{1}{3}$

9 7.5

Exercise 10C

1 $y = 12x - 22$

2 $y = 1$

3 $y = -\frac{1}{2}x + \frac{9}{2}$

4 a Tangent: $z = 26x - 43$, normal: $w = -\frac{1}{26}x + 9\frac{1}{13}$

b Tangent: $z = x - 9$, normal: $w = -x + 27$

5 $w = -\frac{1}{4}(x-2) + 4$ and $z = \frac{1}{4}(x+2) + 4$, meet at $x = 0$

6 $a = 1, b = 9$

7 $k = 1, b = 5$

8 $a = 2, b = -5$

Exercise 10D

1 a $\frac{3}{4}$ b 2.0986

2 a −0.786 max, 2.12 min

b 0 max, 2 min

c No stationary points

3 a $x = \frac{1}{2}$, minimum

b $x = -2$ is a maximum and $x = 2$ is a minimum

c $x = -\frac{1}{2}$ is a maximum and $x = \frac{1}{2}$ is a minimum

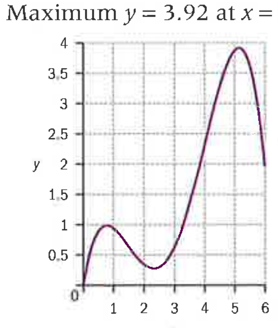
4 a i $P(5) = 4.67$

ii €52 000

iii €46 700

b Strategy **ii**

5 Maximum $y = 3.92$ at $x = 5.15$



6 0.693

Exercise 10E

1 a $l = 50 - x$

b **Area** $A = x \times l = x(50 - x)$ **m²**

c $50 - 2x$

d Maximum when $\frac{dA}{dx} = 0$, $x = 25, l = 25, A = 25 \times 25 = 625\,m^2$

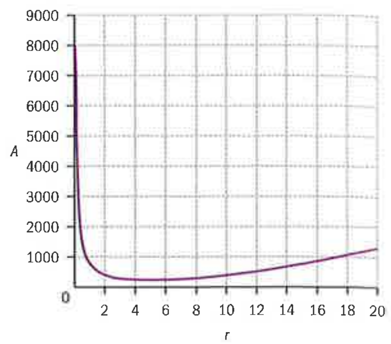
2 a The base of the cylinder is a disc of area $A_b = \pi r^2$, and the volume of a cylinder is then $V = Ah = \pi r^2 h$.

b $h = \frac{400}{\pi r^2}$

c $A = \pi r^2 + 2\pi r\, h$

d $A = \pi r^2 + 2\pi r \times \frac{400}{\pi r^2}$
$= \pi r^2 + \frac{800}{r}$

e



f $A = 238.53$ at $r = 5.03$

3 $h = \frac{2500}{\pi r} - r, r \approx 16.3$ cm, $V = \frac{250\,000}{3\sqrt{3\pi}} \approx 27145$

4 a $V = 6\pi r^2 - \frac{1}{3}\pi r^3$, $\frac{dV}{dr} = 12\pi r - \pi r^2 = \pi r(12 - r)$

b $r = 12$, $V(12) = 288\pi$ but e.g. $V(18) = 0$, so maximum point

5 a Height of box $h = x$, width $w = 20 - 2x$, length $l = 24 - 2x$. Hence, volume is $V = hwl = x(20 - 2x)(24 - 2x) = 4x^3 - 88x^2 + 480x$

b $12x^2 - 176x + 480$

c 3.62, 11.05

d At end points, $V(0) = 0$, $V(10) = 0$

e $V(3.62) = 774.165\,cm^3$

6 a Profit = demand × (price − cost), so $P = \frac{100}{x^2}(x - 0.75) = \frac{100}{x} - \frac{75}{x^2}$

b $\frac{dP}{dx} = -\frac{100}{x^2} + \frac{150}{x^3}$

c $-100x + 150 = 0 \Rightarrow x = 1.5$

d $x = 1 \Rightarrow \frac{dP}{dx} = 50$, $x = 2 \Rightarrow \frac{dP}{dx} = -6.25$, gradient goes from positive to negative, hence a maximum

7 a From the Pythagoras theorem, $x^2 + y^2 = 6^2$. The area of the rectangle is $A = 2x \times 2y = 4x\sqrt{36 - x^2}$ as required.

b $\frac{dA}{dx} = 4\sqrt{36 - x^2} - \frac{4x^2}{\sqrt{36 - x^2}} = 0, 36 - 2x^2 = 0$, $x = 3\sqrt{2}, A = 72$

8 a $A = 2\pi \times 8\sin\theta \times 2 \times 8 \times \cos\theta\,cm^2 = 256\pi \sin\theta\cos\theta\,cm^2$

b $A = 128\pi \sin 2\theta, \frac{dA}{d\theta} = 256\pi\cos 2\theta = 0$ when $\theta = \frac{\pi}{4}$. $A\left(\frac{\pi}{4}\right) = 128\pi$

Exercise 10F

1 a i $= u^3, u = x^2 + 4y$

ii $\frac{dy}{du} = 3u^2 = 3(x^2 + 4)$

b i $y = u^2, u = 5x - 7$

ii $\frac{dy}{du} = 2u = 10x - 14$

c i $y = 2u^4, u = x^3 - 3x^2$

ii $\frac{dy}{du} = 8u^3 = 8(x^3 - 3x^2)^3$

d i $y = u^{\frac{1}{2}}, u = 4x - 5$

ii $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}(4x - 5)^{-\frac{1}{2}}$

e i $y = u^{-2}, u = x^2 + 1$

ii $\frac{dy}{du} = -2u^{-3} = -2(x^2 + 1)^{-3}$

f i $y = 2u^{-\frac{1}{2}}, u = 5x - 2$

ii $\frac{dy}{du} = -u^{-\frac{3}{2}} = -(5x - 2)^{-\frac{3}{2}}$

2 a $f'(x) = 6x^5 + 12x^3 + 6x$

b $g'(x) = \frac{10}{(5x + 2)^{\frac{2}{3}}}$

c $h'(x) = \frac{3(\sqrt{x} - x)^2}{2\sqrt{x}}$

d $s'(t) = 12(t^3 - 2t)$

e $v'(t) = -\frac{20}{(5t - 1)^2}$

f $a'(t) = \frac{3\left(2 - \frac{1}{t}\right)^2}{t^2}$

3 a 1 b −4

Exercise 10G

1 a $2x(3x + 1)$

b $3x^2 + 4x + 3$ (this has imaginary roots)

c $(x + 1)(5x^3 + 3x^2 - 2)$

d $(t - 4)^4(-14t^2 + 16t - 15)$

e $\frac{4(x - 1)(x^2 + x + 1)}{x^2}$

f $2(2t + 1)^3(t + 1)(t^2 - t + 1)(t^3 + 3t^2 + 4)$

2 a $4x^2 + 2 + \frac{4}{x}$

b $\frac{x^2 - 2x - 1}{x^2 + 1}$

c $\frac{x + 2}{(x + 1)^{\frac{3}{2}}}$

d $\frac{4}{(2t + 1)^2}$

e $\frac{2x^3 + 3x^2 + 1}{(x + 1)^2}$

f $-\frac{4}{3t^{\frac{2}{3}}\left(t^{\frac{1}{3}} - 4\right)^2}$

3 a $2(x^2 + 1)^2 x(4x^2 + 1)$

b $-\frac{8}{(2x + 3)^2}$ c $\frac{x + 1}{(2x + 1)^{\frac{3}{2}}}$

d $\frac{12(t - 1)}{\sqrt{2t - 3}}$ e $\frac{3 + 2x}{(3 - 2x)^3}$

f $\frac{4(t + 1)(4t^2 - 9t - 3)}{2t - 3}$

4 a $(2x - 1)(6x - 1)$

b i $y = 5x - 4$

ii $y = -\frac{1}{5}x + \frac{6}{5}$

c 2.6

5 a $y = -1.5x + 5$

b $\left(-\frac{7}{3}, 8.5\right)$

6 $\frac{dy}{dx} = -u\frac{dv}{dx}\frac{1}{v^2} + \frac{du}{dx}\frac{1}{v}$
$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Exercise 10H

1 a $4\cos x$ b $-2\sin x - 2x$

c $\frac{5}{\cos^2 x}$ d $-8\sin 4t$

e $5\cos 5x + 12x^2$

f $3\cos 3t + 6\sin 2t - 2t$

2 a $x\cos x + \sin x$ b $\cos 2x$

c $\frac{2x^2}{\cos^2 x} + 4x\tan x$

d $\frac{t\cos t - \sin t}{t^2}$

e $-\frac{1}{\sin^2 x}$

f $4\cos t - \frac{2}{\cos t} + 2\sin t\tan t$

3 a $2x\cos x^2$ b $6\sin 3x$

c $\frac{3}{\cos^2(3x - 1)}$ d $2\sin t\cos t$

e $-9\cos^2 x\sin x$ f $8\cos 8t$

g $3\sin 6x$

h $-\sin t + 3\cos^2 t\sin t$

4 a $A = 2xy = 10.2x\cos\frac{\pi}{10}x$

b $-1.02\pi x\sin\frac{\pi}{10}x + 10.2\cos\frac{\pi}{10}x$

c $A(2.74) = 18.22$

Exercise 10I

1 a $\frac{2x}{x^2+1}$

b $xe^x + e^x = e^x(1+x)$

c $4xe^{2x^2}$ d $t(1+2\ln t)$

e $4e^2x(x+1)$ f $\frac{1-\ln t}{t^2}$

g $\frac{6t^2}{t^2-2}$ h $\frac{2e^{4x}}{x^2}(4x-1)$

i $e^{3t^2}(1+6t^2)$

2 a $\frac{5}{x}$ b $\frac{8}{2x+3}$

c $\frac{1}{x}+\frac{2}{x-3}$ d $\frac{2}{2x+1}-\frac{1}{x}$

e $2x$ f $\frac{2}{x}-\frac{6}{2x+1}$

3 a $(\ln ax)' = \frac{1}{x}$

b $(\ln ax)' = \frac{1}{ax}\times a = \frac{1}{x}$

4 a i $f'(x) = 20\cos\ln x\times\frac{1}{x}$

ii $x = 0.175$

b i $f'(x) = \frac{1}{\cos x+2}\times(-\sin x)$

ii $x = \pi$

c i $f'(x) = 2\cos x\, e^{2\sin x}$

ii $2\cos x = 1,\ x = \frac{\pi}{3}$

5 a $x = -2.5,\ y = 2e^{-5}$

b $y = 2e^{-5}$

Exercise 10J

1 a $-x\sin x + 2\cos x$

b $\frac{12}{(x-3)^4}$ c $\frac{2}{t}$

2 a $x = 0$ is a maximum,

$x = \left(\frac{6}{5}\right)^{\frac{1}{3}}$ is a minimum

b Inflection point occurs when $x = 0.3^{\frac{1}{3}}$

c $x\in\left(0.3^{\frac{1}{3}},\infty\right)$

3 a $e^x\left(\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x^3}\right)$

b At $x = 1$, $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = e > 0$, so it is a minimum point

c Points of inflection would occur if $\frac{d^2y}{dx^2} = 0$, i.e. if

$\left(\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x^3}\right) = 0,$

$x^2-2x+2 = 0.$

The discriminant is $4-4\times2 = -4 < 0$ so there are no real roots of this equation and hence no points of inflection exist.

4 a $f'(x) = \frac{x-3-x}{(x-3)^2}$

$= -\frac{3}{(x-3)^2} < 0$

for all values of x

b $f''(x) = \frac{6}{(x-3)^3} < 0$

for $x < 3$, $f''(x) > 0$ for $x > 3$ ($f''(x)$ is infinite at $x = 3$), so $f''(x)$ never vanishes.

5 a i $x = -\frac{1}{4}$, minimum

ii $\left(-\frac{1}{4}-2\right)(-3-9)$

iii $x = 2, \frac{1}{2}$

iv Concave down for $x\in\left(\frac{1}{2}, 2\right)$ and concave up otherwise

b i $x = \frac{3\pi}{4}$ is a minimum,

$x = \frac{7\pi}{4}$ is a maximum

ii $e^{-x}2\sin x$

iii $x = 0, \pi, 2\pi$

iv Concave up for $x\in(0, \pi)$ and concave down for $x\in(\pi, 2\pi)$

c i $x = \frac{3}{2}$ is a maximum

ii $e^{-2x}2x(2x^2-6x+3)$

iii $x = 0, \frac{1}{2}(3\pm\sqrt{3})$

iv Concave up for

$x\in\left(0,\frac{1}{2}(3-\sqrt{3})\right)\cup\left(\frac{1}{2}(3+\sqrt{3}),\infty\right)$

and concave down for

$x\in(-\infty,0)\cup\left(\frac{1}{2}(3-\sqrt{3}), \frac{1}{2}(3+\sqrt{3})\right)$

6 a i $p'(t) = \frac{80e^{-2t}}{\left(1+4e^{-2t}\right)^2}$

ii Both numerator and denominator are positive for $t\ge 0$ hence the population is always increasing.

b i $p''(t) = \frac{160e^{-2t}\left(4e^{-2t}-1\right)}{\left(1+4e^{-2t}\right)^3}$

ii ln 2 or 0.693

iii This is the point at which the rate of population growth has reached is maximum.

c i $p(\ln 2) = 5$

ii This is the half the carrying capacity of the population.

Exercise 10K

1 a 2 b 4

2 a $\frac{1}{3}$

b Velocity $= -11$, acceleration $= -14$

c Increasing

d $t = 2$

e i $t = \frac{2}{3}$

ii Point of inflexion.

3 $a\approx -33.54$

4 a $v(t) = y'(t) = -0.4t+2$

b i $v(0) = 2$ ii 6m

iii -2.19

5 a i 1.31 ii $t\approx 3.37$

b $s'(t) = 0.4(1+e^{-0.5t})$

c $s'(0) = 0.8$

d As $t\to\infty$, $s'(t)\to 0.4$

e The marble is first moving within 1% of its terminal velocity when $e^{-0.5t} = 0.01$, $t = -2\log 0.01 = \log 100^2$ ≈ 9.21

6 a $t\approx 65.04$

b $v(t) = 3+\frac{2}{2t+1}$

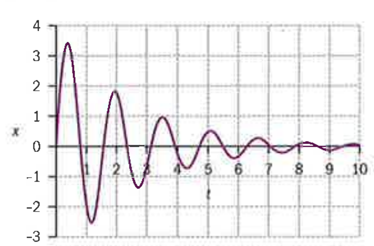
c i $v(0) = 3+2 = 5\,\text{m s}^{-1}$

ii $v(65.04) = 3.02\,\text{m s}^{-1}$

d $a(t) = v(t)$

$= -\frac{4}{(2t+1)^2} < 0\ \forall\, t$

7 a



b $t = 2.84$

c $\dot{x}(t) = 4e^{-0.4t}(-0.4\sin 4t + 4\cos 4t)$

d i $t = \frac{n\pi}{4}$

ii -11.69

8 1.11

Exercise 10L

1 a $\pi r^2\frac{dh}{dt}$ b $2\pi r\frac{dr}{dt}h$

2 $\frac{8}{3}\,\text{m}^2\text{s}^{-1}$

3 $0.75\,\text{rad s}^{-1}$

4 $8\,\text{cm min}^{-1}$

5 $1.15\,\text{cm}^2\text{s}^{-1}$

6 a $f(x) = \frac{\cos x}{\sin x}$,

$f'(x) = \frac{-\sin x\times\sin x-\cos x\times\cos x}{\sin^2 x}$

$= -\frac{1}{\sin^2 x}$

b $87.01\,\text{m s}^{-1}$

Chapter review

1 a 33.5

b (65 536, −1 048 580)

c −60

2 a $f'(x) = -2\sin x + 2\cos x$

b $f'(x) = \frac{1}{\cos^2 x\tan x}$

$= \frac{2}{\sin 2x}$

c $\frac{dy}{dx} = \frac{-\sin^2 x-\cos^2 x}{\sin^2 x}$

$= -\frac{1}{\sin^2 x}$

d $\frac{dy}{dx} = 2xe^{2x}+e^{2x}$

3 a $f'(x) = -\frac{24}{x^3}+\frac{3}{2\sqrt{x}}$,

$f''(x) = \frac{72}{x^4}-\frac{3}{4}x^{-\frac{3}{2}}$

b $f'(x) = 6x-10$, $f''(x) = 6$

c $\frac{dy}{dx} = -\frac{1}{x^2}+\frac{24}{x^3}$,

$\frac{d^2y}{dx^2} = \frac{2}{x^3}-\frac{72}{x^4}$

d $\frac{dy}{dx} = \frac{x^2-24x}{(x-12)^2}$,

$\frac{d^2y}{dx^2} = \frac{288}{(x-12)^3}$

4 $\frac{dy}{dx} = \frac{2e^{2x}}{\left(e^{2x}+1\right)^2}$

Since e^{2x} is always greater than zero so is the gradient.

5 $\left(-\frac{14}{3}, \frac{22}{3}\right)$

a $(1, -7)$ b $\frac{d^2y}{dx^2} = \frac{7}{2x^{\frac{3}{2}}}$

c When $x = 1$ $\frac{d^2y}{dx^2} = \frac{7}{2} > 0$ hence a minimum

6 a $(x(64-x))$

b 32 c $1024\,\text{m}^2$

7 a Right as v is positive

b i To the right when $0 < t < 7$, $t > 10$.

ii To the left when $7 < t < 10$. It depends on the sign of v.

c Positive as the graph of v has a positive gradient.

[0, 4]: v is initially 2 but accelerates at 0.5 m s^{-2}

[4, 6]: constant velocity of 4 m s^{-1}

[6, 8.5]: slowing down at a decreasing rate until 7 seconds whenstarts moving to the left and reaches a maximum speed of 2.2 m s^{-1}at $t = 8.5$s

d [8.5, 10]: slowing down until at 10 seconds it has a velocity of 0

e $t = 7$ because it begins to travel to the left at 7 seconds and stays travelling to the left until 10 seconds.

8 $x = 9.5$. The point of inflexion is (14.5, 35)

9 $\frac{6}{\sqrt{17}}$

Exam-style questions

10 $y = \frac{x}{2}+1$

11 a $\left(1, \frac{25}{6}\right)$ and $\left(\frac{3}{2}, -\frac{145\,833}{100\,000}\right)$

b he gradient is negative between $x = \frac{1}{4}\left(7\pm\sqrt{33}\right)$

12 a The height of the box is $h = x$, the width is $w = 30 - 2x$ and the length is $l = 40 - 2x$. Hence, the volume is $V = hwl = x(30 - 2x)(40 - 2x) = 4x^3 - 140x^2 + 1200x$.

b $\frac{dV}{dx} = 12x^2 - 280x + 1200$

c $\frac{dV}{dx} = 0$ for $12x^2 - 280x + 1200 = 0$, $x^2 - \frac{70}{3}x + 100 = 0$

d Range $x \in (0, 15)$, $V(5.66) = 3032.3$.

13 a $f'(x) = \frac{-2x^4 - 2x}{(2x^3 - 1)^2}$

b $y = -4x + 5$

c $x = 0, x = -1$

d $f(x)$ is increasing for positive gradient, this happens for $x \in (-1, 0)$

14 $\frac{3}{20\pi}$

15 a 1.53

b $\frac{dx}{dt} = -\frac{5}{(t-5)^2} + \frac{2}{1+t} = 0$, stationary at $t = \frac{1}{4}\left(25 - \sqrt{256}\right)$

16 a First, find that

$$\frac{dx}{dt} = e^{-2t}\left(-2\cos\sqrt{12}t - \sqrt{12}\sin\sqrt{12}t\right),$$

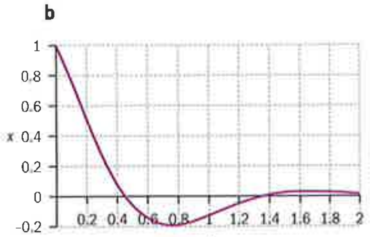
$$\frac{d^2x}{dt^2} = e^{-2t}\left(4\cos\sqrt{12}t + 2\sqrt{12}\sin\sqrt{12}t + 2\sqrt{12}\sin\sqrt{12}t - 12\cos\sqrt{12}t\right)$$

$$= e^{-2t}\left(-8\cos\sqrt{12}t + 4\sqrt{12}\sin\sqrt{12}t\right).$$

Substitute into the given equation:

$$e^{-2t}\left(-8\cos\sqrt{12}t + 4\sqrt{12}\sin\sqrt{12}t - 4\sqrt{12}\sin\sqrt{12}t - 8\cos\sqrt{12}t + 16\cos\sqrt{12}t\right) = 0.$$

b



c $t = 0.75$

d 0.64

Chapter 11

Skills check

1 $\frac{1}{2} \times 1.8(3.2 + 4.8) = 7.2$ cm²

2 a $\frac{dy}{dx} = 9x^2 - x^{-\frac{1}{2}} - 8x^{-3}$ or $\frac{dy}{dx} = 9x^2 - \frac{1}{\sqrt{x}} - \frac{8}{x^3}$

b $f'(x) = -5\sin 5x + 2\sin x\cos x$

c $\frac{ds}{dt} = \frac{1}{t} - 4te^{t^2}$

Exercise 11A

1 a i $y = \sqrt{9 - x^2}, 0 \le x \le 3$

ii $\int_0^3 \sqrt{9 - x^2}\,dx$

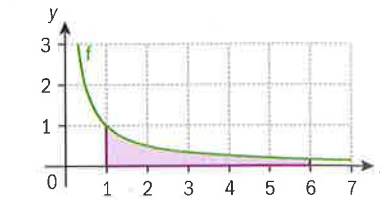
b

x	0	0.5	1.0	1.5	2.0	2.5	3.0
y	3	2.958	2.828	2.598	2.2361	1.658	0

c i 6.139 **ii** 7.639

d 6.889 **e** 3%

2 a



b $\int_1^6 \frac{1}{x}\,dx$

c i 1.45 **ii** 2.283

d $\frac{29}{20} < \int_1^6 \frac{1}{x}\,dx < \frac{137}{60}$

Exercise 11B

1 a i $\int_2^4 x^2\,dx$ **ii** $\frac{56}{3}$

b i $\int_{-1}^1 2^x\,dx$ **ii** 22.4

c i $\int_{-1}^1 \frac{1}{1+x^2}\,dx$ **ii** 1.57

d i $\int_{0.5}^3 \frac{1}{x}\,dx$ **ii** 1.79

e i $\int_0^1 -(x-3)(x+2)\,dx$

ii $\frac{37}{6}$

f i $\int_{-2}^0 -(x-3)(x+2)\,dx$ or $\int_0^3 -(x-3)(x+2)\,dx$

ii $\frac{22}{3}$ or 13.5

g i $\int_{-2}^3 -(x-3)(x+2)\,dx$

ii $\frac{125}{6}$

h i $\int_{-2}^{4.5} -x^2 + 2x + 15\,dx$

ii 80.71

i i $\int_{-\sqrt{17}}^{\sqrt{17}} -x^2 + 2x + 15\,dx$

ii 58.33

j i $\int_{-1}^{\ln(3)} 3 - e^x\,dx$

ii 3.66

k i $\int_{-2-5^{\frac{1}{3}}}^0 (x+2)^3 + 5\,dx$

ii 20.41

2 a −2, 2 **b** (0, 4)

c 20 **d** 5.33

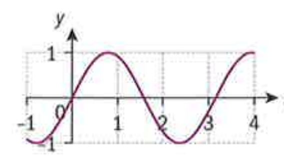
e 20.41

3 3

4 −0.8169

Exercise 11C

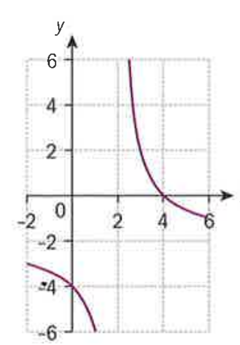
1 a



b i 0.230 **ii** 0.807

iii 1.64

2 a



b The interval [0, 3] contains a discontinuity

c i 0.773 **ii** 1.23

d 2.00

Exercise 11D

1 55

2 18

3 9.85

4 11.553

5 a 5.19 **b** 15

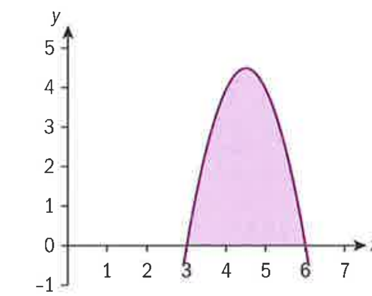
c 12.21 **d** 35

6 a $\frac{1}{2} \times 2\left(4 + 5 + 2(6 + 5.5 + 6.2)\right)$
$= 44.4$

b $\frac{1}{2} \times 2\left(4 + 5 + 2(3 + 2.2 + 3.4)\right)$
$= 26.2$

c $44.4 - 26.2 = 18.2$ km²

7 a

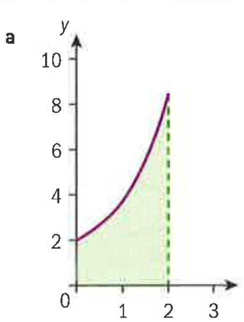


b i $\int_3^6 -2(x-3)(x-6)\,dx$

ii 9

c 8.75 **d** 2.78%

8 a



b i $\int_0^2 (1 + e^x)\,dx$ **ii** 8.389

c 8.474 **d** 1.01%

9 26.5

10 a 51.8 **b** 52.3

c 0.96%

Exercise 11E

1 a $10x + c$ **b** $0.2x^3 + c$

c $\frac{x^6}{6} + c$ **d** $7x - x^2 + c$

e $x + x^2 + c$

f $5x + \frac{x^2}{2} - \frac{1}{9}x^3 + c$

g $-\frac{x^2}{2} - \frac{x^3}{4} + 0.5x + c$

h $x - \frac{x^2}{2} + \frac{x^4}{8} + c$

i $\frac{x^3}{3} - \frac{x^2}{4} + 4x + c$

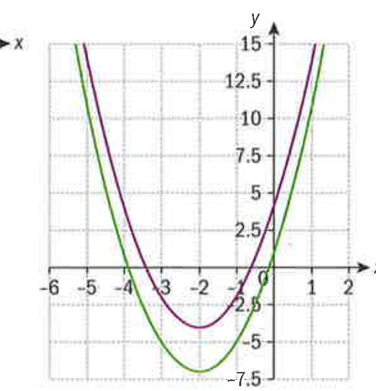
2 a $2x - \frac{1}{3}$

b $\frac{x^3}{3} - \frac{x^2}{6} + 4x + c$

3 a $\frac{1}{2}t^2 - t^3 + c$

b $t^4 - 1.5t^2 + t + c$

4 a



b i $f(-2) = -7$

ii $f(-2) = -4$

5 a $-2\cos x + c$ **b** $4\sin x + c$

c $-\frac{1}{3}\cos x + c$ **d** $3\tan|x| + c$

6 a $2e^x + c$ **b** $3e^x + \ln x + c$

c $4\ln|x| - 3e^x + c$

7 a $\frac{2}{3}e^{3x} + c$ **b** $-\frac{2}{5}\cos 5t + c$

c $4x + \frac{2}{3}e^{-3x} + c$

d $\frac{1}{2}\sin 2t + \frac{1}{3}\cos 3t + c$

8 **a** $\sqrt{5}\times\frac{2}{3}t^{\frac{3}{2}}+c$ **b** $3\pi x^{\frac{1}{3}}+c$

c $\frac{1}{4}a^4-a^3+2a+c$

d $2x^{\frac{1}{2}}+c$

e $\frac{1}{2}x^3+2x+3\ln|x|+c$

f $\frac{1}{2}e^{2x}-\frac{3}{4}\cos 4x+c$

g $\frac{2}{9}x^{\frac{3}{2}}-\frac{1}{12}\sin 4x+c$

9 **a** **i** $\frac{dy}{dx}=-\frac{\cos x}{\sin^2 x}$

ii $c-\frac{1}{\sin x}$

b **i** $\frac{dy}{dx}=\frac{2}{\sqrt{4x-1}}$

ii $\frac{1}{2}\sqrt{4x-1}+c$

10 $y=\frac{x^2}{2}+\frac{x^3}{15}+2x+c$

11 $f(x)=3x-\frac{x^2}{2}+c1$

12 $y=\frac{1}{2}\sin 2x-\cos 2x+4.5$

13 $y=2e^{4x}+x^3+x+2$

Exercise 11F

1 **a** $\frac{1}{12}(4x-1)^3+c$

b $-\frac{1}{4x-1}+c$

c $\frac{1}{2}\sqrt{4x-1}+c$

d $\frac{3}{4}\ln|4x-1|+c$

2 **a** $4x^2+12x+c$

b $e^{1+x^2}+c$

c $2\tan(4+2x)+c$

d $\frac{1}{2}\ln|2x+3|+c$

e $2\sin(x^2+2)+c$

f $\frac{1}{3}\sin^3 x+c$

3 $y=\frac{1}{2}+\ln|x+2|+3$

4 **a** $\frac{1}{3}e^{\sin 3x}+c$

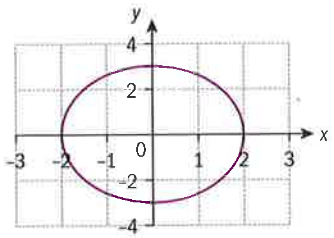
b $\frac{1}{12}\left(x^2-6x+4\right)^6+c$

5 **a** $-\ln|\cos x|+c$

b $y=-\ln|\cos x|+2x+c$, $c=4$

6 **a** $\frac{y^2}{9}+\frac{x^2}{4}=1$

b $x=0$, $y=\pm3$



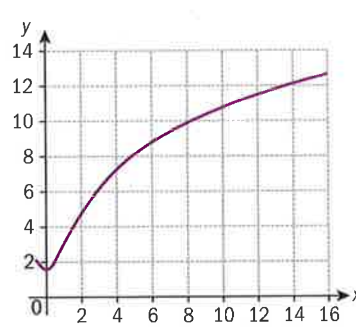
c 18.85

d 6π

7 **a** $C(n)=2\ln(n^2+1)+1.5$

b 9.3

c As n increases, the cost is growing slower and slower, and the shape of the curve becomes more and more logarithmic.



8 **a** $R(t)=\frac{3}{5\left(1+12e^{-0.5t}\right)}+c$

b 9.6 hours

c 75%

9 **a** $0<t<5$

b

$$T(t)=\frac{5}{2}\left(\ln\left(2t^2+1\right)+240\cos\left(\frac{t}{120}\right)\right)-575$$

c $t=5$, $T(5)=34\,m^3$

Exercise 11G

1 **a** $\frac{1}{2}(e^2+1)$

b $2\ln 2$

c $\frac{1}{3}\left(\sqrt{11^3}-8\right)$

2 $\int_1^4\frac{x}{x^2+2}dx=\int_3^{18}\frac{du}{2u}=\left[\frac{1}{2}\ln|u|\right]_3^{18}$

$=\frac{1}{2}\ln 6$,

where $u=x^2+2$

3 $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos 2x dx=\left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$=\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)$

4 Energy

$$E=\int_0^{t_1}\frac{4t}{t^2+2}dt=2\int_2^{t_1^2+2}\frac{du}{u}=2\ln\frac{t_1^2+2}{2}$$

5 27.27

Exercise 11H

1 **a** −1.875 **b** $\frac{81}{4}$ **c** 1.15

d $\frac{32}{3}$ **e** 4.24 **f** 5.52

2 3.296

3 4

4 **a** $a=4$, $b=\frac{\pi}{5}$

b $y=4\sin\frac{\pi}{5}x-1$

c $V=69\,m^3$

5 **a** **i** $p_a=\frac{1}{2}\ln(a+1)$

ii $A_1=\frac{a}{2}-\frac{1}{2}\ln(a+1)$

b 0

Exercise 11I

1 **a** $\frac{\pi}{3}$ **b** $\frac{\pi}{5}$ **c** $\frac{\pi}{7}$

d $\frac{\pi}{2}(e^2-1)$

2 **a** $\frac{\pi}{3}$ **b** $\frac{\pi}{2}$ **c** $\frac{3\pi}{5}$

b $\pi(e-2)$

3 **a** $\frac{300}{10000}x^3-\frac{x^5}{50000}+c$

b 1018 mm³ **c** £580

4 **a** **i** $V=\int_0^3\pi\left(4-\frac{4}{3}x\right)^2dx$

ii $V=\int_0^4\pi\left(3-\frac{3}{4}y\right)^2dy$

b **i** 16π

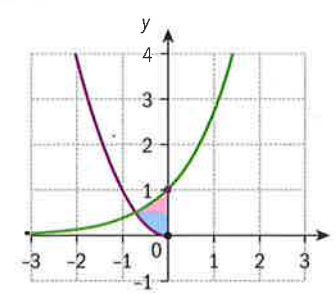
ii 12π

5 **a** Hany's model is best. 279 units³

b **i** 110 inches³

ii 1803 cm³

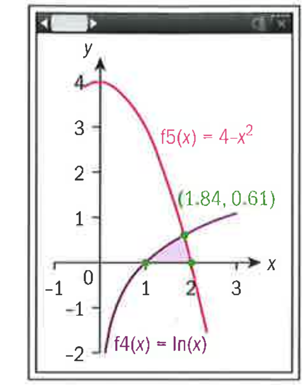
6 **a** **i**



ii $V=\int_{-0.70}^{0}\pi\left(e^{2x}-x^4\right)dx$

iii 1.1

b **i**

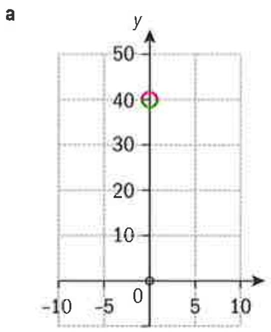


ii Volume $=\pi\int_1^{1.84}\left(\ln(x)\right)^2dx$

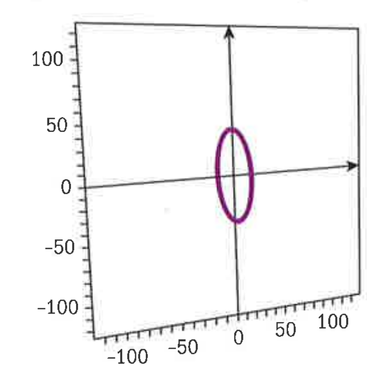
$+\pi\int_{1.84}^{2}\left(4-x^2\right)^2dx$

iii 0.46

7 **a**



b The shape is a torus.



c Volume =

$\pi\int_{-2}^{2}\left(40+\sqrt{4-x^2}\right)^2$

$-\left(40-\sqrt{4-x^2}\right)^2dx$

= 3158 cubic units

8 Consider triangles CAD and DAB. $AC=AB=r$, $CD=DB$ and AD is common. Hence the triangles are congruent.

Hence, $\angle ADB=\angle ADC=\frac{\pi}{2}$.

9 **a** 358 m

b $2.7\times10^6\,m^3$

Exercise 11J

1 **a** $v(t)=-\frac{3}{t+1}+3$,

$d(t)=3\left(t-\ln(1+t)\right)$

b 20 s

2 **a** **i** $a(t)=3e^{2\sin 3t}(2\cos^2 3t-\sin 3t)$

ii $t=0.3$

b $s(t)=\frac{1}{6}(e^{2\sin 3t}-1)$

c $s_{max}=\frac{1}{6}(e^2-1)$

3 **a** 1 **b** $k<\frac{1}{8}$

4 **a** Velocity of the projectile $v(t)=\int g dt=gt+v_0$, height of the projectile $s(t)=\int v(t)dt=\frac{1}{2}gt^2+v_0t+y_0$.

b $s(t_0)=y_0-\frac{v_0^2}{2g}$

c $s(t_0)=4913\,m>4\,km$, so rocket will reach the target height

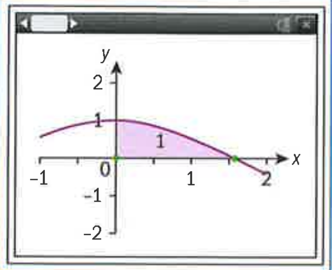
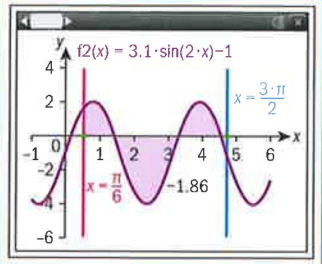
Exercise 11K

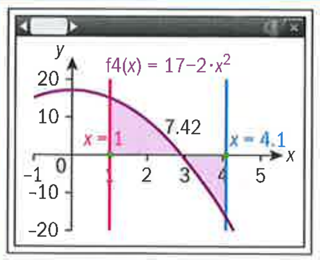
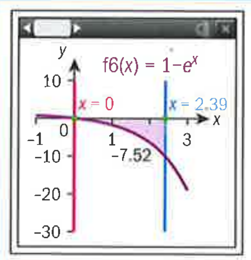
1 **a** −21.3 °C **b** 76.8 °C

2 **a**

Particle	Distance travelled	Displacement
A	50 m	50 m
B	25 m	25 m
C	25 m	0 m
D	25 m	25 m

3

	a	b
Sketch		
Distance travelled	1 km	7.97 km
Displacement	1 km	−1.86 km

	c	d
		
Distance travelled	26 km	7.52 km
Displacement	7.42 km	−7.52 km

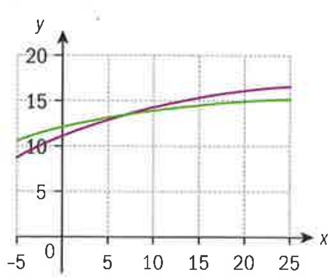
4 a i 20 m **ii** 33.3 m

b 16.45 seconds

5 a −0.015 cm

b 6.53 cm

6 a i



ii 6.49 seconds

b 917.5 m above the ground.

7 $a = \frac{2}{\pi}\text{mJ s}^{-1}$

8 $0.19\,\text{s}^{-1}$

9 a April

b i −13.6 **ii** −5.72

c −9.65

Exercise 11L

1 $y^4 = 2e^{\frac{1}{8}x^2}$

2 $y = 3|x^2 - 1|$

3 a $\frac{dM}{dt} = -kM$, separable equation: $\int \frac{1}{M}dM = \int -k\,dt, \ln M = -kt + c,\ M = Ae^{-kt}$

b $k = \frac{1}{2}\ln 2$

4 a $\frac{dy}{dx} = -\frac{x}{\sqrt{4-x^2}}$

b $-\frac{x}{\sqrt{4-x^2}} = -\frac{x}{\sqrt{4-x^2}}$

c $y = \pm\sqrt{c - x^2}$

5 a $y = -\ln(-c - e^x)$

b $\frac{1}{2}y^2 + \ln y = x + \sin x + c$

6 a $y = \frac{1}{2}\left(x^{\frac{3}{2}} + 7\right)^{\frac{2}{3}}$

b $\frac{1}{2}y^2 = x^2 - \cos x + \frac{3}{2}$

c $y = x$

7 a Initially when the difference between the temperature of the coffee and the ambient temperature is greatest.

b 75 °C

c $T = 24 + 51e^{-kt}$

d $e^{-kt} > 0$

8 a 6.561×10^{151}

b 0.611 hours

9 a $N = N_0 e^{-kt}$ When $t = T$ $N = \frac{N_0}{2}$ Hence $\frac{1}{2} = e^{-kT} \Rightarrow k = -\frac{1}{T}\ln\frac{1}{2} = \frac{\ln 2}{T}$

b 3.90 g

10 a The amount of water leaving the tank every second is $Av = A\sqrt{2gh}$. Hence, rate of change of the total volume of water in the tank is $\frac{dV}{dt} = -A\sqrt{2gh}$.

b i Since $V = \pi r^2 h, \frac{dV}{dt} = -A\sqrt{\frac{2gV}{\pi r^2}} = -k\sqrt{V}$

ii $k = A\sqrt{\frac{2g}{\pi r^2}}$

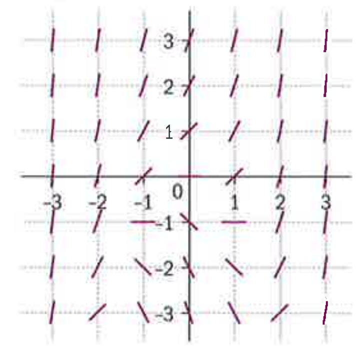
c $t = \frac{2\sqrt{V_0}}{k}$

Exercise 11M

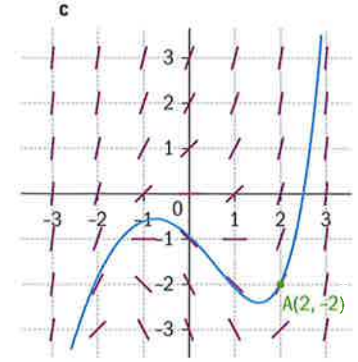
1 a

x \ y	1	2	3
1	2	3	4
2	5	6	7
3	10	11	12

b



c



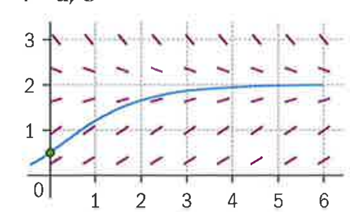
2 a III **b** I **c** IV **d** II

3 a i Along the lines $x + y = \pm\frac{\pi}{2}$.

ii Along the lines $x - y = \pm\frac{\pi}{2}$

b i IB **ii** A

4 a, b



c $2\,\text{cm}^3$

d Maximal size of the tumour agrees with the fact that it grows in the environment where the availability of nutrients is limited.

Exercise 11N

1 4.25

2 −0.71

3 0.274

4 a 0.53

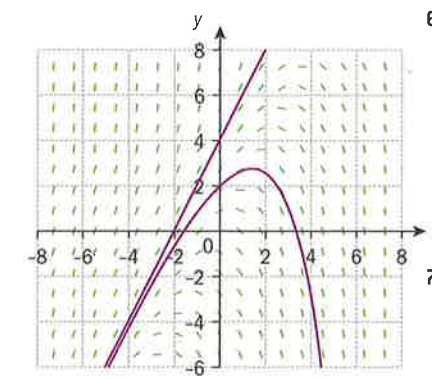
b $y(1.5) = 0.54$, error = 1.9%

5 a 3.99

b $y(1.3) = 4.04$, error = 1.2%

6 a $b = 2, a = 4$

b



c $N_0 = 4$

d $N = 6$

e 4 more meerkats

f 195

Chapter review

1 a For $t \in (0, 30)$, drone moves in positive direction with constant velocity $v = 40$

For $t \in (30, 60)$, drone moves in positive direction with decreasing speed, i.e. with acceleration $v = -\frac{4}{3}$

For $t \in (0, 30)$, drone moves in negative direction with increasing speed, i.e. with acceleration $v = -\frac{4}{3}$

For $t \in (75, 90)$, drone moves in negative direction with constant velocity $v = -20$

b $30 < t < 75$ **c** $30 < t < 60$

d 2250 km **e** 2250 km

2 a 46.7 m **b** 10 seconds

c 30.6 s

3 933.5 m

4 **a**, **b**, **c**, **e**, **f**, **g**, **h** can be found analytically; need technology to find **d**, **i**.

5 a $\frac{dP}{dt} = -aP$

b $P = 600\,000e^{\left(\frac{1}{4}\ln\left(\frac{5}{6}\right)\right)t}$

c 20.1 years

6 a Hemisphere

b $\pi\int_0^r \left(r^2 - x^2\right)^2 dx$

c $V = 2\int_0^r \pi\left(r^2 - x^2\right)dx = 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3$

7 a 3.6, 4.39, 5.45

b $3e^{x^2+2x}$, −0.10103, −0.27, −0.54

8 a $\frac{dV}{dt} = -k\sqrt{h}$

b $k\sqrt{h} = \frac{4\pi}{25}\frac{dh}{dt}$

c 50 minutes

9 a $11\,\text{m}^2$

b $352\,000\,\text{m}^3$

Exam-style questions

10 a 0.347

b $\frac{1}{2}\ln\left(x^2 + 4\right) + c$

c $\frac{\ln 2}{2}$

11 **a** $\frac{1}{3}x^3+\frac{3}{2}x^2+x+c$

b $-\cos x+\frac{1}{2}\sin 2x+c$

c $e^x-e^{-x}+c$

d $\frac{1}{21}(3x+1)^7+c$

e $\frac{1}{2}x^2+x+c$

12 $65\times10^4\,\text{m}^2$

13 **a** $122.98\,\text{m}^2$

b $124.42\,\text{m}^2$

c 1.2%

d Approximating the area using trapezium leaves small area above the trapezium and below the actual curve which is not taken into account while finding the area. Hence, the estimate is smaller than the true value.

14 7.42

15 **a** $v(t)=20(1-e^{-0.2t})$

b $(t)=20t+100(e^{-0.2t}-1)$

c $v(t)\to20$ as $t\to\infty$

d $d(60)=1100\,\text{m}$

16 **a**

x	y
1.25	2.50
1.50	3.28
1.75	4.51
2.00	6.49

b $y=e^{\frac{1}{2}x^2-\frac{1}{2}+\ln 2}$

c $2e^{\frac{3}{2}}$

d 28%

17 **a** $2x\sin x+x^2\cos x$

b $4\sin x+3x^2\sin x+c$

18 **a** **i** Isocline with $k=4$

ii Not an isocline, $\frac{dy}{dx}$ changes with x

iii Isocline with $k=1$

iv Not an isocline because $\frac{dy}{dx}=\frac{2x^2+2x+1}{2x^2+2x}\neq$ const

v Isocline with $k=-1$

b Turning point occurs when $\frac{dy}{dx}=0$. This is impossible for $x\neq0,\ y\neq0$.

Chapter 12

Skills check

1 **a** $\int 6e^{3x}-\frac{1}{e^{2x}}\,dx=2e^{3x}+\frac{1}{2e^{2x}}+c$

b $\int 3\cos(6x)-4\sin(2x)\,dx=\frac{1}{2}\sin(6x)+2\cos(2x)+c$

2 **a** 87.2°

b $\begin{pmatrix}2\\3\\2\end{pmatrix}\times\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}5\\-10\\10\end{pmatrix}$

3 $\lambda=4,-3$. For $\lambda=4$ eigenvectors are multiples of $\begin{pmatrix}1\\1\end{pmatrix}$. For $\lambda=-3$ eigenvectors are multiples of $\begin{pmatrix}-2\\5\end{pmatrix}$.

Exercise 12A

1 **a** 6.71 N, 26.6°

b 5 N, 126.9°

2 **a** $\begin{pmatrix}4.95\\4.95\end{pmatrix}\text{m s}^{-1}$

b $\begin{pmatrix}10.4\\-6\end{pmatrix}\text{m s}^{-1}$

3 $6.08\,\text{m s}^{-1}$, 350.54° clockwise

4 **a** Perpendicular to the force F and displacement r

b 5 N m

5 **a** Resistance force $=\begin{pmatrix}-70\\0\end{pmatrix}$N, force from the first dog $=\begin{pmatrix}147.72\\26.05\end{pmatrix}$N, force from second dog $=\begin{pmatrix}0.97D\\-0.26D\end{pmatrix}$N

b $\begin{pmatrix}77.72+0.97D\\26.05-0.26D\end{pmatrix}$N

c 100.19 N

d $2.91\,\text{m s}^{-2}$

Exercise 12B

1 **a** 3.13 and 2.68

b −0.333 and 5.37

2 **a** 38.3 N

b $0.255\,\text{m s}^{-2}$

3 **a** $\begin{pmatrix}6\\8\end{pmatrix}$

b $9.90\,\text{km h}^{-1}$

4 **a** $\boldsymbol{b}_i=\boldsymbol{a}+(\boldsymbol{b}_i\cdot\boldsymbol{n})\boldsymbol{n}$

b $\boldsymbol{b}_r=\boldsymbol{b}_i-2k\boldsymbol{n}=\boldsymbol{b}_i-2(\boldsymbol{b}_i\cdot\boldsymbol{n})\boldsymbol{n}$

c $\begin{pmatrix}4\\1\\0\end{pmatrix}$

Exercise 12C

1 **a** $\begin{pmatrix}2\\0\end{pmatrix}$m

b $\begin{pmatrix}4\\-1\end{pmatrix}\text{m s}^{-1}$, $4.12\,\text{m s}^{-1}$

c $a=\begin{pmatrix}0\\6\end{pmatrix}\text{m s}^{-2}$, $6\,\text{m s}^{-2}$, 126.9°

2 **a** $\boldsymbol{r}=(2t^2-t+2)\boldsymbol{i}+(-1+5t-t^2)\boldsymbol{j}$

b 9.43 m

3 **a** $v=\begin{pmatrix}2t+2\\t+3\end{pmatrix}\text{m s}^{-1}$

b $r=\begin{pmatrix}t^2+2t+4\\\frac{1}{2}t^2+3t-1\end{pmatrix}$m

4 **a** $t=0.5$

b 76.0°

c $\boldsymbol{r}=(t^2-t+2)\boldsymbol{i}+(t^3-3)\boldsymbol{j}$

d $t=2.61$

5 **a** $v=\begin{pmatrix}3t^2+4\\2t+12\end{pmatrix}\text{m s}^{-1}$

b $t=2\,\text{s}$

6 **a** $r=\begin{pmatrix}2t^2-2.5t+2\\t^3\end{pmatrix}$m

b $v=\begin{pmatrix}6\\4t-8\end{pmatrix}\text{m s}^{-1}$

c $t=4\,\text{s}$

7 $4\,\text{km h}^{-1}$, $t=1.5\,\text{h}$

Exercise 12D

1 **a** $\begin{pmatrix}0\\-9.8\end{pmatrix}\text{m s}^{-2}$

b $v=\begin{pmatrix}5\\2-9.8t\end{pmatrix}\text{m s}^{-1}$

ii $\boldsymbol{r}=\begin{pmatrix}5t\\2t-4.9t^2+1.5\end{pmatrix}$m

c 0.793 s, 3.97 m

2 **a** $v=4\boldsymbol{i}+(2-gt)\boldsymbol{j}\,\text{m s}^{-1}$

b $\boldsymbol{r}=(4t)\boldsymbol{i}+\left(2t-\frac{g}{2}t^2\right)\boldsymbol{j}\,\text{m}$

c 3.40 s

d 13.6 m

3 **a** $\begin{pmatrix}12.7\\7.35\end{pmatrix}\text{m s}^{-1}$

b $v=\begin{pmatrix}12.7\\7.35-9.8t\end{pmatrix}\text{m s}^{-1}$

c 0.75 s

d $r=\begin{pmatrix}12.7t\\7.35t-4.9t^2\end{pmatrix}$

e 2.75 m

f 19.1 mg

g $12.7\,\text{m s}^{-1}$

4 $58.9\,\text{m s}^{-1}$

5 **a** $\begin{pmatrix}5\sqrt{2}\\5\sqrt{2}-gt\end{pmatrix}\text{m s}^{-1}$

b 1.69 m

c 11.5 m

6 **a** $\begin{pmatrix}25\sqrt{3}\\25\end{pmatrix}\text{m s}^{-1}$

b $\begin{pmatrix}25\sqrt{3}t\\25t-\frac{g}{2}t^2\end{pmatrix}$m

c 3.11 s

d 5.18 s, 224 m

Exercise 12E

1 **a** $a=\begin{pmatrix}4e^{2t}\\2e^{2t}\end{pmatrix}$

b $r=\begin{pmatrix}e^{2t}-1\\\frac{1}{2}e^{2t}+2t-\frac{1}{2}\end{pmatrix}$

2 $\begin{pmatrix}2.5e^{2t}-0.5t^2-5t-2.5\\e^{2t}+t^2-2t-1\end{pmatrix}$

3 **a** $r=\begin{pmatrix}-2\cos 2t\\-2\sin 2t\end{pmatrix}$

b 2

c Circle of radius 2 with a centre (0, 0)

d $k=-4$

4 **a** $v=\begin{pmatrix}2e^{2t}\\6e^{2t}\end{pmatrix}$

b $a=\begin{pmatrix}4e^{2t}\\12e^{2t}\end{pmatrix}$

c $k=4$

5 **a** e^{-t}

b Spiral into the point $\begin{pmatrix}2\\5\end{pmatrix}$

c $v=\frac{dr}{dt}=\begin{pmatrix}-4e^{-t}\sin 4t-e^{-t}\cos 4t\\4e^{-t}\cos 4t-e^{-t}\sin 4t\end{pmatrix}$ via the chain and product rules. Write $s=\sin 4t$, $c=\cos 4t$ to simplify the notation. The speed is

$$|v|=e^{-t}\left|\begin{pmatrix}-4s-c\\4c-s\end{pmatrix}\right|$$
$$=e^{-t}\sqrt{(-4s-c)^2+(4c-s)^2}$$
$$=e^{-t}\sqrt{16s^2+8sc+c^2+16c^2-8sc+s^2}$$
$$=e^{-t}\sqrt{17s^2+17c^2}=e^{-t}\sqrt{17}.$$

d 0.693 s

6 **a** $r=\begin{pmatrix}4\sin\left(\frac{\pi}{3}t\right)\\-4\cos\left(\frac{\pi}{3}t\right)\end{pmatrix}$

b $r=\begin{pmatrix}4\sin\left(\frac{\pi}{3}t\right)+5\\-4\cos\left(\frac{\pi}{3}t\right)+4\end{pmatrix}$

c $v=\begin{pmatrix}\frac{4\pi}{3}\cos\left(\frac{\pi}{3}t\right)\\\frac{4\pi}{3}\sin\left(\frac{\pi}{3}t\right)\end{pmatrix}$

Exercise 12F

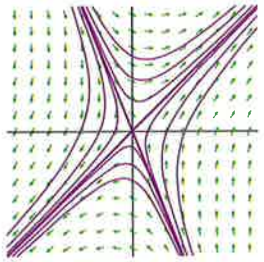
1 **a** **i** $x=2e^{4t}\begin{pmatrix}1\\1\end{pmatrix}+4e^{-3t}\begin{pmatrix}-2\\5\end{pmatrix}$

ii $x=2e^{t}\begin{pmatrix}-2\\5\end{pmatrix}+e^{3t}\begin{pmatrix}1\\1\end{pmatrix}$

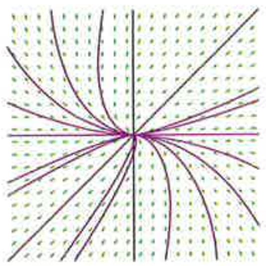
iii $x=3e^{-3t}\begin{pmatrix}2\\-1\end{pmatrix}-4e^{-4t}\begin{pmatrix}1\\-1\end{pmatrix}$

iv $x=3e^{-5t}\begin{pmatrix}-1\\3\end{pmatrix}+2e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$

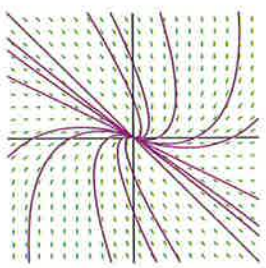
b **i** $y = x, y = -\frac{5}{2}x$



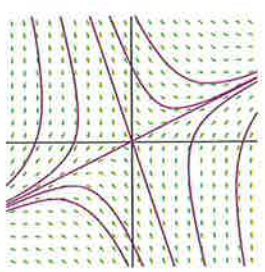
ii $y = x, y = 0$



iii $y = -x, y = -\frac{1}{2}x$



iv $y = -3x, y = \frac{1}{2}x$



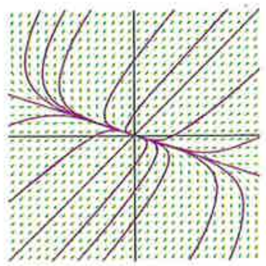
2 **a** D **b** A

c C **d** B

3 **a** $x = Ae^{-t}\begin{pmatrix}-3\\1\end{pmatrix} + Be^{5t}\begin{pmatrix}1\\1\end{pmatrix}$

b Asymptotes are $y = x$ and $y = -3x$. Asymptotic behaviour: $\begin{pmatrix}0\\0\end{pmatrix}$ is an equilibrium point, trajectories on the line $y = -3x$ stay on that line and move away from $\begin{pmatrix}0\\0\end{pmatrix}$ and all other trajectories move away from $\begin{pmatrix}0\\0\end{pmatrix}$ and towards the line $y = x$.

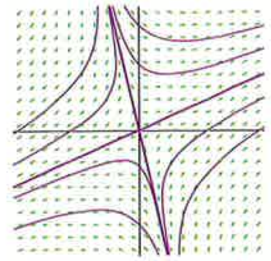
$x = 2e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} - 3e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$



4 **a**

$x = Ae^{-1.45t}\begin{pmatrix}1\\-4.45\end{pmatrix} + Be^{3.45t}\begin{pmatrix}1\\0.449\end{pmatrix}$

b The asymptotes are $y = -4.45x$ and $y = 0.449x$. Asymptotic behaviour: $\begin{pmatrix}0\\0\end{pmatrix}$ is an equilibrium point, trajectories on the line $y = -4.45x$ stay on that line and move towards $\begin{pmatrix}0\\0\end{pmatrix}$ and all other trajectories move away from $\begin{pmatrix}0\\0\end{pmatrix}$ and towards the line $y = 0.449x$.



c $x = -0.112e^{-1.45t}\begin{pmatrix}1\\-4.45\end{pmatrix}$

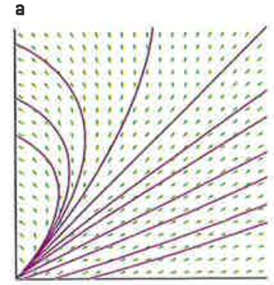
$+1.1 1e^{3}.45t\begin{pmatrix}1\\0.449\end{pmatrix}$

5 **a** When the initial conditions are $x = 0, y = 0$, then $\frac{dx}{dt} = \frac{dy}{dt} = 0$ so trajectory remains at $\begin{pmatrix}0\\0\end{pmatrix}$.

b The condition needed is that $\boldsymbol{M}$ has two distinct real eigenvalues.

c False, this is a sufficient condition for a saddle point instead. We need every point near the equilibrium to tend towards it and not just some of them.

6 **a**



b **i** $\frac{dx}{dy} = 0, \frac{dy}{dt} = 0.1y,$

$y = Be^{0.1t}$ and $x = 0$

ii $\frac{dx}{dt} = 0.4x, \frac{dy}{dt} = 0,$

$x = Ae^{0.4t}$ and $y = 0$

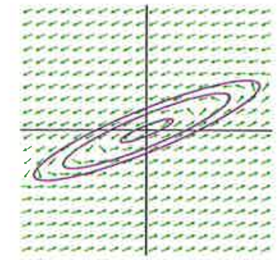
c **i** 2 : 1 **ii** 2 : 1

iii 0 : 1 **iv** 1 : 1

Exercise 12G

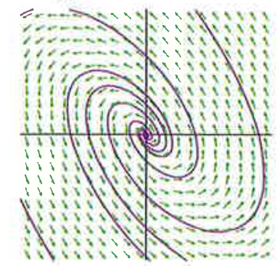
1 **a** **i** $\lambda = \pm i$ **ii** $\frac{1}{2}$

iii



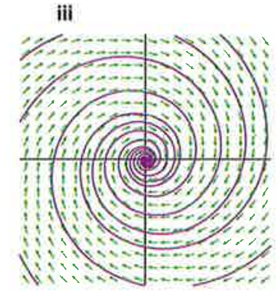
b **i** $1 \pm 2\sqrt{2}i$ **ii** -4

iii



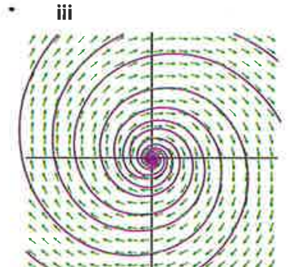
c **i** $\lambda = 0.2 \pm i$ **ii** 5

iii



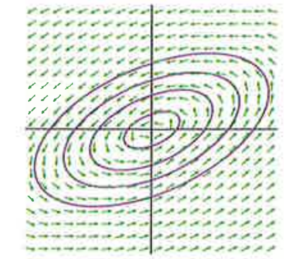
d **i** $\lambda = 1 \pm 5i$ **ii** -5

iii



e **i** $\lambda = \pm 3\sqrt{3}$ **ii** $\frac{4}{3}$

iii



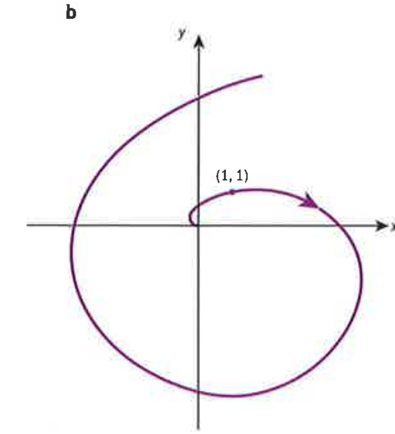
2 **a** $\boldsymbol{M} = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix}$.

$|M - \lambda I| = \begin{vmatrix}2-\lambda & 1\\-1 & 2-\lambda\end{vmatrix}$

$= (\lambda - 2)^2 + 1,$

so $\lambda = 2 \pm i$. The real part is positive, so the satellite spirals away from the Earth at $\begin{pmatrix}0\\0\end{pmatrix}$ as claimed.

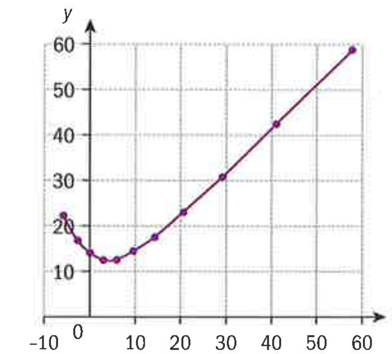
b



Exercise 12H

1 **a** **i** The formulae from the Euler method are
$x_{n+1} = 1.2x_n + 0.2y_n,$
$y_{n+1} = 0.5x_n + 0.9y_n$

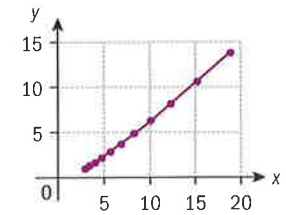
n	x_n	y_n
0	−6	22
1	−2.8	16.8
2	0	13.72
3	2.744	12.348
4	5.7624	12.4852
5	9.41192	14.11788
6	14.11788	17.41205
7	20.42387	22.72979
8	29.0546	30.66874
9	40.99926	42.12917
10	57.62495	58.41588



$\frac{y_{10}}{x_{10}} = 1.01$ and when $t = 1$, $x = 108.8, y = 110.2$. The Euler method severely underestimates the values of x and y.

ii The formulae from the Euler method are
$x_{n+1} = 1.1x_n + 0.2y_n,$
$y_{n+1} = 1.3y_n$

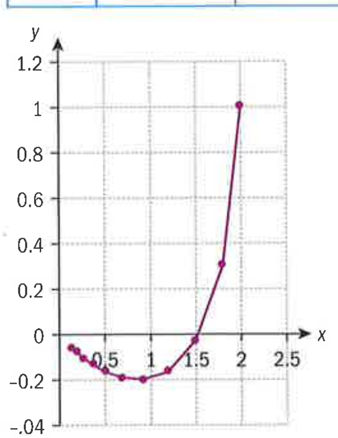
n	x_n	y_n
0	3	1
1	3.5	1.3
2	4.11	1.69
3	4.859	2.197
4	5.7843	2.8561
5	6.93395	3.71293
6	8.369931	4.826809
7	10.17229	6.274852
8	12.44448	8.157307
9	15.32039	10.6045
10	18.97333	13.78585



$\frac{y_{10}}{x_{10}} = 0.73$ and when $t = 1$, $x = 25.52$, $y = 20.09$. The Euler method underestimates the values of x and y and also has a slow rate of convergence of $\frac{y_n}{x_n}$ to 1.

iii The formulae from the Euler method are $x_{n+1} = 0.8x_n + 0.2y_n$, $y_{n+1} = -0.1x_n + 0.95$

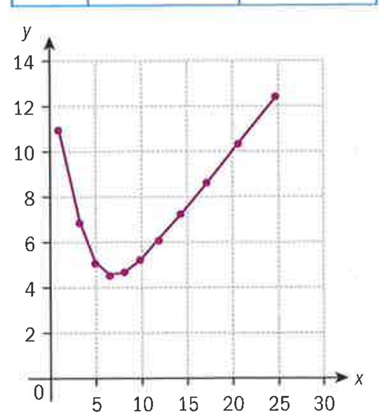
n	x_n	y_n
0	2	1
1	1.8	0.3
2	1.5	−0.03
3	1.194	−0.165
4	0.9222	−0.2019
5	0.69738	−0.19317
6	0.51927	−0.16632
7	0.382151	−0.13509
8	0.278703	−0.10576
9	0.201811	−0.08075
10	0.145299	−0.06056



$\frac{y_{10}}{x_{10}} = -0.42$ and when $t = 1$, $x = 0.23$, $y = -0.08$. The Euler is misestimating the values of x and y, although the ratio of $x:y$ is closer to the limiting value of $-\frac{1}{2}$ than for the exact values.

iv The formulae from the Euler method are $x_{n+1} = 1.1x_n + 0.2y_n$, $y_{n+1} = 0.3x_n + 0.6y_n$

n	x_n	y_n
0	1	11
1	3.3	6.9
2	5.01	5.13
3	6.537	4.581
4	8.1069	4.7097
5	9.85953	5.25789
6	11.89706	6.112593
7	14.30929	7.236674
8	17.18755	8.63479
9	20.63326	10.33714
10	24.76402	12.39226



$\frac{y_{10}}{x_{10}} = 0.5004$ and when $t = 1$, $x = 29.54$, $y = 14.84$. The Euler method has some imprecision with the exact values of x, y, although the ratio converges to the limit of $\frac{1}{2}$ much, much faster than for the exact data.

2 a The formulae from the Euler method are $x_{n+1} = x_n(0.9 + 0.2y_n)$, $y_{n+1} = y_n(1.2 - 0.1x_n)$

n	x_n	y_n
0	1	1
1	1.1	1.1
2	1.232	1.199
3	1.404234	1.291083
4	1.626407	1.368002
5	1.908751	1.419109
6	2.259622	1.432058
7	2.680842	1.394879
8	3.160647	1.29991
9	3.666294	1.149036
10	4.142205	0.957573

b The formulae from the Euler method are $x_{n+1} = x_n(1 + 0.2x_n - 0.1y_n)$, $y_{n+1} = y_n(1 + 0.2x_n - 0.1y_n)$

n	x_n	y_n
0	1	1
1	1.1	1.1
2	1.221	1.221
3	1.370084	1.370084
4	1.557797	1.557797
5	1.80047	1.80047
6	2.12464	2.12464
7	2.576049	2.576049
8	3.239652	3.239652
9	4.289186	4.289186
10	6.128898	6.128898

3 a The formulae from the Euler method are $t_{n+1} = t_n + 0.1$, $x_{n+1} = x_n + 0.1\left(-2t_n x_n + y_n^2\right)$, $y_{n+1} = y_n + 0.1(-3x_n^2 + 3t_n y_n)$

n	t_n	x_n	y_n
0	0	−1	2
1	0.1	0.2	1.7
2	0.2	1.063	1.739
3	0.3	1.927716	1.504349
4	0.4	2.490973	0.524914
5	0.5	2.374356	−1.27358

b The formulae from the Euler method are $t_{n+1} = t_n + 0.1$, $x_{n+1} = x_n + 0.1\left(2t_n x_n + y_n\right)$, $y_{n+1} = y_n + 0.1(-3x_n^2 + t_n y_n)$

n	t_n	x_n	y_n
0	0	0	1
1	0.1	0.1	1
2	0.2	0.202	1.007
3	0.3	0.31078	1.014899
4	0.4	0.430917	1.016371
5	0.5	0.567027	1.001319

Exercise 12I

1 a (0, 0) and (1, 2)

b If $P = 0$ then $\frac{dP}{dt} = 0$ so $P = 0$ forever and $\frac{dQ}{dt} = Q \Rightarrow Q = Ae^t$

c If $Q = 0$ then $\frac{dQ}{dt} = 0$ so $Q = 0$ forever and $\frac{dP}{dt} = -2P \Rightarrow P = Be^{-2t}$

d Approximately 1042 prey and 998 predators

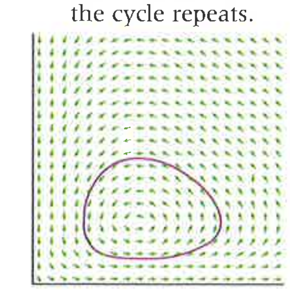
2 a 833 rabbits and 50 foxes

b Both populations are initially decreasing.

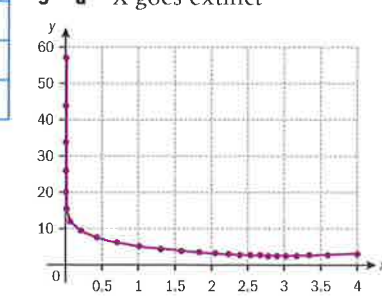
c i 50 foxes

ii 833 rabbits

d The computer generated phase plane sketch shows that the population of both rabbits and foxes decreases until we reach 50 foxes, and then the rabbit population starts to increase, followed by the fox population until the fox population recovers to 50. After this, the rabbit population decreases again until we return to the starting populations, and the cycle repeats.



3 a X goes extinct



b i $\frac{dy}{dt} = 3y$

ii $y = Ae^{3t}$

c When $x = 0$, $\frac{dy}{dt} = 0$ and so $\frac{dy}{dx}$ is ill-defined, so the y-axis is an asymptote.

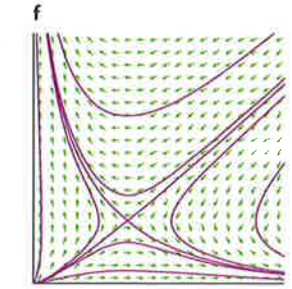
When $y = 0$, $\frac{dy}{dt} = 0$ and so $\frac{dy}{dx} = 0$, so the x-axis is an asymptote.

d i $\frac{dy}{dt} = (3 - x)y < 0$ so y decreases and $\frac{dy}{dt} = (2 - y)x < 0$ so for a while x decreases also. Thus Y decreases for a while.

ii $\frac{dx}{dt} = (2 - y)x < 0$ so x decreases, $\frac{dy}{dt} = (3 - x)y > 0$ so y is increasing. Thus X is always decreasing.

e (0, 0) and (3, 2)

f

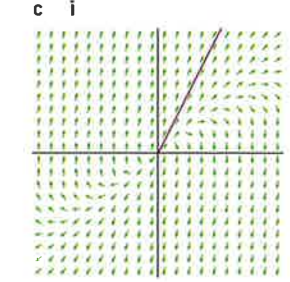


Exercise 12J

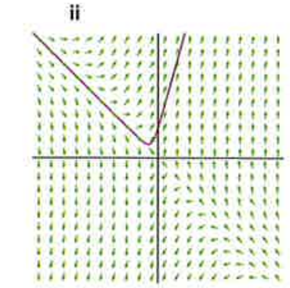
1 a $x = 2e^{2t}$

b $x = 4e^{4t} - e^{-t}$

c i



ii



2 a $\dot{x} = y$, $\dot{y} = -2x - 3y + 6t + 4$

b $x = 6.78$

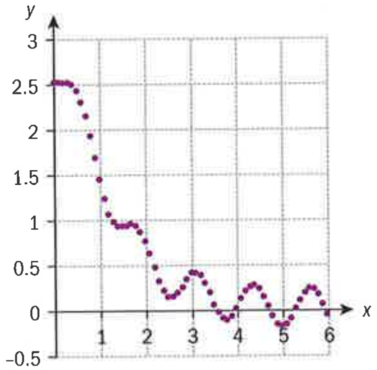
c $x = 5.854$ when $t = 1.6$

d $\frac{dx}{dt} = 3$; $x = -2 + 3t$

3 a

$\frac{dx}{dt} = y, \frac{dy}{dt} = -2x - 3y + 5\cos(5t)$

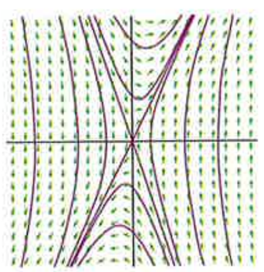
b x decays in bursts, and then has approximate oscillatory behaviour around 0.



Chapter review

1 a $x = Ae^{-3t}\begin{pmatrix}1\\-3\end{pmatrix} + Be^{2t}\begin{pmatrix}1\\2\end{pmatrix}$

b There is exactly one asymptote, $y = 2x$. $y = -3x$ is not an asymptote.



2 b $t = 1.5$ s

c Speed = 2.73 cm s^{-1}

3 a $r = \begin{pmatrix}\frac{4}{3}t^{\frac{3}{2}} + t\\ t^2\end{pmatrix}$

b 2.54

4 a (0, 0), (1, 0) and (2, 2)

b i 20 lions, 2023 zebra

ii 18 lions, 1670 zebra

c converging to (2, 2) or 20 lions, 2000 zebra. The population will eventually be stable in the absence of any external factors.

5 a 49 J

b $W = \boldsymbol{F}\cdot\overrightarrow{AB} = |\boldsymbol{F}|\times|\overrightarrow{AB}|\cos\theta$

$= |\boldsymbol{F}|\times|\overrightarrow{AB}|\cos 0 = |\boldsymbol{F}|\times|\overrightarrow{AB}|$

c 6 810 000 J or 6810 kJ (to 3 s.f.)

6 a $\begin{pmatrix}141\\0\\78\end{pmatrix}$ kN

b $\begin{pmatrix}4.7\\0\\2.6\end{pmatrix}$ m s^{-2}

c i $r = \begin{pmatrix}2.35t^2 + 55t\\0\\1.3t\end{pmatrix}$

ii $r = \begin{pmatrix}11760\\0\\4680\end{pmatrix}$ m

7 a 20 cm b m

c $\dot{x} = \begin{pmatrix}-1.26\sin 6.28t\\1.26\cos 6.28t\end{pmatrix}$ m s^{-1}

d 1.26 m s^{-1}

8 a $r_A = \begin{pmatrix}5t\\3t - t^2\end{pmatrix}$, $r_B = \begin{pmatrix}17 - 4t\\5t - t^2\end{pmatrix}$

b $r_A - r_B = \begin{pmatrix}9t - 17\\-2t\end{pmatrix}$

c 3.69

d $a = 13$, $b = 4$

e 17.1°

9 31.2° 58.8°

Exam-style questions

10 a i $v(t) = \begin{pmatrix}t^2\\2\\4t\end{pmatrix}$

ii $s(t) = \begin{pmatrix}\frac{t^3}{6}\\2t^2\end{pmatrix}$

b i $v(6) = \begin{pmatrix}18\\24\end{pmatrix}$

ii $s(6) = \begin{pmatrix}36\\72\end{pmatrix}$

11 a 26.4 N b 4.8 N

c $-15.84i - 21.12j$ N

12 a i $x = Ae5t + Be{-}2t$,
$x^{\cdot} = 5Ae5t - 2Be{-}2t$ and
$x^{\cdot\cdot} = 25Ae5t + 4Be{-}2t$.
Now $x^{\cdot\cdot} - 3x^{\cdot} - 10x$
$= 25Ae5t + 4Be{-}2t$
$- 15Ae5t + 6Be{-}2t -$
$10Ae5t - 10Be{-}2t = 0$.

ii $x = 2e^{5t} - 2e^{-2t}$

b i $x = A + Be^{-3t}$,
$\dot{x} = -3Be^{-3t}$ and
$\ddot{x} = 9Be^{-3t}$. Now
$\ddot{x} + 3x^{\cdot} = 9Be^{-3t} - 9Be^{-3t}$
$= 0$.

ii $x = 3 - 3e^{-3t}$

iii $x(t) \to 3$ as $t \to \infty$ since $e^{-3t} \to 0$ as $t \to \infty$.

13 a $x = 4e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + 2e^{2t}\begin{pmatrix}1\\0\end{pmatrix}$

b 160, 000 Homosapiens and 15, 000 Neanderthals.

c i $x \approx 2e^{2t}$ as $t \to \infty$

ii Dies out

14 a $a = \begin{pmatrix}315\pi^2\sin 3\pi t\\-315\pi^2\cos 3\pi t\end{pmatrix}$ cm s^{-2}

b $s = \begin{pmatrix}-35\sin 3\pi t\\35\cos 3\pi t + 40\end{pmatrix}$ cm

c $\begin{pmatrix}-35\\75\end{pmatrix}$ cm and $\begin{pmatrix}35\\75\end{pmatrix}$ cm

d 90

15 a Both eigenvalues are positive so the solutions move away from 0

b Both eigenvalues are negative and the solutions move towards 0

c Eigenvalues are $2 \pm i$, real part is positive so solutions spiral away from the origin

d Eigenvalues are $-2 \pm i$, real part is negative so solutions spiral towards the origin

16 a (0, 0) and (2, 1)

b Euler's method gives $x_{n+1} = x_n(1.25 - 0.25y_n)$ and $y_{n+1} = y_n(0.5 + 0.25x_n)$ with initial conditions $x_0 = 1$ and $y_0 = 0.5$. Tabulating the results below we get approximately 3716 lemmings and 31 owls:

n	x_n	y_n
0	1	0.5
1	1.125	0.375
2	1.300781	0.292969
3	1.530704	0.241756
4	1.820866	0.213393
5	2.178943	0.203836
6	2.612642	0.212955
7	3.126708	0.245571
8	3.716428	0.314743

c It is to 3 decimal places for x and seems a reasonable level of accuracy. However, Euler's method with step size $t = 0.25$ does lead to an unphysical negative value of x_{15}, and is somewhat suspect.

17 Let θ be the angle between a and b, and n a unit vector perpendicular to both a and b, so that $|n| = 1$. We have $a \cdot b = |a||b|\cos\theta$ and $a \times b = |a||b|\sin\theta n$.

Therefore

$\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}\right)^2 + \left(\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{b}|}\right)$

$= (|\boldsymbol{a}|\cos\theta)^2 + (|\boldsymbol{a}|\sin\theta|\boldsymbol{n}|)^2$

$= |\boldsymbol{a}|^2(\cos^2\theta + \sin^2\theta) = |\boldsymbol{a}|^2$

as claimed.

18 We have

$\frac{dx}{dt} = 0x + 1y, \frac{dy}{dt} = -cx - by$.

In the usual notation $\boldsymbol{M} = \begin{pmatrix}0 & 1\\-c & -b\end{pmatrix}$. Now

$|M - \lambda I| = \begin{vmatrix}-\lambda & 1\\-c & -b-\lambda\end{vmatrix}$

$= \lambda^2 + b\lambda + c$.

Write λ_1 and λ_2 for the roots $\frac{-b + \sqrt{b^2 - 4c}}{2}$ and $\frac{-b - \sqrt{b^2 - 4c}}{2}$ respectively.

Two eigenvectors that correspond to each eigenvalue are $\begin{pmatrix}1\\\lambda_1\end{pmatrix}$ and $\begin{pmatrix}1\\\lambda_2\end{pmatrix}$ respectively, since λ_1 and λ_2 are both roots of the quadratic $\lambda^2 + b\lambda + c = 0$. Then

$\boldsymbol{x} = Ae^{\lambda_1 t}\begin{pmatrix}1\\\lambda_1\end{pmatrix} + Be^{\lambda_2 t}\begin{pmatrix}1\\\lambda_2\end{pmatrix}$,

so that $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ as claimed.

Chapter 13

Skills check

1 2.5 2 $x \approx 1.39119$

Exercise 13A

1 None of the tables can represent a discrete probability distribution because

a the sums of probabilities is greater than 1;

b -0.2 is not a probability;

c Discrete probability distribution

2 a $P(A = 12) = 0.11$ b 0.14 c 0.59

d 0.41 e 0.281 f 7.78

3 a $P(B = 1) = 0.001$, $P(B = 2) = 0.00009999$, $P(B = 3) = 0.0000998$

b To win on your b-th crisp packet, you need to have had $(b-1)$ losses and then a win on the b-th try, so $P(B = b) = P(\text{lose})^{b-1} \times P(\text{win}) = 0.0001 \times (0.9999)^{b-1}$

c $b \in \mathbb{Z}^+$ d 0.000 999 55

4 a

c	1	2	3	4	5
$P(C = c)$	0.07	0.02	0.17	0.46	0.28

b 3.86

5 a

d	1	2	3	4	6	8	9	12	16
$P(D = d)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

b $\frac{2}{5}$

6 a $E(M) = \frac{3y+32}{8}$ b $y = 8$

7 a

f	0	1	2
$P(F=f)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

b $\frac{6}{7}$

8 a i $f(g) = \frac{10-g}{45}$

ii $E(G) = \frac{11}{3}$

b $f(g) = 0.2 \times 0.8^{g-1}$

c $E(G) = 5$

9 a

$$P(B \geq x+y \mid B \geq x) = \frac{P(B \geq x+y)}{P(B \geq x)}$$
$$= \frac{1-P(B \leq x+y-1)}{1-P(B \leq x-1)}$$
$$= \frac{1-\left(1-(1-p)^{x+y-1}\right)}{1-\left(1-(1-p)^{x-1}\right)}$$
$$= (^{1} = 1 - F(y) = P(B \geq y+1)$$

b $P(B \geq 15 | B \geq 10) = P(B \geq 6)$ means that the process has no "memory". The probability of finding a golden ticket after at least 15 trails given that no golden ticket was found after at least 10 has the same probability as starting over. The same logic works for $P(B \geq 6 | B \geq 1) = P(B \geq 6)$.

Exercise 13B

1 a 0.0535 b 0.991

c 0.809 d 0.558

e 0.983

f Not independent

2 0.773

3 a $n = 13$, $\text{Var}(Y) = 3.12$

b $p = 0.2$ or 0.8

4 a 0.00163

b 0.6785

5 a $A \sim B(25, 0.2)$ b 0.617

c 0.220 d 0.234

e 5. This is the number of correct answers Alex would expect to have if he guesses every question.

f 0.383 g 0

h 0.212

6 a $T \sim B(538, 0.91)$

b 9.21×10^{-23}. The probability that all 538 passengers turn up in time to make the flight is negligible.

c 0.000672. It is highly likely that there will be empty seats on the flight.

d $n = 551$

e $n = 591$

f 0.0573; 0.468; it is very likely that the plane will be overbooked.

7 $p = 0.5$

8 a

Number of 1s thrown	Frequency observed	Frequency expected if fair
0	79	102.4
1	83	76.8
2	9	19.2
3	29	1.6

$O \sim B(3, 0.2)$

b There are significant differences between what is observed from the experiment and what is expected if the dice were fair.

Exercise 13C

1 a 0.0390 b 0.414 c 0.727 d 0.259 e 0.184

2 1.20

3 2.99 and 0.179. In both cases, $P(Y = 1)$ is an unlikely event.

4 27 5 £20420

6

Number of calls	0	1	2	3	4	5 or more
Observed Number of hours	15	30	28	14	7	8
Expected number of hours	14.9	28.7	27.6	17.7	8.48	3.26

This appears to be modelled well by the Poisson distribution except for the frequencies for 5 or more.

7 a 0.00000750.

b 0.00000750

8 a 0.0504 b 1

c 0.00142

9 a 2.25 b 1.56

Exercise 13D

1 a 0.5 b 0.16 c 0.16

d 0.819 e 0.00135 f 0.857

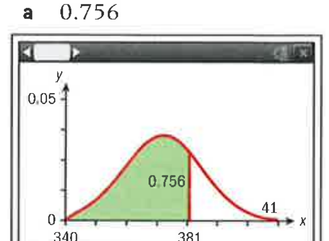
2 a 0.483 b 0.184

c 0.0829 d 0.525 e 0.887

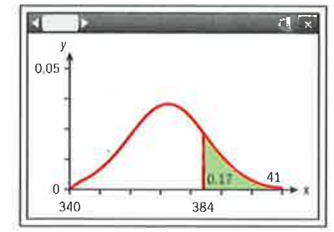
3 404

4 0.767

5 a 0.756



b 384



6 a 0.212 b 0.959

c 0.0446 d 446

7 a 0.170 b 21.6

c 15145

8 35 minutes

9 a Route A is on average shorter, but has more variability so a greater risk of taking longer than route B. Route B takes on average longer but has less variability so the average time is more reliable to predict.

b Route A

c i 0.113 ii 0.759

iii 0.101

Exercise 13E

1 a $\frac{5}{3}$ b $\frac{7}{4}$

2 a 2.6 b 6.22

3 a 17.02 b 31.14

c 2.43 d 3.6

e 11.82 f 7.0771

4 $a = 1.29$, $b = -0.07689$

5 $a = 0.165$, $b = 0.496$

6 . $a = 1.29$, $b = -0.143$

7 a

t	1	2
$P(T=t)$	$\frac{1}{3}$	$\frac{2}{3}$

t	3	4	5
$P(T=t)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

b i $E(T) = \frac{5}{3}$ ii $E(H) = \frac{17}{4}$

c $E(S) = \frac{71}{12}$

d

s	4	5	6	7
$P(S=s)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$

8 In the first game,

$E(Z_1 + Z_2) = E(Z_1) + E(Z_2) = \frac{17}{2}$

and $\text{Var}(Z_1 + Z_2)$

$= \text{Var}(Z_1) + \text{Var}(Z_2) = \frac{11}{8}$.

In the second game,

$E(2Z) = 2E(Z) = \frac{17}{2}$ and

$\text{Var}(2Z) = 4\text{Var}(Z) = \frac{11}{4}$.

Hence Zeinab is correct – since the variances differ, so do the distributions. Also, each game has a different set of outcomes.

Exercise 13F

1 0.7275 2 0.1943

3 0.0639 4 0.334

5 a 0.5 b 0.5

6 a 0.000878 b 0.387 c 6

7 0.186

8 Argentina: 0.0161, Egypt 0.000178

9 25

Chapter review

1 a $p = 0.25$, $q = -0.1$ b 0.5

2 a

t	4	5	6	7	8	9	10
$P(T=t)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{8}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

b 18

3 a 16

b i 0.0309 ii 0.03465

iii 0.996 iv 0.00617

4 a 0.123 b 0.210

c 0.0281

5 a 0.325 b 14.8 c 81.0

d 0.756 e 325

6 a 0.998 b 0.0000313

7 793

8 a C does not follow the Poisson distribution since $E(C) \neq \text{Var}(C)$, D does not follow the Poisson distribution since its domain does not include values less than 9, E does follow the Poisson distribution and F does not follow the Poisson distribution since $E(F) \neq \text{Var}(F)$.

b 14 batteries should be in stock

c 2.2888

9 a $E(T) = 12.04$, $\text{Var}(T) = 16.9$

b 0.8496 c 66

Exam-style questions

10 a i 0.15 ii 0.4

b 2

11 a 0.1755 b 0.4405

c 0.5898 d 0.4557

e i 23 ii 61

f Does not satisfy Poisson distribution as $E(Z) = \text{Var}(Z)$

12 **a** 0.2503 **b** 0.7224

c 19

13 **a** **i** 0.6827 **ii** 0.9545

b **i** 1.6449 **ii** 1.96

14 **a** 0.8321 **b** 0.1336

15 **a** 10.57% **b** 0.4575

16 **a** 6

b Mode occurs at $n = \mu$

17 **a** **i** $\frac{1}{4}$ **ii** $\frac{3}{16}$

b **i** $\frac{1}{4}$ **ii** $\frac{3}{1600}$

c 0.1020 **d** 0.1020

e **i** $T \sim \mathrm{B}\left(100, \frac{1}{4}\right)$

ii 0.1038

18 **a** $\frac{5}{6}$

b **i** $\frac{5}{3}$ **ii** $\frac{5}{9}$

c **i** $\frac{20}{3}$ **ii** $\frac{80}{9}$

d **i** 5 **ii** $\frac{5}{3}$

e **i** $\frac{13}{18}$

ii Not true because $\frac{13}{18} \neq \frac{1}{\frac{5}{3}} = \frac{3}{5}$

Chapter 14

Skills check

1 $\mathrm{P}(S) = \frac{n(\{1,4\})}{n(\{1,2,3,4,5,6\})} = \frac{1}{3}$,

$\mathrm{P}(E) = \frac{n(\{2,4,6\})}{n(\{1,2,3,4,5,6\})} = \frac{1}{2}$,

$\mathrm{P}(E \cap S) = \frac{n(\{4\})}{n(\{1,2,3,4,5,6\})} = \frac{1}{6}$

$= \frac{1}{3} \times \frac{1}{2} = \mathrm{P}(S) \times \mathrm{P}(E)$

so independent

2 0.6306 **3** $r_s = 0.7719$

Exercise 14A

1 **a** 1 **b** 0.99

c −1 **d** 0

2 **a** $r_S = -0.9636$

b 0.8929

3 **a** Not appropriate as the plot indicates that the relationship is not linear.

b −0.9605

c Indicates a strong inverse relationship between velocity and force. From the scatter graph we can see that the result is valid, though the actual relationship between the data is lost and when the values for force are very close for high velocities small changes could affect the value of r_s.

4 **a** $r = 0.6699$, indicates students who do better in Maths tend to do better in English

b −0.942

c Strong inverse relationship between velocity and force.

d From the scatter graph we can see that the result is valid, though the actual relationship between the data is lost and when the values for force are very close for high velocities small changes could affect the value of r_s.

5 **a** Because the ranks are given rather than quantifiable data

b −0.8857, generally the more expensive ones are preferred

6 **a** **i** $r = 0.9462$

ii $r = 0.6021$

b **i** $r_s = 0.6242$

ii $r_s = 0.4833$

c Spearman's correlation coefficient is affected less than the PMCC by an outlier.

Exercise 14B

1 **a** $\mathrm{P}(X \geq 6) = 0.0386$, $0.0386 < 0.05$, significant so reject H_0

b $\mathrm{P}(X \leq 6) = 0.250$, $0.250 > 0.10$, not significant so no reason to reject H_0

c $\mathrm{P}(X \geq 9) = 0.149$, $0.149 > 0.10$, not significant so no reason to reject H_0

2 **a** $\mathrm{H}_0: p = 0.6$ $\mathrm{H}_1: p < 0.6$

b Critical region is $X \leq 13$

c $13 < 14$, not in the critical region so not significant, therefore no reason to reject H_0.

3 **a** $X \geq 4$

b Binomial may not be appropriate as tripping on one fence may mean tripping on others more (i.e. not independent)

4 $\mathrm{H}_0: p = 0.3$ $\mathrm{H}_1: p > 0.3$
Let X be the number of objects remembered $\mathrm{P}(X \geq 8) = 0.2277$, not significant so no reason to reject H_0

5 $\mathrm{P}(X \geq 2) = 0.0815$, $0.0815 > 0.05$. The result is not significant at the 5% level so there is not sufficient evidence to reject H_0

Exercise 14C

1 **a** $\mathrm{P}(X \leq 5) = 0.276$, $0.276 > 0.05$ not significant so no evidence to reject H_0

b $\mathrm{P}(X \geq 10) = 0.0772$, $0.0772 > 0.05$ not significant so no evidence to reject H_0

c $\mathrm{P}(X \geq 23) = 0.00588$, $0.00588 < 0.01$, significant so reject H_0

2 $\mathrm{H}_0: \lambda = 3.6$, $\mathrm{H}_1: \lambda < 3.6$; $\mathrm{P}(X \leq 2) = 0.303$; $0.303 > 0.05$, not significant so no evidence to reject H_0 that the mean is still 1.8 times per week.

3 **a** $\mathrm{H}_0: \lambda = 11.0$, $\mathrm{H}_1: \lambda < 11.0$

b $X \leq 5$, 3.75%

c A constant rate is assumed for the Poisson distribution: this may not be appropriate as different games will be different, so the rate is likely to vary.

4 **a** $\mathrm{H}_0: \lambda = 4.0$, $\mathrm{H}_1: \lambda > 4.0$; Let X be the number of cases of infection. Critical region is $X \geq 9$.

b Need to assume that the infections occur independently

Exercise 14D

1 **a** **i** 0.961

ii p-value $= 0.1789 > 0.1$ so there is insufficient evidence to reject the null hypothesis $\rho = 0$

b **i** 0.7246

ii p-value $= 0.103 > 0.1$ so there is insufficient evidence to reject the null hypothesis $\rho = 0$

c **i** 0.718

ii p-value $= 0.0860 < 0.1$ so we reject the null hypothesis that there is no correlation between the two variables.

iii $y = 42.8 + 0.521x$

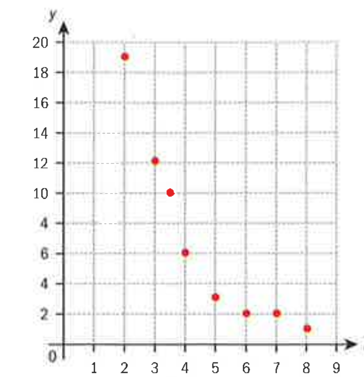
2 **a** $r = -0.901$

b **i** $\mathrm{H}_0: \rho = 0$ and $\mathrm{H}_1: \rho < 0$; p-value $= 0.00112 < 0.05$. The result is highly significant and so the null hypothesis is rejected and the alternative that there is a negative correlation between price and number of sales is accepted.

ii $s = 20.2 - 2.76p$

iii \$8.19

c



Relationship is not linear so an exponential or power regression might be more appropriate.

3 **a** $r = 0.8683$, highly significant so reason to reject the null hypothesis

b $x = 77.7334 + 3.9522y$

c €192.35

d 35 °C is outside the domain of the data provided, so not suitable to extrapolate

Exercise 14E

1 **a** $0.08281 < 0.025$, not significant so no reason to reject H_0

b $0.01511 < 0.025$, significant so reject H_0

c $0.2061 < 0.05$, not significant so no reason to reject H_0

2 **a** The sample size is large enough for the central limit theorem to apply.

b $\mathrm{H}_0: \mu = 24$, $\mathrm{H}_1: \mu < 24$

c p-value $= 0.0139 < 0.05$ significant, so reject H_0 that the average coverage of the paint is 24 m².

d Critical region is $\bar{X} \leq 23.4766$, confirming the result of the test

3 **a** $\mathrm{H}_0: \mu = 8.3$, $\mathrm{H}_1: \mu > 8.3$

b 8.56

c $\bar{X} \sim \mathrm{N}(8.3, 0.882)$

d 0.391

e $0.391 > 0.05$ and 0.01 so the result is not significant even at the 10% level so insufficient evidence to reject the null hypothesis that the mean time between buses is 8.3 minutes.

f p-value $= 0.391$

g $\bar{X} < 6.46$, $\bar{X} > 9.8448$

Exercise 14F

1 **a** p-value $= 0.100 > 0.05$ not significant, so do not reject H_0

b p-value $= 0.42104 > 0.025$ not significant, so do not reject H_0

c p-value $= 0.0239 < 0.05$ significant, so reject H_0

2 **a** p-value $= 0.0288 > 0.05$ not significant, so do not reject H_0

b p-value $= 0.0331 > 0.025$ not significant, so do not reject H_0

c p-value $= 0.0476 < 0.05$ significant, reject H_0

3 **a** The sample size is large enough for the central limit theorem to apply.

b $\mathrm{H}_0: \mu = 28.2$, $\mathrm{H}_1: \mu > 28.2$

c p-value $= 0.233 > 0.05$ not significant, so insufficient evidence to reject H_0

4 **a** $\mathrm{H}_0: \mu = 83$, $\mathrm{H}_1: \mu > 38$

b **i** 86.0 **ii** 3.41 **iii** 3.81

c $0.0765 > 0.05$, not significant so insufficient evidence to reject the null hypothesis

Exercise 14G

1 a (10.4, 14.4)
b (61.9, 62.7)
c (4.58, 8.02)
d (2.10, 2.50)
e (4.47, 4.73)

2 a (10.24, 14.28)
b (−7.85, 6.89)

3 a i (19.76, 22.64)
ii (20.30, 22.10)
iii (20.67, 21.73)
iv (20.83, 21.57)
b The larger the sample the smaller the width of the confidence interval.

4 a (11.57, 15.23)
b The population can be modelled by a normal distribution.
c 15.3 is outside the range for the confidence interval so is unlikely to be the population mean.

Exercise 14H

1 p-value = 0.251; 0.251 > 0.10, not significant, do not reject the null hypothesis. There is no difference in the weights of the apples.

2 p-value = 0.005 39; 0.00539 < 0.01, significant, so reject the null hypothesis. Those using the new remedy do lose more weight.

Exercise 14I

1 $\chi^2 = 9.5218 > 9.210$, significant so reject H_0

2 a H_0: GPA is independent of number of hours on social media, H_1: GPA is not independent of number of hours on social media
b 31.2

3 a Let μ_D be the mean difference. H_0: $\mu_D = 0$
H_1: $\mu_D < 0$
p-value = 0.000127 < 0.05 so the result is significant and the drug has a positive effect.
b H_0: $\mu_D = 0.7$ H_1: $\mu_D < 0.7$; p-value = 0.0285 > 0.05 so the result is significant and there evidence that the drug has a positive effect.
c i Yes because there is some good evidence the drug is working
ii Larger sample, group chosen to eliminate other factors e.g. age, gender.

Exercise 14J

1 0.008 56 < 0.01, significant so reject the null hypothesis that favourite chocolate and gender are independent.

2 a H_0: GPA is independent of number of hours on social media.
H_1: GPA is not independent of number of hours on social media.
b $\frac{85}{270} \times \frac{99}{270} \times 270$
$= 31.166... \approx 31.2$
c Degrees of freedom $(3-1) \times (3-1) = 2 \times 2 = 4$
d $\chi^2 = 78.5167$, $P(\chi_4^2 > 78.5167) = 3.33 \times 10^{-16}$
e 78.5167 > 7.779. Significant so reject the null hypothesis that GPA is independent of number of hours on social media.

3 H_0: number of people walking their dog is independent of the time of day.
H_1: number of people walking their dog is not independent of the time of day.
Either: The χ^2 test statistic = 5.30; 5.30 < 9.488 so the result is not significant so no reason to reject the null hypothesis or: the p-value = 0.257; 0.257 > 0.05 so the result is not significant so no reason to reject the null hypothesis

4 a H_0: the type of degree that a person has is independent of their annual salary
H_1: the type of degree that a person has is not independent of their annual salary.
b BA earning less than \$60 000
c Combine the first and second row to contain salaries < \$120 000
d p-value = 0.00403
e 0.00403 < 0.01 The result is significant so reject the null hypothesis that type of degree and salary are independent.

Exercise 14K

1 a

Colour	Frequency
Yellow	120
Orange	120
Red	120
Purple	120
Green	120

b 4
c H_0: the colours follow a uniform distribution; H_1: the colours do not follow a uniform distribution; χ^2_{calc} = 10.45 10.45 > 9.488; The result is significant so reject the null hypothesis that the distribution is normal.

2 a All are 50
b H_0: the last number follows a uniform distribution; H_1: the last number does not follow a uniform distribution. $\chi^2 = 9.08$, $P(\chi^9 > \chi^2 = 0.4299 \nless 0.1$ not significant so no reason to reject the null hypothesis that the last number on the lottery tickets follows a normal distribution.

3 a

Expected frequency	6.8	68.92	148.56	68.92	6.8

b H_0: the grades fit a normal distribution with mean of 65% and standard deviation of 7.5%. H_1: the grades do not fit a normal distribution with mean of 65% and standard deviation of 7.5%. The p-value = 0.947; 0.947 > 0.10 so the result is not significant and hence there is no reason to reject the null hypothesis that the exam results follow the given distribution.

4 a 21.6, 59.6, 87.6, 59.6, 21.6
b H_0: the heights fit a N(250, 11^2) distribution. H_1: the heights do not fit a N(250, 11^2) distribution. p-value = 0.0906; 0.0906 > 0.05 not significant so no reason to reject the null hypothesis that the sample of elephants was taking from a population whose heights are fit a N(250, 11^2) distribution.

5 a H_0: the scores are normally distributed with mean of 100 and standard deviation of 10. H_1: the scores are not normally distributed with mean of 100 and standard deviation of 10.
b Because if the expected values for the scores outside the range of the observed data are not zero then this would contribute to the test statistic.
c

Score, x	Probability	Expected score
$x < 90$	0.1586	31.7
$90 \le x < 100$	0.3414	68.3
$100 \le x < 110$	0.3414	68.3
$110 \le x < 120$	0.1358	27.2
$120 \le x < 130$	0.0214	4.28
$130 \le x$	0.00140	0.28

d

Score, x	Probability	Expected score
$x < 90$	0.1586	31.7
$90 \le x < 100$	0.3414	68.3
$100 \le x < 110$	0.3414	68.3
$110 \le x$	0.1586	31.7

e 3
f $\chi^2_{calc} = 54.8$ and p-value = 7.61×10^{-12}
58.4 > 6.251 so result is significant and the null hypothesis that the IQs of the students have been taken from this distribution should be rejected.
g i Mean is 105.9, standard deviation is 11.7554
ii The data is fairly symmetrical and most of the data is within two standard deviations of the mean which indicates a normal distribution is possible. The high value of the chi-squared statistic is probably due more to the mean and the standard deviation being quite far from those being tested.
iii Redo the test with a different mean and standard deviation.

Exercise 14L

1 a i 0.694, 0.278, 0.0278
ii 173.61, 69.45, 6.95
b H_0: the number of 6s fits a $B\left(2, \frac{1}{6}\right)$ distribution (or the dice are fair
H_1: the dice are not fair; $\chi^2 = 28.1 > 5.991$. 28.1 > 5.991, result is significant so there is strong evidence to reject the null hypothesis

2 a

Goals	0	1	2	3	4	5	≥6
Probability	0.0907	0.218	0.261	0.209	0.125	0.0602	0.0357
Expected frequency	4.53	10.89	13.07	10.45	6.27	3.01	1.78

b H_0: The number of goals in a football match follows a Poisson distribution with a mean of 2.4.
H_1: The number of goals in a football match does not follow a Poisson distribution with a mean of 2.4. Join together the final three columns to get:

Goals	≥4
Probability	0.2213
Expected frequency	11.06

4 degrees of freedom, p-value = 0.770. 0.770 > 0.10, the result is not significant and so there is insufficient evidence to reject the null hypothesis that the sample is taken from a population which follows a Po(2.4) distribution.

3 **a** 0.0156, 0.1406, 0.4219, 0.4219

b H_0: The number of seeds germinating fit a B(3.0.75) distribution.

H_1: The number of seeds germinating does not fit a B(3.0.75) distribution.

c

Number of seeds germinating	0	1	2	3
Expected frequency	0.78	7.03	21.1	21.1

d 2 as two columns need to be combined to give probability 0.1562 and expected frequency 7.81

e p-value $= 0.01472 < 0.05$ The result is significant so reject the null hypothesis that the seeds fit a B(3.0.75) distribution.

4 **a**

Number of people	Observed frequency	Expected frequency
0	5	0.90
1	7	3.78
2	6	7.94
3	7	11.11
4	8	11.67
5	6	9.80
6	4	6.86
7	8	4.12
≥ 8	9	3.84

Combine the first three and last two columns.

Number of people	Probability	Expected frequency
≥ 2	0.2103	12.62
3	7	11.11
4	8	11.67
5	6	9.80
6	4	6.86
≥ 7	0.1325	7.95

p-value $= 0.00306 < 0.05$, significant so reject the null hypothesis that the number of people joining the queue follows a Po(4.2) distribution.

b **i** Mean $= 4.23$, variance 6.81

ii The parameter for the Poisson is the mean and the mean of the sample is close to the mean for the distribution. For the Poisson distribution the mean and variance are close together, that is not the case in the sample which indicates the distribution is not Poisson.

5 **a** B(5, 0.25)

b 118.65, 197.75, 131.85, 43.95, 7.3, 0.5

c H_0: The students are guessing randomly. H_1: The students are not guessing randomly. Combine 4 and 5 to get 0.0157, 7.85. There are 4 degrees of freedom. p-value $= 0.05177 >$ 0.05 not significant so insufficient evidence to reject the null hypothesis that the students are guessing randomly.

Exercise 14M

1 **a** Standard deviation $= 121$

b

Lifespan, h hours	Expected frequency
$h < 1000$	19.7
$1000 \leq h < 1100$	62.0
$1100 \leq h < 1200$	118.3
$1200 \leq h < 1300$	118.3
$1300 \leq h < 1400$	62.0
$1400 \leq h$	19.7

c 4

d H_0: The lifespan of lightbulbs follows an N(1200, 121^2) distribution.

H_1: The lifespan of lightbulbs does not follow an N(1200, 121^2) distribution.

p-value $= 5.83 \times 10^{-7} < 0.05$ This result is significant and so we reject the null hypothesis

2 **a** $\bar{x} = 1.5319$, $p = 0.5106$

b H_0: The number of boys in a family has a B(3, 0.5106) distribution.

H_1: The number of boys in a family does not have a B(3, 0.5106) distribution.

Expected values:

Number of boys	Expected frequency
0	11.02
1	34.39
2	35.98
3	12.51

Degrees of freedom $= 2$, p-value $= 0.05715. > 0.01$ The result is not significant and so there is not sufficient evidence to reject the null hypothesis

3 **a** $\bar{x} = 2.85$

b H_0: The number of fish caught has a Po(2.85) distribution.

H_1: The number of fish caught does not have a Po(2.85) distribution.

Expected values:

Number of fish	Expected frequency
≤ 1	17.8
2	18.8
3	17.9
4	12.7
5	7.3
≥ 6	5.6

Degrees of freedom $= 4$, p-value $= 0.677 > 0.05$. The result is not significant so no reason to reject the null hypothesis

4 **a** $\bar{x} = 52.6$, $s_{n-1} = 5.15$

b H_0: The weights of the children have an N(52.6, 5.15^2) distribution

H_1: The weights of the children do not have a N(52.6, 5.15^2) distribution.

Expected values:

Weight, w kg	Expected frequency
$w < 45$	14.12
$45 \leq w < 50$	47.58
$50 \leq w < 55$	74.54
$55 \leq w < 60$	48.84
$60 \leq w$	14.92

Degrees of freedom $= 2$, p-value $= 0.0000517 < 0.05$. The result is significant so the sample is very unlikely to have come from a population with a normal distribution.

c The data has two peaks, which suggests two populations, it is possible that the sample included a mixture of boys and girls.

5 **a** B(50, p)

b $\bar{x} = 1.9 \Rightarrow p = 0.03167$

c H_0: The sample is from a B(50, 0.03167) population

H_1: The sample is not from a B(50, 0.03167) population

Expected values

Number of prizes	Expected frequency
0	12.01
1	19.63
2	15.73
≥ 3	12.63

Degrees of freedom $= 3$, p-value $= 0.0470 < 0.05$

The result is significant so reason to reject the null hypothesis

6 **a** H_0: The sample is from a Po(2.0).

H_1: The sample is not from a Po(2.0) population.

Expected values:

Number of calls / 5 minute	Expected frequency
0	8.1
1	16.2
2	16.2
3	10.8
≥ 4	8.6

Degrees of freedom $=$ 4, p-value $= 0.01329 <$ 0.05 significant, so reject the null hypothesis that the data follows a Po(2) distribution.

b $\bar{x} = 2.57$; H_0: The sample is from a Po(2.57). H_1: The sample is not from a Po(2.57) population.

Expected values:

Number of calls / 5 minute	Expected frequency
0	4.6
1	11.8
2	15.2
3	13.0
4	8.3
≥ 5	7.1

Degrees of freedom $= 4$, p-value $= 0.333 > 0.05$, hence the result is not significant and there is no reason to reject the null hypothesis.

Exercise 14N

1 **a** The first data shows random error and the second shows systematic error.

b The second data would have a correlation of 1, the first data would be close to but not equal to 1.

c A high correlation normally means a line of regression is useful. In this case the systematic error has resulted in a perfect correlation but the line of regression would be increasingly inaccurate as x increases.

2 **a** $P(\chi^2_2 > 7.0142)$ $= 0.0300 < 0.05$ so significant and the two factors are not independent.

b $\binom{10}{2} = 45$ **c** 0.9006

d Need to know if there are other reasons to suspect the two attributes given in part a were not independent. If there are no other reasons, need to do further tests as not enough evidence otherwise.

3 **a** **i** Taking a random or stratified sample sample. If measuring which school is better it is important the sample is representative of the school.

ii The samples from the two schools should be as equal as possible between girls and boys, so the improvement is due to the teaching method and not to the gender of the student.

iii A sample of girls should be compared with a sample of boys in each of the schools. The samples could also be pooled so a mixture of boys / girls from both schools. But to avoid the results being affected by teaching methods there should be equal numbers of boys and girls from each of the schools.

b A t-test on the difference between the average improvement in each school to see if there is a significant difference. Assume the populations are normally distributed or the sample size is large enough for the central limit theorem to apply.

4 **a** Good points include – the census contains details of most of the population, it will be relatively cheap and easy to collect, because it contains other information focused sampling could take place if required.

Bad points include – the data is 6 years out of date, it records who is living in the house on that day so if taken in the holidays might include a lot of students who do not normally live at home.

b **i** $P(140 \le X \le 160)$
$= P(X \le 160) - P(X \le 139)$
$= 0.648$

ii 1.00

The survey is almost certain to find the correct proportion to within 0.05, but there is a about a 0.35 chance it will not be within 0.1. If this level of accuracy is needed more households will need to be surveyed.

c 'Do you have any children who have left school? If so how many live at home and how many live away from home? (A single question such as 'Do you have any children who are still living at home?' is ambiguous as it is answered 'no' both by those with no children and by those with children who are not living at home).

5 **a** The method of obtaining the sample, to ensure it is appropriate (the two groups are as uniform as possible) and unbiased. The size of the sample to ensure a t-test can be performed (greater than 30).

b Let X be the number of significant results in the 6 tests. $X \sim B(8, 0.05)$. The probability of at least one significant result by chance is $P(X \ge 1) = 1 - P(X = 0) = 0.337$.

Exercise 14O

1 **a** **i** Two outcomes, each trial independent and identical.

ii Events are independent and occur at a uniform average rate during the period of interest.

iii Data comes from a normal population or the sample size is large

b Chi-squared goodness of fit test

2 **a** The test involves the sum of Poisson distributions (as total infected equals the sum of those infected by the individuals with the infection), this assumes the numbers of infections passed on by each of those infected is independent of the others. This is unlikely as they will be moving in close proximity and so may be meeting the same people. A Poisson distribution assumes a uniform rate. This is unlikely, as the days progress more people become infected so the numbers who are susceptible goes down and the rate will decrease. The numbers infected by an individual might not be independent, contact with one person might make contact with a second more likely, for example if a patient is receiving treatment or has plenty of visitors.

b A chi-squared goodness of fit. A significant result might be because the distribution is not Poisson or because the rate of reaction is not r.

3 **a** Chi-squared or test for $\rho = 0$

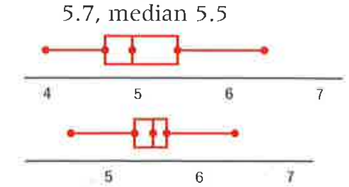
b Chi-squared, as we do not know if distance or weekly allowance are normally distributed. If it is suspected they might be normally distributed a test should be done to see.

c The distances and allowances need to be categorised into groups so that the expected values all greater than 5.

4 **a** Difficult to find parallel forms for questions about food. Test–retest will also indicate any change over time as well as considering the reliability of the data.

b 0.907, which indicates the survey is reliable as a source of information.

c Test 1 quartiles 4.5 and 5.5, median 4.9
Test 2 quartiles 5.2 and 5.7, median 5.5



d Boxplots indicate there has been an improvement as the median and lower quartile have improved considerably not much change for upper quartile and maximum score.

e Box plots are reasonably symmetrical so no reason to assume not normal, so reasonable to use the t-test.

H_0: There has been no improvement in the average score awarded to the canteen.

H_1: There has been an improvement.

p-value $= 0.00528 < 0.05$ so the result is significant at 5% level so significant evidence to say the canteen has improved.

5 **a** p-value $= 0.008783 < 0.05$, significant so very strong evidence to reject H_0

b Male/female mix implies that heights are not likely to have been normal.

Females on average shorter than males so extra factor in that has not been tested for. The positive relationship between height and salary might be a reflection on women getting paid less than men. Better to have all female or all male group and alter hypothesis accordingly.

c Group the sample into salary bands and use a χ^2 test for independence between gender and salary.

Exercise 14P

1 **a** 0.608 **b** 0.281

2 **a** **i** 0.0435 **ii** 0.792

b **i** 0.0386 **ii** 0.9351

3 **a** **i** 0.0458 **ii** 0.9245

b **i** 0.0421 **ii** 0.377

4 **a** Critical regions $\bar{X} < 48.3$, $\bar{X} > 53.68$

b $P\left(48.3 < \bar{X} < 53.7 \mid \mu = 51.5\right)$
$= 0.9346$

5 **a** $\bar{X} > 53.121$ **b** 0.322

c Increase the sample size, (or increase the significance level of the test)

6 **a** H_0:number of people with a car follows a B(50. 0.3) distribution

b Let X be the number of people in the sample who own a car. $X \ge 21$

c **i** 0.561 **ii** 0.101

7 **a** The hours chosen for the sample need to be independent. If they were adjacent to each other an event such as a large car crash might cause several of the hours to be larger than usual.

b H_0: $\mu = 82$, H_1: $\mu < 82$

c The mean of the sample is less than 7.5 but the average number of patients has not dropped from 8.2, and so the phone line is continued unnecessarily.
$P(X \le 75 \mid \mu = 82) = 0.239$

d $P(X \ge 76 \mid \mu = 78) = 0.605$. This is the probability of discontinuing the phone line even though it has had a positive effect. This might be justified if it is felt that the reduction is too small to cover the extra costs involved.

8 **a** H_0: $\mu = 1.2$, H_1: $\mu \ne 1.2$

b 0.05

c Critical regions are $\bar{X} < 1.1562$ and $\bar{X} > 1.2438$,
$P\left(1.1562 < \bar{X} < 1.2438 \mid \mu = 1.17\right)$
$= 0.731$

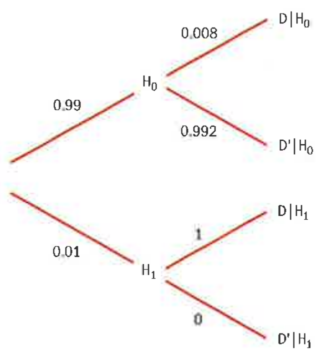
Exercise 14Q

1 **a** Under the null hypothesis that the shapes are guessed randomly, the probability of 3 correct is $0.2^3 = 0.008$.

b The probability he is quoting is the probability of not getting all three cards right if guessing randomly, not the probability he is not just guessing randomly.

c Other evidence would also have to be taken into account, including how many previous tests this person had taken and the results of those, what other evidence is there for the existence of ESP. The evidence from the test does indicate that he was unlikely to be guessing randomly, but all the alternatives, including the possibility of tricks need to be considered, and the likelihood of each compared.

d



e **i** $P(D) = 0.99 \times 0.008 + 0.01 \times 1 = 0.01792$

ii $P(H_1|D) = \dfrac{P(H_1)P(D|H_1)}{P(D)}$

$= \dfrac{0.01 \times 1}{0.01792}$

$= 0.558$

f **i** The researchers believe that the probability of ESP existing is 0.01

ii $P(D) = 0.999 \times 0.008 + 0.001 \times 1 = 0.008992$

$P(H_1|D) = \dfrac{P(H_1)P(D|H_1)}{P(D)}$

$= \dfrac{0.001 \times 1}{0.008992}$

$= 0.111$

2 **a** **i** Let X be the number of patients in the sample with the infection $X \sim B(100, 0.01)$ $P(X \geq 3) = 1 - P(X \leq 2) = 0.07937$

ii No, more information is needed.

b **i** 0.583

ii The probability of bad practice is just under 60%. Though not significant, further checks might be advisable.

3 **a** $\overline{X} > 4.71$ **b** 0.361

c 0.1089 **d** 0.413

Chapter review

1 **a** −0.9525

b The value is strong and negative – so, the taller the person is the less time they take to run the 100 metres.

2 **a** H_0: The colour of the eggs is independent from the type of hen.

H_1: The colour of the eggs is not independent from the type of hen.

b $\dfrac{30 \times 42}{90} = 14$

c 2

d $\chi^2 = 21.7$, p-value = 0.0000194

e 21.7 > 5.991 or 0.0000194 < 0.05 so reject the null hypothesis. The colour of the eggs is not independent from the type of hen.

3 **a** Probability of tossing a tail $= \dfrac{1}{2}$, so the probability of tossing 2 tails is $\dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4}$,

$60 \times \dfrac{1}{4} = 15$

b

Number of tails	0	1	2
Frequency	15	30	15

c 2

d H_0: The data fits a binomial distribution.

H_1: The data does not fit a binomial distribution.

$\chi^2 = 1.2$ or p-value = 0.549

e 0.1.2< 5.991 or 0.549 > 0.05, so not significant and no reason to reject the null hypothesis that the data fits a binomial distribution.

4 **a** H_0: $\mu_1 = \mu_2$ H_1: $\mu_1 \neq \mu_2$

b two-tailed

c p-value = 0.3391 > 0.05, not significant so no reason to reject the null hypothesis that there is no difference between the two groups.

5 **a** (1.3733, 1.741)

b (21.611, 22.589)

6 H_0: $\lambda = 15.0$, H_1: $\lambda > 15.0$; $P(X \geq 19) = 0.181$. $0.181 > 0.05$ not significant so insufficient evidence to reject H_0 that the number of hurricanes is still 7.5 on average.

7 **a** $B(42, 0.82)$ Just two possible outcomes and recovery of patients is likely to be independent of all other patients.

b H_0: $p = 0.82$, H_1: $p < 0.82$

c $X \leq 29$ **d** 2.93%

8 **a** **i** 0.05 **ii** 0, 01

b **i** Critical region $\overline{X} > 32.47$. Probability of a type II error = 0.623

ii Critical region $\overline{X} > 33.49$. Probability of a type II error = 0.840

c **i** Decrease **ii** Increase

iii Decrease

9 **a** Test–retest means the same test is given to the same people after a period of time. If the test is reliable there should be a good correlation between the two sets of results.

b $r = 0.968$ which indicates that the test is very reliable.

c Let μ_D be average difference between the scores on the first and second tests. H_0: $\mu_D = 0$, H_1: $\mu_D \neq 0$.

p-value = 0.133

Not significant at 10% so no reason to reject H_0 that there has been no increase in the overall level of satisfaction.

Assumptions: The differences can be modelled by a normal distribution and the responses of those surveyed were independent of each other.

Exam-style questions

10 **a** 0.0626 **b** 0

11 **a** H_0: favourite TV channel is independent of age, H_1: favourite TV channel isn't independent of age, Degrees of freedom = 4, $P(\chi^2_4 > 21.2774) = 0.000279 < 0.01$, significant so reject the null hypothesis. Favourite TV channel is not independent of age.

b Values are valid as they are all > 5

Expected	Alpha	Beta	Peppa
Up to 5 years old	$\frac{60}{200} \times \frac{40}{200} \times 20 = 12$	$\frac{60}{200} \times \frac{70}{200} \times 20 = 21$	$\frac{60}{200} \times \frac{90}{200} \times 20 = 27$
Between 6 and 10 years	$\frac{70}{200} \times \frac{40}{200} \times 20 = 14$	$\frac{70}{200} \times \frac{70}{200} \times 20 = 24.5$	$\frac{70}{200} \times \frac{90}{200} \times 20 = 31.5$
Between 11 and 15 years	$\frac{70}{200} \times \frac{40}{200} \times 20 = 14$	$\frac{70}{200} \times \frac{70}{200} \times 20 = 24.5$	$\frac{70}{200} \times \frac{90}{200} \times 20 = 31.5$

12 **a** **i** 173.75 **ii** 177.8

b 2-sample t-test since variance is unknown

H_0: $\mu_{Welsh} = \mu_{Scottish}$

H_1: $\mu_{Welsh} < \mu_{Scottish}$

$p = 0.0214$

0.0214 < 0.05 so we reject the null hypothesis and conclude that there is sufficient evidence at the 5% level to conclude that Welsh policemen are shorter than Scottish policemen.

c 0.0214 > 0.01 so at the 1% level we would accept H_0.

13 One tailed test. $H_0: \lambda = 20.0$, $H_1: \lambda < 20.0$, assume H_0 is true, then

$P(X \leq 100) = \sum_{i=0}^{100} \dfrac{120^i e^{-120}}{i!}$,

$= 0.0347 < 0.05$

significant so sufficient evidence to reject H_0, suggesting that Narcissus is exaggerating.

14 **a** If X has a mean of μ and a standard deviation of σ then the mean of a sample that is sufficiently large (> 30), has distribution $\overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$

b (4.804, 5.1960)

c 97

15 A paired t-test is used because we wish to directly compare measurements associated with the same cars. Let μ_D be average difference between the scores on the first and second tests.

Differences	0.2	0	0.4	-0.1	0.6	0	0.2	0.1	−0.1	0.3

$H_0: \mu_D = 0$, $H_1: \mu_D > 0$

$$\bar{d} = \frac{0.2 + 0 + 0.4 + \cdots + 0.1 + (-0.1) + 0.3}{10} = 0.16$$

$$s_n = \sqrt{\frac{0.2^2 + 0^2 + 0.4^2 + \cdots + 0.1^2 + (-0.1)^2 + 0.3^2}{10} - 0.16^2} = 0.2154,$$

$$T = \frac{\bar{d}}{s_n \div \sqrt{n-1}} = \frac{0.16}{0.2154 \div \sqrt{9}} = 2.2284$$

$P(t_9 > T) = 0.0264 < 0.05$, significant at 5% so evidence to reject H_0 that front wheels wear at the same rate as the rear wheels.

16 p-value is $0.7996 \nless 0.05$ so not significant, so not enough evidence to reject the null hypothesis that there is a linear relationship between a female's height and the number of pets she owns.

17 Use a Chi squared goodness of fit test. H_0: toys appear in the colour ratio 3:4:2:1, H_1: toys don't appear in the distribution stated. Degrees of freedom = 3 because once you know three of the probabilities, you know the fourth by definition.

Colour	Blue	Pink	Purple	Green
Observed	32	37	23	8
Expected	30	40	20	10

$$X^2 = \frac{(32-30)^2}{30} + \dots + \frac{(8-10)^2}{10}$$
$$= 1.2083$$

$P(\chi_3^2 > 1.2083) = 0.7510$

$0.7510 \not< 0.05$. Not significant so do not reject the null hypothesis. The colour of the toys follows the stated distribution

Chapter 15

Skills check

1 a $\begin{pmatrix} -1 & -2 & -1 \\ 3 & -1 & 3 \\ -3 & 1 & 4 \end{pmatrix}$

b $\begin{pmatrix} -2 & 3 & -9 \\ -15 & -2 & 6 \\ -6 & 9 & 22 \end{pmatrix}$

2 a $3y = 2x$ Combine with $x + y = 1$ to give $x = 0.6$ and $y = 0.4$

b As an example $P^{10} = \begin{pmatrix} 0.6004 & 0.5994 \\ 0.3996 & 0.4006 \end{pmatrix}$

c 0.4

Exercise 15A

1 All except **c**

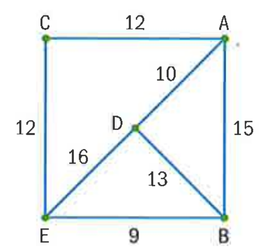
2 a

	A	B	C	D	E
A		1	1	1	
B	1		1		
C	1	1		1	
D	1		1		1
E				1	

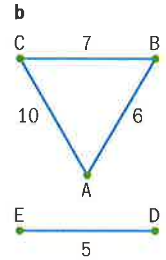
b

	P	Q	R	S	T
P		1	1		
Q	1		1	1	1
R	1	1			
S		1			2
T		1		2	

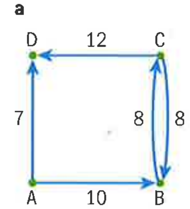
3 a



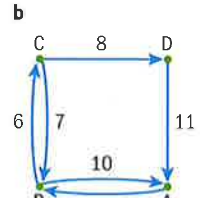
b



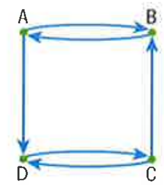
4 a



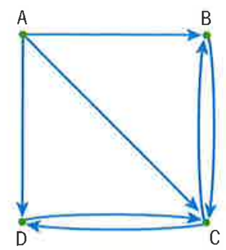
b



5 a



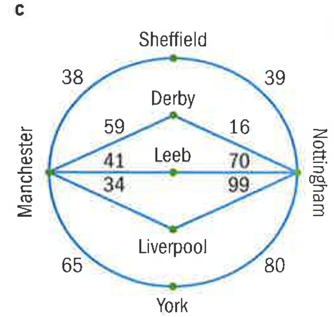
b



6 a Sheffield to Manchester is 38 miles, Sheffield to Nottingham is 39 miles

b One route from Manchester to Nottingham will be via Sheffield = 77 miles, a route via any town more than 77 miles from Manchester will necessarily be longer.

c



d 38 + 59 + 16 + 39 = 152 miles

Exercise 15B

1 **a** and **c** are undirected, **b** and **d** are directed.

2 a i

	A	B	C	D
A		1		
B	1			1
C	1			1
D			1	

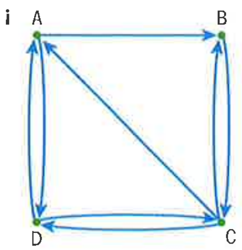
ii

	A	B	C	D
A				
B	8			
C	9			4
D		7	5	

b i Strongly connected

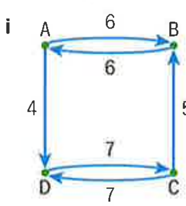
ii Connected but not strongly connected

3 a i



ii A in-degree 2 out degree 2; B in-degree 2-out degree 1; C in-degree 2 out-degree 3; D in-degree 2 out-degree 2

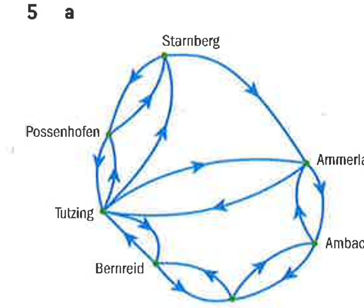
b i



ii A in-degree 1 out-degree 2; B in-degree 2 out-degree 1; C in-degree 1 out-degree 2; D in-degree 2 out-degree 1

4 A in-degree 1 out-degree 3; B in-degree 2 out-degree 1; C in-degree 2 out-degree 2; D in-degree 2 out-degree 2; E in-degree 3 out-degree 2

5 a



b It is easier to see which towns are connected, and the routes between the towns

c Starnberg and Tutzing and Starnberg and Ammerland

d Starnberg, Ammerland, Ambach

Starnberg Possenhofen, Tutzing, Ammerland, Ambach

Starnberg Possenhofen, Tutzing, Bernreid, Seeshaupt, Ambach

Starnberg Ammerland, Tutzing, Bernreid, Seeshaupt, Ambach

e All, except Starnberg Ammerland, Tutzing, Bernreid, Seeshaupt, Ambach

Exercise 15C

1 a

	A	B	C	D
A	0	1	0	1
B	1	0	1	1
C	0	1	0	1
D	1	1	1	0

b

	A	B	C	D
A	0	1	0	0
B	0	0	0	1
C	0	1	0	0
D	1	0	1	0

c

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	1
C	0	1	0	1	0
D	0	1	1	0	1
E	1	1	0	1	0

d

	A	B	C	D	E	F
A	0	1	0	0	1	0
B	0	0	1	0	0	0
C	0	0	1	1	0	0
D	0	1	0	0	0	0
E	1	0	0	1	0	1
F	0	0	0	0	0	0

2 The adjacency matrix consists only of zeros and ones and has zeros on the diagonal.

3 a Sum of entries in either row or column headed by that vertex

b i Sum of elements in the column headed by that vertex

ii Sum of elements in the row headed by that vertex

4 a i Everyone knows their own name

ii It is possible for someone to know the name of someone else without them knowing their name.

b i 3 **ii** 3

iii C **iv** E

c D

Exercise 15D

1 a i 2 ABDA, ADBA

ii 5 CBAD, CDBD, CDAD, CDCD, CBCD

b i 1 ABD **ii** 0

c i 2 ABEA, AEBA

ii 5 CBED, CBCD, CDCD, CDBD, CDED

d i 0 **ii** 1 CCCD

2 DA entry in M^2 is 0

3 a i $M^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

$M^3 = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$

ii 1

iii $S_3 = \begin{pmatrix} 4 & 5 & 6 & 2 \\ 5 & 4 & 6 & 2 \\ 6 & 6 & 5 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix} = 6$

iv S_1 contains 0s but S_2 does not, so diameter is 2

b i
$$M^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 2 & 0 & 4 & 1 & 1 \\ 0 & 4 & 2 & 4 & 4 \\ 1 & 1 & 4 & 2 & 3 \\ 1 & 1 & 4 & 3 & 2 \end{pmatrix}$$

ii 1 iii 1

iv S_1 and S_2contains 0s but S_3 does not, so diameter is 3.

4 a 5

b Castlebay, Lochboisdale and Tobermory

c
$$S_5 = \begin{pmatrix} 57 & 2 & 56 & 20 & 10 & 47 & 58 \\ 2 & 3 & 6 & 4 & 8 & 2 & 2 \\ 56 & 6 & 43 & 28 & 8 & 36 & 56 \\ 20 & 4 & 28 & 11 & 14 & 18 & 20 \\ 10 & 8 & 8 & 14 & 7 & 4 & 10 \\ 47 & 2 & 36 & 18 & 4 & 32 & 47 \\ 58 & 2 & 56 & 20 & 10 & 47 & 57 \end{pmatrix}$$

the largest number of trips needed to get between any two of the ports is 5. S_4 has a zero element at Eigg-Tiree intersection, so this is the route that takes five trips.

5 a
$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Town order is Starnberg, Possenhofen, Tutzing, Ammerland, Bernried, Ambach, Seeshaupt

b Starnberg to Bernreid
Starnberg to Seeshaupt
Possenhofen to Seeshaupt
Possenhofen to Ambach
Seeshaupt to Starnberg
Seeshaupt to Possenhofen
Ambach to Starnberg
Ambach to Possenhofen

Exercise 15E

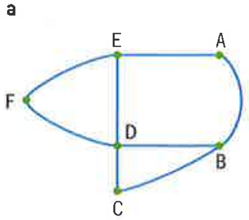
1 a i
$$\begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix}$$

ii
$$\begin{pmatrix} 0.2 \\ 0.3 \\ 0.2 \\ 0.3 \end{pmatrix}$$

b i
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

ii
$$\begin{pmatrix} \frac{6}{13} \\ \frac{4}{13} \\ \frac{1}{13} \\ \frac{2}{13} \end{pmatrix} = \begin{pmatrix} 0.462 \\ 0.308 \\ 0.0769 \\ 0.154 \end{pmatrix}$$

2 a



b
$$\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{3} & 0 \end{pmatrix}$$

c 0.222

d i B and E ii 18.75%

3 a
$$T = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 \end{pmatrix}$$

b
$$T^3 = \begin{pmatrix} 0 & 0.5 & 0.25 & 0.125 \\ 0.75 & 0.125 & 0.375 & 0.625 \\ 0.25 & 0 & 0.125 & 0.125 \\ 0 & 0.375 & 0.25 & 0.125 \end{pmatrix}$$

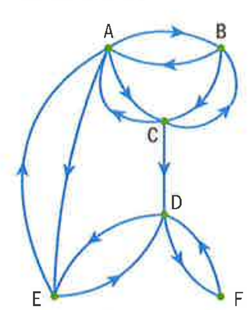
Hence not possible to go from B to C in 3 steps or from A to A and A to D.

c i $A\left(\frac{5}{19}\right), B\left(\frac{8}{19}\right), C\left(\frac{2}{19}\right), D\left(\frac{4}{19}\right)$

ii $\frac{8}{19}$

d B, A, D, C

4 a



b It would only be possible if George ended on Frances' page, otherwise he would need to visit Dawn's page twice. This is not possible as you cannot pass through all the other pages and end at Dawn's page without repeating some pages.

c Yes, if you begin on Frances' page; Frances, Dawn, Emil, Antoine, Bella and Charles

d
$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

e $\frac{85}{648} \approx 0.131$

f $\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{72}$

g i Most likely to visit Dawn's page

ii Least likely to visit Belle's page

5 a Because two states are absorbing states and the probability of entering each of them will depend on the starting position.

b 1 c 3

d
$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix}$$

e 0.125

Exercise 15F

1 a Minimum weight is 35, AF, FE, FD, DC, AB

b Minimum weight is 67, AB, AF, FE, ED, DC, DG

c Minimum weight is 23, AF, FC, CB, CD, FE

d Minimum weight is 39, AB, BF, FE, BG, FC, CD (or FC, CD, BG)

2 a Minimum weight is 35, EF, FD, DC, FA, AB

b Minimum weight is 67, CD, AB (or GD), GD (or AB), AF, FE, ED

c Minimum weight is 23, BC, AF, FC (or CD), CD (or FC), FE

d Minimum weight is 39, EF, CD (or FB), FB, (or CD), CF (or BG), AB, BG

3 You begin with a single vertex. Every time you add a new vertex to the tree you also add an edge. You need to add $v-1$ vertices and so the spanning tree will have $v-1$ edges.

4 a AE, CD, ED (or CB), CB (or ED), AF; $380, 000

b i Begin by connecting B and D and apply the algorithm from this point.

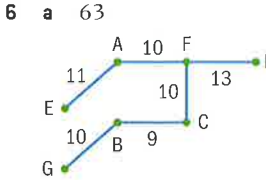
ii BD, DC, DE, EA, AF; $400 000

5 a 75

b 17

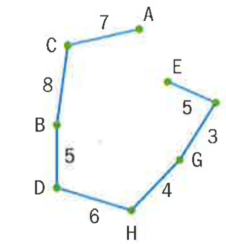
c Because it is difficult to check whether adding an edge will form a cycle.

6 a 63



b 10

7 a i



ii 38

b i 47 ii 41

Exercise 15G

1 a A route is ABFCFEFGBCDEGA, repeated edges CF and EF, weight 103

b A route is ABCDEFBFDBA, repeated edges AB and BF, weight 340

c A route is ABCDEFBECEFA, repeated edges CE and EF (or CB and BF) weight 66

2 a There are two vertices of odd degree

b i $160+\frac{30}{3}=170$ minutes, GF and FE

ii because this could be shown on a graph by having two edges between each vertex, and hence every vertex would have an even degree. Any route which traverses each edge of the graph twice e.g. ABCDEFGABGBFBCF CECDEFGA

3 a $59

b The cost will go down as all vertices now have even degree and so no routes need be repeated. There is an additional cost of $7 but a saving of $9. Total cost is $57.

Exercise 15H

1 a HB 4 FD 5; HF 6 BD 5; HD 6 BF 7; repeat edges HB, DI and IF

b BC 8 FE 11; BF 13 CE 13; BE 21 CF 7; repeat edges BC and FE

2 a IF 100 DC 70; ID 110 CF 90; IC 110 DF 70; need to repeat IH, HF, DG, GC; Possible route AIHFHIBHGFE DGCDGCBA, length 650 + 170 = 820

b Either I and C or I and F

3 a Odd vertices are A, B, D and H; AB 38 DH 49; AD 43 BH 73; AH 35 BD 88; need to repeat AB, DG and GH. A possible route is ABCDGABDGFDEFHGHA. Weight is 526 + 38 + 49 = 613 m.

b He would have to repeat BD which is 88 m. Previously he has had to repeat AB and DH which total 87 m

c As he needs to return to A the best place to be picked up is at B so he will only need to repeat DH, so length of repeated roads will be 49 m.

Exercise 15I

1 a i

	A	B	C	D	E
A	0	9	14	8	10
B	9	0	7	4	7
C	14	7	0	6	9
D	8	4	6	0	3
E	10	7	9	3	0

ii Nearest neighbour algorithm gives ADEBCA, actual route is ADEDBCDA, weight = 39

b i

	A	B	C	D	E	F
A	0	6	7	12	10	4
B	6	0	3	8	11	8
C	7	3	0	5	8	5
D	12	8	5	0	7	10
E	10	11	8	7	0	6
F	4	8	5	10	6	0

ii Nearest neighbour algorithm gives AFCBDEA, actual route is AFCBCDEFA, weight = 37

c i

	A	B	C	D	E	F
A	0	6	6	5	4	9
B	6	0	4	6	9	8
C	6	4	0	2	5	4
D	5	6	2	0	3	6
E	4	9	5	3	0	5
F	9	8	4	6	5	0

ii Hamiltonian cycle AEDCBFA or AEDCBFA, routes AEDCB**CF**EA 30 or AEDCFCBA 27, weights 30 or 27, 27 is the better upper bound

d i

	A	B	C	D	E
A	0	40	45	25	10
B	40	0	25	15	30
C	45	25	0	20	35
D	25	15	20	0	15
E	10	30	35	15	0

ii Hamiltonian cycle AEDBCA, route AEDBC**DE**A, weight 110

2 a Because it does not follow the triangle inequality the upper bound produced by the NNA is far higher than the solution to the TSP.

b Once the algorithm reaches E it is not possible to directly reach another vertex that has not already been passed.

Exercise 15J

1 a i 30

ii Deleting D gives a lower bound of 32

iii The solution to the TSP is between 30 and 39

b i 29

ii Deleting D or E will give a lower bound of 30, deleting C will give a lower bound of 31

iii Solution to the TSP is between 30 or 31 and 37

c i 22

ii Deleting B will give a lower bound of 23, and deleting C gives a lower bound of 24

iii Solution to the TSP is between 23 or 24 and 27

d i 85

ii Deleting D gives a lower bound of 95

iii Solution to the TSP is between 95 and 110

2 a Lower bound is 115

b

	A	B	C	D	E
A	0	35	20	25	10
B	35	0	25	35	15
C	20	25	0	30	10
D	25	35	30	0	20
E	10	15	10	20	0

c AECEBEDA 105

d Lower than lower bound; because shortest route between adjacent vertices is not always the direct route (the triangle inequality does not hold on the graph)

3 52 hours

4 a ADBCEA upper bound = 23 min

b Deleting A or E gives a lower bound of 21 minutes

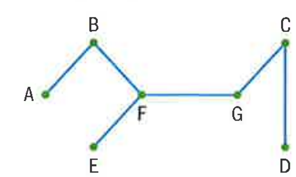
c The minimum weight of two of the edges adjacent to C is 9 and the minimum weight of the two edges adjacent to E is 11. This adds up to 20. Five edges are needed for a cycle and as the smallest is 3 then the solution to the TSP must be at least 23.

d Start at A and move to the nearest vertex which has not already been visited but not B. When all other vertices have been visited go to B then to A. An alternative is to go to B first and then use the NNA as usual. The route would then be the reverse of the one obtained.

Chapter review

1 a There are two odd vertices, C and F

b e.g. CDECBEFBAF

2 

Order of edge selection: EF(CD), CD(EF), CG, BF, FG(AB), AB(FG); total weight 21

3 Possible route ABCDEAEDFEB-FCA. Repeat AE and ED. Total weight 472

4 a i No, it is not symmetric

ii No, it contains multiple edges

iii No, for example, there is no edge between A and E

iv No, it contains a circuit, for example BCB

b A4, B3, C4, D4, E1

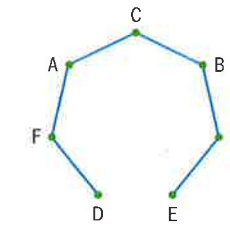
c i No, it has a vertex of degree 1

ii No, not all vertices are even

iii Yes, it has 2 odd vertices

d No, there is no walk of length 1 or length 2 between B and E. The matrix $M + M^2$ has zero entries for these values.

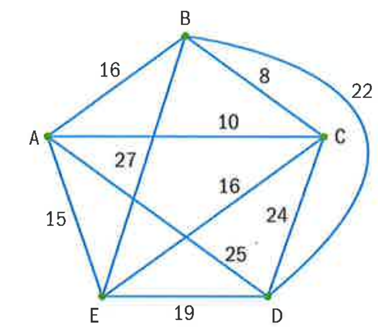
5 a Total weight 65



b 65 + 9 + 11 = 85

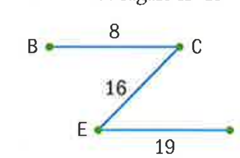
6 a i The graph is complete.

ii



iii 74

b i Weight is 43



ii 68

7 a i Yes, as the edges are directed.

ii No, as it contains a loop.

iii Yes, as it contains a circuit that includes all the vertices.

b

	A	B	C	D
In-degree	2	2	1	1
Out-degree	2	1	1	2

c i No

ii For each vertex the in-degree must equal the out-degree

iii Yes, DAABCDB

iv All vertices must have in-degree equal to out-degree except for one vertex which has an out-degree one more than its in-degree and another vertex which has an in-degree one more than its out-degree.

d

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

e $\begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$ In a random walk approximately equal amounts of time will be spent at each vertex.

8 a C, D, F, G

b CD and FG 1 + 4 = 5; CF and DG 5 + 4 = 9; CG and DF 5 + 4 = 8; possible walk is ABCDCEDGEFEGBFA; weight is 32

c 28: begin at F and end at G, or begin at G and end at F

9 a The cycle would have to pass through vertex C twice to return to the starting point.

b For example ABCDE

c

	A	B	C	D	E
A	-	13	5	12	9
B	13	-	8	22	12
C	5	8	-	7	16
D	12	15	7	-	10
E	9	12	16	10	-

d ACDEBA e 3

Exam-style questions

10 a Every vertex is of even order.

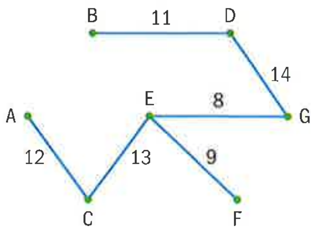
b One possibility is ACEFDEBDCBA

11 a AF, AB, BC, DF, DE; 17

b Algorithm attempts to find the 'optimum choice' at each stage

12 a EG, EF, BD, AC, CE, DG

b £6700



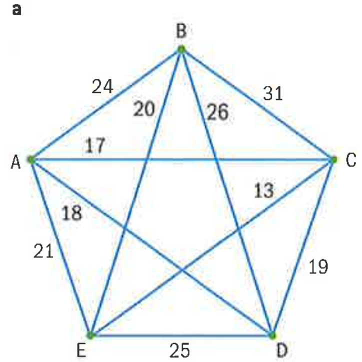
13 a E.g. FEDBAC

b For a Eulerian circuit, all vertices must be even, and there are two odd vertices here (B and E).

c BF, EF, AB or AC: This would then leave a vertex of degree 1, so no Hamiltonian cycle would be possible.

14 Weight of route is 48. One possible route is AFACABFEBCDEDA

15 a



b CE, AC, AD, BE **c** 68

16 a $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

b i $M^3 = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$

ii 1

c i $= \begin{pmatrix} 1 & 3 & 2 & 1 \\ 3 & 3 & 4 & 3 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$

ii 4

iii $B \to C$, $B \to D \to C$, $B \to A \to B \to C$, $B \to C \to B \to C$

17 a $T = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \end{pmatrix}$

b $T^3 = \begin{pmatrix} \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{3}{8} \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{3}{8} \\ \frac{5}{24} & \frac{1}{4} & \frac{1}{6} & \frac{1}{4} \end{pmatrix}$

c $\begin{pmatrix} 0.222 \\ 0.333 \\ 0.222 \\ 0.222 \end{pmatrix}$

d After a large amount of time, a ship is more likely to be based at port B, hence it would likely make the best place for the company's headquarters.

18 a

	A	B	C	D	E	F	G
A	0	4	5	10	9	9	3
B	4	0	3	8	5	5	5
C	5	3	0	5	8	8	2
D	10	8	5	0	6	13	7
E	6	5	8	6	0	7	10
F	9	5	8	13	7	0	10
G	3	5	2	7	10	10	0

b AGCBFEDA

c AGCBFEDCGA

19 a $\begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$

b A = 4 min, B = 19 min, C = 13 min, D = 13 min, E = 11 min

20 a $M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$

b i $M^7 = \begin{pmatrix} 6 & 8 & 7 & 3 & 2 \\ 5 & 4 & 3 & 2 & 2 \\ 5 & 9 & 5 & 5 & 1 \\ 5 & 6 & 3 & 3 & 3 \\ 9 & 8 & 8 & 3 & 3 \end{pmatrix}$

Therefore the required journey is from C to E since there is only one element of '1' here.

ii CBABACDE

21 a AB, DF; Weight of route = 64

b One possible route is ACBABDCGFDFEDA

c If Nasson starts at F, the only possible routes that need to be repeated will either be AB (= 6), BD (= BC + CD = 3) or AD (= 5). BD is the shortest, so this should be repeated. Therefore, given Nasson starts at F, he should finish at A.

22 a Order is ABCDEA. Upper bound is 81.

b By deleting A, Kruskal gives MST for the remainder as BC, CD, CE; weight = 43, lower bound = 75

c By deleting B, Kruskal gives MST for the remainder as CD, CE, CA; weight 47. Lower bound is therefore 47 + (13 + 15) = 75.

d $75 \le L \le 81$

e E.g. tour for original upper bound: ABCDEA

Paper 1

Exam-style questions

1 a $\frac{1}{2}(0.5)^2\frac{3\pi}{2} = 0.589\text{ m}^2$

(2 marks)

b $0.5 \times \frac{3\pi}{2} = 2.36\text{ m}$

(2 marks)

2 Let the triangle be ABC.

a $AC^2 = 5^2 + 6^2 - 2{\times}5{\times}6\cos 100 \Rightarrow AC = 8.4509... = 8.45\text{ km}$

(2 marks)

b $\frac{5}{\sin C} = \frac{8.4509...}{\sin 100} \Rightarrow$ $C = 35.637...$

(2 marks)

Bearing is 360 – 80 + 35.637 = 316°

(2 marks)

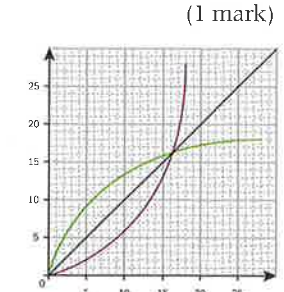
3 a i For each given area A, there is only one possible value for the radius, which is $\sqrt{\frac{A}{\pi}}$. Therefore, since $C = 2\pi r$, C is also unique for that value of A.

(1 mark)

ii For each given circumference C, there is only one possible value for the radius, which is $\frac{C}{2\pi}$. Therefore, $A = \pi r^2$, A is unique for that value of C, so C^{-1} exists.

(1 mark)

b



(2 marks)

c $A = C^{-1}(25) \Rightarrow C(A) = 25$ $\Rightarrow A = 49.7$

(2 marks)

4 a i $T(0) = 86 \Rightarrow 22 + a = 86$ $\Rightarrow a = 64$

(2 marks)

ii $T(0.5) = 28 \Rightarrow 22 + 64 \times 2^{0.5b} = 28$

(1 mark)

$b = -6.83$

(1 mark)

b $T = 22$

(1 mark)

c The temperature of the hot chocolate approaches 22°C as t gets very large, indicating that the temperature of the room is 22°C.

(1 mark)

5 a Discrete

(1 mark)

Number of walks can be counted

(1 mark)

b i 3

(1 mark)

ii $mean = 2.68$ (3 s.f.)

(2 marks)

6 a $\frac{1}{9} + \frac{1}{6} + \frac{1}{12} + \frac{2}{9} + \frac{1}{6} + p = 1$

(1 mark)

$p = \frac{1}{4}$

(1 mark)

b $\frac{1}{9} \times 2 + \frac{1}{6} \times 3 + \frac{1}{12} \times 5 + \frac{2}{9} \times 1 + \frac{1}{6} \times 3 + \frac{1}{4} \times 4 = \frac{103}{36}$ (2.86, 3 sf)

(2 marks)

c RR, BY, YB, PY or YP

$\left(\frac{1}{9}\right)^2 + 4\left(\frac{1}{6}\right)\left(\frac{2}{9}\right) = \frac{13}{81}$

$= 0.160(3sf)$

(2 marks)

7 a $\frac{8}{14}\left(=\frac{4}{7}\right) = 0.571$ (3 sf)

b Let

i Let X be the event that Jeanny picks a pen from the box. Then in 5 days, $X \sim B\left(5, \frac{8}{14}\right)$

(1 mark)

$P(X=2) = \binom{5}{2}\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)^3$

0.257 (3sf)

(1 mark)

ii $P(X \ge 2) = 1 - P(X \le 1)$ $= 0.889$ (3sf)

(2 marks)

8 a $AE = \sqrt{(2-4)^2 + (9-6)^2}$ $= 3.61$ (3 s.f.)

(2 marks)

b Attempt to find perpendicular bisector of BE.

(1 mark)

One technique is shown here:

$PB = PE \Rightarrow \text{sqrt}((x-2)^2 + (y-3)^2) = \text{sqrt}((x-4)^2 + (y-6)^2)$

(1 mark)

Attempt to expand

(1 mark)

$x^2 - 4x + 4 + y^2 - 6y + 9 = x^2 - 8x + 16 + y^2 - 12y + 36$

(1 mark)

$4x + 6y - 39 = 0$

(1 mark)

c The cell corresponds to the region of the park that has E as the closest well.

(1 mark)

9 a $10\,000\,(1.04)^5 = £12\,166.53$

(2 marks)

b $10\,000\,(1.04)^n > 15\,000 \Rightarrow n = 11$

(2 marks)

c £15 394.54 (2dp)

(1 mark)

10 a $P(X \le 19) = 0.0219$ (3sf)

(2 marks)

b $1 - P(X \le 35) = 0.157$ (3*sf*)
(2 marks)

c Let W be the number of emails Katherine receives in a week.
$W \sim P_0(210)$ $P(W \le 200) = 0.258$ (3*sf*)
(2 marks)

d Let $T = X + R$.
Then $T \sim P_0(50)$ $P(T \le 50) = 0.538$ (3*sf*)
(2 marks)

11 a $h(0) = \dfrac{6}{1+2} = 2$ m
(2 marks)

b $\lim_{t\to\infty} h(t) = \dfrac{6}{1} = 6$ m
(2 marks)

c $h(4) = 4.27$ m (3 s.f.)
(2 marks)

d $\dfrac{6}{1 + 2e^{-0.4n}} > 5 \Rightarrow n = 6$
(2 marks)

12 a 10
(1 mark)

b $n = 10e^3 = 201$ (nearest integer)
(2 marks)

c $1000 = 10e^{\frac{t}{2}} \Rightarrow t = 9.2103...$ hrs = 9 hours 13 mins
(3 marks)

d n must be an integer, however this model uses a continuous function
(1 mark)
Eventually the number of bacteria predicted by the model becomes far too large to be realistic.
(1 mark)

13 a $h(0) = 40$ cm
(1 mark)

b period $= \dfrac{2\pi}{2\pi} = 1$ s
(2 marks)

c since $-1 \le \sin \le +1$
i max is 35 + 40 = 75
ii min is −35 + 40 = 5

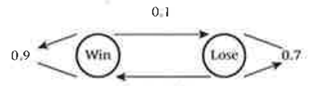
d 73.3 cm
(1 mark)

e $\sin(2\pi t) = \dfrac{4}{7} \Rightarrow 0.608...,$ 2.533...
(1 mark)
$\Rightarrow t = 0.096805...,$ 0.403194...
(1 mark)
more than 60 cm between 0.096805...and 0.403194...
Time interval is 0.403194...−0.096805... = 0.306 s
(1 mark)

14 a $\int \frac{1}{P}\,dP = \int \frac{-1}{V}\,dV \Rightarrow \ln P = -\ln V + c$
(2 marks)
$\ln P + \ln V = \ln PV = c$
(2 marks)
$PV = e^c = A$ (a constant)
(1 mark)
$A = 1000 \times 200 \Rightarrow PV = 200\,000$
(1 mark)

b $P \times 150 = 200\,000 \Rightarrow P = 1333.33... = 1.33 \times 10^3$ Pascals
(2 marks)

15 a $\begin{pmatrix} p_{n+1} \\ 1-p_{n+1} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}\begin{pmatrix} p_n \\ 1-p_n \end{pmatrix} \Rightarrow M = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$
(2 marks)

b


(2 marks)

c $\begin{pmatrix} r \\ 1-r \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}\begin{pmatrix} r \\ 1-r \end{pmatrix} \Rightarrow r = 0.9r + 0.3(1-r)$
(2 marks)
$r = \dfrac{3}{4}$ $V = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$
(2 marks)

16 a [112.92, 117.08]
(3 marks)

b t-distribution since σ is unknown
(2 marks)

17 a only (ii)

b $m = kw^3 \Rightarrow \log m = \log kw^3 = \log k + 3\log w$
(2 marks)
Gradient is 3
(1 mark)

Paper 2

Exam-style questions

1 a **i** $S = 2 \times (19.7 \times 6.2 + 19.7 \times 3.2 + 6.2 \times 3.2) = 410.04$ cm^2
(accept 410 cm^2 (3 s.f.))
(2 marks)
ii $V_{box} = 19.7 \times 6.2 \times 3.2 = 390.848$
(accept 391 cm^3 (3 s.f.))
(1 mark)

b $6.2 \div 0.6 = 10.3 \Rightarrow 10$ columns of pencils will fit
$3.2 \div 0.6 = 5.3 \Rightarrow 5$ rows of pencils will fit
(1 marks)
$N = 50$ pencils if we put them inside the box in 5 layers of 10 pencils
(1 marks)

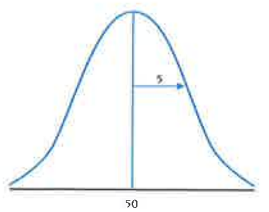
c $V_{pencils} = 50 \times 19.5 \times \pi \times 0.3^2 = 275.67...\text{cm}^3$
(2 marks)
$\dfrac{V_{pencils}}{V_{box}} \times 100\,\% \; \dfrac{275.67}{390.85} \times 100\% = 70.5\%$
(2 marks)

d $\dfrac{dP}{dx} = 0 \Rightarrow x = 1200$
(2 marks)

e $\int -x + 1200\,dx = -\dfrac{x^2}{2} + 1200x + C$
(2 marks)
$P(3) = 1500 \Rightarrow -\dfrac{3^2}{2} + 1200 \times 3 + C = 1500$
C = −2095.5
(2 marks)
$P(x) = -\dfrac{x^2}{2} + 1200x - 2095.5$
(1 mark)

f $P(x) = 0 \Rightarrow x = 1.74...$ or $x = 2398.2...$
(2 marks)

1748 boxes
(1 mark)

2 a


(3 marks)

b $P(45 < t < 55) = 0.683$ (3*sf*)
(2 marks)

c $P(t < 40) = 0.0228$ (3*sf*)
(2 marks)

d $P(t < M) = 0.75 \Rightarrow M = 53.4$(3*sf*) mins
(2 marks)

e $P(t < 40 \mid \text{gained medal} = \dfrac{P(t < 40 \cap \text{gained medal})}{P(\text{gained medal})} = \dfrac{0.02275...}{0.75}$
$= 0.303$(3sf)
(2 marks)

f $10\,000 \times P(t < 33) = 3.37...$
so 3 competitors
(1 mark)

3 a $Q_1 = 6$, $Q_3 = 9 \Rightarrow IQR = 3$
(2 marks)
$6 - 1.5 \times 3 = 1.5$ so value of h for student K is an outliner
(2 marks)

b $r = 0.816$ (3*sf*)
(2 marks)

c $s = 3.79h + 53.6$ (3*sf*)
(2 marks)

d $3.79 \times 5.5 + 53.6 = 74$ to the nearest integer
(2 marks)

e

	A	B	C	D	E	F	G	H	I	J	K
Hour rank	1	2	3	4	5	6	7	8	9	10	11
Score rank	1	3	2	4	6	5	7	8	9	11	10

(2 marks)

f $r_s = 0.973$ (3*sf*)
(2 marks)

g Spearman's is less sensitive to outliers like K, which distort the data, so Spearman's shows greater correlation.
(1 mark)

4 a $|a| = \dfrac{\max - \min}{2} = \dfrac{23 - 9}{2} = 7$
$x_{\min} < x_{\max}$
$a = -7$ so this is a negative sine curve
(1 mark)

b **i** $T = 2 \mid x_{\min} - x_{\min} \mid = 2 \mid 2 - 14 \mid = 24$
$b = \dfrac{2\pi}{24}$
(2 marks)
ii $c = 2 - \dfrac{|x_{min} - x_{max}|}{2} = -4$
(2 marks)
iii $d = \dfrac{y_{max} + y_{min}}{2} = \dfrac{23 + 9}{2} = 16$
(1 mark)

c **i** $t = 4.995 \Rightarrow 5$ hours, 0 minutes
(2 marks)
ii $t = 1.785 \Rightarrow 1$ hours, 47 minutes
(2 marks)
iii $8.206 - 1.785 = 6.421$
(2 marks)
6 hours and 25 minutes
(1 mark)

5 a $r = \begin{pmatrix} 150 \\ 100 \\ 50 \end{pmatrix} + t\begin{pmatrix} -5 \\ -10 \\ -1 \end{pmatrix}, t \ge 0$
(2 marks)

b $\begin{pmatrix} 100 \\ e \\ h \end{pmatrix} = \begin{pmatrix} 150 \\ 100 \\ 50 \end{pmatrix} + t\begin{pmatrix} -5 \\ -10 \\ -1 \end{pmatrix} \Rightarrow$
$t = 10$, $e = 0$ and $h = 40$
(2 marks)
As $e = 0$, the drone is directly north from launching point.
(1 mark)
As $h = 40$, the drone is 40m high. As t=10, the drone flies over the dog at 10 seconds past 1pm.
(1 mark)

c **i** $\sqrt{100^2 + 0^2 + 40^2} =$

$107.7\ldots \approx 108$ metres

(2 marks)

ii $v = \frac{d}{t} = \frac{\sqrt{100^2+0^2+40^2}}{5} =$

$21.54\ldots \text{m s}^{-1}$

(2 marks)

iii $\begin{pmatrix}100\\0\\40\end{pmatrix} + \frac{(t-10)}{5}\begin{pmatrix}-100\\0\\-40\end{pmatrix}$,

$t \geq 10$

(2 marks)

6 **a** $\mathbf{T} = \begin{pmatrix}0.6 & 0.2\\0.4 & 0.8\end{pmatrix}$

(2 marks)

b **i** Either:

$\begin{vmatrix}0.6-\lambda & 0.2\\0.4 & 0.8-\lambda\end{vmatrix}$

$= 0 \Rightarrow \lambda = 1, \lambda = 0.4$

Or

1 and $tr(\mathbf{T}) - 1 = 0.4$

(2 marks)

ii $\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow$

$\begin{pmatrix}0.6 & 0.2\\0.4 & 0.8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

(1 mark)

$\begin{cases}0.4x + 0.2y = 0\\0.4x - 0.2y = 0\end{cases} \Rightarrow y = 2x$

$\Rightarrow \mathbf{u} = a\begin{pmatrix}x\\y\end{pmatrix}$

(2 marks)

$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow$

$\begin{pmatrix}0.6 & 0.2\\0.4 & 0.8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$= 0.4\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{cases}0.2x + 0.2y = 0\\0.4x - 0.4y = 0\end{cases} \Rightarrow y =$

$-4 \Rightarrow \mathbf{v} = \beta\begin{pmatrix}1\\-1\end{pmatrix}$

(1 mark)

The eigenvectors are $\mathbf{u} =$

$\alpha\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathbf{v} =$

$\beta\begin{pmatrix}1\\-1\end{pmatrix}$

c $\mathbf{M} = \begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$

(2 marks)

$\mathbf{D} = \begin{pmatrix}1 & 0\\0 & 0.4\end{pmatrix}$

(2 marks)

d Population after n years

$\mathbf{P}_n = \mathbf{T}^n\mathbf{P}_0 = (\mathbf{MD}^n\mathbf{M}^{-1})\mathbf{P}_0$

Where the initial population is

$P_0 = \begin{pmatrix}4200\\6000\end{pmatrix}$

$\mathbf{M}^{-1} = \frac{1}{3}\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}$

(2 marks)

$\mathbf{D}^n = \begin{pmatrix}1 & 0\\0 & (0.4)^n\end{pmatrix}$

(1 mark)

$P_n = \begin{pmatrix}1600 + 2600 \times (0.4)^n\\3200 - 2600 \times (0.4)^n\end{pmatrix}$

(1 mark)

so $1600 + 2600 \times (0.4)^n$ birds at A, and $3200 - 2600 \times (0.4)^n$ birds at B.

e $\mathbf{P}_\infty = \begin{pmatrix}1600\\3200\end{pmatrix}$

(1 mark)

f The assumption that birds move only between two locations may not be realistic; no deaths or births are considered.

(2 marks)

7 **a** $\frac{d}{dt}$

$\left(\ln\frac{B+R}{B-R} + 2\arctan\frac{B}{R} - 4R^3kt\right)$

$= \frac{d}{dt}(c)$

(1 mark)

$$\frac{\frac{d}{dt}\left(\frac{B+R}{B-R}\right)}{\frac{B+R}{B-R}} + 2\frac{\frac{d}{dt}\left(\frac{B}{R}\right)}{+1\left(\frac{B}{R}\right)}$$

$-4R^3K = 0$

(2 marks)

$$\frac{-2R\frac{dB}{dt}}{B^2-R^2} + \frac{2R\frac{dB}{dt}}{B^2-R^2} = 4R^3k$$

(2 marks)

$\frac{dB}{dt} = -k(B^4 - R^4)$

(1 mark)

b **i** $B(0.5) = 22.7°$ C

(1 mark)

Note: 1 mark for attempt to Euler method

t	B	$\frac{dB}{dt}$
0	35	-67.0313
0.1	28.29688	-24.0571
0.2	25.89117	-14.4686
0.3	24.4443	-9.85175
0.4	23.45913	-7.1432
0.5	**22.74481**	

ii $\ln\frac{B+20}{B-20} + 2\arctan$

$\frac{B}{20} - 1.6t = c$ and (t, B)

$= (0, 35)$

(1 mark)

gives $c = 3.40258$

(1 mark)

So $\ln\frac{B+20}{B-20} + 2\arctan$

$\frac{B}{20} - 1.6t = 3.40$

(1 mark)

Paper 3

Exam-style questions

1 **a** **i** 7 edges

(1 mark)

vertex	a	b	c	d	e	f	g	h
degree	1	1	3	2	4	1	1	1

(1 mark)

ii 8 edges

(1 mark)

vertex	a	b	c	d	e	f	g
degree	3	1	5	1	4	1	1

(1 mark)

iii 7 edges

(1 mark)

vertex	a	b	c	d	e	f
degree	2	3	2	2	3	2

(1 mark)

iv $\begin{pmatrix}n\\2\end{pmatrix} = \frac{n(n-1)}{2}$ edges

all of degree $n - 1$

(2 marks)

b **i** (a)(i) $14 = 2 \times 7$

(a)(ii) $16 = 2 \times 8$

(a)(iii) $14 = 2 \times 7$

(a)(iv) $n(n-1) = 2 \times \frac{n(n-1)}{2}$

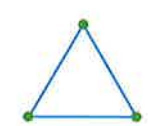
So true in all cases.

(3 marks)

ii Each edge is incident with 2 vertices, so each edge contributes 2 to the sum of the degrees.

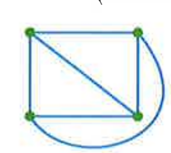
(2 marks)

c **i**



(1 mark)

ii



(2 marks)

iii K_5 is simple and connected

(1 mark)

$v = 5 \quad e = \frac{5 \times 4}{2} = 10$

(2 marks)

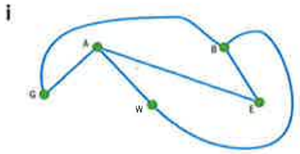
Suppose K_5 is planar $\Rightarrow$

$10 \leq 3 \times 5 - 6 = 9$, which is a contradiction

So K_5 is not planar.

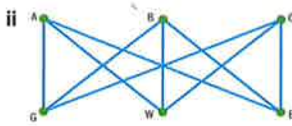
(2 marks)

d **i**



(2 marks)

ii



(2 marks)

iii $v = 6 \quad e = 9$

(2 marks)

iv The graph is simple, connected and does not contain K^3 as a subgraph.

(1 mark)

Suppose it is planar $\Rightarrow 9 \leq 2 \times 6 - 4 = 8$, which is a contradiction.

So the graph cannot be planar.

(2 marks)

2 **a** **i** $\mu = np = 4$

(1 mark)

ii

x	≤1	2	3	4	5	6	≥7
Expected frequency	8.05	14.2	20.0	20.6	16.5	10.7	9.95

(3 marks)

iii 6 degrees of freedom

$p = 0.986 > 0.1$

(3 marks)

so we accept H_0

(1 mark)

b **i**

x	≤1	2	3	4	5	6	≥7
Expected frequency	9.16	14.7	19.5	19.5	15.6	10.4	11.1

(3 marks)

ii 6 degrees of freedom

$p = 0.951 > 0.1$

(3 marks)

so we accept H_0

(1 mark)

c For n large and p small, the binomial distribution can be approximated by the Poisson distribution with the same mean.

(1 mark)

d **i** $\mu = np = 20$

$\sigma^2 = npq = 10$

(1 mark)

ii $P(Y = 22, 23 \text{ or } 24)$

$= 0.241$

(1 mark)

iii

y	$y < 15.5$	$15.5 < y < 18.5$	$18.5 < y < 21.5$	$21.5 < y < 24.5$	$24.5 < y$
Expected frequency	7.69	24.1	36.4	24.1	7.69

(1 mark)

iv 4 degrees of freedom

$p = 0.922 > 0.1$

(3 marks)

so we accept H_0

(1 mark)

e **i** $P(21.5 < Y < 24.5) = 0.240$

(1 mark)

ii

y	$y < 15.5$	$15.5 < y < 18.5$	$18.5 < y < 21.5$	$21.5 < y < 24.5$	$24.5 < y$
Expected frequency	7.74	24.1	36.5	24.1	7.74

(1 mark)

iii 4 degrees of freedom

$p = 0.916 > 0.1$

(3 marks)

so we accept H_0

(1 mark)

f For n large and p not close to 1 or 0, the binomial distribution can be approximated by the normal distribution with the same mean and variance, even though the binomial is a discrete distribution and the normal is a continuous distribution.

(1 mark)