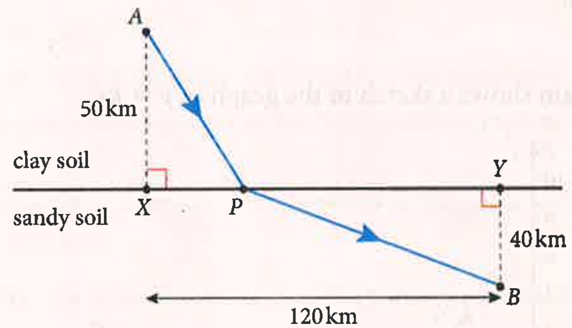
16. A company wants to find the minimum cost of building an oil pipeline from point A to point B across two types of soil. The first type of soil is made of clay and the cost of building the pipeline through this type of soil is \$6 million per km. The second type of soil is sandy and the cost of building the pipeline is \$2 million per km. The boundary XY between the two types of soil is a straight line running from east to west and point A lies 50 km directly north of it. Point B is 120 km to the east of point A and 40 km directly south of the boundary. This is shown in the diagram along with a possible route APB, where P is x km east of X.



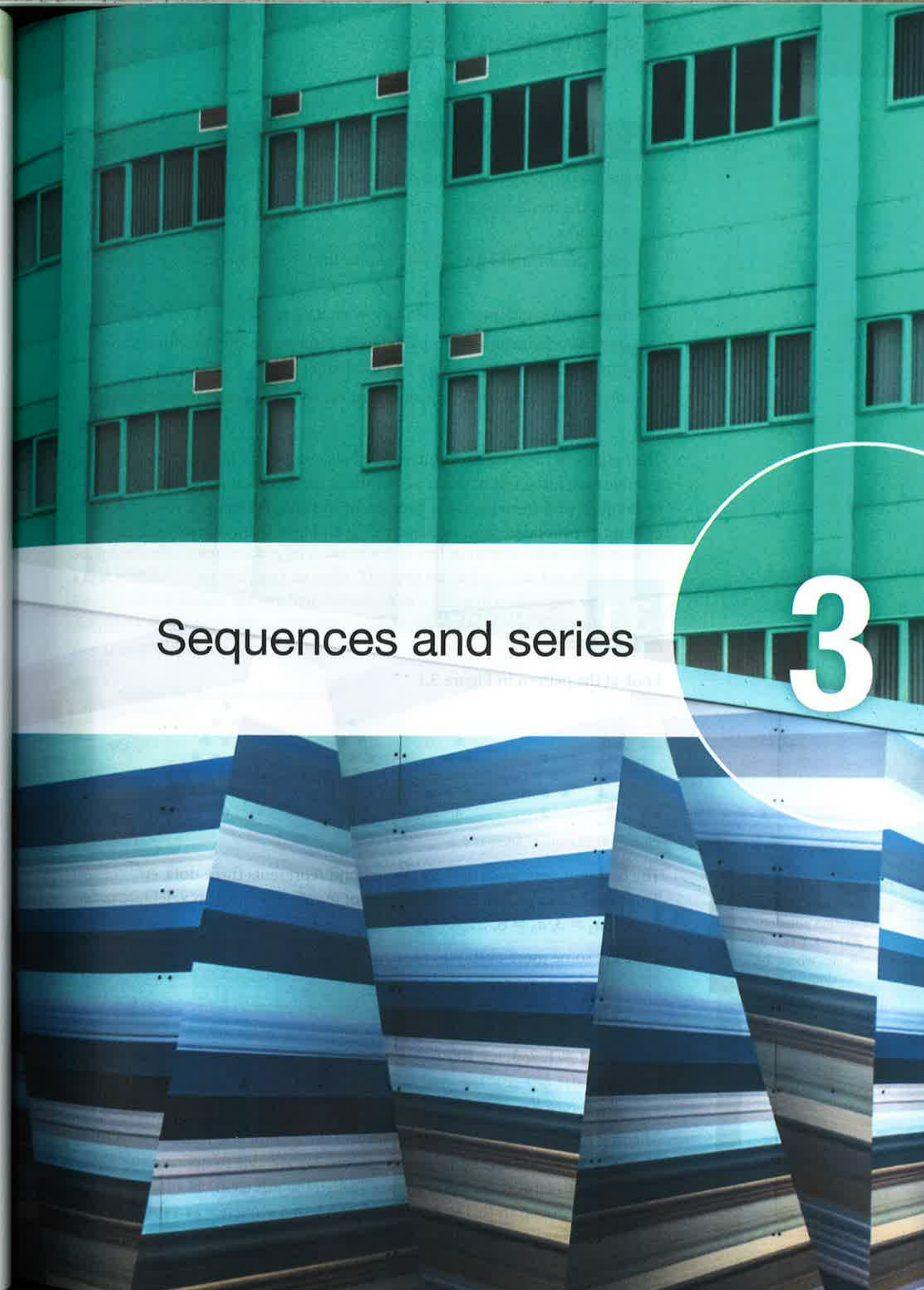
(a) Find a function to model the cost of the pipeline in terms of x.

(b) Find the minimum cost of building the pipeline.

17. Let $f(x) = \ln x$. The graph of f is transformed into the graph of the function g by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, followed by a reflection in the x-axis. Find an expression for $g(x)$, giving your answer as a single logarithm.

18. The function f is defined by $f(x) = 2x^2 - 3x + 1$

The graph of $y = f(x)$ is transformed by a reflection in the x-axis, followed by a translation of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, followed by a horizontal stretch with scale factor 2. Find the equation of the transformed graph. Give your answer in the form $f(x) = ax^2 + bx + c$



3 Sequences and series

Learning objectives

By the end of this chapter, you should be familiar with...

- using the formulae for the nth term for arithmetic and geometric sequences
- using the formulae for the sum of n terms for arithmetic and geometric series
- using sigma notation for the sums of sequences
- applications, including simple interest, compound interest, population growth, annual depreciation and modelling
- solving problems involving amortisation and annuities using technology.

The heights of consecutive bounces of a ball, compound interest, and Fibonacci numbers are just a few of the applications of sequences and series that you may be familiar with from previous courses. In this chapter you will review these concepts, consolidate your understanding and take them one step further.

3.1 Sequences

Look at the pattern in Figure 3.1

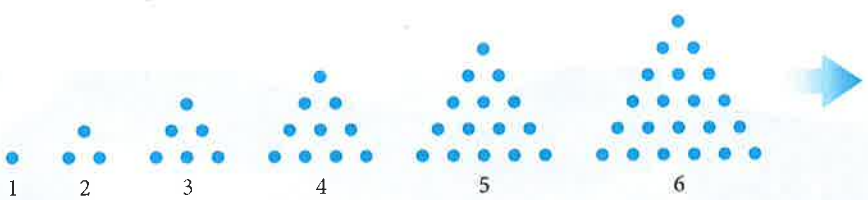


Figure 3.1 Triangular pattern of dots

The first unit represents one dot, the second represents three dots, etc. This pattern can be represented as a list of numbers written in a definite order: $a_1 = 1, a_2 = 3, a_3 = 6, \dots$

The pattern can also be described, for example, in function notation: $f(1) = 1, f(2) = 3, f(3) = 6$, etc., where the domain is $\mathbb{Z}^+$

Here are some more examples of sequences:

16, 12, 18, 24, 30

$3, 9, 27, \dots, 3^k, \dots$

$\left\{\frac{1}{i^2}; i = 1, 2, 3, \dots, 10\right\}$

$\{b_1, b_2, \dots, b_n, \dots\}$, sometimes used with an abbreviation (b_n)

The first and third sequences are **finite** and the second and fourth are **infinite**. Note that in the second and third sequences, we are able to define a rule that yields the nth number in the sequence (called the nth term) as a function of n, the term's number. In this sense you can think of a sequence as a **function** that assigns a **unique** number a_n to each positive integer n.

Example 3.1

Find the first five terms and the 50th term of the sequence (u_n) given that $u_n = 2 - \frac{1}{n^2}$

Solution

Since we know an explicit expression for the nth term as a function of its number n, we only need to find the value of that function for the required terms:

$$u_1 = 2 - \frac{1}{1^2} = 1;\ u_2 = 2 - \frac{1}{2^2} = \frac{7}{4};\ u_3 = 2 - \frac{1}{3^2} = \frac{17}{9};$$

$$u_4 = 2 - \frac{1}{4^2} = \frac{31}{16};\ u_5 = 2 - \frac{1}{5^2} = \frac{49}{25};$$

$$\text{and } u_{50} = 2 - \frac{1}{50^2} = \frac{4999}{2500}$$

So, informally, a sequence is an ordered set of real numbers. That is, there is a first number, a second, and so forth. The way we defined the function in Example 3.1 is called the **explicit** definition of a sequence. Another way of defining a sequence is the **recursive** (or **inductive**) definition. Example 3.2 shows how this is used.

Notation for the terms of a sequence
u_1 = first term
u_2 = second term
.
.
.
u_n = nth term

Example 3.2

Find the first five terms and the 20th term of the sequence (b_n) given that $b_n = 2(b_{n-1} + 3)$ and $b_1 = 5$

Solution

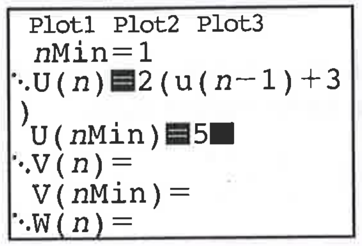
The defining formula for this sequence is recursive. It allows us to find the nth term b_n if we know the preceding term b_{n-1}. So, we can find the second term from the first, the third from the second, and so on. Since we know the first term $b_1 = 5$, we can calculate the the 2nd, 3rd, 4th and 5th terms:

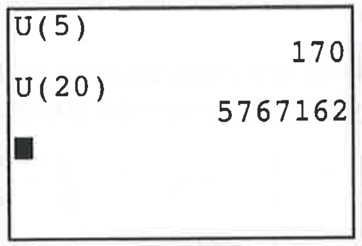
$$b_2 = 2(b_1 + 3) = 2(5 + 3) = 16$$
$$b_3 = 2(b_2 + 3) = 2(16 + 3) = 38$$
$$b_4 = 2(b_3 + 3) = 2(38 + 3) = 82$$
$$b_5 = 2(b_4 + 3) = 2(82 + 3) = 170$$

Recursive sequences are introduced in this chapter to help clarify the underlying concepts. However, note that they do not appear explicitly in this section of the syllabus. One of the main applications of recursive sequences in this course is to solve differential equations using Euler's method, which you will study in chapter 20.

Be aware that not all sequences have formulae, either recursive or explicit. Some sequences are given only by listing their terms. Among the many kinds of sequences that there are, two types are of particular interest to us: arithmetic and geometric sequences, which we will discuss in the next two sections.

So, the first five terms of this sequence are 5, 16, 38, 82, and 170. Although, to find the 20th term, we must first find all 19 preceding terms. This is one of the drawbacks of this type of definition, unless we can change the definition into explicit form. However, we can find a term in a recursive sequence easily using a GDC, as shown below.





Exercise 3.1

1. Find the first five terms of each infinite sequence.

(a) $a_n = 2n - 3$ (b) $b_n = 2^n - 3$

(c) $c_n = 3(2)^{-n}$ (d) $\begin{cases} d_1 = 5 \\ d_n = d_{n-1} + 3, & \text{for } n > 1 \end{cases}$

(e) $e_n = (-1)^n(2^n) + 3$ (f) $\begin{cases} f_1 = 3 \\ f_n = f_{n-1} + 2n, & \text{for } n > 1 \end{cases}$

2. Find the first five terms and the 50th term of each infinite sequence.

(a) $a_n = 2 - 5n$ (b) $b_n = 2 \times 3^{n-1}$

(c) $u_n = (-1)^{n-1}\dfrac{2n}{n^2+2}$ (d) $a_n = n^{n-1}$

(e) $a_n = 2a_{n-1} + 5$ and $a_1 = 3$ (f) $u_{n+1} = \dfrac{3}{2u_n + 1}$ and $u_1 = 0$

(g) $b_n = 3b_{n-1}$ and $b_1 = 2$ (h) $a_n = a_{n-1} + 2$ and $a_1 = -1$

3. Find a recursive definition for each sequence.

(a) $\dfrac{1}{3}, \dfrac{1}{12}, \dfrac{1}{48}, \dfrac{1}{192}, \ldots$

(b) $\dfrac{a}{2}, \dfrac{2a^3}{3}, \dfrac{8a^5}{9}, \dfrac{32a^7}{27}, \ldots$

(c) $a - 5k, 2a - 4k, 3a - 3k, 4a - 2k, 5a - k, \ldots$

4. Find an explicit formula that gives the nth term of each sequence.

(a) 4, 7, 12, 19, ... (b) 2, 5, 8, 11, ...

(c) $1, \dfrac{3}{4}, \dfrac{5}{9}, \dfrac{7}{16}, \dfrac{9}{25}, \ldots$ (d) $\dfrac{1}{4}, \dfrac{3}{5}, \dfrac{5}{6}, 1, \dfrac{9}{8}, \ldots$

5. The Fibonacci sequence F_n is defined by

$$F_1 = 1, F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2}, n > 2$$

A second sequence a_n is defined by

$$a_n = \frac{F_{n+1}}{F_n}, n \geqslant 1,$$

where F_n is a member of the Fibonacci sequence.

(a) Use a spreadsheet to find the first 30 terms of both F_n and a_n

(b) Hence, write down the limit of a_n as n tends to infinity, $\lim\limits_{n\to\infty} a_n$

6. The sequence G_n is defined by $G_1 = G_2 = 1$ and $G_n = 2G_{n-1} + G_{n-2}, n > 2$

A second sequence b_n is defined by $b_n = \dfrac{G_{n+1}}{G_n}, n \geqslant 1$

(a) Use a spreadsheet to find the first 30 terms of both G_n and b_n

(b) Hence, write down the limit of b_n as n tends to infinity, $\lim\limits_{n\to\infty} b_n$

3.2 Arithmetic sequences

Here are three sequences and the most likely recursive formula for each one.

7, 14, 21, 28, 35, 42, ...	$a_1 = 7$ and $a_n = a_{n-1} + 7$, for $n > 1$
2, 11, 20, 29, 38, 47, ...	$a_1 = 2$ and $a_n = a_{n-1} + 9$, for $n > 1$
48, 39, 30, 21, 12, 3, −6, ...	$a_1 = 48$ and $a_n = a_{n-1} - 9$, for $n > 1$

Note that in each case, each term is formed by adding a constant number (which can be negative) to the preceding term. Sequences formed in this manner are called **arithmetic sequences**.

A sequence $a_1, a_2, a_3, \ldots$ is an **arithmetic sequence** if there is a constant d for which

$$a_n = a_{n-1} + d$$

for all integers $n > 1$, where d is the **common difference** of the sequence, and $d = a_n - a_{n-1}$ for all integers $n > 1$

So, for the sequences above, 7 is the common difference for the first, 9 is the common difference for the second, and −9 is the difference for the third.

This description gives us the recursive definition of the arithmetic sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly helps us to see the expression:

$a_2 = a_1 + d$

$a_3 = a_2 + d = a_1 + d + d = a_1 + 2d$

$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d$

and so on.

So you can get to the nth term by adding d to a_1, $(n - 1)$ times.

The **nth term** of an arithmetic sequence, a_n, with first term a_1 and common difference d, may be expressed explicitly as

$$a_n = a_1 + (n - 1)d$$

This result is useful for finding any term of the sequence without knowing all the previous terms.

The arithmetic sequence can be looked at as a linear function as explained in the introduction to this chapter. i.e., for every increase of one unit in n, the value of the sequence will increase by d units. As the first term is a_1, the point $(1, a_1)$ belongs to this function. The constant increase d can be considered to be the gradient (slope) of this linear model, hence the nth term, the dependent variable in this case, can be found by using the point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

$$a_n - a_1 = d(n - 1) \Leftrightarrow a_n = a_1 + (n - 1)d$$

Example 3.3

For the sequence 2, 11, 20, 29, 38, 47, …, find:

(a) a formula for the nth term

(b) the 50th term.

Solution

(a) This is an arithmetic sequence whose first term is 2 and common difference is 9. Therefore,

$$a_n = a_1 + (n - 1)d = 2 + 9(n - 1) = 9n - 7$$

(b) $a_{50} = 9(50) - 7 = 443$

Example 3.4

(a) Find the recursive and the explicit forms of the sequence 13, 8, 3, −2, …

(b) Calculate the value of the 25th term.

Solution

(a) This is clearly an arithmetic sequence, since −5 is the common difference.

The recursive definition is $a_1 = 13$

$a_n = a_{n-1} - 5, n > 1$

The explicit definition is $a_n = 13 - 5(n - 1) = 18 - 5n$

(b) $a_{25} = 18 - 5(25) = -107$

Example 3.5

Find the 20th term of the arithmetic sequence with first term 5 and fifth term 11.

Solution

Since the fifth term is given, using the explicit general form:

$$a_5 = a_1 + (5 - 1)d$$

$$\Rightarrow 11 = 5 + 4d$$

$$\Rightarrow d = \frac{3}{2}$$

This leads to the general term

$$a_n = 5 + \frac{3}{2}(n - 1)$$

Therefore, $a_{20} = 5 + \frac{3}{2}(19) = \frac{67}{2}$

Sometimes a real-life situation can be modelled by an arithmetic sequence, even though it does not follow the sequence exactly. In such cases it is necessary to find an approximation for the common difference. One approach is to use an average of all the differences as an estimate for the common difference. An alternative approach is to use linear regression, which you will meet in Chapter 19. We can use a spreadsheet to calculate differences quickly and then average them.

Example 3.6

An experiment was undertaken to investigate the relationship between the length of a spring and the mass hanging from it, as shown in the diagram.

The table shows the extension of the spring (cm) for each mass (g).

Mass (g)	10	20	30	40	50	60	70	80
Extension (cm)	2.4	4.8	7.0	9.5	11.8	14.2	16.4	18.9

It is believed that the data can be modelled by an arithmetic sequence, according to Hooke's law.

(a) Find an estimate of the common difference, by using an average of the differences.

(b) Find a model for the nth term u_n, where $n = \frac{\text{mass}}{10}$

(c) Use your model to predict the extension for a mass of

(i) 100 g (ii) 150 g

(d) In fact, further experiments found the extension at 150 g to be 30.9 cm. With reference to the spring, give a reason why the model does not give a good prediction in this case.

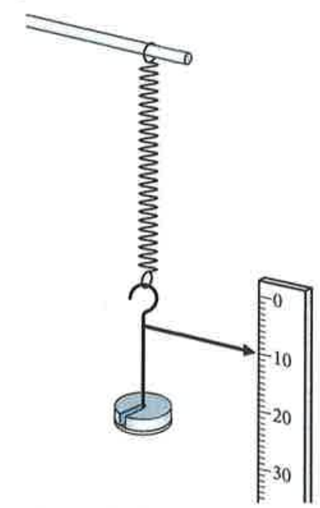


Figure 3.2 Diagram for Example 3.6

Solution

(a) Record the differences in an extra row in the table. We can do this quickly and efficiently using a GDC or spreadsheet.

n	1	2	3	4	5	6	7	8
Extension (cm)	2.4	4.8	7.0	9.5	11.8	14.2	16.4	18.9
Difference (cm)	–	2.4	2.2	2.5	2.3	2.4	2.2	2.5

The average difference, $d = \dfrac{2.4 + 2.2 + 2.5 + 2.3 + 2.4 + 2.2 + 2.5}{7}$

$= 2.36$

(b) Since $u_1 = 2.4$, applying the formula gives $u_n = 2.4 + 2.36(n - 1)$
$= 2.36n + 0.04$

(c) (i) For 100 g, $u_{10} = 2.36 \times 10 + 0.04 = 23.6$ cm

(ii) For 150 g, $u_{15} = 2.36 \times 15 + 0.04 = 35.4$ cm

(d) It is possible that this size mass has caused the spring to become almost fully extended, hence the model is no longer appropriate and cannot be extrapolated to this size mass.

Simple interest

When we invest money in an account, we usually receive interest. When we borrow money then we usually pay interest, for example taking out a loan to buy a car, or a mortgage to buy a house, or using a credit card and not paying the balance off in full each month. Simple interest is interest calculated only on the initial investment. Suppose $2000 is invested in an account paying simple interest at a rate of 5% per year. How much money will there be in the account at the end of 4 years?

Table 3.1 shows how the amount can be calculated for each year.

Time (years)	Amount in the account ($)
0	2000
1	2000 + 2000 × 0.05
2	2000 + 2000 × 0.05 + 2000 × 0.05 = 2000 + 2000 × 0.05 × 2
3	2000 + 2000 × 0.05 × 2 + 2000 × 0.05 = 2000 + 2000 × 0.05 × 3
4	2000 + 2000 × 0.05 × 3 + 2000 × 0.05 = **2000 + 2000 × 0.05 × 4**

Table 3.1 Simple interest calculations

This appears to be an arithmetic sequence with five terms (as both the beginning and the end of the first year are counted).

In general, if a **principal** of $\$P$ is invested in an account with a simple interest rate r (expressed as a decimal) annually, then we can use the arithmetic sequence formula to calculate the **future value** A, which is accumulated after t years.

If we repeat the steps above using the general terms, it becomes easier to develop the formula:

Time (years)	Amount in the account ($)
0	$A_0 = P$
1	$A_1 = P + Pr$
2	$A_2 = A_1 + Pr = P + 2Pr$
⋮	
t	$A_t = P + Prt$

Table 3.2 Developing a formula

Note that since we are counting from 0 to t, we have $t + 1$ terms, and hence using the arithmetic sequence formula,

$$a_n = a_1 + (n - 1)d \Rightarrow A_t = A_0 + (t)A_0 r = P + Prt$$

Example 3.7

$3500 is invested in an account paying simple interest at a rate of 4.2% per year. Interest is added at the end of each year.

(a) Calculate the amount of money in the account after 6 years.

(b) Find the number of years it would take for the amount of money in the account to exceed $6000. (Assume no further money is invested or withdrawn from the account.)

Solution

(a) The interest rate is 4.2%, so $r = 0.042$ and $P = 3500$. After 6 years:

Amount $= 3500 + 3500 \times 0.042 \times 6 = \$4382$

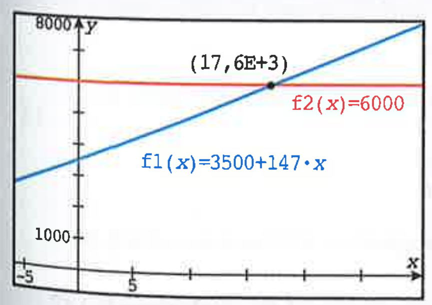
(b) After n years, an expression for the amount in the account is:

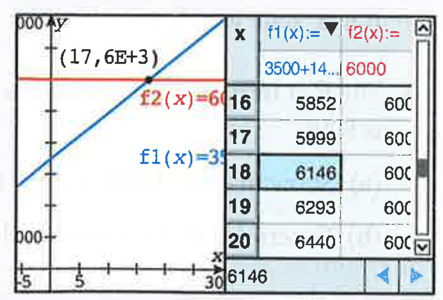
Amount $= 3500 + 3500 \times 0.042 \times n = 3500 + 147n$

We need to solve the inequality:

$$3500 + 147n > 6000$$

This can be solved using a graph or table on the GDC, or algebraically.





We see that $n > 17.6$, so the number of years is 18.

Exercise 3.2

1. State whether each sequence is an arithmetic sequence. If yes, find the common difference and the 50th term. If not, state why not.

   (a) $a_n = 2n - 3$ (b) $b_n = n + 2$

   (c) $c_n = c_{n-1} + 2$, and $c_1 = -1$ (d) 2, 5, 7, 12, 19, …

   (e) 2, −5, −12, −19, …

2. For each arithmetic sequence in parts (a) to (d) find:

   (i) the 8th term

   (ii) an explicit formula for the nth term

   (iii) a recursive formula for the nth term.

   (a) −2, 2, 6, 10, … (b) 10.07, 9.95, 9.83, 9.71, …

   (c) 100, 97, 94, 91, … (d) $2, \frac{3}{4}, -\frac{1}{2}, -\frac{7}{4}, \ldots$

3. In an arithmetic sequence, $a_5 = 6$ and $a_{14} = 42$

   Find an explicit formula for the nth term of this sequence.

4. In an arithmetic sequence, $a_3 = -40$ and $a_9 = -18$

   Find an explicit formula for the nth term of this sequence.

5. For each finite sequence, the first 3 terms and the last term are given.

   Find the number of terms in each sequence.

   (a) 3, 9, 15, …, 525 (b) 9, 3, −3, …, −201

   (c) $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots, 2\frac{5}{6}$ (d) $1 - k, 1 + k, 1 + 3k, \ldots, 1 + 19k$

6. The 30th term of an arithmetic sequence is 147 and the common difference is 4. Find a formula for the nth term.

7. The first term of an arithmetic sequence is −7 and the common difference is 3. Determine if 9803 is a term in this sequence. If it is, state which one.

8. The first term of an arithmetic sequence is 9689 and the 100th term is 8996.

   (a) Show that the 110th term is 8926.

   (b) Determine if 1 is a term in this sequence. If it is, state which term.

9. The first term of an arithmetic sequence is 2 and the 30th term is 147. Determine if 995 is a term in this sequence. If it is, state which term.

10. An experiment was undertaken to investigate the relationship between the length of a spring and the mass hanging from it. The table shows the extension of the spring (cm) for each mass (g).

Mass (g)	10	20	30	40	50	60	70	80
Extension (cm)	4.4	8.7	13.2	17.4	21.7	26.1	30.3	34.7

It is believed that the data can be modelled by an arithmetic sequence, according to Hooke's law.

   (a) Find an estimate of the common difference, by using an average of the differences.

   (b) Find a model for the nth term u_n, where $n = \frac{\text{mass}}{10}$

   (c) (i) Use your model to estimate the extension for a mass of 70 g.

   (ii) Calculate the percentage error in the estimate found in part (i).

11. Marcus is planning to eat a tub of ice cream, one spoonful at a time. He believes that the mass of ice cream remaining can be modelled by an arithmetic sequence. He puts the tub on a set of measuring scales and collects the following data while he is eating.

Number of spoonfuls	1	2	3	4	5	6
Mass remaining (g)	280	255	235	200	175	145

   (a) Find an estimate of the common difference, by using an average of the differences.

   (b) Find a model for the nth term u_n

   (c) (i) Use your model to estimate the original mass of ice cream in the tub.

   (ii) Giving a reason to support your answer, explain if you think this is an overestimate or an underestimate.

   (d) (i) Use your model to estimate the number of spoonfuls required to eat all of the ice cream in the tub.

   (ii) Giving a reason to support your answer, explain if you think this is an overestimate or an underestimate.

12. \$500 is invested in an account paying simple interest at a rate of 3.2% per year. Interest is paid at the end of each year.

   (a) Calculate the amount of money in the account after 5 years.

   (b) Find the number of years it would take for the amount of money in the account to exceed \$2000. (Assume no further money is invested or withdrawn from the account.)

The value of an asset, such as a car, decreases over time due to wear and tear. This reduction in value of an asset over time is called depreciation.

13. The owner of a small company buys a company car for one of his employees. The purchase price of the car is \$16,500. For tax purposes the company can depreciate the value of the car by 10% of its purchase price each year.

(a) Calculate the value of the car to the company after 4 years.

(b) After the initial 4 years, the employee has the option to buy the car from his company by paying 50% of the purchase price. Determine the cost to the company if the employee buys the car.

14. An investment of \$450 is worth \$560 after 7 years of earning an unknown annual interest rate under the simple interest model. Find the interest rate to the nearest tenth of a percent.

15. Nanako takes out a loan of \$16,000 to buy a car. The loan has a simple interest rate of 8% per year. She pays back the loan at a fixed rate of \$250 per month. Find how long it will take for Nanako to repay the loan.

3.3 Geometric sequences

In this section we will consider sequences of the type:

1, 2, 4, 8, 16, ...

4, 12, 36, 108, 324, ...

600, 300, 150, 75, 37.5, ...

3, −12, 48, −192, 768, ...

In this type of sequence, each term is obtained by multiplying the previous term by a fixed constant. For example, in the second sequence, each term is 3 times the previous term. We can write this sequence recursively as

$$u_n = 3u_{n-1} \text{ and } u_1 = 4$$

Sequences of this type are called **geometric sequences.** Note that the ratio of consecutive terms equals a fixed constant. For example in the second sequence, we get

$$\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = \frac{324}{108} = 3$$

The ratio of consecutive terms is a constant equal to 3. This is called the **common ratio** of the geometric sequence.

A sequence $u_1, u_2, u_3, \ldots$ is a **geometric sequence** if there is a constant r for which $u_n = u_{n-1} \times r$ for all integers $n > 1$. The constant, r, is called the **common ratio** of the sequence, and $r = \frac{u_n}{u_{n-1}}$ for all integers $n > 1$.

Example 3.8

For each sequence, state the common ratio and write a recursive formula.

(a) 1, 2, 4, 8, 16, ...

(b) 600, 300, 150, 75, 37.5, ...

(c) 3, −12, 48, −192, 768, ...

Solution

(a) The common ratio is 2 and a recursive formula is $u_n = 2u_{n-1}$, $u_1 = 1$

(b) The common ratio is $\frac{1}{2}$ and a recursive formula is $u_n = \frac{1}{2}u_{n-1}$, $u_1 = 600$

(c) The common ratio is −4 and a recursive formula is $u_n = -4u_{n-1}$, $u_1 = 3$

In practice, the recursive formula is not useful to obtain the terms of a sequence. For example, finding the 100th term in the sequences in Example 3.8 would be laborious, using the recursive formula. If we start with first term u_1 and common ratio r, then we can apply the recursive formula repeatedly, to find an explicit formula for the nth term.

$$u_2 = u_1 \times r$$

$$u_3 = u_2 \times r = u_1 \times r \times r = u_1 \times r^2$$

$$u_4 = u_3 \times r = u_1 \times r^2 \times r = u_1 \times r^3$$

and so on.

So, as you see, you can get to the nth term by multiplying u_1 by r, $(n - 1)$ times.

The general (nth) term of a geometric sequence, u_n with common ratio r and first term u_1 may be expressed explicitly as

$$u_n = u_1 r^{n-1}$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Example 3.9

(a) Find the 12th term in the sequence 3, −12, 48, −192, 768, ...

(b) The third term of a geometric sequence is 5 and the seventh term is 405. Find the two possible values of the tenth term.

Solution

(a) $u_1 = 3$ and $r = -4$, so $u_{12} = 3(-4)^{11} = -12\,582\,912$. Note that when the common ratio is negative, the sign of the terms in the sequence alternates.

(b) $u_3 = 5 \Rightarrow u_1 r^2 = 5$

$u_7 = 405 \Rightarrow u_1 r^6 = 405$

Dividing the second equation by the first gives

$$\frac{u_1 r^6}{u_1 r^2} = \frac{405}{5}$$

Cancelling common factors of u_1, r^2, and 5 gives

$$r^4 = 81 \Rightarrow r = \pm\sqrt[4]{81} \Rightarrow r = \pm 3$$

We can solve for u_1 using the first equation

$$u_1 = \frac{5}{r^2} = \frac{5}{9}$$

So,

$$u_{10} = u_1 r^9 = \frac{5}{9}(\pm 3)^9 = \pm 10\,935$$

The two possible values of the tenth term are 10 935 and −10 935

Compound interest

Interest compounded annually

We have already looked at simple interest in section 3.2, which was calculated only on the initial investment. Compound interest is interest calculated both on the initial investment and also on interest already paid. Suppose $2500 is invested in an account paying compound interest at a rate of 3% per year. How much money will there be in the account at the end of 4 years? It is important to note that the 3% interest is paid annually and is added to the account, so that in the following year it will also earn interest, and so on. Table 3.3 shows how the amount can be calculated for each year.

Time (years)	Amount in the account ($)
0	2500
1	$2500 + 2500 \times 0.03 = 2500(1 + 0.03)$
2	$2500(1 + 0.03) + (2500(1 + 0.03)) \times 0.03 = 2500(1 + 0.03)(1 + 0.03) = 2500(1 + 0.03)^2$
3	$2500(1 + 0.03)^2 + (2500(1 + 0.03)^2) \times 0.03 = 2500(1 + 0.03)^2(1 + 0.03) = 2500(1 + 0.03)^3$
4	$2500(1 + 0.03)^3 + (2500(1 + 0.03)^3) \times 0.03 = 2500(1 + 0.03)^3(1 + 0.03) = 2500(1 + 0.03)^4$

Table 3.3 Compound interest

This appears to be a geometric sequence with 5 terms. Note that the number of terms is 5, as both the beginning and the end of the first year are counted.

In general, if a **principal** of $$P$ is invested in an account that has an interest rate r (expressed as decimal) annually, and this interest is added to the principal at the end of each year, then we can use the geometric sequence formula to calculate the **future value** A_t which is accumulated after t years.

If we repeat the steps above, it becomes easier to develop the formula:

Time (years)	Amount in the account ($)
0	$A_0 = P$
1	$A_1 = P + P \times r = P(1 + r)$
2	$A_2 = A_1(1 + r) = P(1 + r)^2$
⋮	⋮
t	$A_t = P(1 + r)^t$

Table 3.4 Developing a formula

Note that since we are counting from 0 to t, we have $t + 1$ terms, and hence using the geometric sequence formula,

$$u_n = u_1 r^{n-1} \Rightarrow A_t = A_0(1 + r)^t$$

Example 3.10

$1500 is invested in an account paying compound interest at a rate of 4.5% per year. Interest is added at the end of each year.

(a) Calculate the amount of money in the account after 6 years.

(b) Find the number of years it would take for the amount of money in the account to exceed $10,000. (Assume no further money is invested or withdrawn from the account.)

Solution

(a) Each year the amount in the account increases by 4.5%. Calculate the percentage increase by multiplying by 1.045 for each year. After 6 years,

$$\text{Amount} = 1500 \times 1.045^6 = \$1953.39$$

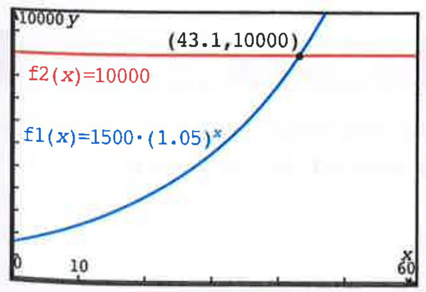
(b) After n years, an expression for the amount in the account is:

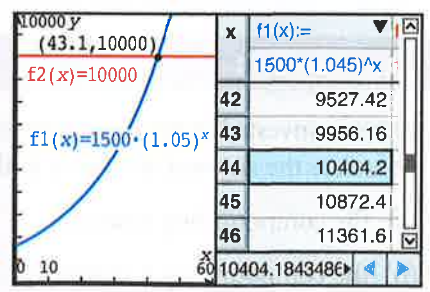
$$\text{Amount} = 1500 \times 1.045^n$$

We need to solve the inequality

$$1500 \times 1.045^n > 10\,000$$

This can be solved using a graph or table on a GDC, or algebraically using logarithms.





We see that $n > 43.1$, so the number of years is 44.

Example 3.10 can also be solved using the built-in financial package on a graphing calculator.

(a) Using the TVM solver, we make the following entries and solve for the future value, FV.

$N = 6$
$I\% = 4.5$
$PV = -1500$
$PMT = 0$
$FV = 1953.39$
$P/Y = 1$
$C/Y = 1$
PMT: END

The number of payments, N, is 6, because interest is compounded once per year for six years. The present value, PV, is negative, because the bank now has the money.

(b) Using the TVM solver, we make the following entries and solve to find the number of payments, N.

$N = 43.1$
$I\% = 4.5$
$PV = -1500$
$PMT = 0$
$FV = 10\,000$
$P/Y = 1$
$C/Y = 1$
PMT: END

A Time-Value-Money (TVM) solver is a built-in financial package on a GDC that allows a user to enter parameters of an investment and then calculate a missing value. The parameters of a TVM solver are as follows.

N = number of payments made in total
$I\%$ = annual interest rate ($\times$ 100)
PV = present value of payments
PMT = value of the regular payment
FV = future value of the payments
P/Y = number of payments per year
C/Y = number of compounding periods per year
PMT: END BEGIN payments made at beginning or end of year

Interest compounded n times per year

Suppose that the principal P is invested as before but the interest is paid n times per year. Then $\frac{r}{n}$ is the interest paid every compounding period. Since every year we have n periods, therefore for t years, we have nt periods. The amount A in the account after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 3.11

€1000 is invested in an account paying compound interest at a rate of 6%. Calculate the amount of money in the account after 10 years if

(a) the compounding is annual

(b) the compounding is quarterly

(c) the compounding is monthly.

Solution

(a) The amount after 10 years compounding annually is

$$A = 1000(1 + 0.06)^{10} = €1790.85$$

(b) The amount after 10 years compounding quarterly is

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{40} = €1814.02$$

(c) The amount after 10 years compounding monthly is

$$A = 1000\left(1 + \frac{0.06}{12}\right)^{120} = €1819.40$$

Example 3.11 can also be solved using the built-in financial package on a graphing calculator.

(a) Using the TVM solver, we make the following entries and solve to find the future value, FV.

$N = 10$
$I\% = 6$
$PV = -1000$
$PMT = 0$
$FV = 1790.85$
$P/Y = 1$
$C/Y = 1$
PMT: END

(b) Using the TVM solver, enter $N = 40$ and payments per year, $PpY = 4$

$N = 40$
$I\% = 6$
$PV = -1000$
$PMT = 0$
$FV = 1814.02$
$P/Y = 4$
$C/Y = 1$
PMT: END

(c) Using the TVM solver, enter $N = 120$ and payments per year, $PpY = 12$

$N = 120$
$I\% = 6$
$PV = -1000$
$PMT = 0$
$FV = 1819.40$
$P/Y = 2$
$C/Y = 1$
PMT: END

Real rate of return

In Examples 3.10 and 3.11, the interest rate used to calculate the future value of an investment was the **nominal rate**. The nominal rate of interest is the interest rate before taking other factors, such as inflation, into account. In practice, the rate of inflation can affect the future value of an investment. In order to account for the rate of inflation, the nominal rate is adjusted to take inflation into account. This adjusted interest rate is called the **real rate of return.** Suppose the interest rate is 6% per year and the inflation rate is 2% per year. Then the **real rate of return** is only 4% per year.

Inflation means an overall rise in the cost of goods and services. This cost increase means that inflation reduces the purchasing power of each unit of currency.

Example 3.12

\$2500 is invested in an account paying an annual interest rate of 5.6%. The interest is compounded half-yearly. The inflation rate during the time of the investment is 2%.

(a) Find the real rate of return per year.

(b) Calculate the real value of the investment after 6 years.

Solution

(a) The **nominal** rate of return per year is

$$\left(1 + \frac{0.056}{2}\right)^2 = 1.056784 \text{ or } 5.6784\%$$

The **real** rate of return per year is

$$5.6784 - 2 = 3.6784\%$$

(b) After 6 years the **real** value, A, of the investment is

$$A = 2500(1 + 0.036784)^6 = \$3105.06$$

Exponential growth and decay

In the section on compound interest, the expression for the amount after t years, $A_t = P(1 + r)^t$, is an example of exponential growth. There are several other applications of geometric sequences, linked to exponential growth or decay.

Example 3.13

In 2016, the population growth rate in Singapore was 1.3% and the size of the population was 5.6 million (source: World Bank). Assuming the population growth rate remains constant, find:

(a) the size of the population in 2021

(b) the number of years until the population exceeds 7 million.

Solution

(a) This can be modelled by a geometric sequence with first term 5.6 million and common ratio of 1.013

In 2021, population size $= 5.6 \times 1.013^5 = 5.97$ million

(b) We need to solve the inequality:

$$5.6 \times 1.013^t > 7$$

We can use a GDC to sketch a graph and/or create a table.

According to the model, the population will exceed 7 million after 17.3 years.

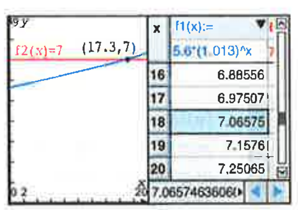


Figure 3.3 Using a GDC to solve the inequality in Example 3.13 (b)

Example 3.14

A company buys a machine for \$60,000. The expected lifetime of the machine is 6 years. At the end of 6 years, the company estimates it can sell the machine for \$8000. Calculate the annual rate of depreciation of the machine.

Solution

The value of the machine can be modelled by a geometric sequence with $u_1 = 60\,000$ and $u_7 = 8000$. Substituting into $u_n = u_1 r^{n-1}$ gives

$$8000 = 60\,000 r^6$$

$$\Rightarrow r = \sqrt[6]{\frac{8000}{60\,000}} = 0.715$$

The annual rate of depreciation is $1 - 0.715 = 0.285$ or 28.5%

Annual depreciation is the reduction in value of an asset at a particular rate per year.

Example 3.14 can also be solved using the built-in financial package on a graphing calculator. Using the TVM solver, we make the following entries and solve for the interest rate, I.

$N = 6$
$I\% = -28.5$
$PV = -60\,000$
$PMT = 0$
$FV = 8000$
$P/Y = 1$
$C/Y = 1$
PMT: END

Note that the negative interest rate corresponds to depreciation.

Exercise 3.3

In each of questions **1–15**:

(a) Determine whether the sequence is arithmetic, geometric, or neither.

(b) Find the common difference for the arithmetic sequences and the common ratio for the geometric sequences.

(c) Find the 10th term for each arithmetic or geometric sequence.

1. $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots$

2. $a_n = 3n - 3$

3. $b_n = 2^{n+2}$

4. $c_n = 2c_{n-1} - 2$, and $c_1 = -1$

5. $u_n = 3u_{n-1}$ and $u_1 = 4$

6. 2, 5, 12.5, 31.25, 78.125, …

7. 2, −5, 12.5, −31.25, 78.125, …

8. 2, 2.75, 3.5, 4.25, 5, …

9. $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$

10. 52, 55, 58, 61, …

11. −1, 3, −9, 27, −81, …

12. 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, …

13. 3, 6, 12, 18, 21, 27, …

14. 6, 14, 20, 28, 34, …

15. 2.4, 3.7, 5, 6.3, 7.6, …

For each arithmetic or geometric sequence in questions **16–32**, find:

(a) the 8th term

(b) an explicit formula for the nth term

(c) a recursive formula for the nth term.

16. $-3, 2, 7, 12, \ldots$

17. $19, 15, 11, 7, \ldots$

18. $-8, 3, 14, 25, \ldots$

19. $10.05, 9.95, 9.85, 9.75, \ldots$

20. $100, 99, 98, 97, \ldots$

21. $2, \frac{1}{2}, -1, -\frac{5}{2}, \ldots$

22. $3, 6, 12, 24, \ldots$

23. $4, 12, 36, 108, \ldots$

24. $5, -5, 5, -5, \ldots$

25. $3, -6, 12, -24, \ldots$

26. $972, -324, 108, -36, \ldots$

27. $-2, 3, -\frac{9}{2}, \frac{27}{4}, \ldots$

28. $35, 25, \frac{125}{7}, \frac{625}{49}, \ldots$

29. $-6, -3, -\frac{3}{2}, -\frac{3}{4}, \ldots$

30. $9.5, 19, 38, 76, \ldots$

31. $100, 95, 90.25, \ldots$

32. $2, \frac{3}{4}, \frac{9}{32}, \frac{27}{256}, \ldots$

33. A geometric sequence has a first term of 3 and a common ratio of 5. Find the eighth term.

34. A geometric sequence has a third term of 7 and a common ratio of $\frac{1}{3}$. Find the first term.

35. A geometric sequence has a third term of 8 and a sixth term of 27. Find the tenth term.

36. A geometric sequence has a fifth term of 6 and a seventh term of 8.64 Find the possible values of the second term.

37. A geometric sequence has a second term of 2, a seventh term of $\frac{243}{512}$ and an nth term of $\frac{19\,683}{131\,072}$. Find the value of n.

38. John buys a car for \$12,000. The value of the car depreciates by 12% each year. Find the value of the car after 11 years.

39. Sarah buys a car for \$14,900. The car depreciates by a fixed rate each year, and after 8 years it is worth \$2300. Find the annual rate of depreciation of the car.

40. In 2016, the UK government estimated the total greenhouse gas emissions in the UK at 483.0 million tonnes carbon dioxide equivalent. This was a decrease of 6.2% from the previous year.

(a) Assuming that the percentage decrease remains constant at 6.2% per year, estimate the UK total greenhouse gas emissions in 2030.

(b) Calculate the percentage decrease in UK total greenhouse gas emissions from 2016 to 2030.

41. Ashish invests \$2500 in an account that pays 6.2% compound interest per year. He does not invest any further amount or withdraw any money from the account.

(a) Find the amount in his account after 7 years.

(b) Find the number of years until the amount in his account exceeds \$5000.

42. At her daughter Jane's birth, Charlotte put \$500 into a savings account. The interest she earned was 4% compounded quarterly. How much money will Jane have on her 16th birthday?

43. Stephen invests \$4000 in an account that pays 4.7% interest compounded monthly. He does not invest any further amount or withdraw any money from the account.

(a) Find the amount in his account after 5 years.

(b) Find the number of months until the amount in his account doubles.

44. In 2016, the population growth rate in India was 1.2% and the population was 1.324 billion. The population growth rate in China was 0.5% and the population was 1.379 billion (source: World Bank). Assume the population growth rate in each country remains constant.

(a) Find the size of the population in India in 2020.

(b) Find the size of the population in China in 2020.

(c) Find when the population of India will exceed that of China.

45. A bank pays interest at a rate of 4% compounded annually. If the inflation rate is 1.7%, calculate the real rate of return per year.

46. A bank pays interest at a rate of 3.5% compounded monthly. If the inflation rate is 2.1%, calculate the real rate of return per year.

3.4 Series

The word 'series' in common language implies much the same thing as 'sequence'. But in mathematics when we talk of a series, we are referring in particular to the sum of terms in a sequence. For example, for a sequence of values a_n, the corresponding series is the sequence S_n with

$$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$$

If the terms are in an arithmetic sequence we call the sums an **arithmetic series**.

Sigma notation

Many of the series we consider in mathematics are **infinite** series. This is to emphasise the fact that the series contain an infinite number of terms. Any sum in the series S_k will be called a **partial sum** and is given by:

$$S_k = a_1 + a_2 + \dots + a_{k-1} + a_k$$

For convenience, this partial sum is written using the sigma notation:

$$S_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + \dots + a_{k-1} + a_k$$

Sigma notation is a concise and convenient way to represent long sums. Here, the symbol Σ is the Greek capital letter sigma. So this expression means the sum of all the terms a_i where i takes the values from 1 to k. We can also write $\sum_{i=m}^{n} a_i$ to mean the sum of the terms a_i where i takes the values from m to n. In such a sum, m is called the lower limit and n the upper limit.

Example 3.15

Write out what is meant by:

(a) $\sum_{i=1}^{5} i^4$ (b) $\sum_{r=3}^{7} 3^r$ (c) $\sum_{j=1}^{n} x_j p_j$

Solution

(a) $\sum_{i=1}^{5} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4$ (b) $\sum_{r=3}^{7} 3^r = 3^3 + 3^4 + 3^5 + 3^6 + 3^7$

(c) $\sum_{j=1}^{n} x_j p_j = x_1 p_1 + x_2 p_2 + \dots + x_n p_n$

Example 3.16

Evaluate $\sum_{n=0}^{5} 2^n$

Solution

$$\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$$

Example 3.17

Write the sum $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \dots + \frac{99}{100}$ in sigma notation.

Solution

The terms in the numerator and denominator are consecutive integers, so they take on the absolute value of $\frac{k}{k+1}$ or any equivalent form.

The signs of the terms alternate and there are 99 terms. To take care of the sign change, we use some power of (-1) that will start with a positive value. If we use $(-1)^k$, then the first term will be negative, hence we can use $(-1)^{k+1}$ instead. We can therefore write the sum as:

$$(-1)^{1+1}\left(\frac{1}{2}\right) + (-1)^{2+1}\left(\frac{2}{3}\right) + (-1)^{3+1}\left(\frac{3}{4}\right) + \dots + (-1)^{99+1}\left(\frac{99}{100}\right)$$

$$= \sum_{k=1}^{99} (-1)^{k+1}\left(\frac{k}{k+1}\right)$$

Properties of the sigma notation

There are a number of useful results that we can obtain using sigma notation.

- For example, suppose we have a sum of constant terms

$$\sum_{i=1}^{5} 2$$

What does this mean? If we write this out in full we get

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 5 \times 2 = 10$$

In general, if we sum a constant n times then we can write

$$\sum_{i=1}^{n} k = k + k + \dots + k = n \times k = nk$$

- Suppose we have the sum of a constant times i. What does this give us? For example,

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5(1 + 2 + 3 + 4 + 5)$$
$$= 75$$

However, this can also be interpreted as follows.

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5(1 + 2 + 3 + 4 + 5)$$
$$= 5\sum_{i=1}^{5} i$$

which implies that

$$\sum_{i=1}^{5} 5i = 5\sum_{i=1}^{5} i$$

In general, we can say

$$\sum_{i=1}^{n} ki = k \times 1 + k \times 2 + \dots + k \times n$$
$$= k \times (1 + 2 + \dots + n)$$
$$= k\sum_{i=1}^{n} i$$

- Suppose we need to consider the sum of two different functions, such as:

$$\sum_{k=1}^{n}(k^2 + k^3) = (1^2 + 1^3) + (2^2 + 2^3) + \ldots + (n^2 + n^3)$$
$$= (1^2 + 2^2 + \ldots + n^2) + (1^3 + 2^3 + \ldots + n^3)$$
$$= \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k^3$$

In general:

$$\sum_{k=1}^{n}(f(k) + g(k)) = \sum_{k=1}^{n}f(k) + \sum_{k=1}^{n}g(k)$$

Arithmetic series

In an arithmetic series, we add the terms of an arithmetic sequence. It is very helpful to be able to find an explicit expression for the partial sum of an arithmetic series.

For example, find the partial sum for the first 50 terms of the series:

$$3 + 8 + 13 + 18 + \ldots$$

We can express S_{50} in two different ways and then add them together:

$$S_{50} = 3 + 8 + 13 + \ldots + 248$$
$$S_{50} = 248 + 243 + 238 + \ldots + 3$$
$$2S_{50} = 251 + 251 + 251 + \ldots + 251$$

There are 50 terms in this sum, and hence

$$2S_{50} = 50 \times 251 \Rightarrow S_{50} = 6275$$

This reasoning can be extended to any arithmetic series in order to develop a formula for the nth partial sum S_n.

Let (a_n) be an arithmetic sequence with first term a_1 and a common difference d. We can construct the series in two ways: forwards, by adding d to a_1 repeatedly and backwards by subtracting d from a_n repeatedly. We get the following two expressions for the sum:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d),$$

and

$$S_n = = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d)$$

By adding, term by term vertically, we get:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d)$$
$$\searrow \quad \searrow \quad \searrow \quad \downarrow$$
$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n)$$

Since we have n terms, we can reduce the expression to $2S_n = n(a_1 + a_n)$

Dividing both sides by 2 gives $S_n = \frac{n}{2}(a_1 + a_n)$, which in turn can be rearranged to give an interesting perspective

of the sum, i.e., $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ is n times the average of the first and last terms.

If we substitute $a_1 + (n-1)d$ for a_n then we arrive at an alternative formula for the sum.

Online

See a geometric derivation for the sum of an arithmetic series formula.

The sum, S_n, of n terms of an arithmetic series with common difference d, first term a_1 and nth term a_n can be expressed explicitly as

$$S_n = \frac{n}{2}(a_1 + a_n) \text{ or } S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Example 3.18

Find the partial sum for the first 50 terms of the series $3 + 8 + 13 + 18 + \ldots$

Solution

Using the second formula for the sum we get:

$$S_{50} = \frac{50}{2}(2 \times 3 + 5(50 - 1)) = 25 \times 251 = 6275$$

Alternatively, to use the first formula we need to know the nth term.

So, $a_{50} = 3 + 49 \times 5 = 248$

$\Rightarrow S_{50} = 25(3 + 248) = 6275$

Geometric series

A geometric series is the sum of the terms in a geometric sequence. As is the case with arithmetic series, it is desirable to find a general expression for the nth partial sum of a geometric series.

For example, find the partial sum for the first 20 terms of the series

$$3 + 6 + 12 + 24 + \ldots$$

Express S_{20} in two different ways and subtract them:

$$S_{20} = 3 + 6 + 12 + \ldots + 1\,572\,864$$
$$2S_{20} = 6 + 12 + \ldots + 1\,572\,864 + 3\,145\,728$$
$$-S_{20} = 3 - 3\,145\,728$$
$$\Rightarrow S_{20} = 3\,145\,725$$

This reasoning can be extended to any geometric series in order to develop a formula for the nth partial sum S_n.

Let (a_n) be a geometric sequence with first term a_1 and a common ratio $r \neq 1$.

The sum, S_n, of n terms of a geometric series with common ratio $r \neq 1$ and first term a_1 can be expressed explicitly as

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

or

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

It is better to use the first version of the formula when $r < 1$ and the second version when $r > 1$. This avoids having too many negative numbers in calculations.

Online

An interactive derivation of the geometric sum formula.

We can construct the series in two ways as before, and using the definition of the geometric sequence, i.e., $a_n = a_{n-1} \times r$, we have:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$rS_n = ra_1 + ra_2 + ra_3 + \ldots + ra_{n-1} + ra_n$$

$$= a_2 + a_3 + \ldots + a_{n-1} + a_n + ra_n$$

Now, subtract the first and last expressions to get

$$S_n - rS_n = a_1 - ra_n \Rightarrow S_n(1 - r) = a_1 - ra_n \Rightarrow S_n = \frac{a_1 - ra_n}{1 - r}, r \neq 1$$

This expression however, requires that r, a_1, as well as a_n be known in order to find the sum. However, using the nth term expression developed earlier, we can simplify this formula for the sum:

$$S_n = \frac{a_1 - ra_n}{1 - r} = \frac{a_1 - ra_1r^{n-1}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}, r \neq 1$$

Example 3.19

Find the partial sum for the first 20 terms of the series 3 + 6 + 12 + 24 + …

Solution

$$S_{20} = \frac{3(1 - 2^{20})}{1 - 2} = \frac{3(1 - 1\,048\,576)}{-1} = 3\,145\,725$$

Example 3.20

In a geometric sequence the second term is 5 and the fifth term is 135. Find the sum of the first 12 terms in the sequence.

Solution

$$u_2 = 5 \Rightarrow u_1 r = 5$$

$$u_5 = 135 \Rightarrow u_1 r^4 = 135$$

Dividing the second equation by the first equation gives

$$\frac{u_1 r^4}{u_1 r} = \frac{135}{5} \Rightarrow r^3 = 27 \Rightarrow r = 3$$

Substituting r into the first equation gives

$$u_1 = \frac{5}{3}$$

Since $r > 1$, it is better to use $S_n = \dfrac{a_1(r^n - 1)}{r - 1}$

$$S_{12} = \frac{\frac{5}{3}(3^{12} - 1)}{3 - 1} = \frac{5}{6}(3^{12} - 1) = \frac{1\,328\,600}{3}$$

Example 3.21

In a geometric sequence the sum of the first three terms is 304 and the sum of the first six terms is 1330. Find the sum of the first eight terms.

Solution

$$S_3 = 304 \Rightarrow \frac{u_1(r^3 - 1)}{r - 1} = 304$$

$$S_6 = 1330 \Rightarrow \frac{u_1(r^6 - 1)}{r - 1} = 1330$$

Dividing the second equation by the first equation:

$$\frac{r^6 - 1}{r^3 - 1} = \frac{1330}{304} = \frac{35}{8} \Rightarrow 8(r^6 - 1) = 35(r^3 - 1)$$

Rearranging:

$$8r^6 - 35r^3 + 27 = 0$$

Solving on a GDC gives the roots of the polynomial as

```
polyRoots(8·x^6-35·x^3+27,x)
                      {1, 3/2}
```

$r = 1$ and $r = \frac{3}{2}$

So $r = \frac{3}{2}$, since $r \neq 1$

Substituting r into the first equation:

$$u_1 = \frac{304\left(\frac{3}{2} - 1\right)}{\left(\left(\frac{3}{2}\right)^3 - 1\right)} = 64$$

Finally,

$$S_8 = \frac{64\left(\left(\frac{3}{2}\right)^8 - 1\right)}{\frac{1}{2}} = 3152.5$$

We will now consider an application of geometric series to the compound interest model. It is often the case that an investor makes regular equal payments into an investment, all of which accumulate interest at an equal annual rate. Such an investment is called an **annuity**. We will consider the case in which equal payments are made annually, and interest is compounded annually at an equal rate at the end of the year. The general case in which payments and interest compounding periods do not coincide will be considered in the next section.

Annuities are equal regular payments that earn an annual interest rate.

Example 3.22

When an annuity is paid at the beginning of the period, it is called **annuity due**.

Christine invests in a retirement plan in which equal payments of \$1200 are made at the beginning of each year. Interest is earned on each payment at an annual rate of 2% per year, compounded annually.

(a) How much is the investment worth at the end of 30 years?

(b) How much interest has been earned on the investment in 30 years?

Solution

(a) Note that in 30 years, Christine has invested $30 \times \$1200 = \$36{,}000$. However, each of the payments earns compound interest at the rate of 2% per year. The investment made in the first year has a value of $1200(1.02)^{30}$ at the end of the investment period; the investment in the second year has a value of $1200(1.02)^{29}$ and so on. The last payment earns interest for 1 year and has the value 1200(1.02). As such, the value of the annuity is the sum of 30 terms of the geometric series with first term \$1200(1.02) and common ratio 1.02.

$$1200(1.02) + 1200(1.02)^2 + 1200(1.02)^3 + \dots + 1200(1.02)^{30}$$
$$= 1200(1.02)[1 + 1.02 + 1.02^2 + \dots + 1.02^{29}]$$
$$= 1200(1.02)\left[\frac{1.02^{30} - 1}{1.02 - 1}\right]$$
$$= \$49{,}655.33$$

(b) The value of the payments made is \$36,000, thus the interest earned is \$13,655.33

Example 3.22 (a) can also be solved using the built-in financial package on a graphing calculator. Using the TVM solver, we make the following entries:

N = 30
I% = 2
PV = 0
PMT = −1200
FV = 49 655.33
P/Y = 1
C/Y = 1
PMT: *BEGIN*

The present value, *PV* = 0, since there is no money invested until the first payment of \$1200 is made. The payment is made at the beginning of each year, so set the *PmtAt* = *BEGIN*. Then solve for future value, *FV*.

Sum to infinity of a geometric series

In this section we consider if it is possible to calculate the sum of infinitely many terms in a geometric series.

First consider the infinite geometric series with first term 1 and common ratio 2.

$$\sum_{k=1}^{\infty} 2^{k-1} = 1 + 2 + 4 + 8 + 16 + \dots$$

We can calculate the partial sums for 10, 50 and 100 terms.

$$\sum_{k=1}^{10} 2^{k-1} = \frac{2^{10} - 1}{2 - 1} = 1023$$

$$\sum_{k=1}^{50} 2^{k-1} = \frac{2^{50} - 1}{2 - 1} = 1\,125\,899\,906\,842\,623$$

$$\sum_{k=1}^{100} 2^{k-1} = \frac{2^{100} - 1}{2 - 1} = 1\,267\,650\,600\,228\,229\,401\,496\,703\,205\,375$$

We observe that as the number of terms in the partial sum increases, the sum also increases. It would appear that the sum to infinity, $\sum_{k=1}^{\infty} 2^{k-1}$, approaches infinity.

Now consider the infinite geometric series with first term 1 and common ratio $\frac{1}{2}$

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

We can calculate the partial sums for 10, 20 and 50 terms.

$$\sum_{k=1}^{10} \left(\frac{1}{2}\right)^{k-1} = \frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}} = 1.998046875$$

$$\sum_{k=1}^{20} \left(\frac{1}{2}\right)^{k-1} = \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} = 1.9999980926514$$

$$\sum_{k=1}^{50} \left(\frac{1}{2}\right)^{k-1} = \frac{1 - \left(\frac{1}{2}\right)^{50}}{1 - \frac{1}{2}} = 1.9999999999999982236$$

We observe that as the number of terms in the partial sum increases, the sum also increases, but it approaches a limiting value of 2. It would appear that the sum to infinity, $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}$, equals 2.

Using the language of limits, we define the sum to infinity, S_∞ as the limit as n tends to infinity of S_n. We write this as

$$S_\infty = \lim_{n \to \infty} S_n$$

For the above example we have

$$S_\infty = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} = \lim_{n \to \infty} \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2, \text{ since } \lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0$$

If $S_\infty = L$, where L is finite, then we say that the infinite series **converges** to L.

So this particular infinite series converges to 2. The first series, $\sum_{k=1}^{\infty} 2^{k-1}$, does not have a limit and we say it **diverges**.

We are now ready to develop a general rule for finding the sum to infinity of a geometric series.

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{u_1(1-r^n)}{1-r}, r \neq 1$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and $S_\infty = \frac{u_1(1-0)}{1-r} = \frac{u_1}{1-r}$

Example 3.23

Find the sum to infinity of the geometric series $45 - 15 + 5 - \frac{5}{3} + \frac{5}{9} - \ldots$

Solution

$u_1 = 45$ and $r = -\frac{1}{3}$

$$S_\infty = \frac{45}{1-\left(-\frac{1}{3}\right)} = \frac{45}{\frac{4}{3}} = \frac{135}{4}$$

Example 3.24

The sum of the first three terms of a geometric series is 140 and the sum to infinity is 160. Find the tenth term.

Solution

$$S_3 = 140 \Rightarrow \frac{u_1(1-r^3)}{1-r} = 140$$

$$S_\infty = 160 \Rightarrow \frac{u_1}{1-r} = 160$$

Substituting the second equation into the first:

$$160(1-r^3) = 140 \Rightarrow 1 - r^3 = \frac{140}{160} = \frac{7}{8}$$

Rearranging:

$$r^3 = \frac{1}{8} \Rightarrow r = \frac{1}{2}$$

Substituting r into the second equation: $u_1 = 80$

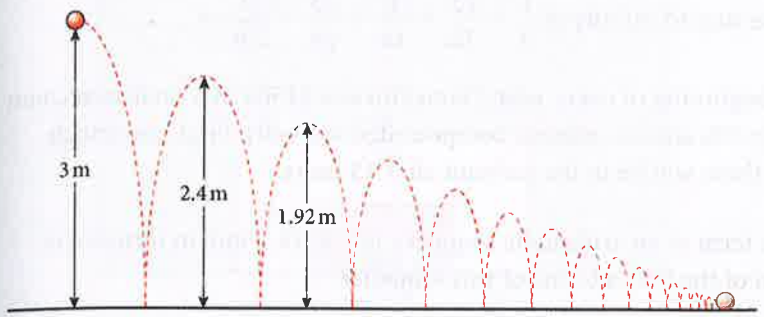
Finally,

$$u_{10} = 80\left(\frac{1}{2}\right)^9 = \frac{5}{32}$$

Example 3.25

If a ball has elasticity such that it bounces up 80% of its previous height, find the total vertical distance travelled down and up by this ball when it is dropped from a height of 3 metres. Ignore friction and air resistance.

Solution



After the ball is dropped the initial 3 m, it bounces up and down a vertical distance of 2.4 m. Each bounce after the first bounce, the ball travels 0.8 times the previous height twice – once upwards and once downwards. So, the total vertical distance is given by

$$h = 3 + 2(2.4 + (2.4 \times 0.8) + (2.4 \times 0.8^2) + \ldots) = 3 + 2 \times l$$

The amount in parentheses is an infinite geometric series with $a_1 = 2.4$ and $r = 0.8$. The value of that quantity is

$$l = \frac{2.4}{1-0.8} = 12$$

Hence the total distance required is

$$h = 3 + 2(12) = 27 \text{ m}$$

Exercise 3.4

1. Find the sum of the arithmetic series $11 + 17 + \ldots + 365$
2. Find the value of $9 + 13 + 17 + \ldots + 85$
3. Find the value of $8 + 14 + 20 + \ldots + 278$
4. Find the value of $155 + 158 + 161 + \ldots + 527$
5. Find the value of $120 + 24 + \frac{24}{5} + \frac{24}{25} + \ldots + \frac{24}{78\,125}$

6. Find the value of $2 - 3 + \frac{9}{2} - \frac{27}{4} + \ldots - \frac{177\,147}{1024}$

7. Find the value of $\sum_{k=0}^{13}(2 - 0.3k)$

8. Find the sum to infinity of $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \ldots$

9. Find the sum to infinity of $\frac{1}{3} + \frac{\sqrt{3}}{12} + \frac{1}{16} + \frac{\sqrt{3}}{64} + \frac{3}{256} + \ldots$

10. At the beginning of every year, Daniel invests \$1500 in a savings account that pays 6% annual interest, compounded annually. Find how much money there will be in the account after 15 years.

11. The kth term of an arithmetic sequence is $2 + 3k$. Find, in terms of n, the sum of the first n terms of this sequence.

12. Find the least number of terms required for the series $17 + 20 + 23 + \ldots$ to exceed 678.

13. Find the least number of terms required for the series $-18 - 11 - 4 - \ldots$ to exceed 2335.

14. Consider the arithmetic sequence 3, 7, 11, …, 999

(a) Find the sum of this sequence.

(b) Every third term of the sequence is removed (i.e. $u_3, u_6, u_9, \ldots$ are removed). Find the sum of the terms of the remaining sequence.

15. The sum of the first 10 terms of an arithmetic sequence is 235, and the sum of the second 10 terms is 735. Find the first term and the common difference.

16. Use your GDC or a spreadsheet to evaluate each sum.

(a) $\sum_{k=1}^{20}(k^2 + 1)$

(b) $\sum_{i=3}^{17}\left(\frac{1}{i^2 + 3}\right)$

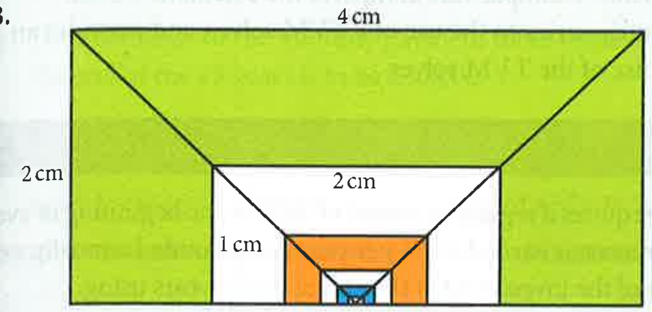
(c) $\sum_{n=1}^{100}(-1)^n\frac{3}{n}$

17. A ball is dropped from a height of 16 m. Every time it hits the ground it bounces 81% of its previous height.

(a) Find the maximum height it reaches after the 10th bounce.

(b) Find the total distance travelled by the ball until it comes to rest. (Assume no friction and no loss of elasticity.)

18.

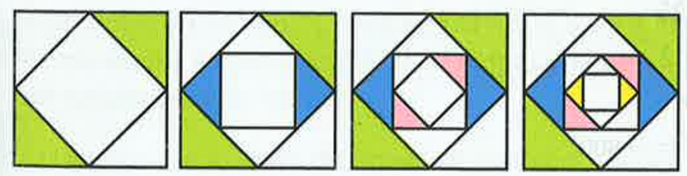


The largest rectangle has dimensions 4 by 2, as shown; another rectangle is constructed inside it with dimensions 2 by 1. The process is repeated. The region surrounding every other inner rectangle is shaded, as shown.

(a) Find the total area for the three regions shaded already.

(b) If the process is repeated indefinitely, find the total area of the shaded regions.

19. The sides of a square are 16 cm long. A new square is formed by joining the midpoints of the adjacent sides and two of the resulting triangles are coloured as shown.



(a) If the process is repeated 6 more times, determine the total area of the shaded region.

(b) If the process is repeated indefinitely, find the total area of the shaded region.

3.5 Annuities and amortisation

In Example 3.22, we applied geometric series to the calculation of annuities, which are equal regular payments that earn an annual interest rate. We also showed how the problem could be solved using the built-in financial package on a graphing calculator. Example 3.26 compares the calculation of an annuity using geometric series to the use of a TVM solver, and provides an introduction to the use of the TVM solver.

Example 3.26

A retirement plan requires a regular payment of $800 at the beginning of every year for 25 years. Interest is earned at 2% per year, compounded annually. Calculate the value of the investment at the end of the 25 years using

(a) a geometric series (b) a TVM solver.

Solution

(a) $800(1.02) + 800(1.02)^2 + \ldots + 800(1.02)^{25}$

$= 800(1.02)[1 + 1.02 + 1.02^2 + \ldots + 1.02^{24}]$

$= 800(1.02)\left[\dfrac{1.02^{25} - 1}{1.02 - 1}\right]$

$= \$26{,}136.72$

(b) Using the TVM solver, we make the following entries:

$N = 25$
$I\% = 2$
$PV = 0$
$PMT = -800$
$FV = 26\,136.72$
$P/Y = 1$
$C/Y = 1$
PMT: BEGIN

Note that because the $800 is being paid, it has a negative sign. However, the value of the annuity at the end of the 25 years is the future value (FV) and is positive, as the investor receives this money upon completion of the payments. Since payments are made at the beginning of the year, BEGIN is highlighted for the PMT option.

The TVM solver is very helpful in comparing financial instruments without having to create and evaluate a geometric series. Example 3.27 demonstrates how the annuity in Example 3.26 can be analysed using the TVM solver.

Example 3.27

Consider the annuity from Example 3.26, in which $800 is invested each year at 2% per year, compounded annually. Using a TVM solver, answer the following questions.

(a) Find the value of the investment after 25 years if the monthly payment is changed to $810.

(b) Find the value of the monthly payment if the value of the investment at the end of the 25 years is to be $30,000.

(c) In Example 3.26, the annuity had a value of $26,136.72 after 25 years. If the payment per year is reduced to $761, find how much longer it takes for the investment to reach a value of $26,136.72

Solution

(a) The following entries are input into the TVM solver:

$N = 25$
$I\% = 2$
$PV = 0$
$PMT = -810$
$FV = 26\,463.43$
$P/Y = 1$
$C/Y = 1$
PMT: BEGIN

This means that within 25 years, an additional $250 is invested in payments, but an increase in $326.71 in future value is obtained due to the accumulation of interest.

(b) In this case, we solve for the unknown payment. The entries in the TVM solver are as follows:

$N = 25$
$I\% = 2$
$PV = 0$
$PMT = -918.25$
$FV = 30\,000$
$P/Y = 1$
$C/Y = 1$
PMT: BEGIN

Thus, the payments need to be increased to $918.25 per year to have $30,000 after 25 years.

(c) In this example, we must find the number of payments. Since $P/Y=1$, the number of payments is given by N.

$N = 26$
$I\% = 2$
$PV = 0$
$PMT = -761$
$FV = 26\,136.72$
$P/Y = 1$
$C/Y = 1$
PMT: $BEGIN$

Thus, reducing the yearly payment to $761 results in an additional year of investment in order to obtain the same future value of $26,136.72

So far, we have considered examples in which money is invested over time and accumulates interest according to the compound interest model. We now consider the situation in which one borrows an amount of money and repays it in regular payments that earn interest. We call this sum of money a **loan** and the process of repaying the loan is called **amortisation.** Example 3.28 demonstrates the amortisation of loans and the use of the TVM solver in comparing loan amortisation schemes.

Example 3.28

In order to pay for a university course, a student takes a loan of $20,000. The unpaid balance on the loan has an interest rate of 3% per year, compounded annually. Use a TVM solver to answer the following questions about the amortisation of the loan.

(a) The loan is to be repaid in payments of $1200 per year. Find how long it takes to repay the loan, if payments are made at the end of each year.

(b) Hence, calculate how much has been paid in total in amortising the loan.

(c) If the loan is to be amortised in 10 years, find the annual payment, and how much has been paid in total.

Solution

(a) Use the TVM solver to find the value of N.

$N = 23.4$
$I\% = 3$
$PV = 20000$
$PMT = -1200$
$FV = 0$
$P/Y = 1$
$C/Y = 1$
PMT: END

The present value is positive, as the student receives this money from an institution. As the payments are outgoing, they are negative in value. The future value is zero, as the student wants to pay off the loan. According to the TVM solver, the loan is amortised in about 23 years with payments of $1200 per year.

(b) $1200 × 23.4 = $28,080

Thus, the student pays $28,080 in amortising the loan.

(c) Using $N = 10$, the inputs for the TVM solver are as follows:

$N = 10$
$I\% = 3$
$PV = 20\,000$
$PMT = -2344.61$
$FV = 0$
$P/Y = 1$
$C/Y = 1$
PMT: END

To amortise the loan in 10 years, the student must pay $2344.61 annually. However, the total amount paid in 10 years would be $2344.61 × 10 = $23,446.10

Example 3.28 shows that if the annual payment increases, the time taken to amortise a loan decreases, and consequently, the total amount paid to amortise the loan decreases.

We will now use the TVM solver to analyse a loan in which the accumulation of interest does not coincide with the regular payments.

Example 3.29

A young family takes a loan of $100,000 to buy a house. Interest on the loan accumulates at the rate of 1.5% per year, compounded semi-annually. The family will amortise the loan in monthly payments, paid at the beginning of each month.

(a) The loan is to be amortised in 25 years. Find the monthly payment.

(b) Calculate how much the family has paid in total after 25 years.

(c) Explain the benefits to the family of increasing the monthly payment by $75. Justify your answer.

Solution

(a) Since payments are made monthly, $N = 25 \times 12 = 300$ and $P/Y = 12$
In addition, since interest is compounded semi-annually, $C/Y = 2$
Entering the values into the TVM solver:

$N = 300$
$I\% = 1.5$
$PV = 100\,000$
$PMT = -399.22$
$FV = 0$
$P/Y = 12$
$C/Y = 2$
PMT: $BEGIN$

The monthly payment is \$399.22

(b) At the end of 25 years, the family has paid \$399.22 × 300 = \$119,766

(c) Using \$474.22 for payment in the TVM Solver and solving for N gives $N = 244.4$, so about 245 months in total.
Thus, the family amortises the loan 55 months earlier than in part (a).
Using $N = 245$, the total amount paid in amortising the loan is 245 × \$475 = \$116,375. If the family can afford to pay an additional \$75 per month, they finish paying the loan more than four and a half years earlier, and pay \$3391 less in total.

Exercise 3.5

1. An investment of \$300 per year is made for 15 years, earning 1% per year, compounded annually. Find the value of the investment using:
   (a) geometric series (b) a TVM solver.

2. Explain the financial calculation being completed in each set of the TVM solver parameters.

(a) $N = 30$, $I\% = 1.5$, $PV = 0$, $PMT = -60$, $FV = 2286.11$, $P/Y = 1$, $C/Y = 1$, PMT: $BEGIN$

(b) $N = 15$, $I\% = 3$, $PV = 0$, $PMT = -52.20$, $FV = 1000$, $P/Y = 1$, $C/Y = 1$, PMT: $BEGIN$

(c) $N = 21.76$, $I\% = 1.2$, $PV = 0$, $PMT = -100$, $FV = 2500$, $P/Y = 1$, $C/Y = 1$, PMT: $BEGIN$

(d) $N = 30$, $I\% = 4$, $PV = 0$, $PMT = -50$, $FV = 1579.38$, $P/Y = 12$, $C/Y = 2$, PMT: $BEGIN$

3. At the beginning of each year Peter invests \$1800 in a savings account that pays 5% annual interest, compounded annually. Find how many years it will take until Peter has \$50,000 in his account.

4. At the beginning of each year Anjali invests \$2000 in a savings account that pays annual interest, compounded annually. After 15 years she has \$60,000 in her account. Find the annual interest rate.

5. (a) An investment of \$720 is made at the beginning of every year for 12 years. Interest accumulates at the rate of 3% per year, compounded annually. Use the TVM solver to find the value of the investment at the end of the 12 years.
   (b) A second investor invests \$60 at the beginning of every month for 12 years, and interest accumulates at the rate of 3% per year, compounded annually. Use the TVM solver to find the value of the investment at the end of the 12 years. Compare your answer to your answer in part (a).

6. (a) A teacher invests \$50 at the beginning of every month into an investment fund that earns 2% per year, compounded semi-annually. Find the value of the investment at the end of 30 years.
   (b) The teacher is given the option to invest \$75 per month instead at a rate of 1.7% per year, compounded semi-annually. Compare the value of this investment at the end of the 30 years to the answer in part (a).
   (c) Another teacher would like to have \$28,000 at the end of the 30 years. She invests monthly at a rate of 2% per year, compounded semi-annually. Calculate how much she must invest per month.

7. A family has taken out a loan of \$250,000 to purchase a house. They agree to pay the bank \$2100 at the end of every month to amortise the loan, and interest accumulates on the balance at a rate of 1.3% per year, compounded annually.
   (a) Find how long it takes to pay back the loan, in years and months.
   (b) Calculate how much have they paid in total in amortising the loan.
   (c) The family considers increasing the monthly payment to \$2300. By completing appropriate calculations with a TVM solver, justify this decision.

Chapter 3 practice questions

1. The seventh, third and first terms of an arithmetic sequence form the first three terms of a geometric sequence. The arithmetic sequence has first term a and non-zero common difference d.

   (a) Show that $d = \frac{a}{2}$

   The seventh term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 200.

   (b) Find the least value of n for which this occurs.

2. **(a)** Find the sum of all integers, between 10 and 200, which are divisible by 7.

   (b) Express the sum found in part (a) using sigma notation.

   (c) An arithmetic sequence has first term 1000 and common difference −6. The sum of the first n terms of this sequence is negative. Find the least value of n.

3. The sum of the first two terms of a geometric series is 10 and the sum of the first four terms is 30.

   (a) Show that the common ratio r satisfies $r^2 = 2$

   (b) Given $r = \sqrt{2}$

   (i) find the first term **(ii)** find the sum of the first ten terms.

4. The fourth term in an arithmetic sequence is 34 and the tenth term is 76.

   (a) Find the first term and the common difference.

   (b) The sum of the first n terms exceeds 5000. Find the least possible value of n.

5. A geometric sequence has first term a, common ratio r and sum to infinity 76. A second geometric sequence has first term a, common ratio r^3 and sum to infinity 36. Find r.

6. **(a)** The arithmetic sequence $(u_n: n \in \mathbb{Z}^+)$ has first term $u_1 = 1.6$ and common difference $d = 1.5$. The geometric sequence $(v_n: n \in \mathbb{Z}^+)$ has first term $v_1 = 3$ and common ratio $r = 1.2$

   Find an expression for $u_n - v_n$ in terms of n.

   (b) Determine the set of values of n for which $u_n > v_n$

   (c) Determine the greatest value of $u_n - v_n$

   Give your answer correct to 3 significant figures.

7. A metal rod 1 metre long is cut into 10 pieces, the lengths of which form a geometric sequence. The length of the longest piece is 8 times the length of the shortest piece. Find, to the nearest millimetre, the length of the shortest piece.

8. **(a)** Each time a ball bounces, it reaches 95% of the height reached on the previous bounce. Initially, it is dropped from a height of 4 metres.

   What height does the ball reach after its fourth bounce?

   (b) How many times does the ball bounce before it no longer reaches a height of 1 metre?

   (c) What is the total vertical distance travelled by the ball?

9. **(a)** A geometric sequence, $u_1, u_2, u_3, \ldots$ has $u_1 = 27$ and sum to infinity $\frac{81}{2}$

   Find the common ratio of the geometric sequence.

   (b) An arithmetic sequence $v_1, v_2, v_3, \ldots$ is such that $v_2 = u_2$ and $v_4 = u_4$.

   Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$

10. The sum, S_n, of the first n terms of a geometric sequence, whose nth term is u_n, is given by

$$S_n = \frac{7^n - a^n}{7^n} \text{ where } a > 0$$

   (a) Find an expression for u_n

   (b) Find the first term and common ratio of the sequence.

   (c) Consider the sum to infinity of the sequence.

   (i) Determine the values of a such that the sum to infinity exists.

   (ii) Find the sum to infinity when it exists.

11. A rope of length 81 metres is cut into n pieces of increasing lengths that form an arithmetic sequence with a common difference of d metres. Given that the lengths of the shortest and longest pieces are 1.5 metres and 7.5 metres respectively, find the values of n and d.

12. **(a)** A bank offers loans of \$$P$ at the beginning of a particular month at a monthly interest rate of I. The interest is calculated at the end of each month and added to the amount outstanding. A repayment of \$$R$ is required at the end of each month. Let S_n denote the amount outstanding, in dollars, immediately after the nth monthly repayment.

   (i) Find an expression for S_1 and show that:

$$S_2 = P\left(1 + \frac{I}{100}\right)^2 - R\left(1 + \left(1 + \frac{I}{100}\right)\right)$$

   (ii) Determine a similar expression for S_n

   Hence show that:

$$S_n = P\left(1 + \frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1 + \frac{I}{100}\right)^n - 1\right)$$

(b) Sue borrows \$5000 at a monthly interest rate of 1% and plans to repay the loan in 5 years (60 months).

(i) Calculate the required monthly repayment, giving your answer correct to 2 decimal places.

(ii) After 20 months, Sue inherits some money and she decides to repay the loan completely at that time. Giving your answer correct to the nearest dollar, how much will she have to repay?

13. Phil takes out a bank loan of \$150,000 to buy a house, at an annual interest rate of 3.5%. The interest is calculated at the end of each year and added to the amount outstanding.

To pay off the loan, Phil makes annual deposits of \$$P$ at the end of every year in a savings account, paying an annual interest rate of 2%. He makes his first deposit at the end of the first year after taking out the loan.

David visits a different bank and makes a single deposit of \$$Q$ into a new savings account, the annual interest rate being 2.8%.

(a) Find the amount Phil would owe the bank after 20 years. Give your answer to the nearest dollar.

(b) Show that the total value of Phil's savings after 20 years is:

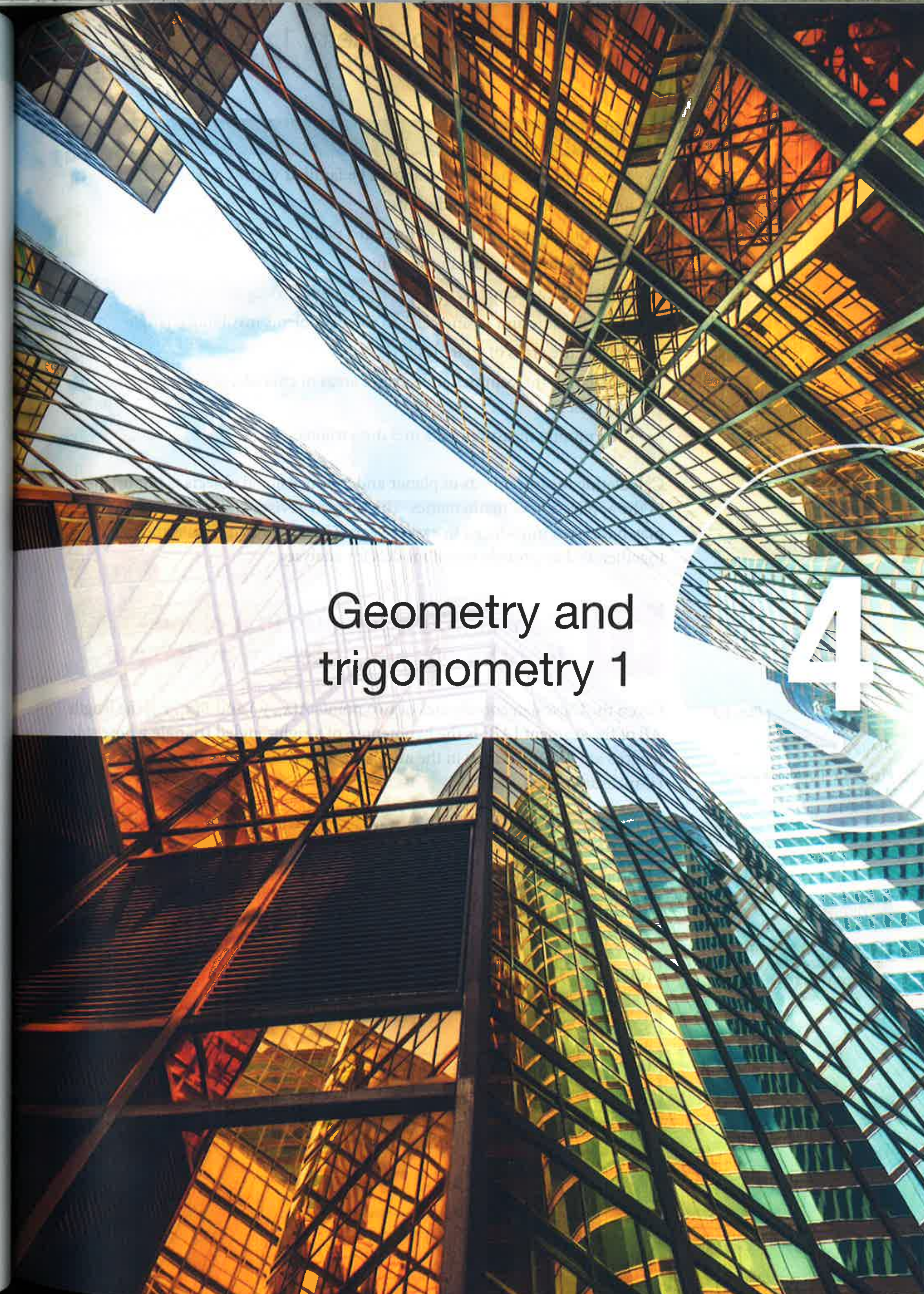
$$\frac{(1.02^{20} - 1)P}{(1.02 - 1)}$$

(c) Given that Phil's aim is to own the house after 20 years, find the value for P to the nearest dollar.

(d) **(i)** David wishes to withdraw \$5000 at the end of each year for a period of n years. Show that an expression for the minimum value of Q is

$$\frac{5000}{1.028} + \frac{5000}{1.028^2} + \dots + \frac{5000}{1.028^n}$$

(ii) Hence or otherwise, find the minimum value of Q that would permit David to withdraw annual amounts of \$5000 indefinitely. Give your answer to the nearest dollar.



Geometry and trigonometry 1

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