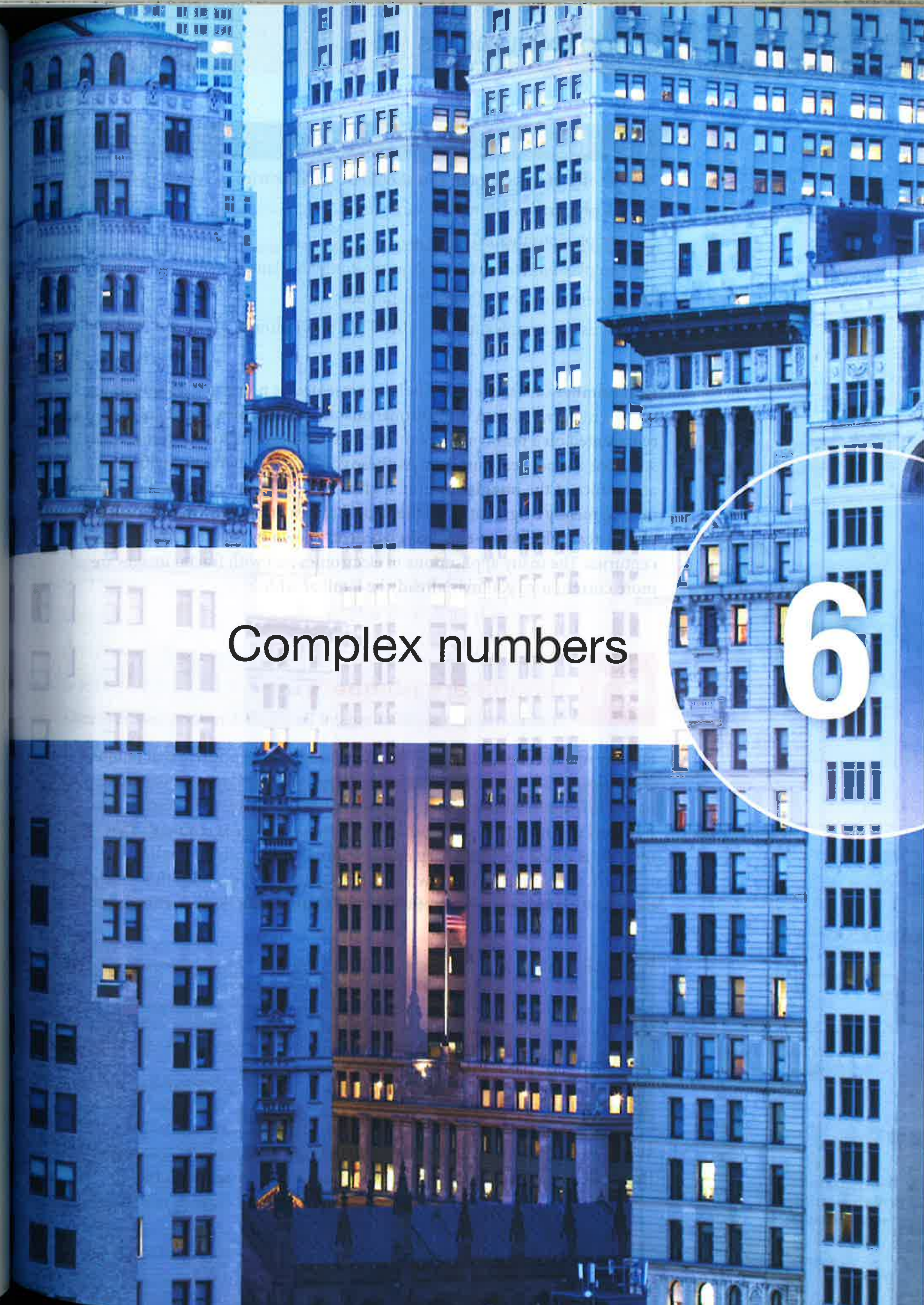
16. Voronoi diagrams of regular polygons:

(a) Sites A, B, and C are located at the vertices of an equilateral triangle. Sketch the Voronoi diagram for these sites.

(b) Sites A, B, C, and D are located at the vertices of a square. Sketch the Voronoi diagram for these sites.

(c) Give a description for the Voronoi diagram for a regular n-gon including the position of the edges and the number of vertices.



6 Complex numbers

Learning objectives

By the end of this chapter, you should be familiar with...

- the properties of complex numbers
- complex number calculations
- complex numbers as vectors in the complex plane
- powers and roots of complex numbers
- the use of complex numbers in STEM applications.

Real numbers are those that can be represented as points on a number line, such as integer, rational, or irrational quantities. **Complex numbers** cannot be represented on a number line, but are analytic solutions to equations whose solutions are not real numbers. As these complex numbers start with the acceptance of the **imaginary** unit, it gives the mistaken notion that a solution may be merely a figment of mathematical imagination; however, the relevance of non-real solutions to equations has been known by mathematicians for centuries. The many applications in electronics and with fractal images are more current uses you may already be familiar with.

6.1 Imaginary numbers

For any real number a, its square is guaranteed to be a positive real number. If a is positive, a^2 is positive; if a is negative, a^2 is positive again. So, what happens mathematically when you need a solution to $x^2 = -1$? Since real numbers will not give a solution, an imaginary number is required. This number is called the imaginary unit i, where $i^2 = -1$, which implies that $i = \sqrt{-1}$. Like other well-known constants, such as π, ϕ, and e, this constant i has many applications, some of which will be presented later in this chapter.

> There is a reason why we do not define $i = \sqrt{-1}$. It is the convention in mathematics that when we write $\sqrt{9}$ then we mean the non-negative square root of 9, namely 3. We do not mean -3.
>
> i does not belong to this category since we cannot say that i is the positive square root of -1, i.e., $i > 0$. If we do, then $-1 = i \cdot i > 0$, which is false, and if we say $i < 0$, then $-i > 0$, and $-1 = -i \cdot -i > 0$, which is also false. Actually $-i$ is also a square root of $\sqrt{-1}$ because $-i \cdot -i = i^2 = -1$
>
> With this in mind, we can use a convention which calls i the **principal** square root of -1 and write $i = \sqrt{-1}$

When we isolate the factor of -1 when there are square roots in an expression, then we can express all negative radicands using the imaginary unit i.

Example 6.1

Express each value using the imaginary unit i.

(a) $\sqrt{-16}$ (b) $\sqrt{-18}$ (c) $\sqrt{-27}$ (d) $\sqrt{-4} \cdot \sqrt{-9}$

Solution

(a) $\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1} = 4i$

(b) $\sqrt{-18} = \sqrt{18} \cdot \sqrt{-1} = 3\sqrt{2}\,i$

(c) $\sqrt{-27} = \sqrt{27} \cdot \sqrt{-1} = 3\sqrt{3}\,i$

(d) $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{9} \cdot \sqrt{-1}$
$= 2i \cdot 3i = 6i^2 = -6$

> Be very careful with the last example.

Given $i = \sqrt{-1}$ and $i^2 = -1$, consider further powers of i:

The first four:	The next four:
$i = \sqrt{-1}$	$i^5 = i^4 \cdot i = 1 \cdot i = i$
$i^2 = -1$	$i^6 = i^4 \cdot i^2 = 1 \cdot i^2 = -1$
$i^3 = i^2 \cdot i = -i$	$i^7 = i^4 \cdot i^3 = 1 \cdot i^3 = -i$
$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$	$i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$

> **Online**
> See how multiplying by i can be interpreted geometrically.

Note that every power k that is an integer multiple of 4 produces $i^k = 1$ and the pattern repeats.

Example 6.2

Express each in its simplest form.

(a) i^{31} (b) i^{125}

Solution

Divide 31 and 125 by 4 and look for the remainder.

(a) $i^{31} = i^3 = -i$

(b) $i^{125} = i^1 = i$

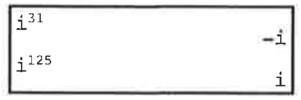


Figure 6.1 GDC screen for the solution to Example 6.2

Complex numbers add another dimension to real numbers represented by points on a number line. This new dimension contains values that are not real, and contains all real numbers as well as imaginary numbers. This superset of numbers is called the set of **complex numbers.** A complex number has the form $z = a + bi$ where a is the real component and b is the imaginary component.

> If $z = a + bi$, then a is called the real part of z and written as Re z, and b is called the imaginary part of z and written as Im z.

The zeros of a function are the solutions to $y = 0$. Graphically, the zeros of a function are the x-intercepts, where $y = 0$.

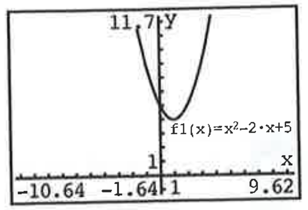


Figure 6.2 GDC screen for the solution to Example 6.3

Example 6.3

Find the zeros of $y = x^2 - 2x + 5$

Solution

As the graph of $y = x^2 - 2x + 5$ shows, there are no real zeros.

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2}$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

$$\Rightarrow x = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Exercise 6.1

1. Express each value using the imaginary unit i.

(a) $\sqrt{-36}$ **(b)** $\sqrt{-12}$ **(c)** $\sqrt{-63}$ **(d)** $\sqrt{-8} \cdot \sqrt{-18}$

2. Express each number as a complex number in the form $a + bi$

(a) $4 + \sqrt{-9}$ **(b)** $-3 - \sqrt{-4}$ **(c)** $-18 + \sqrt{-18}$

(d) $4\sqrt{2} - \sqrt{-8}$ **(e)** $\sqrt{-4}$ **(f)** $12 + \sqrt{-12}$

(g) $-\sqrt{49}$ **(h)** $(2 + \sqrt{-4})\,i$

3. Express each number in its simplest form.

(a) i^7 **(b)** i^{66} **(c)** i^{721} **(d)** i^{-24}

4. Find the zeros of each function.

(a) $y = x^2 + 2x + 10$ **(b)** $y = x^2 - 4x + 7$

(c) $y = 2x^2 + 4x + 5$ **(d)** $y = x^2 - 2x + 10$

(e) $y = x^2 + 6x + 10$ **(f)** $y = x^2 + 3x + 3$

5. $x^k - 1 = 0$ has k roots. For the value of k indicated, find them by factorising $x^k - 1$. Knowing that $(x - 1)$ is a factor may be useful. There will be an alternative strategy presented later in this chapter.

(a) $k = 4$ **(b)** $k = 3$

6.2 Operations with complex numbers

Since a complex number has the form $z = a + bi$ that separates the real component a from the imaginary component b, the addition and subtraction of complex numbers is very straightforward: combine the components separately.

Example 6.4

Find each sum.

(a) $(1 + 3i)$ and $(4 + i)$ (b) $(5 - 2i)$ and $(-8 + 5i)$

(c) $(2\sqrt{3} - 4i)$ and $(\sqrt{3} + i)$

Solution

(a) $\begin{cases} 1 + 4 = 5 \\ 3i + i = 4i \end{cases}$

The sum is $5 + 4i$

(b) $\begin{cases} 5 - 8 = -3 \\ -2i + 5i = 3i \end{cases}$

The sum is $-3 + 3i$

(c) $\begin{cases} 2\sqrt{3} + \sqrt{3} = 3\sqrt{3} \\ -4i + i = -3i \end{cases}$

The sum is $3\sqrt{3} - 3i$

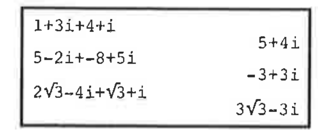


Figure 6.3 GDC output for the solutions to Example 6.4

The product of complex numbers is nothing more than the product of two binomials, found in much the same way as the product $(3x - 4y)(5x + 2y)$, for instance. We can find the product $(3 - 4i)(5 + 2i)$ in a similar way.

$$\begin{aligned}(3 - 4i)(5 + 2i) &= 3 \cdot 5 + 3 \cdot 2i + (-4i) \cdot 5 + (-4i) \cdot (2i) \\ &= 15 + 6i - 20i - 8i^2 \\ &= 15 + 8 + (6 - 20)i \quad \text{since } i^2 = -1 \\ &= 23 - 14i\end{aligned}$$

Before we divide complex numbers, we need a reminder on the simplification of rational expressions containing irrational denominators.

Example 6.5

Express each fraction with a rational number denominator.

(a) $\frac{6}{\sqrt{2}}$ (b) $\frac{12}{3 - \sqrt{5}}$

Solution

(a) $\frac{6}{\sqrt{2}} \cdot \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

(b) We need the conjugate of $3-\sqrt{5}$ which is $3+\sqrt{5}$, so

$$\frac{12}{3-\sqrt{5}}=\frac{12}{3-\sqrt{5}}\cdot\left(\frac{3+\sqrt{5}}{3+\sqrt{5}}\right)$$

$$=\frac{12(3+\sqrt{5})}{9-5}=\frac{12(3+\sqrt{5})}{4}=3(3+\sqrt{5})=9+3\sqrt{5}$$

By considering any complex number $a+bi$ as $a+b\sqrt{-1}$, it essentially has the same form as the binomial $3-\sqrt{5}$ above. The **complex conjugate** of $z=a+bi$ is called $z^*=a-bi$ and serves a similar purpose to the binomials. Consider the product of $z=a+bi$ and its complex conjugate, $z^*=a-bi$:

$$(a+bi)(a-bi)$$
$$=a^2-(bi)^2$$
$$=a^2-b^2i^2$$
$$=a^2+b^2$$

Just as binomial conjugates produce rational products, the multiplication of a complex number by its complex conjugate produces an entirely real value.

Example 6.6

Express each rational expression in the form $a+bi$

(a) $\dfrac{6}{1+i}$ (b) $\dfrac{50}{-4+3i}$ (c) $\dfrac{9}{2+\sqrt{5}i}$ (d) $\dfrac{2-3i}{3+2i}$

Solution

(a) $\dfrac{6}{1+i}=\dfrac{6}{1+i}\cdot\left(\dfrac{1-i}{1-i}\right)=\dfrac{6(1-i)}{1+1}=\dfrac{6(1-i)}{2}=3(1-i)=3-3i$

(b) $\dfrac{50}{-4+3i}=\dfrac{50}{-4+3i}\cdot\left(\dfrac{-4-3i}{-4-3i}\right)=\dfrac{50(-4-3i)}{16+9}=\dfrac{50(-4-3i)}{25}$

$=2(-4-3i)=-8-6i$

(c) $\dfrac{9}{2+\sqrt{5}i}=\dfrac{9}{2+\sqrt{5}i}\cdot\left(\dfrac{2-\sqrt{5}i}{2-\sqrt{5}i}\right)=\dfrac{9(2-\sqrt{5}i)}{4+5}=2-\sqrt{5}i$

(d) $\dfrac{2-3i}{3+2i}=\dfrac{2-3i}{3+2i}\cdot\left(\dfrac{3-2i}{3-2i}\right)=\dfrac{(2-3i)(3-2i)}{9+4}=\dfrac{-13i}{13}=-i$

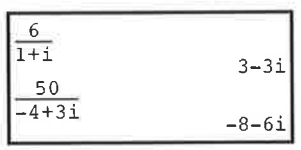


Figure 6.4 GDC output for the solutions to Example 6.6 (a) and (b)

Consider the roots, $x=1\pm 2i$, of the equation $y=x^2-2x+5$ in Example 6.3 which we found using the quadratic formula. They are complex conjugates of each other. We can generate the original equation from the roots. In general:

- when a quadratic polynomial in x has zeros r_1 and r_2 then

  $(x-r_1)$ and $(x-r_2)$ are its factors

- when $(x-r_1)$ and $(x-r_2)$ are its factors then the quadratic is

  $(x-r_1)(x-r_2)=x^2-(r_1+r_2)x+r_1r_2$

In other words, the **negative of the sum of the zeros** is the coefficient of x, and the **product of the zeros** is the constant term.

Example 6.7

Show how the original function $y=x^2-2x+5$ can be found when only its zeros, $x=1\pm 2i$, are known.

Solution

Since the complex conjugates $(1+2i)$ and $(1-2i)$ have a sum of 2 and a product of 5, the original function would be $y=x^2-2x+5$

A polynomial has **zeros**, a polynomial equation has **roots**.

Example 6.8

A quadratic equation of the form $ax^2+bx+c=0$ has one known root, $x=3-4i$

If a, b, and c are all real numbers, then what is the other root?

Solution

In Example 6.7, it was shown that the sum of the roots would be $-b$ and the product of the roots would be c. For b and c to be real numbers, the other root must be the complex conjugate of the root given.

Hence, the other root is $x=3+4i$

Example 6.9

Find the equation with the roots given in Example 6.8

Solution

Given that $3-4i$ and $3+4i$ are the roots, their sum is 6 and their product is 25. If we assume that $a=1$, then the required equation is $x^2-6x+25=0$. However, as there is no specific value given for a, the other coefficients are multiples of a, and the general solution is $ax^2-6ax+25a=0$

Remember that the coefficient of the quadratic term need not be 1.

Exercise 6.2

1. Find the sum of each pair of complex numbers.
   (a) $(2 - 4i)$ and $(-3 + 2i)$ (b) $(-1 + i)$ and $(3 - 2i)$
   (c) $(2\sqrt{2} + i)$ and $(-\sqrt{2} - 2i)$

2. Find each product.
   (a) $(1 - 4i)(1 + 4i)$ (b) $(2\sqrt{3} + i)(-2\sqrt{3} + i)$
   (c) $(3 + 4i)(3 - 4i)$

3. Find the product of each given complex number and its conjugate.
   (a) $4 - 3i$ (b) $-5 + 12i$ (c) $-4 - 2\sqrt{5}i$

4. Express each quotient in $a + bi$ form.
   (a) $\frac{5}{2 - i}$ (b) $\frac{1 - 2i}{1 + 2i}$ (c) $\frac{2 - 4i}{-3 + 2i}$

5. A quadratic function $y = ax^2 + bx + c$ has real coefficients a, b, and c. Find the function when one of the zeros is:
   (a) $1 - i$ (b) $-7 + i$ (c) $-2\sqrt{3} + \sqrt{3}i$

6. Find the quadratic function whose zeros are:
   (a) $x = 2 + \sqrt{3}, x = 2 - \sqrt{3}$ (b) $x = \frac{1 \pm \sqrt{5}}{2}$
   (c) $x = -1 \pm 2i$ (d) $x = \frac{3}{2} \pm \frac{5}{2}i$
   (e) $x = \frac{1 \pm \sqrt{5}i}{2}$ (f) $x = -2\sqrt{3} \pm \sqrt{3}i$

7. Find the quadratic function of the form $y = x^2 + bx + c$ whose zeros are:
   (a) $(5 + 2i)$ and $(3 - i)$ (b) $(3 + 2i)$ and $(-3 - 2i)$
   (c) $(3 + \sqrt{2}i)$ and $(-3 - \sqrt{2}i)$

8. Let $z = a + bi$. Find the values of a and b if $(2 + 3i) \cdot z = 7 + i$

9. $(2 + yi)(x + i) = 1 + 3i$, where x and y are real numbers. Solve for x and y.

10. Consider the complex number $z = 1 + \sqrt{3}i$
    (a) Evaluate z^3
    (b) Prove that $z^{6n} = 8^{2n}$, where $n \in \mathbb{Z}^+$
    (c) Hence, find z^{48}

11. Consider the complex number $z = -\sqrt{2} + \sqrt{2}i$
    (a) Evaluate z^2
    (b) Prove that $z^{4k} = (-16)^k$, where $k \in \mathbb{Z}^+$
    (c) Hence, find z^{46}

12. Given that z is a complex number such that $|z + 4i| = 2|z + i|$ find the value of $|z|$.

13. Write the complex number $z = 3 + \frac{2i}{2 - \sqrt{2}i}$ in the form $a + bi$

14. Find the values of the two real numbers x and y if $(x + yi)(4 - 7i) = 3 + 2i$

15. Find the complex number z and write it in the form $a + bi$:
    (a) $(z + 1)i = 3z - 2$ (b) $z^2 = 3 - 4i$

6.3 The complex plane

Real numbers can be found on a number line, but imaginary numbers cannot. However, purely imaginary numbers are ordered in the same way as real numbers.

For example, with real numbers: $1 < \sqrt{2} < 2 < e < 3 < \pi$

and with imaginary numbers: $i < \sqrt{2}i < 2i < ei < 3i < \pi i$

There are operations such as those in section 6.2 where we take two purely imaginary numbers or two complex numbers and produce a real number.

If we use a separate number line to show the imaginary component of a complex number, and put it at right angles to the real number line, we have the **complex plane** (Figure 6.5). This is similar to the x- and y-axes we use for graphs of functions. Complex numbers of the form $a + bi$ can be represented as a coordinate point measuring a units along the horizontal x-axis, and b units along the vertical y-axis. This complex plane is called the **Argand plane** (also known as an Argand diagram).

Since this plane is essentially the Cartesian plane, there are many ways we can use this representation. The distance of a number z from the origin on the Argand plane is known as the **absolute value** or **modulus** of the complex number, $|z|$.

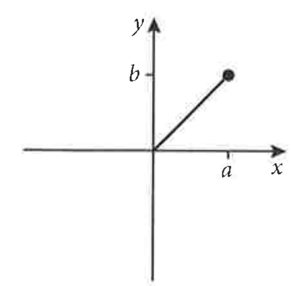


Figure 6.5 $a + bi$ in the complex plane

Example 6.10

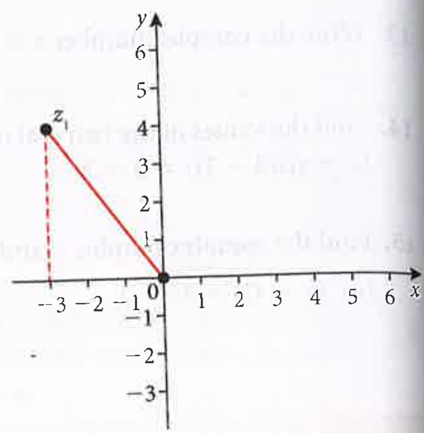
Consider the complex number $z_1 = -3 + 4i$

(a) Sketch it and find its distance from the origin.

(b) Compare this distance to $\sqrt{z_1 \cdot z_1^*}$

Solution

(a) The complex number $z_1 = -3 + 4i$ is shown in the Argand diagram. Its distance from the origin is the length of the hypotenuse of the right-angled triangle that could be drawn, and is 5 units long.



(b) As $z_1 = -3 + 4i$, its conjugate is $z_1^* = -3 - 4i$, and their product is $9 + 16 = 25$

Hence, $\sqrt{z_1 \cdot z_1^*} = 5$ and is exactly the same as the answer above.

In general, $z \cdot z^* = |z|^2$

Example 6.11

Given the complex numbers $z_1 = -3 + 4i$ and $z_2 = 5 + i$:

(a) plot them on the complex plane

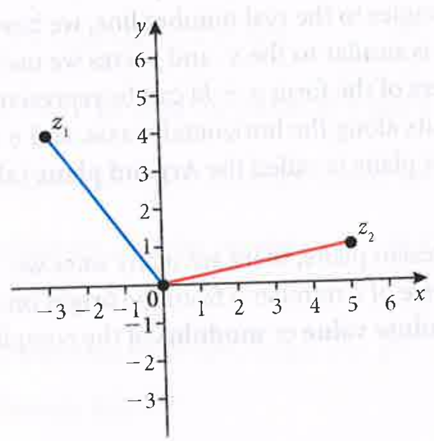
(b) find the modulus of each number (the distance from the origin)

(c) find their sum, and call it z_3

(d) plot z_3 in relation to z_1 and z_2. What do you notice?

Solution

(a)



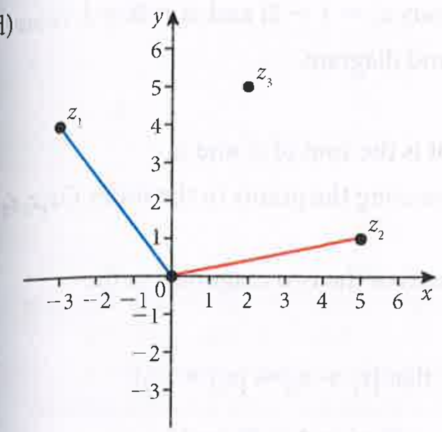
(b) $|z_1| = \sqrt{(-3)^2 + 4^2} = 5$

$|z_2| = \sqrt{5^2 + 1^2} = \sqrt{26}$

(c) $z_3 = z_1 + z_2$

$= (-3 + 4i) + (5 + i) = 2 + 5i$

(d)



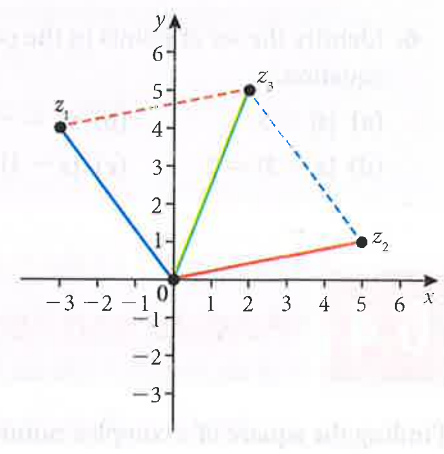
The real and imaginary parts of z_3 are the sums of the real and imaginary parts of z_1 and z_2. This produces a parallelogram when plotted.

Also, the modulus of z_3 is not the sum of the moduli of z_1 and z_2.

Compare $3 - 4i$ and $-3 + 4i$

Is one bigger than the other?

As complex numbers are represented by points representing two different components, it should be easy to see why complex numbers are not ordered.



Online

Visualise the sum of two complex numbers.

Exercise 6.3

1. Graph these complex numbers in the same complex plane:
$3 + 4i$, $4 + 3i$, $-3 + 4i$, $-4 + 3i$, $5i$, -5
   (a) What is common to all of them?
   (b) Name another complex number in quadrant 3 with the same property.
   (c) Does $-2\sqrt{3} + \sqrt{13}i$ have the same property? Why?
   (d) Name both complex numbers whose imaginary part is $3\sqrt{2}$ that fit the pattern.

2. Find the modulus of each complex number.

(a) $1 + i$ (b) $\sqrt{3} + i$ (c) $2i$

(d) $-2i$ (e) $-5 - 12i$ (f) $-5 + 12i$

(g) $-21 + 20i$ (h) $2\sqrt{3} + 4\sqrt{6}i$

3. Consider the two complex numbers $z_1 = 1 + 2i$ and $z_2 = 5 + i$

(a) Plot these points on an Argand diagram.

(b) Find their moduli.

(c) Plot the point z_3, given that it is the sum of z_1 and z_2

(d) Draw a quadrilateral by connecting the points in the order $Oz_1z_3z_2$ then back to O.

(e) What is the significance of each of the two diagonals in the quadrilateral drawn?

4. Use an Argand diagram to show that $|z_1 + z_2| \leq |z_1| + |z_2|$

5. The complex numbers $z_1 = 2\sqrt{3} - 2i$, $z_2 = 2 + 2i$, and $z_3 = (2\sqrt{3} - 2i)(2 + 2i)$ represent the vertices of a triangle in an Argand diagram. Find its area.

6. Identify the set of points in the complex plane that correspond to each equation.

(a) $|z| = 3$ (b) $z^* = -z$ (c) $z + z^* = 8$

(d) $|z - 3| = 2$ (e) $|z - 1| + |z - 3| = 2$

6.4 Powers and roots of complex numbers

Finding the square of a complex number, such as $1 + \sqrt{3}i$, takes only a little effort, since by binomial multiplication:

$$\begin{aligned}(1 + \sqrt{3}i)^2 &= (1 + \sqrt{3}i)(1 + \sqrt{3}i)\\ &= 1 + \sqrt{3}i + \sqrt{3}i + 3i^2\\ &= -2 + 2\sqrt{3}i\end{aligned}$$

However, it is much more difficult to expand $(1 + \sqrt{3}\,i)^6$, unless we use some trigonometry.

The polar form of complex numbers

Consider the complex number $z = a + bi$ in the Argand diagram.

By drawing a segment from the origin to the complex number represented as a point, a right-angled triangle is identified with an angle, its adjacent side, its opposite side, and hypotenuse (Figure 6.6).

The angle, shown in red, is the **argument** or **position angle**, θ, and the distance r from the origin to the point is its absolute value, otherwise known as its **modulus** or **magnitude.** This description is called its **modulus-argument** or **polar form**.

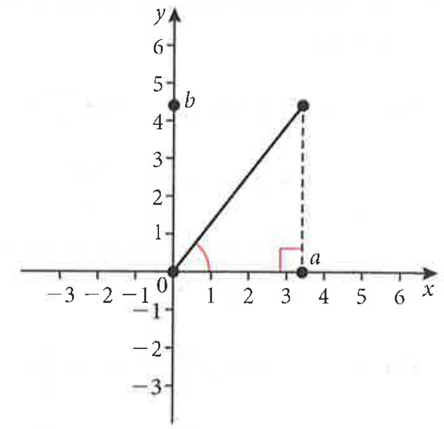


Figure 6.6 A right-angled triangle is identified

In the right-angled triangle with (position) angle θ and hypotenuse r;

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

$$a + bi = r\cos\theta + r\sin\theta \cdot i = r(\cos\theta + i \cdot \sin\theta)$$

This expression is often written in shorthand as $r \operatorname{cis} \theta$

Although the addition and subtraction of complex numbers is simple in $z = a + bi$ form, the polar form is often easier for multiplication and division, when we use trigonometric identities. Consider the multiplication of two complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$

In polar form, they are $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$

Their product is

$$z_1z_2 = (r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2)$$

$$= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) + i^2\sin\theta_1\sin\theta_2)$$

$$= r_1r_2(\cos\theta_1\cos\theta_2 + i \cdot \cos\theta_1\sin\theta_2 + i \cdot \sin\theta_1\cos\theta_2, \text{ but } i^2 = -1, \text{ so}$$

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

Since $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1 + \theta_2)$ and

$\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 = \sin(\theta_1 + \theta_2)$

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = r_1r_2[\operatorname{cis}(\theta_1 + \theta_2)]$$

Simply put, the product of two complex numbers in polar form contains the product of their moduli (r_1r_2) and the sum of their arguments $(\theta_1 + \theta_2)$ Geometrically, when $z_1 = r_1\operatorname{cis}\theta_1$ is multiplied by z_2, the complex number z_1 is rotated by θ_2 and stretched by the scalar r_2.

Example 6.12

Find the products of the complex numbers in polar form.

(a) $z_1 = 2 \operatorname{cis}\frac{\pi}{6}$ and $z_2 = 4\operatorname{cis}\frac{5\pi}{6}$ (b) $z_1 = 6\operatorname{cis}\frac{\pi}{3}$ and $z_2 = 10\operatorname{cis}\frac{\pi}{6}$

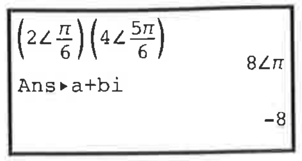


Figure 6.7 GDC output for the solution to Example 6.12 (a)

Solution

(a) $z_1 \cdot z_2 = \left(2 \operatorname{cis}\frac{\pi}{6}\right)\cdot\left(4 \operatorname{cis}\frac{5\pi}{6}\right)$

$= 2 \cdot 4 \operatorname{cis}\left(\frac{\pi}{6} + \frac{5\pi}{6}\right)$

$= 8 \operatorname{cis}\pi = 8(\cos\pi + \mathrm{i}\sin\pi)$

$= -8$

(b) $z_1 \cdot z_2 = \left(6 \operatorname{cis}\frac{\pi}{3}\right)\cdot\left(10 \operatorname{cis}\frac{\pi}{6}\right)$

$= 60 \operatorname{cis}\frac{\pi}{2} = 60\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$

$= 60\mathrm{i}$

Now, consider the quotient of two complex numbers $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$

Their quotient is

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis}\theta_1}{r_2 \operatorname{cis}\theta_2}$$

$$= \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)}$$

Multiply both numerator and denominator by z_2^*

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)}\cdot\frac{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2 - \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 - \mathrm{i}^2\sin\theta_1\sin\theta_2)}{\cos^2\theta_2 - \mathrm{i}^2\sin^2\theta_2}$$

but $\mathrm{i}^2 = -1$,

$$\text{so } \frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

Since $\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2)$, and

$\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 = \sin(\theta_1 - \theta_2)$, while

$\cos^2\theta_2 + \sin^2\theta_2 = 1$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot[\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2}\cdot\operatorname{cis}(\theta_1 - \theta_2)$$

The quotient of two complex numbers in polar form uses the quotient of their moduli $\frac{r_1}{r_2}$ and the difference of their arguments $(\theta_1 - \theta_2)$. Intuitively, the quotient should involve the opposite operations from the product.

Example 6.13

Find the quotient $\frac{z_1}{z_2}$. State your answer in $a + bi$ form.

(a) $z_1 = 12 \operatorname{cis}\frac{2\pi}{3}$ and $z_2 = 4 \operatorname{cis}\frac{\pi}{6}$

(b) $z_1 = 8 \operatorname{cis}\frac{5\pi}{6}$ and $z_2 = 2 \operatorname{cis}\frac{\pi}{6}$

Solution

(a) $\frac{z_1}{z_2} = \frac{12 \operatorname{cis}\frac{2\pi}{3}}{4 \operatorname{cis}\frac{\pi}{6}} = \frac{12}{4}\operatorname{cis}\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) = 3 \operatorname{cis}\frac{\pi}{2} = 3\mathrm{i}$

(b) $\frac{z_1}{z_2} = \frac{8 \operatorname{cis}\frac{5\pi}{6}}{2 \operatorname{cis}\frac{\pi}{6}} = 4 \operatorname{cis}\frac{2\pi}{3} = -2 + 2\sqrt{3}\mathrm{i}$

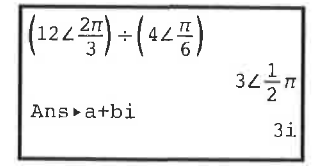


Figure 6.8 GDC output for the solution to Example 6.13 (a)

Powers of complex numbers

Consider the square of $z = a + bi = r \operatorname{cis}\theta$:

$(a + bi)^2 = (r \operatorname{cis}\theta)^2 = (r \operatorname{cis}\theta)(r \operatorname{cis}\theta) = r^2 \operatorname{cis} 2\theta$

Exploring higher powers in this fashion,

$(a + bi)^3 = (a + bi)^2 \cdot (a + bi) = (r^2 \operatorname{cis} 2\theta)\cdot(r \operatorname{cis}\theta) = r^3 \operatorname{cis} 3\theta$

To generalise, $(a + bi)^n = r^n \operatorname{cis} n\theta$. This is known as **de Moivre's theorem**.

Example 6.14

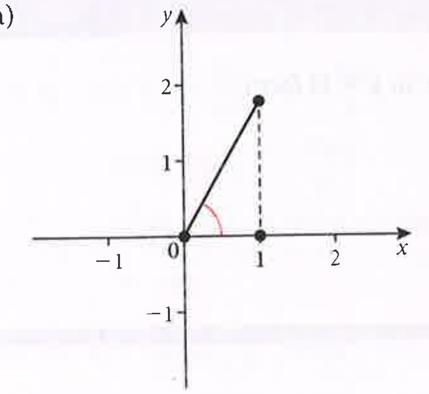
For each expression, raise the complex number to the power indicated and state the result in $a + bi$ form.

(a) $(1 + \sqrt{3}\mathrm{i})^6$ (b) $(2 + 2\mathrm{i})^4$

Solution

It is helpful to show the number on an Argand diagram.

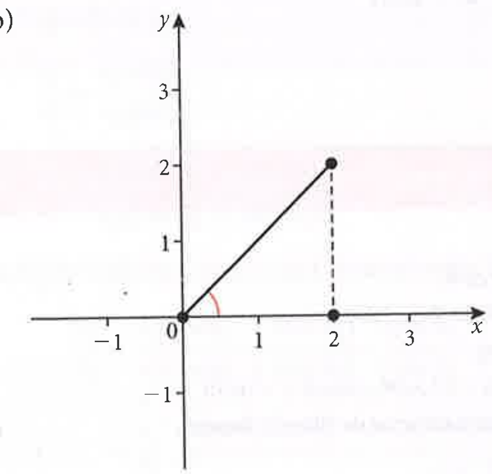
(a)



$1 + \sqrt{3}\,\mathrm{i}$: $r = \sqrt{1^2 + \sqrt{3}^2} = 2$ and $\theta = \arctan\frac{\sqrt{3}}{1} = 60°$ or $\frac{\pi}{3}$ radians, so

$$(1 + \sqrt{3}\,\mathrm{i})^6 = \left(2\operatorname{cis}\frac{\pi}{3}\right)^6 = 2^6 \operatorname{cis} 2\pi$$
$$= 64(\cos 2\pi + \mathrm{i}\sin 2\pi) = 64$$

(b)



$2 + 2\mathrm{i}$: $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and

$\theta = \arctan 1 = 45°$ or $\frac{\pi}{4}$ radians, so

$$(2 + 2\mathrm{i})^4 = \left(2\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^4 = (2\sqrt{2})^4 \operatorname{cis}\frac{4\pi}{4}$$
$$= 64(\cos\pi + \mathrm{i}\sin\pi) = -64$$

Roots of complex numbers

Now, take a close look at part (a) of Example 6.14

If $(1 + \sqrt{3}\,\mathrm{i})^6 = 64$ and $(2 + 2\mathrm{i})^4 = -64$, then $\sqrt[6]{64} = 1 + \sqrt{3}\,\mathrm{i}$ and $\sqrt[4]{-64} = 2 + 2\mathrm{i}$

We can use de Moivre's theorem to verify that the results are correct. Apply de Moivre's theorem, which is assumed to be true for rational numbers $n = \frac{1}{k}$ as well, to verify that $\sqrt[6]{64} = 1 + \sqrt{3}\,\mathrm{i}$

This topic is not required for exams. It is mentioned here as an application of what you learned.

64 as a complex number is $64 + 0\mathrm{i}$ which is merely a point on the x-axis.

$$\Rightarrow r = 64 \text{ and } \theta = 0, \text{ so } \sqrt[6]{64} = (64 \operatorname{cis} 0)^{\frac{1}{6}} = 64^{\frac{1}{6}}(\cos 0 + \mathrm{i}\sin 0) = 2$$

This is disappointing. What happened to $\sqrt[6]{64} = 1 + \sqrt{3}\,\mathrm{i}$?

A little bit of trigonometry will help. Recall that $\sin\theta$ and $\cos\theta$ are periodic functions. Adding a multiple of 360° or 2π just produces another co-terminal angle.

Thus, $\sin\theta = \sin(\theta + 2k\pi)$ and $\cos\theta = \cos(\theta + 2k\pi)$ where $k \in \mathbb{Z}$

$$\sqrt[6]{64} = (64 \operatorname{cis} 0)^{\frac{1}{6}} = (64 \operatorname{cis}(0 + 2k\pi))^{\frac{1}{6}} = 64^{\frac{1}{6}}\left(\cos\frac{0 + 2k\pi}{6} + \mathrm{i}\sin\frac{0 + 2k\pi}{6}\right)$$

$= 2\left(\cos\frac{k\pi}{3} + \mathrm{i}\sin\frac{k\pi}{3}\right)$ for $k \in \{0, 1, 2, 3, 4, 5\}$ which produces six possibilities

$$= \{2, 1 + \sqrt{3}\mathrm{i}, -1 + \sqrt{3}\,\mathrm{i}, -2, -1 - \sqrt{3}\,\mathrm{i}, 1 - \sqrt{3}\mathrm{i}\}$$

Online

Visualise the roots of a complex number.

We can also analyse $\sqrt[4]{-64}$ in a similar way.

-64 as a complex number is $-64 + 0\mathrm{i}$ which is also a point on the x-axis in the opposite direction, but the distance is still $64 \Rightarrow r = 64$ and $\theta = \pi$

$$\sqrt[4]{-64} = 64 \operatorname{cis} \pi^{\frac{1}{4}} = (64 \operatorname{cis}(\pi + 2k\pi))^{\frac{1}{4}}$$
$$= 64^{\frac{1}{4}}\left(\cos\frac{\pi + 2k\pi}{4} + \mathrm{i}\sin\frac{\pi + 2k\pi}{4}\right)$$

$= 2\sqrt{2}\left(\cos\frac{\pi + 2k\pi}{4} + \mathrm{i}\sin\frac{\pi + 2k\pi}{4}\right)$ for $k \in \{0, 1, 2, 3\}$ which produces

$$= \left\{\begin{matrix} 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right), 2\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right), \\ 2\sqrt{2}\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right), 2\sqrt{2}\left(\cos\frac{7\pi}{4} + \mathrm{i}\sin\frac{7\pi}{4}\right) \end{matrix}\right\}$$

$$= \{2 + 2\mathrm{i}, -2 + 2\mathrm{i}, -2 - 2\mathrm{i}, 2 - 2\mathrm{i}\}$$

The Euler form of complex numbers

Euler form is also known as Exponential form.

There is yet another form that complex numbers can take: the **Euler form.** Every complex number can be expressed as $z = r \cdot e^{\mathrm{i}\theta}$. Unfortunately, the change from $r \operatorname{cis} \theta$ to $r \cdot e^{\mathrm{i}\theta}$ requires some understanding of infinite series, which is not part of this course.

If a complex number is in Euler form, a GDC will readily convert it into Cartesian form. Consider the complex number $z = 1 + \mathrm{i}$

A quick mental sketch should show an isosceles right-angled triangle with $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$; hence, $z = \sqrt{2}e^{\frac{i\pi}{4}}$

On a GDC, enter $z = \sqrt{2}e^{\frac{i\pi}{4}}$, then press ENTER:

√2e^(iπ/4)
1+1i

Now, consider $z = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^3$

$= 2\sqrt{2}e^{\frac{3i\pi}{4}}$ but a GDC is faster

By default, your GDC is set to accept input in $r \cdot e^{i\theta}$ form and produce results in $a + bi$ form.

(√2e^(iπ/4))^3
-2+2i

If you wish to go from $a + bi$ form to Euler form quickly, some GDCs allow you to change the settings to make this possible.

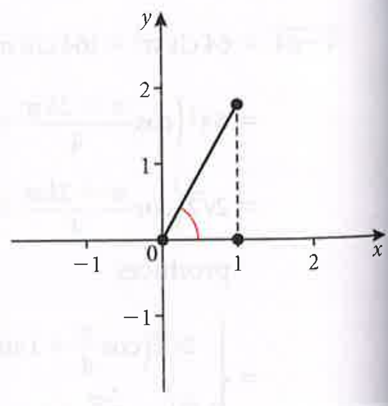
Example 6.15

(a) Convert $1 + \sqrt{3}\mathrm{i}$ to the forms: $r \operatorname{cis}\theta$ and $r \cdot e^{i\theta}$

(b) Use the Euler form to find $(1 + \sqrt{3}\mathrm{i})^6$

Solution

First of all, an Argand diagram is useful.



$r = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\theta = \arctan\frac{\sqrt{3}}{1} = 60°$ or $\frac{\pi}{3}$ radians

(a) $1 + \sqrt{3}\mathrm{i} = 2\operatorname{cis}\frac{\pi}{3}$ or $2e^{\frac{i\pi}{3}}$

(b) $(1 + \sqrt{3}\mathrm{i})^6 = \left(2e^{\frac{i\pi}{3}}\right)^6 = 64e^{2\pi i} = 64$

Consider how much easier it would be to explain the multiplication and division of complex numbers when r and θ are known. No trigonometric identities are required, just basic exponent rules.

Given $\begin{cases} z_1 = r_1 \operatorname{cis}\theta_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 \operatorname{cis}\theta_2 = r_2 e^{i\theta_2}, \end{cases}$ their product and quotient respectively are

$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 \cdot e^{i(\theta_1 + \theta_2)}$ and $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$

Exercise 6.4

1. Express each complex number in polar form.

(a) $3 + 3\mathrm{i}$ (b) $-3 + 3\mathrm{i}$ (c) $3 - 3\mathrm{i}$

(d) $-3 - 3\mathrm{i}$ (e) $5 + 5\sqrt{3}\mathrm{i}$ (f) $5\sqrt{3} + 5\mathrm{i}$

(g) $-5 + 5\sqrt{3}\mathrm{i}$ (h) $-5\sqrt{3} - 5\mathrm{i}$

2. Find the products of each pair of complex numbers. State your answer in $a + bi$ form.

(a) $z_1 = 5\operatorname{cis}\frac{\pi}{3}$ and $z_2 = 3\operatorname{cis}\frac{\pi}{6}$

(b) $z_1 = 4\operatorname{cis}\frac{2\pi}{3}$ and $z_2 = 2\operatorname{cis}\frac{2\pi}{3}$

3. Find the quotient $\frac{z_1}{z_2}$. State your answer in $a + bi$ form.

(a) $z_1 = 6\operatorname{cis}\frac{\pi}{2}$ and $z_2 = 2\operatorname{cis}\frac{\pi}{6}$

(b) $z_1 = 16\operatorname{cis}\frac{3\pi}{2}$ and $z_2 = 4\operatorname{cis}\frac{\pi}{6}$

(c) $z_1 = 8\operatorname{cis}\frac{\pi}{4}$ and $z_2 = 2\operatorname{cis}\frac{\pi}{2}$

(d) $z_1 = 16\operatorname{cis}\frac{\pi}{6}$ and $z_2 = 2\operatorname{cis}\frac{\pi}{3}$

4. For each expression, raise the complex number to the power indicated and state the result in $a + bi$ form.

(a) $(-1 + \mathrm{i})^5$ (b) $(-1 - \mathrm{i})^4$ (c) $(-\sqrt{3} + \mathrm{i})^4$ (d) $(-\sqrt{3} - \mathrm{i})^6$

5. Find each root.

(a) $\sqrt[3]{8}$ (b) $\sqrt{(1 + \sqrt{3}\,\mathrm{i})}$ (c) $\sqrt[4]{-16}$

6. Write each complex number in Euler form.

(a) $-6 + 6\mathrm{i}$ (b) $2\sqrt{3} + 6\mathrm{i}$ (c) $1 - \sqrt{3}\mathrm{i}$

7. Find the product of the complex numbers in Euler form.

(a) $z_1 = 3e^{\frac{\pi i}{8}}$ and $z_2 = 2e^{\frac{3\pi i}{8}}$ (b) $z_1 = 4e^{\frac{3\pi i}{4}}$ and $z_2 = 3e^{\frac{\pi i}{2}}$

8. Find the quotient $\frac{z_1}{z_2}$ of these complex numbers and give the answer in $a + bi$ form.

(a) $z_1 = 12e^{\frac{3\pi i}{4}}$ and $z_2 = 3e^{\frac{\pi i}{8}}$ (b) $z_1 = 16e^{\frac{3\pi i}{2}}$ and $z_2 = 2e^{\frac{\pi i}{4}}$

9. This chapter started with $i = \sqrt{-1}$. This time, consider $\sqrt{i}$.

   (a) How many roots should there be?

   (b) Find them in $a + bi$ form.

   (c) Are these roots the negative of the roots of $\sqrt{-i}$? Explain your answer.

10. Starting with the complex number $z = -1 + 0i$, show how Euler's formula $e^{i\pi} + 1 = 0$ can be found by expressing in Euler form.

6.5 Applications of complex numbers

The application of complex numbers within mathematics can be found in topics such as differential equations and eigenvalues; however, the discussion of these topics requires an understanding of mathematics beyond what has been covered in the HL syllabus to this point. Complex numbers are used in fluid dynamics, control theory, quantum mechanics, and Fourier transforms, for example. We will look at the use of complex numbers in vectors and alternating current (AC) electrical circuits. Vectors are covered in chapter 8. We will consider impedance in AC circuits here.

Consider an AC waveform, commonly known as a **sinusoidal** curve (Figure 6.9).

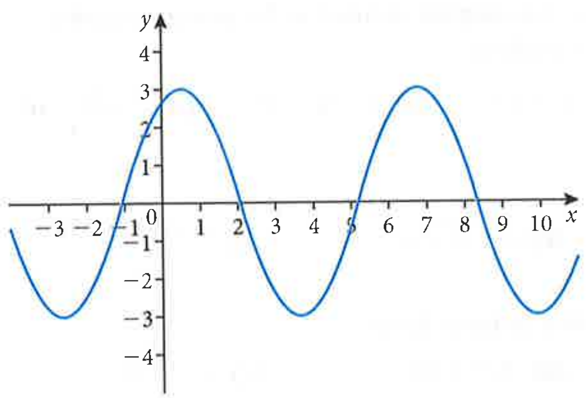


Figure 6.9 Sinusoidal curve

All AC waveforms have three distinguishing features. Each waveform:

- is periodic, repeating after a cycle is completed, over a span of n degrees or radians. The number of cycles per second (frequency) is given in hertz (Hz).
- has an **amplitude**, which is the measure of its potential difference or current
- has a specific starting point, called the **phase**, typically measured in degrees relative to its basic waveform which is generally set at the origin.

The frequency for AC circuits is constant and is generally set at 50 Hz or 60 Hz, and can be taken out of consideration. However, in circuits, both the potential difference/current and phase do vary. Since there are then two components to be considered, AC circuits can and are modelled well by complex numbers.

Consider the potential difference given as a sinusoidal function, $V = 10\cos\left(\omega t + \frac{\pi}{4}\right)$ whose potential difference (amplitude) is 10, with a phase angle of $\frac{\pi}{4}$. These characteristics can be illustrated by the Argand diagram shown.

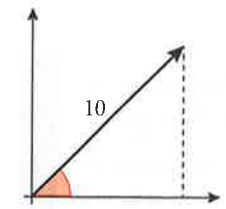


Figure 6.10 Potential difference in the complex plane

In electrical notation, this is stated as $V = 10\angle 45°$ where the phase angle is always measured in degrees. In Euler form, it is $V = 10e^{45°i}$. As you can see from the simple right-angled triangle diagram above, the real component is $5\sqrt{2}$ as is the imaginary component. The potential difference with phase shift is described by the complex number $5\sqrt{2} + 5\sqrt{2}i$

Multiple sinusoidal functions

When more than one AC source is placed in a series circuit, the difference in phase between them can be resolved by considering the complex number representation of each source, then added.

Example 6.16

Three AC sources are placed in series, with potential differences $V_1 = \cos\omega t$, $V_2 = 4\cos\left(\omega t + \frac{\pi}{4}\right)$, and $V_3 = 6\left(\cos\omega t + \frac{\pi}{2}\right)$

Find the components of the total, $V = V_1 + V_2 + V_3$

Solution

In electrical notation, $V_1 = 1\angle 0°$, $V_2 = 4\angle 45°$, and $V_3 = 6\angle 90°$

$$\Rightarrow V = 1 + (2\sqrt{2} + 2\sqrt{2}i) + 6i = (1 + 2\sqrt{2}) + (6 + 2\sqrt{2})i$$

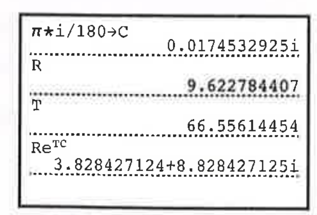
$$\Rightarrow |V| = \sqrt{(1 + 2\sqrt{2})^2 + (6 + 2\sqrt{2})^2} \approx 9.62$$

$$\text{and } \arctan\left(\frac{6 + 2\sqrt{2}}{1 + 2\sqrt{2}}\right) \approx 66.6°$$

$$\therefore V = 9.62\angle 66.6° \text{ or } 9.62e^{66.6°i}$$

To convert your answer to $a + bi$ form, use your GDC.

In $a + bi$ form, $V \approx 3.83 + 8.83i$



Impedance – complex variables used in electrical theory

Physicists use j to represent the imaginary unit i to avoid confusing i with I, the current in a circuit.

Resistance (R) is a measure used in direct current (DC) circuits. In AC circuits, **impedance** (Z) is the measure of resistance that includes reactance due to capacitance (X_C) and inductance (X_L). The formula $V = IR$ becomes $V = IZ$ with AC circuits.

Potential difference across a resistor is noted as V_R, across a capacitor, V_C, and across an inductor, V_L. Both V_C and V_L are considered imaginary components, with the symbol j used to denote the imaginary unit in electrical theory.

Potential difference across a resistor V_R is in phase with the current; however, potential difference across a capacitor V_C lags, and potential difference across an inductor V_L leads. Hence, impedance creates a shift measured in degrees, named the **phase angle**, with $\theta = \arctan \frac{X_L - X_C}{R}$. Note how θ is calculated as $\arctan\left(\frac{\text{imaginary component}}{\text{real component}}\right)$ just as for complex numbers in the complex plane. In electrical notation, Z represents impedance, and $Z = R + \mathrm{j}(X_L - X_C)$. This is basically the form $z = a + b\mathrm{i}$

Example 6.17

A particular AC circuit has a resistor of 6 Ω, a reactance across an inductor of 11 Ω and a reactance across a capacitor of 3 Ω.

(a) Express the impedance of the circuit as a complex number in Cartesian form.

(b) Express the impedance in Euler form, with θ given in degrees, correct to 3 significant figures.

Solution

(a) $R = 6, X_L = 11, X_C = 3$, so $Z = 6 + \mathrm{j}(11 - 3) = 6 + 8\mathrm{j}$ (this compares to $z = 6 + 8\mathrm{i}$)

(b) $|Z| = \sqrt{6^2 + 8^2} = 10$ and $\theta = \arctan \frac{8}{6} \approx 53.1°$. Hence, $Z = 10\,\mathrm{e}^{53.1°\mathrm{i}}$

This angle measurement may not work with some calculators which expect the angle measurement in radians. Here is a possible conversion:

Store a simple conversion factor into your GDC to convert from degrees to radians. The imaginary value was stored in C and can be reused.

```
π*i/180→C
                0.01745329251i
```

As a simple example, take $1 + \mathrm{i}$ which has $r = \sqrt{2}$ and $\theta = 45°$, so $z = \sqrt{2}\mathrm{e}^{45°\mathrm{i}}$

```
π*i/180→C
                0.01745329251i
√2e^45C
                          1+1i
```

Example 6.18

A resistor, an inductor, and a capacitor are connected in series in an AC circuit, with potential differences across them of 8.0 V, 10.5 V, and 4.5 V respectively. What is the potential difference (EMF) of the source?

Solution

$V_R = 8,\ V_L = 10.5$, and $V_C = 4.5$

Since $V = V_R + \mathrm{j}\,(V_L - V_C)$

$= 8.0 + (10.5 - 4.5)\mathrm{j} = 8 + 6\mathrm{j}$

The magnitude of the source potential difference is $|V| = \sqrt{8^2 + 6^2} = 10$, that is, 10 V.

Example 6.19

The current in a given AC circuit is $4.1 - 5.3\mathrm{j}$ A and the impedance is $6.2 + 2.3\mathrm{j}$ Ω. What is the magnitude of the potential difference?

Solution

Since $V = IZ$, $|V| = |I| \cdot |Z| = \sqrt{4.1^2 + 5.3^2} \cdot \sqrt{6.2^2 + 2.3^2} \approx 44.3$ V

Impedance in parallel circuits

The work with complex numbers becomes much more useful when **parallel** circuits are analysed. The resistance of three resistors connected in parallel is given by: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. So, in AC circuits, impedance is defined as $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$

Consider when there are two impedances in a parallel circuit, Z_1 and Z_2:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \Rightarrow \frac{1}{Z} = \frac{Z_1 + Z_2}{Z_1 Z_2} \Rightarrow Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$ which is the product of two complex numbers divided by their sum. The addition of complex numbers is easiest in $a + bi$ form, but their multiplication would be simpler in polar or Euler form.

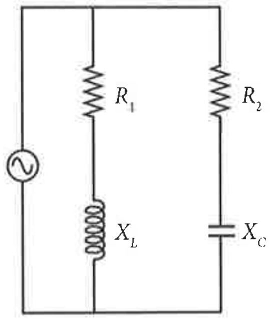


Figure 6.11 Circuit diagram for Example 6.20

Example 6.20

A parallel AC circuit has two loops. The first has a resistor with $R_1 = 80\,\Omega$ and an inductor with $X_L = 60\,\Omega$, while the second has a resistor with $R_2 = 12\,\Omega$ and a capacitor with $X_C = 5\,\Omega$.

Find the combined impedance.

Solution

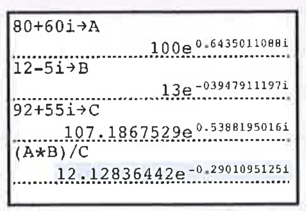
In the first loop, $Z_1 = 80 + 60j$, and in the second, $Z_2 = 12 - 5j$

The combined impedance is $Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$, so we need to find their product and their sum. Again, since the sum is much easier in $a + bi$ form, start with the denominator. $Z_1 + Z_2 = 92 + 55j$

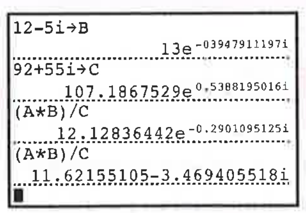
Next convert Z_1, Z_2, and $Z_1 + Z_2$ with your GDC into Euler form:

Change the output mode to Euler form, then enter the impedances.

Note that the angle is in radians, but that setting is kept until the final conversion.



Now, reset the mode to its default to produce an answer in $a + bi$ form.



To 3 significant figures, this complex number is $11.6 - 3.47i$

i.e. $Z = 11.6 - 3.47j$

Exercise 6.5

1. A series AC circuit has a resistor of $R = 12\,\Omega$, a reactance across an inductor of $X_L = 12\,\Omega$ and a reactance across a capacitor of $X_C = 3\,\Omega$.
   (a) Express the impedance of the circuit as a complex number in Cartesian form.
   (b) Express the impedance in Euler form, with θ given in degrees to 3 significant figures.
2. Find the impedance of a series AC circuit with $R = 4\,\Omega$, $X_L = 2\,\Omega$, and $X_C = 5\,\Omega$ in Cartesian form.
3. A resistor, an inductor, and a capacitor in a series AC circuit have potential differences across the of 9.0 V, 15.0 V, and 3.0 V respectively. What is the potential difference of the source?
4. The potential differences across a resistor ($V_R = 6$ V), an inductor ($V_L = 11.5$ V), and capacitor ($V_C = 3.5$ V) are individually measured in a series circuit. Find the potential difference at the source.
5. The current in a given AC circuit is $6 - 3j$ A and the impedance is $8 + 4j\,\Omega$. What is the magnitude of the potential difference? [Remember $|V| = |I| \cdot |Z|$]
6. The potential difference across a given AC circuit is 100 V and the impedance is $4 - 3j\,\Omega$. What is the magnitude of the current?
7. A parallel AC circuit has two loops. The first has a resistor with $R_1 = 12\,\Omega$ and an inductor with $X_L = 5\,\Omega$, while the second has a resistor with $R_2 = 8\,\Omega$ and a capacitor with $X_C = 6\,\Omega$
   Find the combined impedance.
8. In the circuit in question 7, $R_1 = R_2 = 8\,\Omega$, $X_L = 6\,\Omega$ and $X_C = 6\,\Omega$
   What is the impedance?

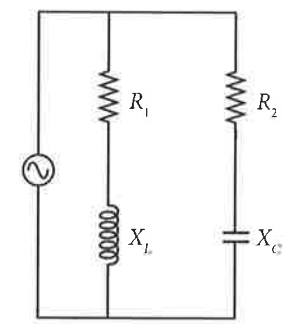


Figure 6.12 Circuit diagram for question 7

Chapter 6 practice questions

1. Express each value using the imaginary unit i.
   (a) $\sqrt{-49}$ **(b)** $\sqrt{-18}$
   (c) $\sqrt{-9} \cdot \sqrt{-1}$ **(d)** $\sqrt{-12} \cdot \sqrt{-27}$
2. Express each as a complex number in the form $a + bi$
   (a) $\sqrt{-16}$ **(b)** $25 + \sqrt{-25}$ **(c)** $5 + \sqrt{-25}$
   (d) $-3\sqrt{2} + \sqrt{-18}$ **(e)** $2\sqrt{3} - \sqrt{-12}$ **(f)** $-\sqrt{i^2}$
   (g) $-\sqrt{i^4}$ **(h)** $(-3 + \sqrt{-9})\,i$

3. Express each value in simplest form.

(a) i^{22} (b) i^{21} (c) i^{20} (d) i^{19}

4. Find the zeros of each function.

(a) $y = x^2 + 4x + 8$ (b) $y = x^2 - 6x + 10$

(c) $y = x^2 - 8x + 25$ (d) $y = x^2 - 4x + 8$

(e) $y = x^2 - 10x + 29$ (f) $y = x^2 + 8x + 32$

5. Find the sums of the complex numbers.

(a) $(5 - i)$ and $(-4 + 3i)$ (b) $(4 + 2i)$ and $(-2 + i)$

(c) $(\sqrt{5} - 3i)$ and $(2\sqrt{5} + i)$

6. Find the products of the complex numbers.

(a) $(6 - 4i)(6 + 4i)$ (b) $(7 + 2i)(-7 + 2i)$

(c) $(3\sqrt{3} + i)(3\sqrt{3} - i)$

7. Find the product of each complex number and its conjugate.

(a) $9 + 12i$ (b) $6 - 8i$ (c) $3 - 3\sqrt{2}i$

8. Express each rational expression in $a + bi$ form.

(a) $\frac{10}{3 + i}$ (b) $\frac{14}{\sqrt{3} - 2i}$ (c) $\frac{1 - 3i}{-3 + 9i}$

9. A quadratic function $y = ax^2 + bx + c$ has real coefficients a, b, and c. Find the function if one of the zeros is known.

(a) $1 + 2i$ (b) $4 + 3i$ (c) $-3\sqrt{2} + \sqrt{2}i$

10. Find the quadratic function of the form $y = x^2 + bx + c$ with the given zeros.

(a) $(-1 - i)$ and $(-1 + i)$ (b) $(2 + \sqrt{2}i)$ and $(2 - \sqrt{2}i)$

(c) $(3 + 2i)$ and $(4 - i)$

11. Plot a complex number in each of the four quadrants, each with a modulus of 10.

12. If $2\sqrt{2}$ is the imaginary component of a complex number also with a modulus of 10, what are the possible values of its real component?

13. On the complex plane, connect the origin O to any two points $a + bi$ and $c + di$ then construct a parallelogram with sides parallel to those segments. What is the significance of the two diagonals of the parallelogram?

14. Express each complex number in polar form.

(a) $2 - 2i$ (b) $-2 - 2i$ (c) $2 + 2i$

(d) $-2 + 2i$ (e) $2 + 2\sqrt{3}i$ (f) $2 - 2\sqrt{3}i$

(g) $-2 + 2\sqrt{3}i$ (h) $-2 - 2\sqrt{3}i$

15. Find the products of these complex numbers. State your answer in the form $a + bi$

(a) $z_1 = 6 \text{ cis} \frac{\pi}{2}$ and $z_2 = 2 \text{ cis} \frac{3\pi}{4}$ (b) $z_1 = 8 \text{ cis} \frac{\pi}{3}$ and $z_2 = 2 \text{ cis} \frac{\pi}{6}$

16. Find the quotient $\frac{z_1}{z_2}$ of these complex numbers and give your answer in the form $a + bi$

(a) $z_1 = 9 \text{ cis} \frac{3\pi}{4}$ and $z_2 = 3 \text{ cis} \frac{\pi}{4}$ (b) $z_1 = 10 \text{ cis} \frac{5\pi}{4}$ and $z_2 = 2 \text{ cis} \frac{\pi}{2}$

(c) $z_1 = 2 \text{ cis} \frac{\pi}{3}$ and $z_2 = 4 \text{ cis} \frac{\pi}{6}$ (d) $z_1 = 12 \text{ cis} \frac{\pi}{4}$ and $z_2 = 3 \text{ cis} \frac{\pi}{2}$

17. Raise the complex number to the power indicated and state the result in the form $a + bi$

(a) $(1 + i)^4$ (b) $(1 - i)^4$ (c) $(-1 + \sqrt{3}i)^3$ (d) $(2\sqrt{3} + 2i)^4$

18. Find the indicated roots.

(a) $\sqrt[3]{-8}$ (b) $\sqrt[3]{64}$ (c) $\sqrt{(-3 + 3\sqrt{3}i)}$

19. Write each complex number in Euler form.

(a) $1 - i$ (b) $-2 - 2i$ (c) $-\sqrt{3} + 3i$

20. Find the products of these complex numbers and give the answer in the form $a + bi$

(a) $z_1 = 2e^{\frac{\pi i}{6}}$ and $z_2 = 5e^{\frac{3\pi i}{2}}$ (b) $z_1 = 2e^{\frac{2\pi i}{3}}$ and $z_2 = 4e^{\frac{\pi i}{6}}$

21. Find the quotients $\frac{z_1}{z_2}$ of these complex numbers and give the answer in the form $a + bi$

(a) $z_1 = 10e^{\frac{3\pi i}{2}}$ and $z_2 = 2e^{\frac{\pi i}{4}}$ (b) $z_1 = 12e^{\frac{\pi i}{2}}$ and $z_2 = 3e^{\frac{3\pi i}{4}}$

22. In a series AC circuit there is a resistor of $R = 12\,\Omega$, a reactance across an inductor of $X_L = 3\,\Omega$ and a reactance across a capacitor of $X_C = 8\,\Omega$.

(a) Express the impedance of the circuit as a complex number in the form $a + bi$

(b) Express the impedance in Euler form, with θ given in radians to 3 significant figures.

23. Determine the impedance in $a + bi$ form of a series AC circuit if it contains a resistor, inductor, and capacitor with $R = 4\,\Omega$, $X_L = 7\,\Omega$, and $X_C = 3\,\Omega$.

24. A resistor, an inductor, and a capacitor in a series AC circuit have potential differences across them of 12.0 V, 4.5 V, and 9.5 V respectively. What is the potential difference of the source?

25. The potential differences across a resistor, an inductor, and capacitor are individually measured in a series circuit and found to be $V_R = 8$ V, $V_L = 4.5$ V, and $V_C = 10.5$ V. Find the potential difference at the source.

26. The current in a given AC circuit is $I = 2 + \text{j}$ A. If the impedance is $X_L = 4 + 2\text{j}\ \Omega$, what is the magnitude of the potential difference?

27. The potential difference across a given AC circuit is 65 V and the impedance is $12 - 5\text{j}\ \Omega$. What is the magnitude of the current?

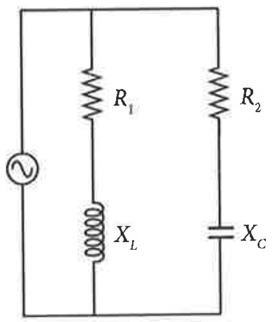


Figure 6.13 Circuit diagram for question 28

28. Consider the parallel circuit shown in the diagram. A 6 Ω resistor and 8 Ω inductor are in the first loop, and a 6 Ω resistor and a 3 Ω capacitor are in the second loop. Find their combined impedance.

29. Consider the complex numbers $u = 2 + 3\text{i}$ and $v = 3 + 2\text{i}$

(a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{10}{w}$, express w in the form $a + b\text{i}$, $a, b \in \mathbb{R}$

(b) Find w^* and express it in the form $r \cdot e^{\text{i}\theta}$

30. **(a)** Find three distinct roots of the equation $8z^3 + 27 = 0$, $z \in \mathbb{C}$ giving your answers in modulus-argument form.

(b) The roots are represented by the vertices of a triangle in an Argand diagram. Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$

31. Let $w = \cos\frac{2\pi}{7} + \text{i}\sin\frac{2\pi}{7}$

(a) Verify that w is a root of the equation $z^7 - 1 = 0$, $z \in \mathbb{C}$

(b) **(i)** Expand $(w - 1)(1 + w + w^2 + w^3 + w^4 + w^5 + w^6)$

(ii) Hence deduce that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

(c) Write down the roots of the equation $z^7 - 1 = 0$, $z \in \mathbb{C}$ in terms of w and plot these roots on an Argand diagram.

Consider the quadratic equation $z^2 + bz + c = 0$ where $b, c \in \mathbb{R}$, $z \in \mathbb{C}$. The roots of this equation are α and α^* where α^* is the complex conjugate of α.

(d) **(i)** Given that $\alpha = w + w^2 + w^4$, show that $\alpha^* = w^6 + w^5 + w^3$

(ii) Find the value of b and the value of c.

(e) Using the values for b and c obtained in part **(d) (ii)**, find the imaginary part of α, giving your answer in surd form.

32. One root of the equation $x^2 + ax + b = 0$ is $2 + 3\text{i}$ where $a, b \in \mathbb{R}$ Find the value of a and the value of b.