24. A resistor, an inductor, and a capacitor in a series AC circuit have potential differences across them of 12.0 V, 4.5 V, and 9.5 V respectively. What is the potential difference of the source?

25. The potential differences across a resistor, an inductor, and capacitor are individually measured in a series circuit and found to be $V_R = 8$ V, $V_L = 4.5$ V, and $V_C = 10.5$ V. Find the potential difference at the source.

26. The current in a given AC circuit is $I = 2 + \mathrm{j}$ A. If the impedance is $X_L = 4 + 2\mathrm{j}\ \Omega$, what is the magnitude of the potential difference?

27. The potential difference across a given AC circuit is 65 V and the impedance is $12 - 5\mathrm{j}\ \Omega$. What is the magnitude of the current?

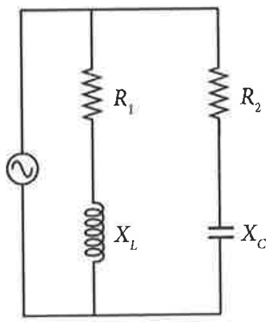


Figure 6.13 Circuit diagram for question 28

28. Consider the parallel circuit shown in the diagram. A 6 Ω resistor and 8 Ω inductor are in the first loop, and a 6 Ω resistor and a 3 Ω capacitor are in the second loop. Find their combined impedance.

29. Consider the complex numbers $u = 2 + 3\mathrm{i}$ and $v = 3 + 2\mathrm{i}$

 (a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{10}{w}$, express w in the form $a + b\mathrm{i}$, $a, b \in \mathbb{R}$

 (b) Find w^* and express it in the form $r \cdot e^{\mathrm{i}\theta}$

30. **(a)** Find three distinct roots of the equation $8z^3 + 27 = 0$, $z \in \mathbb{C}$ giving your answers in modulus-argument form.

 (b) The roots are represented by the vertices of a triangle in an Argand diagram. Show that the area of the triangle is $\frac{27\sqrt{3}}{16}$.

31. Let $w = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$

 (a) Verify that w is a root of the equation $z^7 - 1 = 0$, $z \in \mathbb{C}$

 (b) **(i)** Expand $(w - 1)(1 + w + w^2 + w^3 + w^4 + w^5 + w^6)$

 (ii) Hence deduce that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

 (c) Write down the roots of the equation $z^7 - 1 = 0$, $z \in \mathbb{C}$ in terms of w and plot these roots on an Argand diagram.

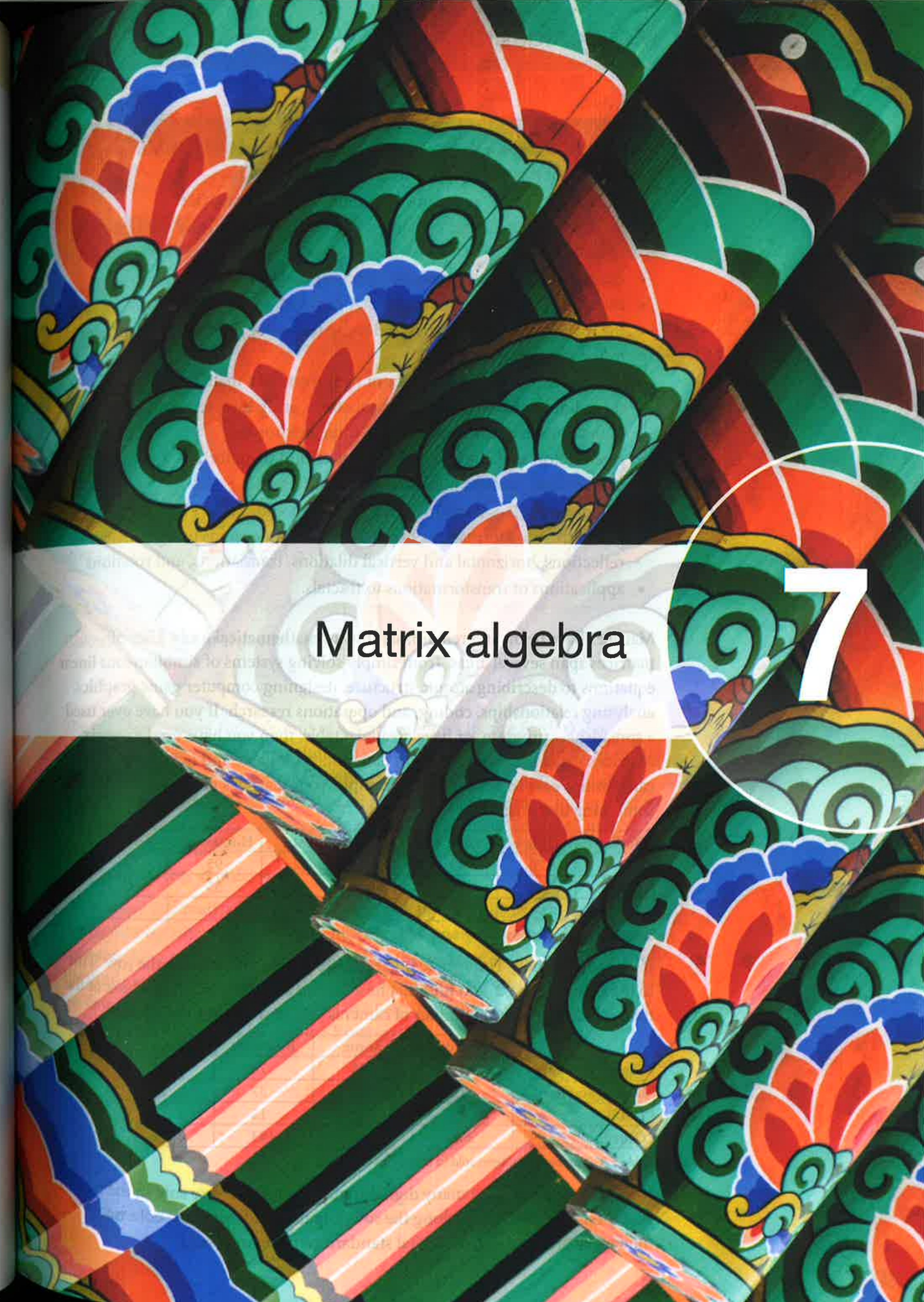
 Consider the quadratic equation $z^2 + bz + c = 0$ where $b, c \in \mathbb{R}$, $z \in \mathbb{C}$. The roots of this equation are α and α^* where α^* is the complex conjugate of α.

 (d) **(i)** Given that $\alpha = w + w^2 + w^4$, show that $\alpha^* = w^6 + w^5 + w^3$

 (ii) Find the value of b and the value of c.

 (e) Using the values for b and c obtained in part **(d)** **(ii)**, find the imaginary part of α, giving your answer in surd form.

32. One root of the equation $x^2 + ax + b = 0$ is $2 + 3\mathrm{i}$ where $a, b \in \mathbb{R}$ Find the value of a and the value of b.



Learning objectives

By the end of this chapter, you should be familiar with...

- a matrix, its order, and elements; identity and zero matrices
- the algebra of matrices: equality, addition, subtraction, and multiplication by a scalar
- multiplying matrices manually and using technology
- calculating the determinant of a 2×2 and a 3×3 square matrix
- the inverse of a 2×2 matrix and using technology to find the inverse of $n \times n$ matrices
- the conditions for the existence of the inverse of a matrix
- the solution of systems of linear equations using inverse matrices (a maximum of three equations in three unknowns)
- eigenvectors and eigenvalues and how to find them for 2×2 matrices
- characteristic polynomials for 2×2 matrices
- diagonalising 2×2 matrices and applying to powers of such matrices
- geometric transformations of points in two dimensions using matrices: reflections, horizontal and vertical dilations, translations, and rotations
- applications of transformations to fractals.

Matrices have been, and remain, significant mathematical tools. Uses of matrices span several areas, from simply solving systems of simultaneous linear equations to describing atomic structure, designing computer game graphics, analysing relationships, coding, and operations research. If you have ever used a spreadsheet program, or have created a table, then you have used a matrix. Matrices make the presentation of data understandable and help make calculations easy to perform. For example, your teacher's grade book may look something like this:

Student	Quiz 1	Quiz 2	Test 1	Test 2	Homework	Grade
Tim	70	80	86	82	95	A
Maher	89	56	80	60	55	C
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 7.1 Example of teacher's grade book

If we want to know Tim's grade on Test 2, we simply follow along the row 'Tim' to the column 'Test 2' and find that he achieved a mark of 82. Take a look at the matrix below about the number of cameras sold at shops in four cities.

	Venice	Rome	Budapest	Prague
Digital compact	153	98	74	56
Digital standard	211	120	57	29
DSLR	82	31	12	5
Other	308	242	183	107

Table 7.2 Number of cameras sold in four cities

If we want to know how many digital standard cameras were sold in the Budapest shop, we follow along the row 'Digital standard' to the column 'Budapest' and find that 57 digital standard cameras were sold.

7.1 Matrix definitions and operations

What is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers.

Matrix A is denoted by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{matrix} \Bigg\} m \text{ rows}$$

$$\underbrace{\uparrow \quad \uparrow \quad \cdots \quad \uparrow}_{n \text{ columns}}$$

Row i of A has n elements and is $(a_{i1} \quad a_{i2} \quad \cdots \quad a_{in})$

Column j of A has m elements and is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

The number of rows and columns of a matrix defines its size (order). So, a matrix that has m rows and n columns is said to have an $m \times n$ (m by n) order. A matrix A with $m \times n$ order is sometimes denoted as $[A]_{m \times n}$ or $[A]_{mn}$ to show that A is a matrix with m rows and n columns. (Sometimes $[a_{ij}]$ is used to represent a matrix.) The camera sales matrix has a 4×4 order. When $m = n$, the matrix is said to be a square matrix with order n, so the camera sales matrix is a square matrix of order 4.

Every entry in a matrix is called an **entry** or **element** of the matrix and is denoted by a_{ij}, where i is the row number and j is the column number of that element. The ordered pair (i, j) is also called the **address** of the element. So, in the grade book matrix example, the entry (2, 4) is 60, the student Maher's grade on Test 2, while (2, 4) in the camera sales matrix example is 29, the number of digital standard cameras sold in the Prague shop.

Vectors

A vector is a matrix that has only one row or one column. There are two types of vector: row vectors and column vectors.

Row vector

If a matrix has one row, it is called a row vector.

$B = (b_1 \quad b_2 \quad \dots \quad b_m)$ is a row vector with **dimension** m.

$B = (1 \quad 2)$ could represent the position of a point in a plane and is an example of a row vector of dimension 2.

Column vector

If a matrix has one column, it is called a column vector.

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ is a column vector with dimension n.

$C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ again could represent the position of a point in a plane and is an example of a column vector of **dimension** 2.

Vectors can be represented by row or column matrices.

Submatrix

If some row(s) and/or column(s) of a matrix A are deleted, the remaining matrix is called a **submatrix** of A.

For example, if we are interested in the sales of only the three main types of camera and only in Italian cities, we can represent them with the following submatrix of the original matrix:

$$\begin{pmatrix} 153 & 98 \\ 211 & 120 \\ 82 & 31 \end{pmatrix}$$

Submatrix

$$\begin{pmatrix} 153 & 98 & 74 & 56 \\ 211 & 120 & 57 & 29 \\ 82 & 31 & 12 & 5 \\ 308 & 242 & 183 & 107 \end{pmatrix}$$

Original matrix

Zero matrix

A zero matrix is one for which all entries are equal to zero, ($a_{ij} = 0$ for all i and j)

Some zero matrix examples: $(0 \quad 0)$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Diagonal matrix

In a square matrix, the entries $a_{11}, a_{22}, \dots, a_{mm}$ are called the **diagonal elements** of the matrix. Sometimes the diagonal of the matrix is also called the **principal** or **main diagonal** of the matrix.

What is the diagonal in our camera sales matrix?
Here $a_{11} = 153$, $a_{22} = 120$, $a_{33} = 12$, and $a_{44} = 107$

$$\begin{pmatrix} 153 & 0 & 0 & 0 \\ 0 & 120 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 107 \end{pmatrix}$$

Triangular matrix

You can use a matrix to show distances between different cities.

	Graz	Salzburg	Innsbruck	Linz
Vienna	191	298	478	185
Graz		282	461	220
Salzburg			188	135
Innsbruck				320

Table 7.3 Distance (in km) between Austrian cities.

The data in Table 7.3 can be represented by a triangular matrix. It is an upper triangular matrix, in this case.

$$\begin{pmatrix} 191 & 298 & 478 & 185 \\ 0 & 282 & 461 & 220 \\ 0 & 0 & 188 & 135 \\ 0 & 0 & 0 & 320 \end{pmatrix}$$

In a triangular matrix, the entries on one side of its diagonal are all zero.

A triangular matrix is a square matrix with order n for which $a_{ij} = 0$ when $i > j$ (upper triangular) or alternatively when $i < j$ (lower triangular).

Another way of representing the distance data is given by the following matrix.

	Vienna	Graz	Salzburg	Innsbruck	Linz
Vienna	0	191	298	478	185
Graz	191	0	282	461	220
Salzburg	298	282	0	188	135
Innsbruck	478	461	188	0	320
Linz	185	220	135	320	0

Again, the data in the table can be represented by a matrix called a **symmetric** matrix. In such matrices, $a_{ij} = a_{ji}$ for all i and j. All symmetric matrices are square.

$$\begin{pmatrix} 0 & 191 & 298 & 478 & 185 \\ 191 & 0 & 282 & 461 & 220 \\ 298 & 282 & 0 & 188 & 135 \\ 478 & 461 & 188 & 0 & 320 \\ 185 & 220 & 135 & 320 & 0 \end{pmatrix}$$

Matrix operations

Equal matrices

Two matrices A and B are equal if the orders of A and B are the same (number of rows and columns are the same for A and B) and $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $\begin{pmatrix} 2 & x \\ x^2 - 4 & 7 \end{pmatrix}$ are equal only if $x = 3$ and $x^2 - 4 = 5$ which can only be true if $x = 3$

Adding and subtracting matrices

We can add two matrices A and B only if they are the same size. If C is the sum of the two matrices, then $C = A + B$ where $c_{ij} = a_{ij} + b_{ij}$, so we add corresponding terms, one by one.

For example

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} x & y \\ a & b \end{pmatrix} = \begin{pmatrix} 2 + x & 3 + y \\ 5 + a & 7 + b \end{pmatrix}$$

We carry out subtraction in a similar way

$$\begin{pmatrix}2 & 3 & 1\\ 5 & 7 & 0\end{pmatrix} - \begin{pmatrix}x & y & 8\\ a & b & 2\end{pmatrix} = \begin{pmatrix}2-x & 3-y & -7\\ 5-a & 7-b & -2\end{pmatrix}$$

The operations of addition and subtraction of matrices obey all rules of algebraic addition and subtraction.

Multiplying a matrix by a scalar

A scalar is any object that is not a matrix. You multiply each term of the matrix by the scalar.

A is an $m \times n$ matrix, and c is a scalar. The scalar product of c and A is another matrix $B = cA$, such that every entry b_{ij} of B is a multiple of its corresponding entry in A. So, for every entry in B, we have $b_{ij} = c \times a_{ij}$

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the matrix below, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{pmatrix}\frac{1}{2} & -\frac{3}{2}\\ \frac{5}{2} & \frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -3\\ 5 & 1\end{pmatrix}$$

Matrix multiplication

At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

$A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix. The product AB is an $m \times p$ matrix $AB = [c_{ij}]$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

For the product of two matrices to be defined, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

A	B	$= AB$
$m \times n$	$n \times p$	$m \times p$

equal

order of AB

This definition means that each entry with an address ij in the product AB is obtained by multiplying the entries in the **ith row** of A by the corresponding entries in the **jth column** of B and then adding the results:

$$c_{ij} = (a_{i1} \quad a_{i2} \quad \ldots \quad a_{in})\begin{pmatrix}b_{1j}\\ b_{2j}\\ \vdots\\ b_{nj}\end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 7.1

Find $C = AB$ when $A = \begin{pmatrix}3 & -5 & 2\\ 2 & 1 & 7\end{pmatrix}$ and $B = \begin{pmatrix}3 & -2 & 1 & 5\\ 5 & 8 & -4 & 0\\ -9 & 10 & 5 & 3\end{pmatrix}$

Solution

A is a 2×3 matrix, B is a 3×4 matrix, so the product will be a 2×4 matrix. Every entry in the product is the result of multiplying the entries in the rows of A and columns of B. For example

$$c_{12} = \sum_{k=1}^{3} a_{1k}b_{k2} = (a_{11} \quad a_{12} \quad a_{13})\begin{pmatrix}b_{12}\\ b_{22}\\ b_{32}\end{pmatrix} = (3 \quad -5 \quad 2)\begin{pmatrix}-2\\ 8\\ 10\end{pmatrix}$$

$$= 3 \times (-2) - 5 \times 8 + 2 \times 10 = -26$$

and

$$c_{23} = \sum_{k=1}^{3} a_{2k}b_{k3} = (a_{21} \quad a_{22} \quad a_{23})\begin{pmatrix}b_{13}\\ b_{23}\\ b_{33}\end{pmatrix} = (2 \quad 1 \quad 7)\begin{pmatrix}1\\ -4\\ 5\end{pmatrix}$$

$$= 2 \times 1 + 1 \times (-4) + 7 \times 5 = 33$$

Repeat the operation for each entry in the solution matrix to get:

$$C = AB = \begin{pmatrix}-34 & -26 & 33 & 21\\ -52 & 74 & 33 & 31\end{pmatrix}$$

We can also use our GDC to find the product.

```
[A][B]
[[-34 -26 33 21…
 [-52 -74 33 31…
```

Figure 7.1 Using a GDC to find a matrix product

Here are some examples of matrix multiplication. Multiplying a 2×3 matrix by a 3×2 matrix results in a 2×2 product matrix.

$$\begin{pmatrix}5 & 0 & 3\\ -2 & 1 & 2\end{pmatrix}\begin{pmatrix}-2 & 4\\ 1 & -1\\ 3 & -2\end{pmatrix} = \begin{pmatrix}-1 & 14\\ 11 & -13\end{pmatrix}$$

$2 \times 3 \quad 3 \times 2 \quad 2 \times 2$

When matrices are the same size, the product is the same size.

$$\begin{pmatrix}4 & -5\\ 1 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}4 & -5\\ 1 & -7\end{pmatrix}$$

$2 \times 2 \quad 2 \times 2 \quad 2 \times 2$

$$\begin{pmatrix}5 & 0 & 3\\ -2 & 1 & 2\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}-\frac{1}{7} & -\frac{3}{7} & \frac{3}{7}\\ -\frac{10}{7} & -\frac{9}{7} & \frac{16}{7}\\ \frac{4}{7} & \frac{5}{7} & -\frac{5}{7}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$3 \times 3 \quad 3 \times 3 \quad 3 \times 3$

When a matrix of order 2 is multiplied by the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, the product is the original matrix. The matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is called the **identity** matrix of order 2.

The **identity matrix** of order n is a diagonal matrix where $a_{ii} = 1$

Two further identity matrices are $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

Sometimes, the identity matrix is denoted simply by $\boldsymbol{I}$, or by $\boldsymbol{I}_n$, where n is the order. So, the identity matrix with three rows and columns is $\boldsymbol{I}_3$, and the identity matrix with four rows and columns is $\boldsymbol{I}_4$.

Example 7.2

Let $\boldsymbol{A} = (2 \quad -1 \quad 3)$ and $\boldsymbol{B} = \begin{pmatrix}2\\ 5\\ 4\end{pmatrix}$

Calculate:

(a) $\boldsymbol{AB}$ (b) $\boldsymbol{BA}$

Solution

(a) $(2 \quad -1 \quad 3)\begin{pmatrix}2\\ 5\\ 4\end{pmatrix} = 2 \times 2 + (-1) \times 5 + 3 \times 4 = 11$

(b) $\begin{pmatrix}2\\ 5\\ 4\end{pmatrix}(2 \quad -1 \quad 3) = \begin{pmatrix}2 \times 2 & 2 \times (-1) & 2 \times 3\\ 5 \times 2 & 5 \times (-1) & 5 \times 3\\ 4 \times 2 & 4 \times (-1) & 4 \times 3\end{pmatrix} = \begin{pmatrix}4 & -2 & 6\\ 10 & -5 & 15\\ 8 & -4 & 12\end{pmatrix}$

Note that the order of multiplication affects the product. Matrix multiplication, in general, is **not commutative**. It is usually not true that $\boldsymbol{AB} = \boldsymbol{BA}$.

Let $\boldsymbol{A} = \begin{pmatrix}3 & 6\\ 5 & 2\end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix}-2 & 3\\ 1 & 5\end{pmatrix}$, then $\boldsymbol{AB} = \begin{pmatrix}3 & 6\\ 5 & 2\end{pmatrix}\begin{pmatrix}-2 & 3\\ -1 & 5\end{pmatrix} = \begin{pmatrix}-0 & 39\\ -8 & 25\end{pmatrix}$

but $\boldsymbol{BA} = \begin{pmatrix}-2 & 3\\ 1 & 5\end{pmatrix}\begin{pmatrix}3 & 6\\ 5 & 2\end{pmatrix} = \begin{pmatrix}9 & -6\\ 28 & 16\end{pmatrix} \Rightarrow \boldsymbol{AB} \neq \boldsymbol{BA}$

However, there are some special cases where matrix multiplication is commutative. For example

$\boldsymbol{A} = \begin{pmatrix}3 & 6\\ 5 & 2\end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix}2 & 6\\ 5 & 1\end{pmatrix}$, then $\boldsymbol{AB} = \begin{pmatrix}3 & 6\\ 5 & 2\end{pmatrix}\begin{pmatrix}2 & 6\\ 5 & 1\end{pmatrix} = \begin{pmatrix}36 & 24\\ 20 & 32\end{pmatrix}$ and

$\boldsymbol{BA} = \begin{pmatrix}2 & 6\\ 5 & 1\end{pmatrix}\begin{pmatrix}3 & 6\\ 5 & 2\end{pmatrix} = \begin{pmatrix}36 & 24\\ 20 & 32\end{pmatrix} \Rightarrow \boldsymbol{AB} = \boldsymbol{BA}$

Multiplying by an identity matrix is also commutative.

$$\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$$

Example 7.3

Use the information given in the table to set up a matrix to find the camera sales in each city.

	Venice	Rome	Budapest	Prague
Digital compact	153	98	74	56
Digital standard	211	120	57	29
DSLR	82	31	12	5
Other	308	242	183	107

The average selling price for each type of camera is as follows:

Digital compact €1200; Digital standard €1100; DSLR €900; Other €600

Solution

We set up a matrix multiplication in which the individual camera sales are multiplied by the corresponding price. Since the rows represent the sales of the different types of camera, create a row matrix of the different prices and perform the multiplication.

$$(1200 \quad 1100 \quad 900 \quad 600)\begin{pmatrix}153 & 98 & 74 & 56\\ 211 & 120 & 57 & 29\\ 82 & 31 & 12 & 5\\ 308 & 242 & 183 & 107\end{pmatrix}$$

$$= (674\,300 \quad 422\,700 \quad 272\,100 \quad 167\,800)$$

So, the sales (in euros) from each city are:

	Venice	Rome	Budapest	Prague
Sales	674 300	422 700	272 100	167 800

Remember that we are multiplying a 1×4 matrix with a 4×4 matrix and hence we get a 1×4 matrix.

Exercise 7.1

1. Consider the matrices

$$A = \begin{pmatrix} -2 & x \\ y-1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} x+1 & -3 \\ 4 & y-2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2x & -1 \\ 2 & 3 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 \\ 2x & 3 \\ -1 & 0 \end{pmatrix}$$

(a) Evaluate:

(i) $A + B$ (ii) $3A - B$ (iii) $A + C$

(b) Find x and y such that $A = B$

(c) Find x and y such that $A + B$ is a diagonal matrix.

(d) Find AB and BA

(e) Find x and y such that $C = D$

2. Solve for the variables:

(a) $\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & p \\ 3 & q \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -6 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 6 & -4 \end{pmatrix}$

3. The diagram shows the major highways connecting some European cities: Vienna (V), Munich (M), Frankfurt (F), Stuttgart (S), Zurich (Z), Milan (L), and Paris (P).

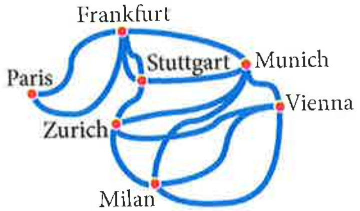


Figure 7.2 Diagram for question 3

The partially completed matrix below shows the number of direct routes between these cities.

(a) Use the diagram to copy and complete the matrix.

$$\begin{array}{c} \\ V \\ M \\ F \\ S \\ Z \\ L \\ P \end{array} \begin{array}{c} \begin{array}{ccccccc} V & M & F & S & Z & L & P \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \end{array}$$

(b) Multiply the matrix from part (a) by itself and interpret what it signifies.

4. Consider the matrices

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 0 & -3 & 2 \\ 7 & 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} m & -2 \\ 3m & -1 \\ 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} x-1 & 5 & y \\ 0 & -x & y+1 \\ 2x+y & x-3y & 2y-x \end{pmatrix}$$

(a) Find $A + C$

(b) Find AB

(c) Find BA

(d) Solve for x and y if $A = C$

(e) Find $B + C$

(f) Solve for m if $3B + 2\begin{pmatrix} -1 & m^2 \\ -5 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 17 & 1 \\ 2m+2 & 7 \end{pmatrix}$

5. Find a, b, and c so that the following equation is true.

$$2\begin{pmatrix} a-1 & b \\ c+2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 8 & c+9 \end{pmatrix}$$

6. Find x and y so that the following equation is true.

$$\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} x-11 & 1-x \\ -5 & x+2y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7. Find m and n so that the following equation is true.

$$\begin{pmatrix} m^2-1 & m+2 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 3 & n+1 \\ 5 & n-5 \end{pmatrix}$$

8. There are two shops in your area. Your shopping list consists of 2 kg of tomatoes, 500 g of meat, and 3 litres of milk. Prices differ between the different shops, and it is difficult to switch between shops to make certain you are paying the least amount of money. A better strategy is to check where you pay less on average. The prices of the different items are given in the table. Which shop should you go to?

Product	Price in shop A	Price in shop B
Tomatoes	€1.66/kg	€1.58/kg
Meat	€2.55/100 g	€2.6/100 g
Milk	€0.90/litre	€0.95/litre

9. Consider the matrices

$$A = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix} \quad C = \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix}$$

(a) Find $A + (B + C)$ and $(A + B) + C$

(b) Make a conjecture about the addition of 2×2 matrices observed in part **(a)** and prove it.

(c) Find $A(BC)$ and $(AB)C$

(d) Make a conjecture about the multiplication of 2×2 matrices observed in part **(c)** and prove it.

10. A company sells air conditioning units, electric heaters and humidifiers. Row matrix A represents the number of units sold of each appliance last year, and matrix B represents the profit margin for each unit. Find AB and describe what this product represents.

$$A = \begin{pmatrix} 235 & 562 & 117 \end{pmatrix} \quad B = \begin{pmatrix} €120 \\ €95 \\ €56 \end{pmatrix}$$

11. Find r and s such that $rA + B = A$ is true, where

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \quad B = \begin{pmatrix} -12 & -18 \\ s-8 & -42 \end{pmatrix}$$

12. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(a) Find:

(i) A^2 **(ii)** A^3 **(iii)** A^4 **(iv)** A^n

Let $B = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$

(b) Find:

(i) B^2 **(ii)** B^3 **(iii)** B^4 **(iv)** B^n

13. Solve for x and y such that $AB = BA$ when

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}$$

14. Solve for x and y such that $AB = BA$ when

$$A = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix}$$

15. Solve for x such that $AB = BA$ when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$$

16. Solve for x and y such that $AB = BA$ when

$$A = \begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$$

7.2 Applications to systems

There is a wide range of applications of matrices in solving systems of equations.

Recall from algebra that the equation of a straight line can take the form

$ax + by = c$ where a, b, and c are constants, and x and y are variables.

We say this is a linear equation in two variables. Similarly, the equation of a plane in three-dimensional space has the form

$ax + by + cz = d$ where a, b, c, and d are constants, and x, y, and z are variables.

We say that this is a linear equation in three variables.

A solution of a linear equation in n variables (in this case 2 or 3) is an ordered set of real numbers (x_0, y_0, z_0) so that the equation in question is satisfied when these values are substituted for the corresponding variables. For example, the equation $x + 2y = 4$ is satisfied when $x = 2$ and $y = 1$

Some other solutions are: $x = -4$ and $y = 4$
$x = 0$ and $y = 2$
$x = -2$ and $y = 3$

The set of all solutions of a linear equation is its solution set, and when this set is found, the equation is said to have been solved. To describe the entire solution set we often use a **parametric representation**, as illustrated in the following examples.

Example 7.4

Solve the linear equation $x + 2y = 4$

Solution

To find the solution set of an equation in two variables, we solve for one variable in terms of the other. For instance, if we solve for x, we obtain

$$x = 4 - 2y$$

In this form, y is free, as it can take on any real value, while x is not free, since its value depends on that of y. To represent this solution set in general terms, we introduce a third variable, for example t, called a parameter, and by letting $y = t$ we represent the solution set as

$$x = 4 - 2t, y = t, t \text{ is any real number}$$

Particular solutions can then be obtained by assigning values to the parameter t. For instance, $t = 1$ yields the solution $x = 2$ and $y = 1$, and $t = 3$ yields the solution $x = -2$ and $y = 3$

Note that the solution set of a linear equation can be represented parametrically in several ways. For instance, in Example 7.4, if we solve for y in terms of x, the parametric representation would take the form:

$$x = m, y = 2 - \frac{1}{2}m, m \text{ is a real number}$$

Also, by choosing $m = 2$, one particular solution is $(x, y) = (2, 1)$, and when $m = -2$, another particular solution is $(-2, 3)$.

Example 7.5

Solve the linear equation $3x + 2y - z = 3$

Solution

Choosing x and y as the free variables, we solve for z.

$$z = 3x + 2y - 3$$

Letting $x = p$ and $y = q$, we obtain the parametric representation:

$$x = p, y = q, z = 3p + 2q - 3, \text{ where } p \text{ and } q \text{ are any real numbers}$$

A particular solution is $(x, y, z) = (1, 1, 2)$

Parametric representation is very important when we study vectors and lines later on in the book.

Systems of linear equations

A system of k **equations** in n variables is a set of k linear equations in the same n variables. For example

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

is a system of two linear equations in two variables, while

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \end{aligned}$$

is a system with two equations and three variables, and

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

is a system with three equations and three variables.

A solution of a system of equations is an ordered set of numbers $x_0, y_0, \ldots$ which satisfy every equation in the system. For example $(3, -1)$ is a solution of

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

Both equations in the system are satisfied when $x = 3$ and $y = -1$ are substituted into the equations. However, $(0, 1)$ is not a solution of the system; it satisfies the first equation, but it does not satisfy the second.

In this chapter, we will use matrix methods to solve systems of equations.

Taking our example above, we can write the system of equations in matrix form:

$$\begin{cases} 2x + 3y = 3 \\ x - y = 4 \end{cases} \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The representation of the system of equations this way enables us to use matrix operations in solving systems of equations. This matrix equation can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \mathbf{AX} = \mathbf{C}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{X}$ is the variable matrix, and $\mathbf{C}$ is the constant matrix. However, to solve this equation, the inverse of a matrix has to be defined as the solution of the system in the form

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}$$

where $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$.

Matrix inverse

To solve the equation $2x = 6$ for x, we need to multiply both sides of the equation by $\frac{1}{2}$:

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 6 \Rightarrow x = 3 \qquad \text{This is so, because } \frac{1}{2} \times 2 = 2 \times \frac{1}{2} = 1$$

$\frac{1}{2}$ is the multiplicative inverse of 2. The inverse of a matrix is defined in a similar manner and plays a similar role in solving a matrix equation, such as $\boldsymbol{AX} = \boldsymbol{C}$

A square matrix $\boldsymbol{B}$ is the inverse of a square matrix $\boldsymbol{A}$ if $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$ where $\boldsymbol{I}$ is the identity matrix.

Note that only square matrices can have multiplicative inverses.

The notation $\boldsymbol{A}^{-1}$ is used to denote the inverse of a matrix $\boldsymbol{A}$. Thus, $\boldsymbol{B} = \boldsymbol{A}^{-1}$

Example 7.6

Are the matrices $\boldsymbol{A} = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}$ multiplicative inverses?

Solution

$$\boldsymbol{AB} = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 21-20 & -35+35 \\ 12-12 & -20+21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 21-20 & 15-15 \\ -28+28 & -20+21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $\boldsymbol{A}$ and $\boldsymbol{B}$ are multiplicative inverses.

We can also find the inverse using a GDC.

```
[A]-1
        [[3  -5]
         [-4  7 ]]
[A]-1[A]
        [[1  0]
         [0  1]]
```

Figure 7.3 GDC screen for the solution to Example 7.6

We will now find the general form for the inverse of a matrix.

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume $\boldsymbol{A}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and then solve the following matrix equation for e, f, g, and h in terms of a, b, c, and d.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we can set up two systems to solve for the required variables:

$$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{array}{l} ae+bg=1 \\ ce+dg=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} dae+dbg=d \\ bce+bdg=0 \end{array}\right\} \Rightarrow e = \frac{d}{ad-bc}, g = \frac{-c}{ad-bc}$$

$$\left.\begin{array}{l} af+bh=0 \\ cf+dh=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} daf+dbh=0 \\ bcf+bdh=b \end{array}\right\} \Rightarrow f = \frac{-b}{ad-bc}, h = \frac{a}{ad-bc}$$

In a matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $ad - bc \neq 0$, then its inverse $\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$

or $\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 7.7

Find the inverse of $\boldsymbol{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$

Solution

Here $a = 4$, $b = 7$, $c = 3$, and $d = 5$, so $ad - bc = -1$

Thus $\boldsymbol{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 5 & -7 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 7 \\ 3 & -4 \end{pmatrix}$

```
[A]
        [[4  7]
         [3  5]]
[A]-1
       [[-5  7]
        [3  -4]]
```

Figure 7.4 GDC screen for the solution to Example 7.7

The number $ad - bc$ is called the **determinant** of the 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The notation we will use for this number is **det** $\boldsymbol{A}$ or $|\boldsymbol{A}|$, so we write this as:

$$\det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$

The determinant plays an important role in determining whether or not a matrix has an inverse.

When the determinant is zero ($ad - bc = 0$), the matrix does not have an inverse. A matrix that does not have an inverse is called a **singular matrix**; a matrix that does have an inverse is called a **non-singular** matrix.

Example 7.8

Solve the system of equations using matrices.

$$2x + 3y = 3$$
$$x - y = 4$$

Solution

In matrix form, the system can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Write the equation in the form $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{C}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

```
[A]-1[C]
           [[3 ]
            [-1]]
```

Figure 7.5 GDC screen for the solution to Example 7.8

Find A^{-1}, then substitute into the equation and simplify

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -15 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

In general, a system of equations can be written in matrix form as $AX = B$

There is a solution to the system when A is non-singular, which is $X = A^{-1}B$

If $B = 0$, the system is **homogeneous**. A homogeneous system will always have a solution, called the **trivial** solution, $X = 0$ when A is non-singular. When A is singular then the system has an infinite number of solutions.

We use a similar procedure to solve systems of equations in three variables. However, we will use a GDC to find the inverse of a 3×3 matrix. As in the case of a 2×2 matrix, the existence of an inverse for a 3×3 matrix depends on the value of its determinant.

There are two methods for calculating the determinant of a 3×3 matrix A:

```
[A]
     [[5  1  -4]
      [2  -3 -5]
      [7  2  -6]]
det([A])
              17
```

Figure 7.6 Method 1

Method 1

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \det A = a(ei - fh) - b(di - fg) + c(dh - eg)$$

For example, if $A = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}$

then $\det A = 5(18 + 10) - 1(-12 + 35) - 4(4 + 21) = 17$

Method 2

Use a special set up as follows:

$$\det A = \begin{vmatrix} + & + & + & & \\ a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ - & - & - & & \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$$

This is done by copying the first two columns and adding them to the end of the matrix, multiplying down the main diagonals and adding the products, and then multiplying up the second diagonals and subtracting them from the previous product as shown. For example:

$$\begin{matrix} + & + & + & & \\ 5 & 1 & -4 & 5 & 1 \\ 2 & -3 & -5 & 2 & -3 \\ 7 & 2 & -6 & 7 & 2 \\ - & - & - & & \end{matrix}$$

$= 5\cdot(-3)(-6) + 1\cdot(-5)\cdot 7 + (-4)\cdot 2\cdot 2 - 7(-3)(-4) - 2(-5)\cdot 5 - (-6)\cdot 2\cdot 1$

$= 90 - 35 - 16 - 84 + 50 + 12 = 152 - 135 = 17$

This arrangement is a re-ordering of the calculations involved in the first method.

Example 7.9

Solve the system of equations

$$5x + y - 4z = 5$$
$$2x - 3y - 5z = 2$$
$$7x + 2y - 6z = 5$$

Solution

We write this system in matrix form:

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

Since $\det A = 17 \neq 0$, we can find the solution in the same way we did for the 2×2 matrix:

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

To check our work, using a GDC, we can store the answer matrix as D and then substitute the values into the system

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 - 2 - 8 \\ 6 + 6 - 10 \\ 21 - 4 - 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

```
[A]-1[C]
           [[3 ]
            [-2]
            [2 ]]
```

```
[A][D]
           [[5]
            [2]
            [5]]
```

Figure 7.7 GDC screens for the solution to Example 7.9

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is equal to $\left|\frac{1}{2}|A|\right|$ where

$$A = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$

Area of a triangle

An interesting application of determinants that you may find helpful is finding the area of a triangle whose vertices are given as points in a coordinate plane.

Example 7.10

Find the area of triangle ABC whose vertices are $A(1, 3)$, $B(5, -1)$ and $C(-2, 5)$.

Solution

We let $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (5, -1)$, and $(x_3, y_3) = (-2, 5)$

To find the area, we evaluate the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix} = -4$$

Using this value, we can conclude that the area of the triangle is

$$\text{Area} = \left|\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}\cdot -4\right| = 2$$

Use determinants to find the area of triangle ABC with $A(2, 3)$, $B(12, 3)$, and $C(12, 9)$. Confirm your answer by using the formula for the area of a triangle.

Lines in plane

What happens when the three points are collinear? The triangle becomes a line segment and the area becomes zero. This fact allows us to develop two techniques that are very helpful in dealing with questions of collinearity and equations of lines.

For example, consider the points $A(-2, -3)$, $B(1, 3)$ and $C(3, 7)$. Find the area of 'triangle' ABC.

$$\text{Area} = \left|\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 1 & 3 & 1 \\ 3 & 7 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}\cdot 0\right| = 0$$

This result can be stated in general as a test for collinearity.

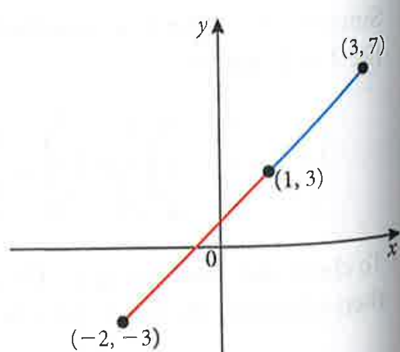


Figure 7.8 Three collinear points

The three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 7.11

Determine if the points $(-2, 3)$, $(2, 5)$ and $(5, 7)$ lie on the same line.

Solution

Set up the matrix as given above:

$$\begin{vmatrix} -2 & 3 & 1 \\ 2 & 5 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 2 \neq 0$$

The points cannot lie on a line because the value of the determinant is not equal to zero.

Two-point equation of a line

The test for collinearity leads us to a method for finding the equation of a line containing two points. Consider two points (x_1, y_1), (x_2, y_2) which lie on a given line. To find the equation of the line through these two points, we introduce a general point (x, y) on the line. These three points (x_1, y_1), (x_2, y_2) and (x, y) are collinear, and hence they satisfy the determinant equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

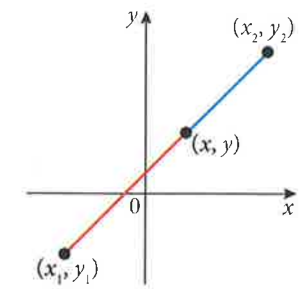


Figure 7.9 (x_1, y_1), (x_2, y_2) and (x, y) are collinear

which gives us the equation of the line in the form:

$$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - y_1x_2) = 0$$

which in turn is of the form: $Ax + By + C = 0$

Example 7.12

Find the equation of the line through $(-2, 3)$ and $(3, 7)$

Solution

Apply the determinant formula for the equation of a line.

$$\begin{vmatrix} x & y & 1 \\ -2 & 3 & 1 \\ 3 & 7 & 1 \end{vmatrix} = (3 - 7)x + (3 + 2)y + (-14 - 9) = 0$$

The equation of the line is $-4x + 5y - 23 = 0$

We can use column vectors instead of row vectors, and the calculation in Example 7.12 becomes:

$$\begin{vmatrix} x & -2 & 3 \\ y & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = (3-7)x + (-14-9) + (3+2)y = 0$$

$$-4x + 5y - 23 = 0$$

The matrix $A' = \begin{pmatrix} x & -2 & 3 \\ y & 3 & 7 \\ 1 & 1 & 1 \end{pmatrix}$ is called the transpose of the matrix $A = \begin{pmatrix} x & y & 1 \\ -2 & 3 & 1 \\ 3 & 7 & 1 \end{pmatrix}$

where each row of the matrix becomes a column of the transpose. The transpose has the same determinant as the matrix itself.

Coding and decoding messages: cryptography

Data encryption is essential in applications such as online banking. Encryption uses encoding-decoding methods in which matrices play a very important role.

The methods included here are not secure enough to use for applications such as internet banking, but they result in codes that are not easy to break and provide a good introduction to the ideas of encryption.

The process of encryption is called cryptography. In cryptography, a message that has not yet been encrypted is called plaintext, after the encryption process the encrypted message is called ciphertext. The process of converting the plaintext to ciphertext is called enciphering and the reverse process where the ciphertext is converted to plaintext is called deciphering. One such method, called Hill's method, involves dividing the plaintext message into sets of n letters, each of which is replaced by n cipher letters. This is called a polygraphic system. Hill-ciphers require a matrix based polygraphic system. A system of cryptography in which the plaintext is divided into sets of n letters, each of which is replaced by a set of n cipher letters, is called a polygraphic system. For example, {*mathematics is a great subject*} = {math emat ics⊔ is⊔a ⊔gre at⊔s ubje ct⊔⊔}. In this case we used blocks of 4 letters. [⊔ stands for space]

Modular/clock arithmetic

In order to work efficiently with cryptography, some basic knowledge of modular arithmetic is helpful.

Two integers, a and b are said to be congruent modulo n, written as $a \equiv b \bmod n$, if they leave the same remainder when divided by n. For example $41 \equiv 5 \bmod 12$ or $15 \equiv 3 \bmod 4$. Alternatively, $a \equiv b \bmod n$ also means that $n \mid (a - b)$. Note that $41 - 5 = 36$, and $12 \mid 36$

In calculations, using the same modulus, you can replace any integer by any integer congruent to it. For example, in mod 4, $19 \times 3 \equiv 1 \bmod 4$ because you can write it as $3 \times 3 \equiv 9 \equiv 1 \bmod 4$, or alternatively $19 \times 3 = 57$ which leaves a remainder of 1 when divided by 4. Thus $19 \times 3 = 57 \equiv 1 \bmod 4$.

When replacing numbers by their equivalents, it is a good idea to either add or subtract multiples of the modulus until you reach a number less than the mod. Remember that when dividing by n the possible remainders are $0, 1, \ldots, n-1$. For example, the closest multiple of 4 to 57 is 56. Thus $57 - 56 = 1$, which explains why $57 \equiv 1 \bmod 4$.

Let us describe the process with an example.

Say you want to send the message 'listen to me please!'

The process is as follows:

1. Choose a code table similar to the one below. The table depends on how many letters/symbols you need and what language you use. We are using English here, so we need a table to cater for the whole alphabet at least.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
O	P	Q	R	S	T	U	V	W	X	Y	Z	!	?	.
15	16	17	18	19	20	21	22	23	24	25	26	27	28	29

Table 7.4 Code table

The first cell is for a space.

2. Then, translate the text message into codes from the table.

L	I	S	T	E	N		T	O		M	E		P	L	E	A	S	E	!
12	9	19	20	5	14	0	20	15	0	13	5	0	16	12	5	1	19	5	27

3. Choose a non-singular coding matrix of any order of your choice. We will use a 3×3 matrix. Also, for convenience, we will choose it to have a determinant of 1. For example, we will use the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

An efficient way of getting a matrix with determinant 1 is to start with an identity matrix and then add or subtract rows or multiples of rows (or columns)

4. Subdivide your codes into columns (or rows) of 3 each, including spaces. If you don't have enough letters to fill the columns, use a space. (This is so, because our coding matrix is of order 3.)

$$\begin{pmatrix} 12 & 20 & 0 & 0 & 0 & 5 & 5 \\ 9 & 5 & 20 & 13 & 16 & 1 & 27 \\ 19 & 14 & 15 & 5 & 12 & 19 & 0 \end{pmatrix}$$

5. Now, multiply the coding matrix by the code matrix.

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 12 & 20 & 0 & 0 & 0 & 5 & 5 \\ 9 & 5 & 20 & 13 & 16 & 1 & 27 \\ 19 & 14 & 15 & 5 & 12 & 19 & 0 \end{pmatrix}$$

This is a 3×3 matrix multiplied by a 3×7 matrix, thus the result is a 3×7 matrix.

$$\begin{pmatrix} 31 & 34 & 15 & 5 & 12 & 24 & 5 \\ -3 & -15 & 20 & 13 & 16 & -4 & 22 \\ 47 & 33 & 50 & 23 & 40 & 39 & 27 \end{pmatrix}$$

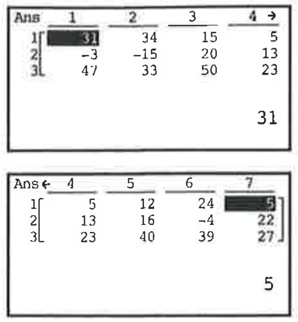


Figure 7.10 GDC output

6. Before we give out the ciphered message, we need to replace the numbers in pink with their congruent numbers mod 30 (This is so, because we are using 30 codes).

The message will now be:

$$\begin{pmatrix} 1 & 4 & 15 & 5 & 12 & 24 & 5 \\ 27 & 15 & 20 & 13 & 16 & 26 & 22 \\ 17 & 3 & 20 & 23 & 10 & 9 & 27 \end{pmatrix}$$

This is equivalent to the message A!QDOCOTTEMWLPJXZIEV!

The GDC output is as shown.

7. The receiver of the message will decipher the message by multiplying the inverse of the coding matrix by the message matrix. In this case, the inverse of the coding matrix is:

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 15 & 5 & 12 & 24 & 5 \\ 27 & 15 & 20 & 13 & 16 & 26 & 22 \\ 17 & 3 & 20 & 23 & 10 & 9 & 27 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 20 & 30 & 0 & 30 & 65 & 5 \\ 39 & 35 & 50 & 13 & 46 & 91 & 27 \\ -11 & -16 & -15 & 5 & -18 & -41 & 0 \end{pmatrix}$$

Now, replace the numbers in pink with their congruent counterparts mod 30

$$\begin{pmatrix} 12 & 20 & 0 & 0 & 0 & 5 & 5 \\ 9 & 5 & 20 & 13 & 16 & 1 & 27 \\ 19 & 14 & 15 & 5 & 12 & 19 & 0 \end{pmatrix}$$

Which is the matrix for the original message.

Example 7.13

You receive the following message.

UHTUWWE??SCVPMALU!TJ.ZMYFLL

You also know from your sender that the coding matrix is

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Use Table 7.4 to decode the message.

Solution

We first use the code table to write out the matrix corresponding to the coded message:

$$\begin{pmatrix} 21 & 21 & 5 & 19 & 16 & 12 & 20 & 26 & 6 \\ 8 & 23 & 28 & 3 & 13 & 21 & 10 & 13 & 12 \\ 20 & 23 & 28 & 22 & 1 & 27 & 29 & 25 & 12 \end{pmatrix}$$

Next, we multiply the inverse of the coding inverse by this matrix

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 21 & 21 & 5 & 19 & 16 & 12 & 20 & 26 & 6 \\ 8 & 23 & 28 & 3 & 13 & 21 & 10 & 13 & 12 \\ 20 & 23 & 28 & 22 & 1 & 27 & 29 & 25 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -25 & -51 & 13 & -10 & -30 & 0 & 0 & -18 \\ -9 & 48 & 79 & -29 & 35 & 45 & -9 & 1 & 30 \\ 12 & 0 & 0 & 19 & -12 & 6 & 19 & 12 & 0 \end{pmatrix}$$

Now, we replace the numbers that are less than 0 or over 30 with their congruent counterparts mod 30.

$$\begin{pmatrix} 5 & 5 & 9 & 13 & 20 & 0 & 0 & 0 & 12 \\ 21 & 18 & 19 & 1 & 5 & 15 & 21 & 1 & 0 \\ 12 & 0 & 0 & 19 & 18 & 6 & 19 & 12 & 0 \end{pmatrix}$$

Now, replacing the ciphers with letters

EULER IS MASTER OF US ALL

If numbers are to be included in messages, then you can extend the code table by 10 to represent the integers from 0 to 9. Your mod will then be 40.

Exercise 7.2

1. Consider the matrix M which satisfies the matrix equation

$$\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix} M = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

(a) Find the inverse of matrix $\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}$

(b) Hence, write M as a product of two matrices.

(c) Evaluate M.

(d) Now consider the equation containing the matrix N:

$$N\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

(i) Express N as a product of two matrices.

(ii) Evaluate N.

(e) Write a short paragraph describing your work on this problem.

2. Find the matrix E in the following equation.

$$\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} E \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$$

3. (a) Prove that the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{pmatrix}$ should have an inverse.

(b) Write out A^{-1}.

(c) Hence, solve the system of equations

$$\begin{cases} 2x - 3y + z = 4.2 \\ x + y - 3z = -1.1 \\ 3x - 2y - 3z = 2.9 \end{cases}$$

4. Find the inverse for each matrix:

(a) $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

(b) $B = \begin{pmatrix} a & 1 \\ a + 2 & \frac{3}{a} + 1 \end{pmatrix}$

5. For what values of x is the following matrix singular?

$$A = \begin{pmatrix} x + 1 & 3 \\ 3x - 1 & x + 3 \end{pmatrix}$$

6. Find n such that $\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix}$ is the inverse of $\begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix}$

7. Consider the two matrices $A = \begin{pmatrix} 4 & 2 \\ 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$

(a) Find X such that $XA = B$

(b) Find Y such that $AY = B$

(c) Is $X = Y$? Explain.

8. Consider the two matrices

$$P = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 5 & 4 \\ 1 & 0 & -1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 3 & -1 & 1 \\ 4 & 0 & 0 \\ 1 & -2 & -1 \end{pmatrix}$$

(a) Find PQ and QP.

(b) Find:

(i) P^{-1} (ii) Q^{-1} (iii) $P^{-1}Q^{-1}$

(iv) $Q^{-1}P^{-1}$ (v) $(PQ)^{-1}$ (vi) $(QP)^{-1}$

(c) Write a few sentences about your observations in parts (a) and (b).

9. Consider the matrices

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -4 & 1 & -3 \\ 1 & -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -29 \\ 37 \\ -24 \end{pmatrix}$$

(a) Find the matrix C where $AC = B$

(b) Solve the system of equations

$$\begin{cases} 3x - 2y + z = -29 \\ 4x - y + 3z = -37 \\ -x + 5y - z = 24 \end{cases}$$

10. Solve the matrix equation:

$$\begin{pmatrix} 2 & 2 + x \\ 5 & 4 + x \end{pmatrix}\begin{pmatrix} 3 & x \\ x - 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & x \\ x - 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 + x \\ 5 & 4 + x \end{pmatrix}$$

11. Consider the matrices A and B. Find x and y such that $AB = BA$

(a) $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 - x & 1 \\ 5x & y \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 - x & x \\ 5x & y \end{pmatrix}$

(c) $A = \begin{pmatrix} 3 + x & 1 \\ -5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} y - x & x \\ 5x - y + 1 & y + x \end{pmatrix}$

12. Use matrix methods to find an equation of a line that contains the given points.

(a) $A(-5, -6)$, $B(3, 11)$ (b) $A(5, -2)$, $B(3, -2)$

(c) $A(-5, 3)$, $B(-5, 8)$

13. Find the area of the parallelogram with the given points as three of its vertices.

(a) $A(-5, -6)$, $B(3, 11)$, $C(8, 1)$ (b) $A(3, -5)$, $B(3, 11)$, $C(8, 11)$

(c) $A(4, -6)$, $B(-3, 9)$, $C(7, 7)$

14. Find x such that the area of triangle ABC is 10 square units.

(a) $A(x, -6)$, $B(3, 11)$, $C(8, 3)$

(b) $A(-5, x)$, $B(3, x+2)$, $C(x^2+2x-3, 1)$

15. Find the value of k such that the points P, Q, and R are collinear.

(a) $P(2, -5)$, $Q(4, k)$, $R(5, -2)$ **(b)** $P(-6, 2)$, $Q(-5, k)$, $R(-3, 5)$

$f(x)$ is called the characteristic polynomial of A.

16. Consider the matrix $A = \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix}$. Define $f(x) = \det(xI - A)$ where x is any real number and I is the identity matrix.

(a) Find $\det(A)$.

(b) Expand $f(x)$ and compare the constant term to your answer in (a).

(c) How is the coefficient of x in the expansion of $f(x)$ related to A?

(d) Find $f(A)$ and simplify it.

(e) Now repeat parts (a)–(d) with matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

17. Consider the matrix $A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 3 & 2 \\ 5 & 5 & -4 \end{pmatrix}$

Define $f(x) = \det(xI - A)$, where x is any real number and I is the identity matrix.

(a) Find $\det(A)$.

(b) Expand $f(x)$ and compare the constant term to your answer in (a).

(c) How is the coefficient of x^2 in the expansion of $f(x)$ related to A?

(d) Find $f(A)$ and simplify it.

(e) Now repeat parts (a)–(d) with matrix $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

18. **(a)** Use Table 7.4 to decode the following message, given that ⊔ stands for a space.

S.TPEHZO?WPOSWYSFPV!FRGIUTGMEBSH

The coding matrix is $\begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

(b) Choose an appropriate matrix of your own, code your answer and decode it.

7.3 Further properties and applications

In question 8 of Exercise 7.2, you were asked to make some observations concerning the answers to parts (a) and (b). The question shows some properties of inverse matrices.

You should have found out that:

$$P^{-1}Q^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix} \qquad Q^{-1}P^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$$

$$(PQ)^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix} \qquad (QP)^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$$

So $(PQ)^{-1} \neq P^{-1}Q^{-1}$, but $(PQ)^{-1} = Q^{-1}P^{-1}$

and $(QP)^{-1} = P^{-1}Q^{-1}$

This leads to the following general result.

When A and B are non-singular matrices of order n, then AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

The proof of this theorem is straightforward:

To show that $B^{-1}A^{-1}1$ is the inverse of AB, we need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A-1 = A(I)A^{-1} = AA^{-1} = I$

Similarly, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B-1(I)B = B-1B = I$

Hence, AB is non-singular and its inverse is $B^{-1}A^{-1}$

Non-singular matrices also have these properties:

$(A^{-1})^{-1} = A$

$(cA)^{-1} = \frac{1}{c}A^{-1}; c \neq 0$

$\det(AB) = \det A \cdot \det B$

$\det A^{-1} = \frac{1}{\det A}$

We can prove the last property using the third property.

Since $AA^{-1} = I$, then

$$\det(AA^{-1}) = \det I \Rightarrow \det A \cdot \det A^{-1} = 1 \Rightarrow \det A^{-1} = \frac{1}{\det A}$$

In Section 7.2, we solved a system of equations using inverse matrices. The method works only when the system has a unique solution. In many cases, there is either an infinite number of solutions or the system is inconsistent. We can use another method of solution.

Some terminology

In Section 7.2 we learned how to solve a system of equations by writing the system in matrix form. When the system has a unique solution then it can be solved. However, the method is limited and it has a strict constraint. If we use a slightly different arrangement, we can use **matrices to** find the solution whether it is unique, there are an infinite number of **solutions,** or simply no solution. We write the system as follows.

$$\left(\begin{array}{ccc|c} 2 & 3 & -4 & 8 \\ 0 & 2 & 4 & -3 \\ 1 & 0 & -2 & 4 \end{array}\right)$$

This called the **augmented** matrix of the system. It is customary to put a bar between the coefficients and the answers. However, this bar is not necessary and we will not be using it in this book. Just remember that the last column is the answers' column.

Gauss-Jordan elimination

The idea behind this method is very simple. We apply certain simple operations to the system of equations to reduce them into a special form that is easy to solve. We keep applying the operations until we have a form that is easy to solve. The operations are called **elementary row operations** and they can be applied to the system without changing the solution to the system. That is, the solution to the reduced system (**reduced row echelon form**) is the same as that for the original system. We can apply the operations either to the system itself or to its augmented matrix. As it is easier to work with the augmented matrix, we recommend that you first write the augmented matrix, reduce it, and then write the equivalent system to read the solution from.

There are three types of elementary row operations:

- multiply any row by a non-zero real number
- interchange any two rows
- add a multiple of one row to another row.

Note that the order in which we apply the operations is not unique.

We will demonstrate the method with an example.

Example 7.14

Consider the following system with its augmented matrix and simplify it to its reduced row echelon form.

$$\begin{cases} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ 2x + 4y + 6z = 6 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 4 & 6 & 6 \end{array}\right)$$

Switch row 1 and row 2:

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + y - z = 2 \\ 2x + 4y + 6z = 6 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 2 & 4 & 6 & 6 \end{array}\right)$$

Multiply row 3 by $\frac{1}{2}$:

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + y - z = 2 \\ x + 2y + 3z = 3 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & 3 & 3 \end{array}\right)$$

Multiply row 1 by -2 and add it to row 2, and multiply row 1 by -1 and add it to row 3 (we replace the second row with the result):

$$\begin{cases} x + 3y + 2z = 1 \\ -5y - 5z = 0 \\ -y + z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -5 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Note that row 1 did not change and rows 2 and three were replaced with the result of the elementary operation.

Multiply row 2 by $-\frac{1}{5}$

$$\begin{cases} x + 3y + 2z = 1 \\ y + z = 0 \\ -y + z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Now, add row 2 to row 3, and multiply row 2 by -3 and add it to row 1:

$$\begin{cases} x - z = 1 \\ y + z = 0 \\ 2z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

Now multiply row 3 by $\frac{1}{2}$:

$$\begin{cases} x - z = 1 \\ y + z = 0 \\ z = 1 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Lastly, add row 3 to row 1, and multiply row 3 by -1 and add it to row 2:

$$\begin{cases} x = 2 \\ y = -1 \\ z = 1 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

As you can see, this last system is very easy to read the solution from. You can verify that this solution is also the solution to the original system.

```
[A]
   [[2  1  -1  2]
    [1  3  2   1]
    [2  4  6   6]]
```

```
rref([A])
   [[1  0  0  2 ]
    [0  1  0  -1]
    [0  0  1  1 ]]
```

Figure 7.11 Carrying out the operation on a GDC

The simplified matrix is in its reduced row echelon form.

Of course, when we do the work, we do not have to show the processes in parallel. We just perform the operation on the matrix and then translate it into the equation form at the end.

You can carry out this whole operation easily using a GDC.

Example 7.15

Solve the system of equations:

$$\begin{cases} x + y + 2z = 1 \\ x \quad + z = 2 \\ \quad y + z = 0 \end{cases}$$

Solution

The augmented matrix is

$$\begin{cases} x + y + 2z = 1 \\ x \quad + z = 2 \\ \quad y + 2z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Multiply row 1 by -1 and add to row 2:

$$\begin{cases} x + y + 2z = 1 \\ \quad -y - \ z = 1 \\ \quad y + \ z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Add row 2 to row 1 and row 2 to row 3:

$$\begin{cases} x \quad + z = 2 \\ \quad -y - z = 1 \\ \quad\quad 0 = 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

At this stage, work can stop because if you write the last row as an equation, then it reads

$$0x + 0y + 0z = 1$$

This statement cannot be true for any value, and hence the system is inconsistent.

You can also use a GDC.

```
[B]
   [[1  1  2  1]
    [1  0  1  2]
    [0  1  1  0]]
```

```
rref([A])
   [[1  0  1  0]
    [0  1  0  0]
    [0  0  0  1]]
```

Figure 7.12 GDC screens for the solution to Example 7.15

Example 7.16

Solve the system of equations:

$$\begin{cases} 2x + y - z = 4 \\ x + 3y + 7z = 7 \\ 2x + 4y + 8z = 10 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{cases} 2x + y - z = 4 \\ x + 3y + 7z = 7 \\ 2x + 4y + 8z = 10 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & 3 & 7 & 7 \\ 2 & 4 & 8 & 10 \end{pmatrix}$$

Swap rows 1 and 2:

$$\begin{cases} x + 3y + 7z = 7 \\ 2x + y - z = 4 \\ 2x + 4y + 8z = 10 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 2 & 1 & -1 & 4 \\ 2 & 4 & 8 & 10 \end{pmatrix}$$

Multiply row 2 by -1 and add to row 3; multiply row 1 by -2 and add to row to 2:

$$\begin{cases} x + 3y + 7z = 7 \\ -5y - 15z = -10 \\ 3y + 9z = 6 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 0 & -5 & -15 & -10 \\ 0 & 3 & 9 & 6 \end{pmatrix}$$

Multiply row 2 by $-\frac{1}{5}$; multiply row 3 by $\frac{1}{3}$:

$$\begin{cases} x + 3y + 7z = 7 \\ y + 3z = 2 \\ y + 3z = 2 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Multiply row 2 by -1 and add to row 3; multiply row 2 by -3 and add to row 1

$$\begin{cases} x + 3y + 7z = 7 \\ y + 3z = 2 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the last row is all zeros, there is not much that we can do. The conclusion is that this last row is true for any choice of values for the variables. Now we are left with a system of two equations and three variables.

$$\begin{cases} x \quad -2z = 1 \\ \quad y + 3z = 2 \end{cases}$$

We need to solve for two of the variables in terms of the third. A wise choice here would be to solve for x and y in terms of z. That is

$$x = 1 + 2z, y = 2 - 3z$$

This means that for every choice of a value for z, we have a corresponding solution for the system. For example, if $z = 0$, then the solution would be (1, 2, 0), for $z = 2$, the solution is (5, −4, 2), and so on. This means that we have an infinite number of solutions. So we present the solution in terms of a parameter such as t. We let $z = t$, and our general solution would then be

$$(1 + 2t, 2 - 3t, t)$$

Reduced row echelon form

A matrix is in **reduced row echelon form** when it satisfies the following properties:

- If there are any rows consisting entirely of 0s, they appear at the bottom of the matrix.
- In any non-zero row, the first non-zero entry is 1. This entry is called the **pivot** of the row.
- For any consecutive rows, the pivot of the lower row must be to the right of the pivot of the preceding row.
- Any column that contains a pivot, has zeros everywhere else.

Matrix $\boldsymbol{A}$ is in reduced row echelon form, but matrix $\boldsymbol{B}$ is not.

$$\boldsymbol{A} = \begin{pmatrix} \boxed{1} & 0 & 3 & 0 & 5 & 8 \\ 0 \rightarrow & \boxed{1} & 4 & 0 & 4 & 2 \\ 0 & 0 \rightarrow & 0 \rightarrow & \boxed{1} & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 6 & 7 \\ 0 & 0 & 1 \leftarrow & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Curve fitting

Another application of matrices (systems) is to help fit specific models to sets of points.

Example 7.17

Fit a quadratic model to pass through the points (−1, 10), (2, 4), and (3, 14)

Solution

The problem is to find parameters a, b, and c that will force the curve representing the function $f(x) = ax^2 + bx + c$ to contain the given points. This means $f(-1) = 10$, $f(2) = 4$, and $f(3) = 14$

Since we need to find the three unknown parameters, we need three equations which are offered by the conditions above.

$$f(x) = ax^2 + bx + c$$
$$f(-1) = a - b + c = 10$$
$$f(2) = 4a + 2b + c = 4$$
$$f(3) = 9a + 3b + c = 14$$

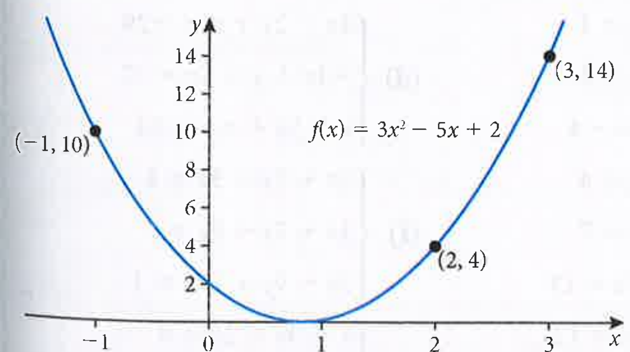
This is clearly a system of three equations which can be solved using matrix methods.

Using the reduced row echelon form, we get the following result

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 14 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Which means that $a = 3$, $b = -5$, and $c = 2$, so the function is

$$f(x) = 3x^2 - 5x + 2$$



Equivalently, we can use the inverse matrix directly:

$$\begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

You can also use a GDC.

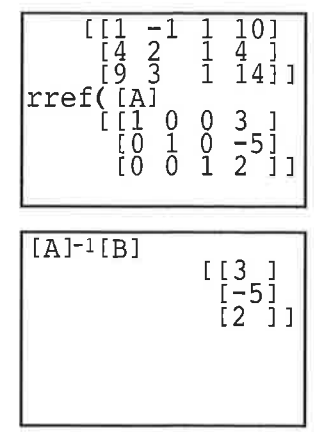


Figure 7.13 GDC screens for the solution to Example 7.17

Exercise 7.3

1. Let the matrix $A = \begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$. Find the values of the real numbers m such that $\det(A - mI) = 0$, where I is the 2×2 multiplication identity matrix.

2. (a) Find the values of a and b given that the inverse of the matrix

$$A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix} \text{ is the matrix } B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

(b) For the values of a and b found in part (a), solve the system of linear equations

$$\begin{cases} x + 2y - 2z = 5 \\ 3x + by + z = 0 \\ -x + y - 3z = a - 1 \end{cases}$$

3. Find the value(s) of m so that the matrix $\begin{pmatrix} 1 & m & 1 \\ 3 & 1 - m & 2 \\ m & -3 & m - 1 \end{pmatrix}$ is singular.

4. Solve each system of equations. If a solution does not exist, justify why not.

(a) $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ (b) $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

(c) $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$ (d) $\begin{cases} 3x - 2y + z = -29 \\ -4x + y - 3z = 37 \\ x - 5y + z = -24 \end{cases}$

(e) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$ (f) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 1 \end{cases}$

(g) $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$ (h) $\begin{cases} x - 3y - 2z = 8 \\ -2x + 7y + 3z = -19 \\ x - y - 3z = 3 \end{cases}$

5. (a) Find the values of k such that the matrix A is not singular.

$$A = \begin{pmatrix} 1 & 1 & k - 1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix}$$

(b) Find the value(s) of k such that A is the inverse of B, where

$$B = \begin{pmatrix} k - 3 & -3 & k \\ 3 & k + 2 & -1 \\ -2 & -4 & 1 \end{pmatrix}$$

(c) For the value of k found in part (b), apply elementary row operations to reduce the matrix $\begin{pmatrix} 1 & 1 & k - 1 & 1 & 0 & 0 \\ k & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & -3 & 0 & 0 & 1 \end{pmatrix}$ into $\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$ where $a, b, c, \dots i$ are to be determined.

6. (a) Find the values of k such that the matrix A is not singular.

$$A = \begin{pmatrix} \frac{2}{5} & \frac{-17}{5} & \frac{k + 9}{5} \\ \frac{-1}{5} & \frac{21}{5} & \frac{-13}{5} \\ k - 2 & 3 & -2 \end{pmatrix}$$

(b) Find the value(s) of k such that A is the inverse of B, where

$$B = \begin{pmatrix} k + 1 & 1 & k \\ 2 & k + 2 & -3 \\ 3 & 6 & -5 \end{pmatrix}$$

(c) For the value of k found in part (b), apply elementary row operations to reduce the matrix $\left(\begin{array}{ccc|ccc} 2 & -17 & k + 9 & 1 & 0 & 0 \\ -1 & 21 & -13 & 0 & 1 & 0 \\ 5(k - 2) & 15 & -10 & 0 & 0 & 1 \end{array}\right)$

into $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{array}\right)$ where $a, b, c, \dots i$ are to be determined.

7. Use elementary row operations to transform the matrix $[A|I]$ to a matrix in the form $[I|B]$. Comment on the relationship between A and B and support your conclusion.

(a) $\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$

(b) $\left(\begin{array}{ccc|ccc} 1 & 4 & 6 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 18 & 16 & 0 & 0 & 1 \end{array}\right)$

8. Determine the function f so that the curve representing it contains the indicated points.

There is more than one curve.

(a) $f(x) = ax^2 + bx + c$ to contain $(-1, 5)$, $(2, -1)$, and $(4, 35)$

(b) $f(x) = ax^2 + bx + c$ to contain $(-1, 12)$ and $(2, -3)$

9. Consider this system of equations.

$$\begin{cases} 2x + y + 3z = -5 \\ 3x - y + 4z = 2 \\ 5x + 7z = m - 5 \end{cases}$$

Find the value(s) of m for which this system is consistent. For the value of m found, find the most general solution of the system.

10. Consider this system of equations.

$$\begin{cases} -3x + 2y + 3z = 1 \\ 4x - y - 5z = -5 \\ x + y - 2z = m - 3 \end{cases}$$

Find the value(s) of m for which this system is consistent. For the value of m found, find the most general solution of the system.

11. Consider the matrix $A = \begin{pmatrix} 3 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$

(a) Find $\det(A)$.

(b) Add a multiple of one row to another row to transform the matrix A into matrix B in triangular form.

(c) Find $\det(B)$.

(d) Use a GDC to find $\det(C)$ for $C = \begin{pmatrix} 2 & 1 & -3 & 5 \\ 4 & 3 & -4 & -6 \\ 6 & -8 & 5 & 7 \\ -6 & -5 & 3 & 4 \end{pmatrix}$

(e) Repeat parts (b) and (c) for C.

7.4 Eigenvectors and eigenvalues

Recall that if a matrix has one column, it is called a column vector. We usually denote vectors with an arrow as shown. (You will learn more about vectors in Chapter 9.)

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ or } \vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

If we multiply an $n \times n$ matrix by an $n \times 1$ vector we will get a new $n \times 1$ vector. In other words, $A\vec{v} = \vec{w}$

However, instead of just getting a brand new vector out of the multiplication, we would like to know if it is possible to instead get:

$$A\vec{v} = \lambda\vec{v}$$

In other words, is it possible, for certain λ and $\vec{v}$, that matrix multiplication is the same as multiplying the vector by a constant? The answer is yes, it is possible for this to happen, but it won't happen for just any value of λ and $\vec{v}$. There is a particular value of λ and $\vec{v}$ for which this works. They always occur in pairs and λ is an **eigenvalue** of A and $\vec{v}$ is an **eigenvector** of A.

Coverage in this course is limited to 2×2 matrices only.

In order to see how we can develop a method for finding eigenvectors and eigenvalues, we start at the original equation $A\vec{v} = \lambda\vec{v}$

Firstly, we must make sure that $\vec{v} \neq \vec{0}$. If $\vec{v} = \vec{0}$, then $A\vec{v} = \lambda\vec{v}$ will be true for any value of λ.

Now, rewriting the original equation and simplifying:

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I\vec{v} = (A - \lambda I)\vec{v} = \vec{0}$$

Note that before we factored out the $\vec{v}$ we added the identity matrix I. This is equivalent to multiplying by 1. We needed to do this because without it we would have had the difference of a matrix, A, and a constant, λ, and this can't be done. We now have the difference of two matrices of the same size which can be done.

In order to find the eigenvectors for a matrix we will need to solve a homogeneous system. If a system is written in matrix form as you recall from section 7.3, we will either have the trivial solution, $\vec{v} = \vec{0}$, or we will have an infinite number of solutions.

A homogeneous solution can have an infinite number of solutions if the coefficient matrix is singular.

So, to solve the equation $(A - \lambda I)\vec{v} = \vec{0}$, the matrix $A - \lambda I$ must be singular, and hence $\det(A - \lambda I) = 0$. This is called the **characteristic equation**. $\det(A - \lambda I)$ is called the **characteristic polynomial**.

Example 7.18

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$

Solution

The first step is to find the eigenvalues.

$$A - \lambda I = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 - \lambda & 7 \\ -1 & -6 - \lambda \end{pmatrix}$$

In particular we need to determine where the value(s) of λ for which the determinant of this matrix is zero.

$$\det(A - \lambda I) = (2 - \lambda)(-6 - \lambda) + 7 = \lambda^2 + 4\lambda - 5$$

Thus, the characteristic polynomial is $\lambda^2 + 4\lambda - 5$

Now, finding the zeros of this polynomial will give us the eigenvalues

$$\lambda^2 + 4\lambda - 5 = (\lambda + 5)(\lambda - 1) = 0$$

$$\Rightarrow \lambda_1 = -5, \lambda_2 = 1$$

Each of these eigenvalues will generate an eigenvector.

For $\lambda_1 = -5$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}\vec{v} = \vec{0}$$

Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 7a + 7b = 0 \\ -a - b = 0 \end{cases}$

We can either use simple addition of 6 times the second equation and add it to the first, or use Gauss-Jordan method:

$$a + b = 0 \Rightarrow b = -a \Rightarrow \vec{v} = \begin{pmatrix} a \\ -a \end{pmatrix}$$

This is a general eigenvector, but in fact we usually need specific vectors, and hence any value for a will suffice, say $a = 1$ here. Therefore $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a first eigenvector. Note that for any other choice of a, the resulting vector will be parallel to this one.

For $\lambda_2 = 1$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 1 & -7 \\ -1 & -7 \end{pmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a + 7b = 0 \\ -a - 7b = 0 \end{cases}$$

Again, this will lead to

$$a + 7b = 0 \Rightarrow a = -7b \Rightarrow \vec{v} = \begin{pmatrix} -7b \\ b \end{pmatrix}$$

Using $b = 1$ (or any other value, not zero, of your choice) we have

$\vec{v} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$ as a second eigenvector.

Note that $\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} = -5\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as required, and

$\begin{pmatrix} -2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ as required.

If A is a square matrix, and if $\vec{v}$ is an eigenvector, then any scalar multiple $k\vec{v}$ is also an eigenvector.
If A is a square matrix and $\vec{v}$ and $\vec{w}$ are eigenvectors, then $\vec{u} + \vec{w}$ is an eigenvector.
If A is a square matrix, and if λ is an eigenvalue and $\vec{v}$ is a corresponding eigenvector, then λ^n is an eigenvalue and $\vec{v}$ is an eigenvector of A^n.

Example 7.19

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ and the matrix A^4.

Solution

The first step is to find the eigenvalues of A.

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{pmatrix}$$

The determinant of this matrix must be zero.

$$\det\begin{pmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{pmatrix} = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 3$$

For $\lambda_1 = 2$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a - b = 0 \\ 2a + 2b = 0 \end{cases}$$

Solving for a and b will yield an infinite number of solutions such that $b = -a$

Thus, any vector of the form $\vec{v} = \begin{pmatrix} t \\ -t \end{pmatrix}$ is an eigenvector. Just for demonstration purposes, we have:

$$A\vec{v} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} 2t \\ -2t \end{pmatrix} = 2\begin{pmatrix} t \\ -t \end{pmatrix}$$

For $\lambda_2 = 3$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a - b = 0 \\ 2a + b = 0 \end{cases}$$

Solving for a and b will yield an infinite number of solutions such that $b = -2a$.

Thus, anyvector of the form $\vec{v} = \begin{pmatrix} t \\ -2t \end{pmatrix}$ is an eigenvector.

Also, for demonstration purposes, we have:

$$A\vec{v} = \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} t \\ -2t \end{pmatrix} = \begin{pmatrix} 3t \\ -6t \end{pmatrix} = 3\begin{pmatrix} t \\ -2t \end{pmatrix}$$

For A^4 the eigenvalues are 2^4 and 3^4 with the same eigenvectors as before.

$$A^4 = \begin{pmatrix} -49 & -65 \\ 130 & 146 \end{pmatrix} \text{ and } \begin{pmatrix} -49 & -65 \\ 130 & 146 \end{pmatrix}\begin{pmatrix} t \\ -t \end{pmatrix} = 16\begin{pmatrix} t \\ -t \end{pmatrix}$$

$$\text{and } \begin{pmatrix} -49 & -65 \\ 130 & 146 \end{pmatrix}\begin{pmatrix} t \\ -2t \end{pmatrix} = \begin{pmatrix} 81t \\ -2t \end{pmatrix}$$

Example 7.20

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$

Solution

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 6 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = 4$$

For $\lambda_1 = -1$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2a + b = 0 \\ 6a + 3b = 0 \end{cases}$$

Solving for a and b will yield an infinite number of solutions such that $b = -2a$

Thus, any vector of the form $\vec{v} = \begin{pmatrix} t \\ -2t \end{pmatrix}$ is an eigenvector. Also, for demonstration purposes, let us choose $t = 1$

$$A\vec{v} = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 4$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\vec{v} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -3a + b = 0 \\ 6a - 2b = 0 \end{cases}$$

Solving for a and b will yield an infinite number of solutions such that $b = 3a$

Thus, any vector of the form $\vec{v} = \begin{pmatrix} t \\ 3t \end{pmatrix}$ is an eigenvector. Also, for demonstration purposes, let us choose $t = 1$

$$A\vec{v} = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

To find eigenvalues and eigenvectors of a square matrix A:
form the matrix $A - \lambda I$

- solve the equation $\det(A - \lambda I) = 0$; the real solutions are the eigenvalues of A
- for each eigenvalue λ_0, form the matrix $A - \lambda_0 I$
- solve the homogeneous system $(A - \lambda_0 I)\vec{X} = \vec{0}$

When the characteristic equation gives distinct real eigenvalues, the corresponding eigenvectors are said to be linearly independent.

The proof is straightforward.

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

Then, $\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ 0 & c-\lambda \end{pmatrix} = 0 \Rightarrow (a-\lambda)(c-\lambda) = 0 \Rightarrow \lambda = a$ or $\lambda = c$

Eigenvalues of triangular matrices
The eigenvalues of a triangular matrix are its diagonal entries.

Diagonalisation

Why is this useful? Suppose you wanted to find A^3. If A can be diagonalised, then:

$$A^3 = (PDP^{-1})^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1})$$
$$= PDP^{-1}PDP^{-1}PDP^{-1} = PD(P^{-1}P)D(P^{-1}P)DP^{-1}$$

But $P^{-1}P = I$, so,

$$A^3 = PD(P^{-1}P)D(P^{-1}P)DP^{-1} = PDDDP^{-1} = PD^3P^{-1}$$

A square matrix A can be diagonalised if it has the property that there exists a matrix P that has an inverse and a diagonal matrix D such that $A = PDP^{-1}$.

In general, when $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$

The connection between eigenvectors, eigenvalues and diagonalisation is given by the diagonalistion theorem which states that an $n \times n$ matrix A is diagonalisable if and **only if A has n linearly independent eigenvectors.**

If $v_1, v_2, \cdots, v_n$ are linearly independent eigenvectors of A and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are their **corresponding eigenvalues, then $A = PDP^{-1}$, where**

$P = (v_1, v_2, \cdots, v_n)$, i.e. P is the matrix whose columns are the eigenvectors, and

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

Example 7.21

Diagonalise the matrix $A = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$

Solution

This is the same matrix as in Example 7.20, so the eigenvalues are -1 and 4. The eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \text{ and so,}$$

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \text{ and } D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$$

Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$$

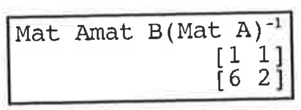


Figure 7.14 GDC screen for the solution to Example 7.21

To verify that the equation above holds, we perform the multiplication on the right-side of the equation:

$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 4 \\ 2 & 12 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{-1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$$

Example 7.22

Given that $A = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$, calculate A^4

Solution

From Example 7.20:

$$A = \begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$$

So

$$A^4 = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}^4\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} (-1)^4 & 0 \\ 0 & 4^4 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{-1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 256 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{-1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 256 \\ -2 & 768 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{-1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 103 & 51 \\ 306 & 154 \end{pmatrix}$$

Example 7.23

Given that A^t is the transpose of a matrix A, prove that A and A^t have the same eigenvalues.

Solution

Recall that the eigenvalues of A are the solutions to its characteristic equation

$$\det(A - \lambda I) = 0$$

In order to find the eigenvalues for A^t, we find its characteristic equation. However, remembering that a matrix and its transpose have equal determinants, we have

$$\det(A - \lambda I) = \det(A - \lambda I)^t = \det(A^t - \lambda I^t) = \det(A^t - \lambda I)$$

Which means that both A and A^t have the same characteristic polynomials. Therefore, A and A^t have the same eigenvalues.

Example 7.24

Let A be the matrix $A = \begin{pmatrix} 1 & 1 & -4 \\ 2 & 0 & -4 \\ -1 & 1 & -2 \end{pmatrix}$

Find a diagonalisation of A.

Solution

The characteristic equation of A is

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & -4 \\ 2 & -\lambda & -4 \\ -1 & 1 & -2-\lambda \end{pmatrix}$$
$$= -(\lambda + 1)(\lambda + 2)(\lambda - 2) = 0$$

And so, the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 2$

We use the Gauss-Jordan method to find the eigenvectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Thus, $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

We have the factorisation:

$$\begin{pmatrix} 1 & 1 & -4 \\ 2 & 0 & -4 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

You can use a GDC.

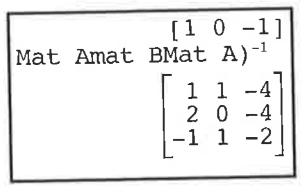


Figure 7.15 GDC screen for the solution to Example 7.24

To diagonalise a square matrix A:

- form the matrix $A - \lambda I$ and find the eigenvalues and eigenvectors as before
- use the eigenvectors found as column vectors to form matrix P
- check that P is non-singular (has an inverse).
- form the diagonal matrix D where the diagonal entries are the eigenvalues found earlier.

Now, $A = PDP^{-1}$ (equivalently, $D = P^{-1}AP$)

Markov chains

This is an introduction to Markov chains. More details will be given in Chapter 15.

Example 7.25

In a rural town, a survey indicated that 85% of the children of university educated parents went to university, while only 35% of the children of parents that do not have university education went to university. What percentage of the second generation went to university if initially 30% of the town parents were university educated?

Solution

The situation described is a part of a sequence of experiments in which the outcomes and their associated chances depend on the outcomes of the preceding experiments. So, for this case, the percentage of university-oriented children, depends on whether their parents went to university, and for those in turn, their parents and so on.

Such a process is called a **Markov process**. The outcome of any experiment is called the **state** of the experiment.

A tree diagram helps to show the situation. Let C stand for university educated person and N for not-university educated.

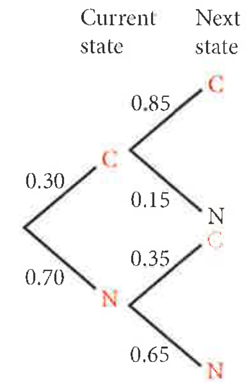


Figure 7.16 Tree diagram for the solution to Example 7.25

The first generation has 30% university educated, and hence 70% are not. In order to find the percentages for the third generation, we first establish the state of the second generation.

The university educated children, C, in this generation could come from university educated parents 0.30×0.85, or from not-university educated parents 0.70×0.35

The proportion of university educated children in the next generation is: $0.30 \times 0.85 + 0.70 \times 0.35 = 0.50$

The children without a university education, N, in this generation could come from university educated parents 0.30×0.15, or from not-university educated parents 0.70×0.65

The proportion of children without a university education for the next generation is: $0.30 \times 0.15 + 0.70 \times 0.65 = 0.50$.

For the third generation we will have to expand every branch with two new ones, i.e. eight separate possible outcomes! You can imagine how the tree will look if you want more generations.

It is very useful to describe the Markov process by a **transition matrix**. The transition matrix gives the proportion going from one state to another. For example, in this question.

Current state

$$\begin{matrix} & \text{C} & \text{N} & \\ & \begin{pmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{pmatrix} & \begin{matrix} \text{C} \\ \text{N} \end{matrix} & \text{Next state} \end{matrix}$$

The initial state is

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0.30 \\ 0.70 \end{pmatrix}$$

And thus, the next generation is

$$\begin{pmatrix} 0.85 & 0.35 \\ 0.15 & 0.65 \end{pmatrix}\begin{pmatrix} 0.30 \\ 0.70 \end{pmatrix} = \begin{pmatrix} 0.85 \times 0.30 + 0.35 \times 0.70 \\ 0.15 \times 0.30 + 0.65 \times 0.70 \end{pmatrix} = \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix}$$

For the second generation, we just multiply the transition matrix by this result

$$\begin{pmatrix}0.85 & 0.35\\0.15 & 0.65\end{pmatrix}\begin{pmatrix}0.50\\0.50\end{pmatrix}=\begin{pmatrix}0.85\times0.50+0.35\times0.50\\0.15\times0.50+0.65\times0.50\end{pmatrix}=\begin{pmatrix}0.60\\0.40\end{pmatrix}$$

Or, put differently

$$\begin{pmatrix}0.85 & 0.35\\0.15 & 0.65\end{pmatrix}^2\begin{pmatrix}0.30\\0.70\end{pmatrix}=\begin{pmatrix}0.50\\0.50\end{pmatrix}$$

We can summarise the steps as follows.

In a two-state Markov process, we let the proportions of moving from one state to the other be given by a transition matrix

From

$$T=T_0\begin{pmatrix}p_{1,1} & p_{2,1}\\p_{1,2} & p_{2,2}\end{pmatrix}$$

where $p_{i,j}$ represents the proportion moving from state i to state j. (from C to N or from N to N.... in the example). Note that the sum of the entries in each column must be 1.
We also let the initial state be represented by

$X_0=\begin{pmatrix}x_0\\y_0\end{pmatrix}$, then the state after n experiments will have a state represented by

$$X_n=T^nX_0$$

For more state cases, the matrices will be of an appropriate order. For example, in a 3-state process

$$T=\begin{pmatrix}p_{1,1} & p_{2,1} & p_{3,1}\\p_{1,2} & p_{2,2} & p_{3,2}\\p_{1,3} & p_{2,3} & p_{3,3}\end{pmatrix}\text{ and }X_0=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$$

This process can also be done with horizontal vectors. That is

$$X_n=X_0T^n=(x_0,y_0,z_0)\begin{pmatrix}p_{1,1} & p_{1,2} & p_{1,3}\\p_{2,1} & p_{2,2} & p_{2,3}\\p_{3,1} & p_{3,2} & p_{3,3}\end{pmatrix}^n$$

This means that the sum of entries in each row must be 1.

Example 7.26

Each year 3% of the population living in a certain city will move to the suburbs and 6% of the population living in the suburbs will move into the city. At present 975 000 people live in the city itself, and 525 000 live in the suburbs. Assuming that the total population of the area does not change, find the distribution of the population:

(a) 1 year from now

(b) 10 years from now.

Solution

(a) The transition matrix is:

Current state

$$\begin{matrix} & C & S & \\ & \begin{pmatrix}0.97 & 0.06\\0.03 & 0.94\end{pmatrix} & \begin{matrix}C\\S\end{matrix} & \text{Next state}\end{matrix}$$

The initial state is:

$$\begin{pmatrix}x_0\\y_0\end{pmatrix}=\begin{pmatrix}975\,000\\525\,000\end{pmatrix}$$

One year from now, is one stage above the initial one

$$X_1=TX_0\Rightarrow\begin{pmatrix}0.97 & 0.06\\0.03 & 0.94\end{pmatrix}\begin{pmatrix}975\,000\\525\,000\end{pmatrix}=\begin{pmatrix}977\,250\\522\,750\end{pmatrix}$$

(b) 10 years from now:

$$X_{10}=TX_0\Rightarrow\begin{pmatrix}0.97 & 0.06\\0.03 & 0.94\end{pmatrix}^{10}\begin{pmatrix}975\,000\\525\,000\end{pmatrix}\approx\begin{pmatrix}990\,264\\509\,736\end{pmatrix}$$

Example 7.27

A taxi company divides the city into three zones: I, II, and III. The company determined from previous records that 60% of passengers picked up in zone I stay in zone I, 30% go to zone II and 10% to zone III. Of those picked up in zone II, 40% go to zone I, 30% to zone II, and 30% to zone III. Of those picked up in zone III, 30% go to zone I, 30% to zone II, and 40% to zone III.

At the beginning of the day, 80% of the taxis are in zone I, 15% are in zone II, and 5% are in zone III.

(a) What is the distribution of the taxis after they have each made one trip?

(b) On average, all taxis make 20 trips per day. What is the distribution by the end of the day?

Solution

$$T=\begin{pmatrix}0.6 & 0.4 & 0.3\\0.3 & 0.3 & 0.3\\0.1 & 0.3 & 0.4\end{pmatrix},\ X_0=\begin{pmatrix}0.80\\0.15\\0.05\end{pmatrix}$$

(a) $X_1=TX_0=\begin{pmatrix}0.6 & 0.4 & 0.3\\0.3 & 0.3 & 0.3\\0.1 & 0.3 & 0.4\end{pmatrix}\begin{pmatrix}0.80\\0.15\\0.05\end{pmatrix}=\begin{pmatrix}0.555\\0.300\\0.145\end{pmatrix}$

(b) $X_{10}=T^{10}X_0=\begin{pmatrix}0.6 & 0.4 & 0.3\\0.3 & 0.3 & 0.3\\0.1 & 0.3 & 0.4\end{pmatrix}^{10}\begin{pmatrix}0.80\\0.15\\0.05\end{pmatrix}\approx\begin{pmatrix}0.471\\0.300\\0.229\end{pmatrix}$

Eigenvalues and eigenvectors can be used to evaluate the powers of transition matrices. The approach is similar to what we did with diagonalising matrices earlier. In Example 7.26, we can diagonalise the transition matrix

$$\begin{pmatrix} 0.97 & 0.06 \\ 0.03 & 0.94 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.91 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$, which will enable us to find its powers more easily.

Exercise 7.4

1. For each matrix:

(i) find the characteristic polynomial

(ii) find the eigenvalues and eigenvectors

(iii) diagonalise the matrix.

(a) $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$ **(c)** $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & -1 \\ -3 & 2 \end{pmatrix}$ **(e)** $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **(f)** $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(g) $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ **(h)** $A = \begin{pmatrix} 7 & -1 \\ 6 & 2 \end{pmatrix}$ **(i)** $A = \begin{pmatrix} 4 & 9 \\ 2 & 7 \end{pmatrix}$

(j) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ **(k)** $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ **(l)** $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

(m) $A = \begin{pmatrix} 0 & a^2 \\ b^2 & 0 \end{pmatrix}$ **(n)** $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ **(o)** $A = \begin{pmatrix} 3 & 7 \\ 5 & 10 \end{pmatrix}$

(p) $A = \begin{pmatrix} 15 & -8 \\ 18 & -15 \end{pmatrix}$ **(q)** $A = \begin{pmatrix} -5 & 4 \\ -8 & 7 \end{pmatrix}$ **(r)** $A = \begin{pmatrix} 8 & 6 \\ -15 & -11 \end{pmatrix}$

2. The transition matrix for a Markov process is given by

State

$$\begin{matrix} & 1 & 2 & \\ & \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix} & & \begin{matrix} \text{State 1} \\ \text{State 2} \end{matrix} \end{matrix}$$

(a) What does the entry 0.3 represent?

(b) The initial-state distribution vector is given by

$$X_0 = \begin{matrix} \text{State 1} \\ \text{State 2} \end{matrix}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$$

Find the distribution of the system after one observation.

3. Kevin is either happy or sad. If he is happy one day, then he is happy the next day four times out of five. If he is sad one day, then he is sad the next day one time out of three. Over the long term, what are the chances that Kevin is happy on any given day?

4. Three grocery chains serve a large area in a certain country. During the year, grocery A expects to retain 80% of its customers, 5% are lost to grocery B, and 15% to grocery C. Grocery B expects to retain 90% of its customers, and loses 5% to each of the other two groceries. Grocery C expects to retain 75% of its customers, and loses 10% to grocery A and 15% to grocery B.

(a) Construct the transition matrix for the Markov chain that describes the change in the market share.

(b) Currently the market share is 0.4 for grocery A, 0.3 for grocery B and 0.3 for grocery C. What share of the market is held by each grocery after 1 year?

(c) Assuming the trend continues, what share does each grocery hold after 2 years?

5. TG Polling conducted a poll 6 months before elections in a country in which a liberal and a conservative were running for president. TG found that 60% of the voters intended to vote for the conservative and 40% for the liberal. In a poll conducted 3 months later, TG found that 70% of those who had earlier stated a preference for the conservative candidate still maintained that preference, whereas 30% of those voters now preferred the liberal candidate. Of those who earlier preferred the liberal, 80% still maintained their preference, whereas 20% switched to the conservative.

(a) If elections were held at this time (after 3 months), who would win?

(b) If the trend continues, which candidate will win the election?

6. Three truck manufacturers A, B, and C share the domestic market in a certain country. Their current market shares are 60%, 30% and 10% respectively. Market studies show that manufacturer A retains 75% of its customers, and loses 15% to manufacturer B and 10% to manufacturer C. Of the customers who buy from manufacturer B, 90% would keep their preference, while 5% go to each of manufacturers A and C. Of the customers who buy from manufacturer C, 85% are retained, while 5% would buy from manufacturer A and 10% from manufacturer B.

(a) Assuming that these sentiments reflect the buying habits of customers in the future, determine the market share that will be held by each manufacturer after 2 years.

(b) Under the same conditions, determine the market share that will be held by each manufacturer after 5 years.

7. By reviewing its donation records, the alumni office of a university finds that 80% of its alumni who contribute to the annual fund one year will also contribute the next year, and 30% of those who do not contribute one year will contribute the next. Consider a new graduate who did not give a donation in the initial year after graduation.

(a) Construct the probable future donation record for three years of such new graduates who did not give a donation in the initial year after graduation.

(b) Consider the situation on the 11th year and conjecture a pattern for the long term.

8. A car rental agency has three rental locations, Zurich (Z), Geneva (G), and Basel (B). A customer may rent a car from any of the three locations and return the car to any of the three locations. The manager finds that customers return the cars to the various locations according to the following probabilities:

Rented from location

$$\begin{array}{c} \quad Z \quad\; G \quad\; B \\ \begin{pmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.6 \\ 0.1 & 0.5 & 0.2 \end{pmatrix} \end{array} \begin{array}{c} Z \\ G \\ B \end{array} \quad \text{Returned to location}$$

Previous records show that 40% of the fleet are rented in Geneva, 35% in Zurich, and 25% in Basel. Find the long-term trend in terms of percentage of cars present at each location.

9. 200 000 people live in a certain city and 25 000 people live in its suburbs. The Regional Planning Commission determines that each year 5% of the city population moves to the suburbs and 3% of the suburban population moves to the city.

(a) Assuming that the total population remains constant, make a table that shows the populations of the city and its suburbs over a five-year period (round to the nearest integer).

(a) Over the long term, how will the population be distributed between the city and its suburbs?

7.5 Matrices and geometric transformations

In Chapter 3 you learned about function transformations: Reflecting in the x-axis, or y-axis, stretching vertically or horizontally, or in both directions or combinations of those. In this chapter we focus on transformations of the plane.

The green triangle in the diagram in Figure 7.17, for example, is reflected in the y-axis to get the red triangle. As the diagram shows, only the position has changed. All angles and sides still have the same measures.

Take a look at the coordinates of the points making the triangles:

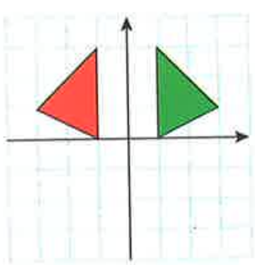


Figure 7.17 The green triangle is reflected in the y-axis to get the red triangle

Green triangle: (1, 0), (3, 1), and (1, 3)

Red triangle: (−1, 0), (−3, 1), and (−1, 3)

That means, any point with coordinates (x, y) is reflected into a point $(-x, y)$.

This process can be achieved by matrix multiplication. Consider the matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and construct a matrix, T with the coordinates of the vertices of the green triangle as its columns $T = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Now multiply M by T: $MT = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -3 & -1 \\ 0 & 1 & 3 \end{pmatrix}$

The columns of the resulting matrix are the coordinates of the red triangle.

The method we can use to find the matrix representing a transformation is given by the following theorem.

Matrix Basis Theorem

Let T be a transformation represented by a matrix M. Then if

$$T: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}, \text{ and } T: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} b \\ d \end{pmatrix}, \text{ then } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Online

Explore matrix transformations visually.

Here is a list of major transformations that you are familiar with and their corresponding matrices.

1. **Reflection in the x-axis:** $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. **Reflection in the y-axis:** $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3. **Reflection in $y = x$:** $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4. **Reflection in $y = -x$:** $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

5. **Horizontal dilation** by a constant k (stretch/shrink):

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow M = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

For example, triangle (1, 1), (2, 2), (1, 4) will be transformed into

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} k & 2k & k \\ 1 & 2 & 4 \end{pmatrix}$$

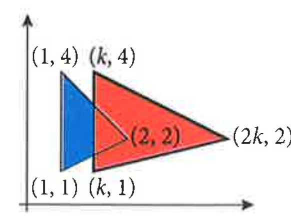


Figure 7.18 Horizontal dilation

6. **Vertical dilation** by a constant m (stretch/shrink):

$$\begin{pmatrix}1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\m\end{pmatrix}\Rightarrow M=\begin{pmatrix}1&0\\0&m\end{pmatrix}$$

For example, triangle (1, 1), (2, 2), (1, 4) will be transformed into

$$\begin{pmatrix}1&0\\0&m\end{pmatrix}\begin{pmatrix}1&2&1\\1&2&4\end{pmatrix}=\begin{pmatrix}1&2&1\\m&2m&4m\end{pmatrix}$$

7. The matrix $M=\begin{pmatrix}k&0\\0&m\end{pmatrix}$ represents a **horizontal dilation of magnitude k and a vertical dilation of magnitude m.** When $k=m$, this is called scaling.

For example, triangle (1, 1), (2, 2), (1, 4) will be transformed into

$$\begin{pmatrix}k&0\\0&m\end{pmatrix}\begin{pmatrix}1&2&1\\1&2&4\end{pmatrix}=\begin{pmatrix}k&2k&k\\m&2m&4m\end{pmatrix}$$

Example 7.28

The triangle ABC has vertices $A(2,-3)$, $B(3,1)$, and $C(-1,4)$.

(a) Find the area of ABC.

(b) Find the coordinates of the image of ABC under each of the following transformations:

(i) reflection in the x-axis

(ii) reflection in the line $y=x$

(c) Find the coordinates of the image of ABC under each of the following transformations, find the area of the image, find the determinant of the transformation matrix, and make a conjecture about the relationship between the three quantities.

(i) dilation of magnitude 2 in the horizontal direction

(ii) horizontal dilation of magnitude k and vertical dilation of magnitude m.

Solution

(a) From section 7.2, the area is given by

$$A=\frac{1}{2}\left\|\begin{matrix}x_1&y_1&1\\x_2&y_2&1\\x_3&y_3&1\end{matrix}\right\|=\frac{1}{2}\left\|\begin{matrix}2&-3&1\\3&1&1\\-1&4&1\end{matrix}\right\|=\frac{19}{2}$$

(b) (i) Reflection in the x-axis: $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}2&3&-1\\-3&1&4\end{pmatrix}$

$$=\begin{pmatrix}2&3&-1\\3&-1&-4\end{pmatrix}$$

that is $(2,3)$, $(3,-1)$, $(-1,-4)$

(ii) Reflection in the line $y=x$: $\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}2&3&-1\\-3&1&4\end{pmatrix}$

$$=\begin{pmatrix}-3&1&4\\2&3&-1\end{pmatrix}$$

that is $(-3,2)$, $(1,3)$, $(4,-1)$

(c) (i) Dilation of magnitude 2 in the horizontal direction:

$$\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}2&3&-1\\-3&1&4\end{pmatrix}=\begin{pmatrix}4&6&-2\\-3&1&4\end{pmatrix}$$

So, image has the vertices $(4,-3)$, $(6,1)$, $(-2,4)$

$$\text{area}=\frac{1}{2}\left\|\begin{matrix}4&-3&1\\6&1&1\\-2&4&1\end{matrix}\right\|=19\text{ and determinant is }\begin{vmatrix}2&0\\0&1\end{vmatrix}=2$$

Thus, area of image = determinant × Area of pre-image. In fact, it must be the absolute value of the determinant.

(ii) Dilations in both directions:

$$\begin{pmatrix}k&0\\0&m\end{pmatrix}\begin{pmatrix}2&3&-1\\-3&1&4\end{pmatrix}=\begin{pmatrix}2k&3k&-k\\-3m&m&4m\end{pmatrix}$$

So, image has the vertices

$(2k,-3m)$, $(3k,m)$, $(-k,4m)$,

$$\text{area}=\frac{1}{2}\left\|\begin{matrix}2k&-3m&1\\3k&m&1\\-k&4m&1\end{matrix}\right\|=\frac{19}{2}|km|$$

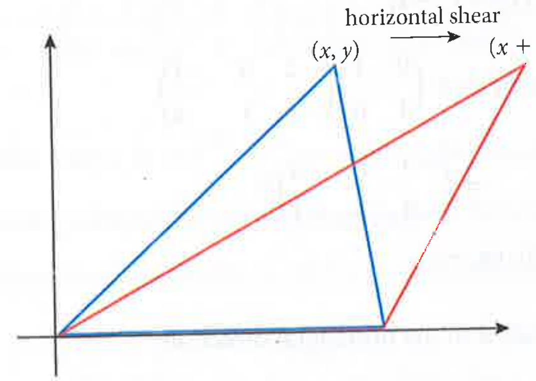
Thus, area of image = |determinant| × Area of object (pre-image)

If a matrix M is a transformation matrix, then Area of image = $|\det M|$ × area of object.

The above transformations are called **affine transformations**. Affine transformations are known to preserve points, (for example, a triangle is mapped into a triangle), also, sets of parallel lines and planes remain parallel after an affine transformation. An affine transformation does not necessarily preserve angles between lines or distances between points, though it does preserve ratios of distances between points lying on a straight line.

Two more transformations worth mentioning, but may not be examined without guidance are **horizontal and vertical shearing**:

A horizontal shearing of magnitude k is given by $\begin{pmatrix}1 & k\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+ky\\y\end{pmatrix}$ and a vertical shearing of magnitude m is given by $\begin{pmatrix}1 & 0\\m & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y+mx\end{pmatrix}$



Two more transformations, that are a special type of affine transformation, are called **isometries: translations** and **rotations**. These transformations preserve angles and distances. That is, in a translation, a triangle is mapped into a congruent triangle. Similarly, for rotations.

Example 7.29

Consider the matrix $A=\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

and the unit square with vertices (0, 0), (1, 0), (1, 1), and (0, 1).

(a) Perform the transformation, sketch graphs showing the unit square and its image, and conjecture what this transformation represents.

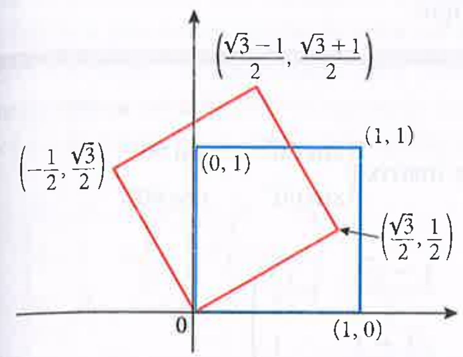
(b) What do A^2 and A^3 represent?

Solution

(a) The transformation mapped

$$\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\end{pmatrix} \text{ to } \begin{pmatrix}0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}-1}{2} & -\frac{1}{2}\\ 0 & \frac{1}{2} & \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

As the figure implies, we have a rotation through an angle of 30°. This is confirmed by observing that the image of $\begin{pmatrix}1\\0\end{pmatrix}$ is $\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$ which is $\begin{pmatrix}\cos 30°\\ \sin 30°\end{pmatrix}$



(b) $A^2=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$ which implies that the image of $\begin{pmatrix}1\\0\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}\cos 60°\\ \sin 60°\end{pmatrix}$

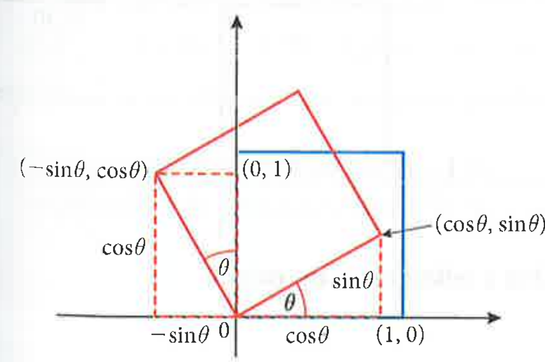
which in turn implies a rotation of 60°.

$A^3=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ implying a rotation of 90°.

We can generalise the above discussion for rotations around the origin through any angle.

As the diagram shows, the rotation maps $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$, thus we can state the following result:

The matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ represents a **rotation** of angle θ around the origin.



Example 7.30

Consider the unit square in Example 7.29. Perform the following two rotations on the square and draw a sketch of the result:

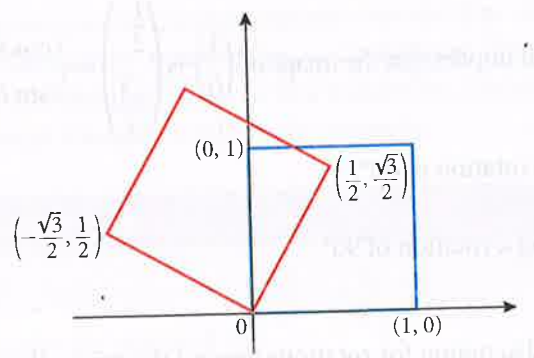
(a) rotation of 60° around the origin

(b) rotation of 180° around the origin.

Solution

(a) We multiply $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ by matrix $\begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ to get $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1-\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}+1}{2} & \frac{1}{2} \end{pmatrix}$

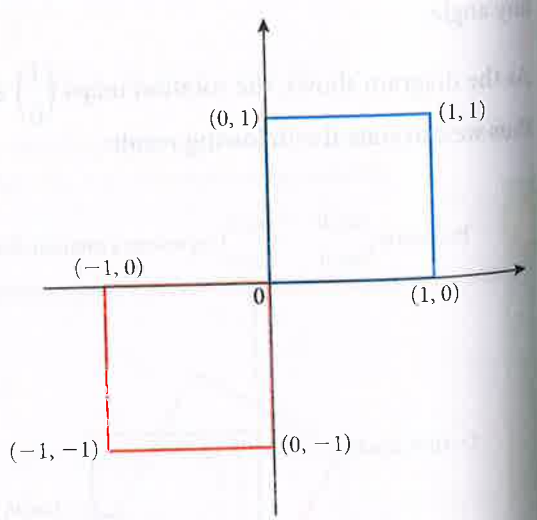


(b) We multiply by

$\begin{pmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to get

$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$



In fact, this is nothing but a reflection in the origin.

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection in the origin.

Translation

A translation of h units horizontally and k units vertically is simply achieved by mapping $\begin{pmatrix} x \\ y \end{pmatrix}$ into $\begin{pmatrix} x+h \\ y+k \end{pmatrix}$. This is called a translation of $\begin{pmatrix} h \\ k \end{pmatrix}$

As is, we cannot perform translation using matrix multiplication. However, we can follow a procedure similar to what we did in using determinants to find the area of a triangle. We introduce a 'new' set of 3D coordinates $(x, y, 1)$, called **homogeneous coordinates** to represent the 2D coordinates (x, y). Translations can then be performed with matrix multiplication in the following manner.

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix}$$

Rotation in this system can also be performed by considering the rotation matrix as

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In fact, all matrices introduced earlier can be written in this manner:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composition of transformations

Let S and T be two transformations. The transformation $S \circ T$ is called the **composition** of the two transformations where T is applied first. It is also written as $S(T(x, y))$

For example, if $S(x, y) = (-x, y)$ and $T(x, y) = (-y, x)$, then:

$$S \circ T(x, y) = S(T(x, y)) = S(-y, x) = (y, x)$$

and $$T \circ S(x, y) = T(-x, y) = (-y, -x)$$

Composition of transformations can be achieved with matrix multiplication.

If M is the matrix representation of a **transformation** M, and N is that of transformation N, then the product MN is the matrix representation of $M \circ N$

Composition of transformations is not commutative. That is, in general $M \circ N \neq N \circ M$

Example 7.31

Consider the unit square in Example 7.29. Perform the following transformations on the square.

(a) a translation of $\begin{pmatrix}2\\3\end{pmatrix}$

(b) a translation of $\begin{pmatrix}2\\3\end{pmatrix}$ followed by a rotation of 180°

(c) a rotation of 180° followed by a translation of $\begin{pmatrix}2\\3\end{pmatrix}$

Solution

(a) $\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\end{pmatrix} \Rightarrow \begin{pmatrix}0+2 & 1+2 & 1+2 & 0+2\\0+3 & 0+3 & 1+3 & 1+3\end{pmatrix} = \begin{pmatrix}2 & 3 & 3 & 2\\3 & 3 & 4 & 4\end{pmatrix}$

It can be done by $\begin{pmatrix}1 & 0 & 2\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 3 & 3 & 2\\3 & 3 & 4 & 4\\1 & 1 & 1 & 1\end{pmatrix}$

(b) Translation is done as above. Applying rotation to the result

$$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}2 & 3 & 3 & 2\\3 & 3 & 4 & 4\end{pmatrix} = \begin{pmatrix}-2 & -3 & -3 & -2\\-3 & -3 & -4 & -4\end{pmatrix}$$

This can be done by the following matrix multiplication

$$\begin{pmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}-2 & -3 & -3 & -2\\-3 & -3 & -4 & -4\\1 & 1 & 1 & 1\end{pmatrix}$$

(c) Rotation: $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}0 & -1 & -1 & 0\\0 & 0 & -1 & -1\end{pmatrix}$, applying translation to the result $\begin{pmatrix}0+2 & -1+2 & -1+2 & 0+2\\0+3 & 0+3 & -1+3 & -1+3\end{pmatrix}$

$= \begin{pmatrix}2 & 1 & 1 & 2\\3 & 3 & 2 & 2\end{pmatrix}$

This can be done by the following matrix multiplication

$$\begin{pmatrix}1 & 0 & 2\\0 & 1 & 3\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 0\\0 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1 & 1 & 2\\3 & 3 & 2 & 2\\1 & 1 & 1 & 1\end{pmatrix}$$

Note in Example 7.31 that a translation followed by a rotation is not the same as a rotation followed by a translation.

Example 7.32

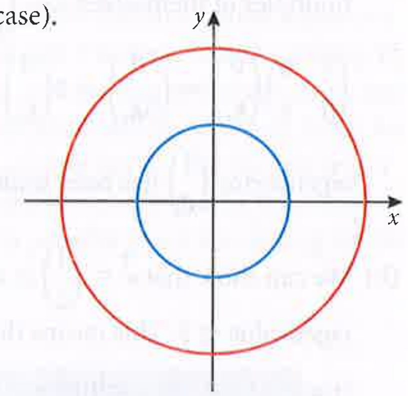
Describe the effect of each transformation on any point (x, y) in the plane and on a circle with equation $x^2 + y^2 = 16$

(a) $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$ (b) $\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}$

Solution

(a) This is a simple dilation (scale in this case).

$$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\2y\end{pmatrix}$$



The new coordinates are

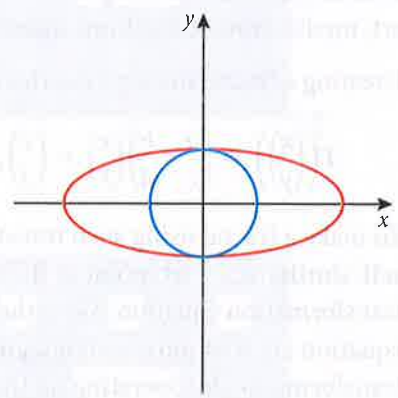
$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}2x\\2y\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{x_1}{2}\\\frac{y_1}{2}\end{pmatrix}$$

so, the equation of the image is

$x^2 + y^2 = 16 \Rightarrow \left(\frac{x_1}{2}\right)^2 + \left(\frac{y_1}{2}\right)^2 = 16 \Rightarrow x_1^2 + y_1^2 = 64$, which is another circle with a larger radius (which you can write as $x^2 + y^2 = 64$). Note that the new circle has an area 4 times the original. The determinant of the matrix is 4.

(b) This is a dilation in the x-direction.

$$\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x\\y\end{pmatrix}$$



The new coordinates are

$$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}3x\\y\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{x_1}{3}\\y_1\end{pmatrix}$$

so the equation of the image is

$$x^2 + y^2 = 16 \Rightarrow \left(\frac{x_1}{3}\right)^2 + y_1^2 = 16$$

$\Rightarrow \frac{x_1^2}{144} + \frac{y_1^2}{16} = 1 \left(\text{or } \frac{x^2}{144} + \frac{y^2}{16} = 1\right)$ which is an ellipse.

Example 7.33

Consider each transformation matrix.

Find an eigenvector and interpret its meaning.

(a) $\begin{pmatrix}3 & 0\\0 & 5\end{pmatrix}$ (b) $\begin{pmatrix}3 & 0\\8 & -1\end{pmatrix}$

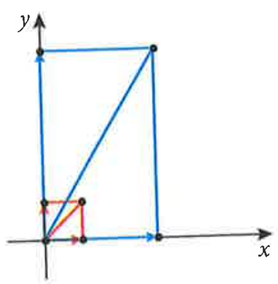


Figure 7.19 Diagram for solution to Example 7.33 (a)

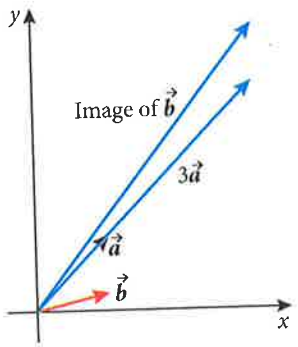


Figure 7.20 Diagram for solution to Example 7.33 (b)

Solution

(a) From section 7.4, you know that the characteristic equation is $(\lambda - 3)(\lambda - 5) = 0$ implying that the eigenvalues are 3 and 5 with corresponding eigenvectors $\begin{pmatrix} k_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ k_2 \end{pmatrix}$. This means that vectors along the x-axis and along the y-axis will have their images multiples of themselves. i.e., $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} k_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3k_1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} k_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5k_2 \end{pmatrix} = 5\begin{pmatrix} 0 \\ k_2 \end{pmatrix}$. Note from the graph that an eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has been multiplied by 3 and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by 5.

(b) We can show that $\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to an eigenvalue of 3. This means that only vectors parallel to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are transformed into multiples of themselves. See the diagram where we show another vector with its image.

Fractals

Fractal mathematics is a growing field of study. Fractals are applied in today's art, media, communications industry, and science.

Creating a fractal involves a series of affine transformations of the form

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$, where a, b, c, d, e and f are scalars.

To make a fractal using such transformations, we begin with an initial **self-similar** set. Each point in the set is transformed as dictated by the transformation equation. Since the only operations present in transformation equation are rotations, scalings, and translations, one can visually arrive at the transformation by operating on the whole set rather than one point at a time. Below is one of the most famous fractals, the **Sierpinski carpet**.

The 8 transformation equations used to generate this image are:

1. $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. $T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$
3. $T_3\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$
4. $T_4\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

5. $T_5\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$
6. $T_6\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$
7. $T_7\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}$
8. $T_8\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$

Start with the unit square, call it U_0, then apply the transformation equations to the square, you will get U_1. Notice how square 1 is the result of applying T_1, with no translation, T_2 has a translation of 1:3 in the horizontal direction, and so on. The square in the middle marked with white will be removed. Next, for each new image, apply the equations again, and we get U_2, the squares left in the centre of each square will be removed, and so on. Eventually, with many iterations, obviously done with software, it is possible to get something similar to U_n and more!

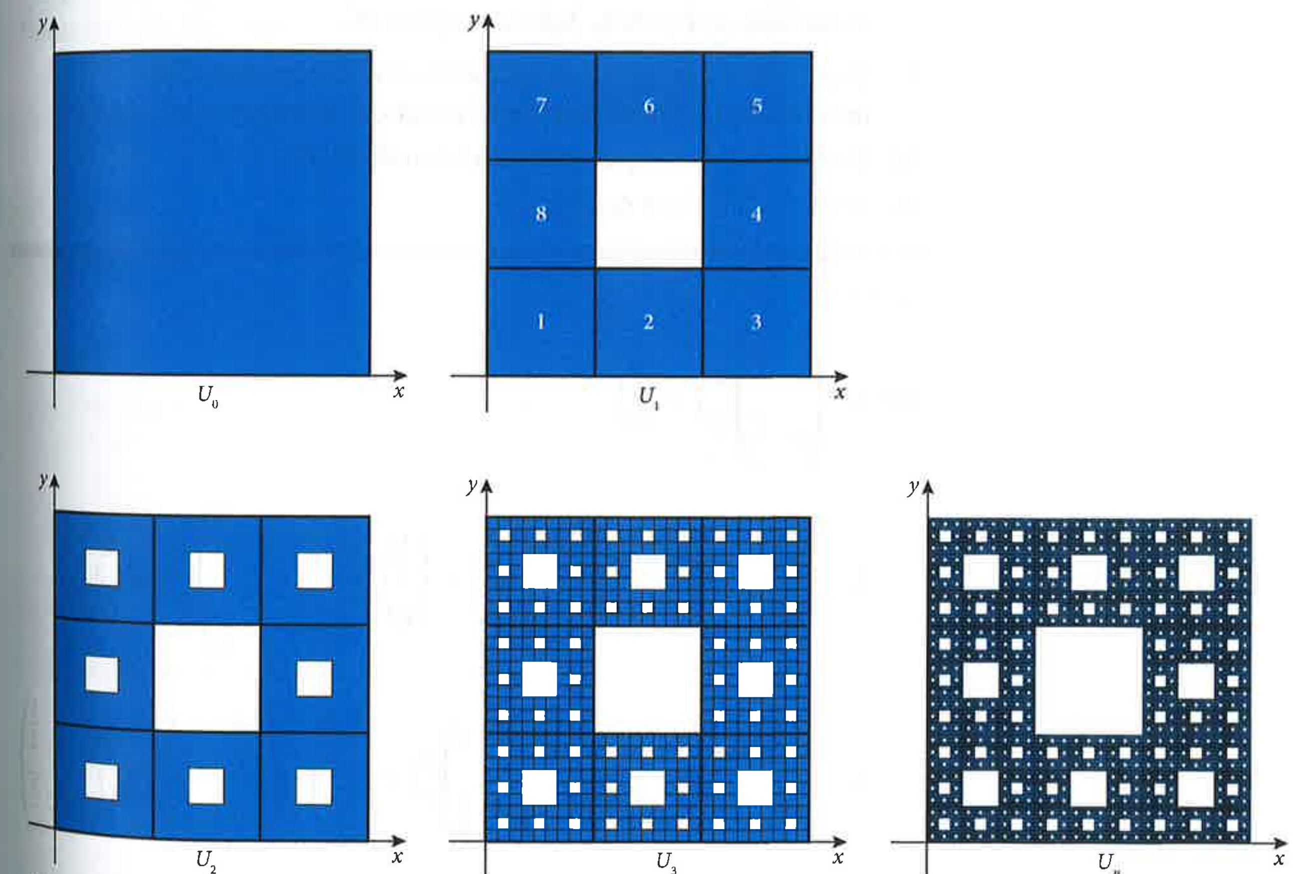
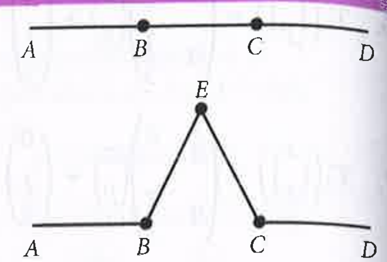


Figure 7.21 Sierpinski carpet

Example 7.34

The **Koch curve** is constructed by removing the middle third of a line segment (say length 1 unit) and replacing it with two sides of an equilateral triangle. Here is the first iteration.



The Koch curve can be created with four transformations:

1. The first shrinks the whole segment to one third of its original size. This maps AD to AB.
2. The second is a dilation of ratio 1:3 followed by a rotation of 60° and a horizontal translation of 1:3. This maps AD to BE.
3. The third transformation is a dilation of ratio 1:3 followed by a rotation of 260° and a translation of $\frac{1}{2}$ in the horizontal direction and $\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6}$ in the vertical direction. This maps AD to EC.
4. The fourth transformation is a dilation of ratio 1:3 followed by a translation of 2:3 in the horizontal direction. This maps AD to CD.

(a) Find the matrices representing each transformation.

(b) Draw the first three iterations.

Solution

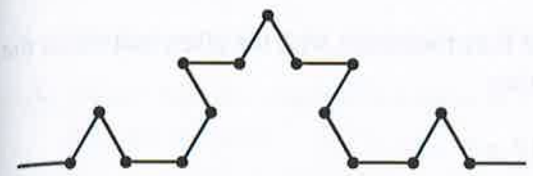
(a) 1. $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2. $\begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$

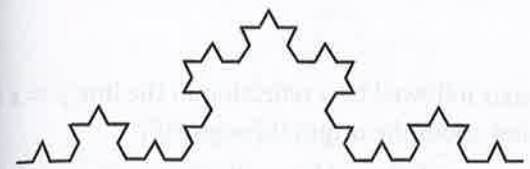
3. $\begin{pmatrix} \cos(-60) & -\sin(-60) \\ \sin(-60) & \cos(-60) \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$

4. $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$

(b) For the second iteration, we apply the same procedure on each segment created in the first iteration:



For the third iteration, we apply the same procedure on each segment created in the second iteration:



Exercise 7.5

1. Find the image of $\triangle ABC$ with vertices $A\,(0, 0)$, $B\,(3, 0)$, and $C\,(3, 1)$ under each of the following transformations, then describe the effect of the transformation in words.

   (a) $T(x, y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **(b)** $T(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

   (c) $T(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **(d)** $T(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   (e) $T(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ **(f)** $T(x, y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

2. Describe the effect of applying the transformation $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ on objects in the plane.

3. Determine whether the matrix A below may represent a rotation about the origin. Explain your answer: $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

4. Describe the effect upon the line with equation $3x + 2y = 6$ of the transformation:

   (a) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

5. Show that a reflection in the x-axis followed by a reflection in the y-axis is equivalent to a rotation of 180° about the origin.

6. Describe the effect of the transformation with the given matrix on the graph of the given equation:

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$; $x^2 + (y-1)^2 = 9$

(b) $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$; $3x + 2y = 6$

7. Show that:

(a) a reflection in the x-axis followed by a reflection in the line $y = x$ is equivalent to a rotation about the origin through 90°

(b) a reflection in the line $y = x$ followed by a reflection in the x-axis is equivalent to a rotation about the origin through −90°

8. Consider the unit square with vertices (0, 0), (1, 0), (1, 1), and (0, 1).

Find the image of this square under the following transformations and sketch the graphs of the final images:

(a) a translation of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ followed by a dilation $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

(b) a dilation $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ followed by a translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

9. Consider the transformation $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and the line $3x + y = 6$

Find the equation of the image of the line under the transformation. Then choose three points on the line and find their images and check the correctness of your equation.

10. Consider each of the transformation matrices, find the eigenvectors for each and interpret them in terms of transformations.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}$

11. A rotation about the origin of $\alpha°$ followed by a rotation of $\beta°$ is equivalent to a rotation of $\alpha° + \beta°$

(a) Write down the matrix for a rotation of $\alpha° + \beta°$

(b) By considering rotation of $\alpha°$ followed by a rotation of $\beta°$ as a composition of transformations, find the matrix for this transformation.

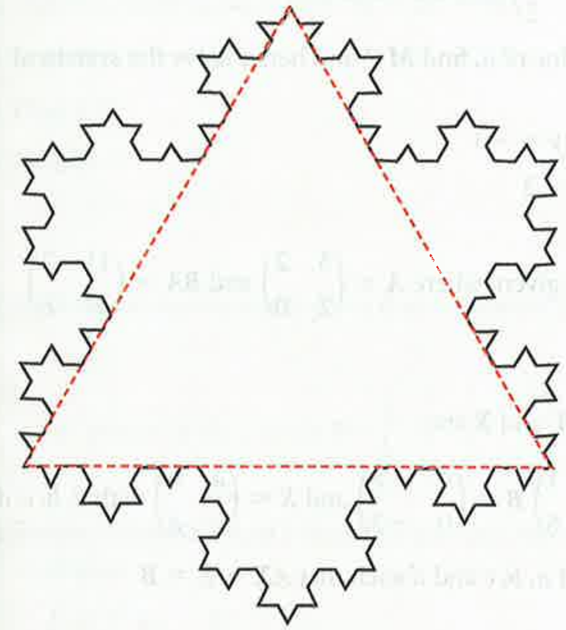
(c) Compare your answers to **(a)** and **(b)**.

12. Consider a Koch curve as described in Example 7.34, where the length of each side is 1 unit.

(a) Find an expression for the length of the nth iterated curve.

(b) Hence, find the length of the curve as it is iterated an infinite number of times.

If instead of starting with a line, you start with an equilateral triangle, you get a **Koch snowflake**.



(c) Find an expression for the number of small sides after n iterations.

(d) Find the length of each side after n iterations.

(e) Find the perimeter after n iterations, and hence the perimeter of the fractal.

(f) Find the number of smaller triangles (each missing one side).

(g) Find the area of each triangle.

(h) Find the total area of the triangles after n iterations, and hence the area of the fractal.

Chapter 7 practice questions

1. If $A = \begin{pmatrix} 2x & 3 \\ -4x & x \end{pmatrix}$ and $\det A = 14$, find the value of x.

2. Let $M = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}$ where $a \in \mathbb{Z}$

   (a) Find M^2 in terms of a.

   (b) $M^2 = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$ find the value of a.

   (c) Using this value of a, find M^{-1} and hence solve the system of equations

   $$\begin{cases} -x + 2y = -3 \\ 2x - y = 3 \end{cases}$$

3. Two matrices are given, where $A = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$

   Find B.

4. The matrices A, B, and X are:

   $$A = \begin{pmatrix} 3 & 1 \\ -5 & 6 \end{pmatrix} B = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix} \text{ and } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a, b, c, d \in \mathbb{Q}$$

   Find the values of a, b, c and d such that $AX + X = B$

5. $A = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$ is a 2×2 matrix.

   (a) Write out A^{-1}

   (b) (i) If $XA + B = C$, where B, C, and X are 2×2 matrices, express X in terms of A^{-1}, B, and C.

   (ii) Find X when $B = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$

6. Let $A = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix}$

   (a) Find $A + B$

   (b) Find AB

7. (a) State the inverse of the matrix $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$

   (b) Hence solve the system of simultaneous equations

   $$\begin{cases} x - 3y + z = 1 \\ 2x + 2y - z = 2 \\ x - 5y + 3z = 3 \end{cases}$$

8. Let $C = \begin{pmatrix} -2 & 4 \\ -1 & 7 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$

   The matrix Q is given such that $3Q = 2C - D$

   (a) Find Q

   (b) Find CD

   (c) Find D^{-1}

9. (a) Find the values of a and b given that the matrix $A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & -4 \end{pmatrix}$ is the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$

   (b) For the values of a and b found in part (a), solve the system of linear equations:

   $$\begin{cases} x + 2y - 2z = 5 \\ 3x + by + z = 0 \\ -x + y - 3z = a - 1 \end{cases}$$

10. (a) Given matrices A, B, C for which $AB = C$ and $\det A \neq 0$, express B in terms of A and C.

    (b) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$

    (i) Find the matrix DA

    (ii) Find B when $AB = C$

    (c) The following three equations represent three planes that intersect at a point. Find the coordinates of this point.

    $$x + 2y + 3z = 5, \; 2x - y + 2z = 7 \text{ and } 3x - 3y + 2z = 10$$

11. **(a)** Find the determinant of the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}$

(b) Find the value of λ for which the following system of equations can be solved.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ \lambda \end{pmatrix}$$

(c) For this value of λ, find the general solution to the system of equations.

12. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$

13. At a large private university, the student loan program evaluates the payment status of student loans. The loans are divided into three categories: loans paid within 14 days are up to date, D, loans paid within 15–60 days are considered late, L, and those paid after 60 days are labelled as problematic, P. Each year, some of the students change categories because they get behind in payments or catch up. The table shows the fraction of students that change from one category to another or stay in the same category.

		Move from category		
		D	L	P
Move to category	D	0.86	0.62	0.17
	L	0.08	0.29	0.37
	P	0.06	0.09	0.46

One year, the fraction of students in each category was 0.8 in D, 0.11 in L, and 0.09 in P.

(a) Find the fraction in each category the next year.

(b) Find the fraction in each category three years later.

14. **(a)** Diagonalise the matrix $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

(b) Hence or otherwise find A^8

15. A country is divided into three demographic regions. It is found that each year 5% of the residents of region 1 move to region 2, and 5% move to region 3. Of the residents of region 2, 15% move to region 1 and 10% move to region 3. And of the residents of region 3, 10% move to region 1 and 5% move to region 2. What percentage of the population resides in each of the three regions after a long period of time?

16. Two competing television channels, channel 1 and channel 2, each have 50% of the viewer market at some initial point in time. Over each one-year period, channel 1 captures 10% of channel 2's share, and channel 2 captures 20% of channel 1's share.

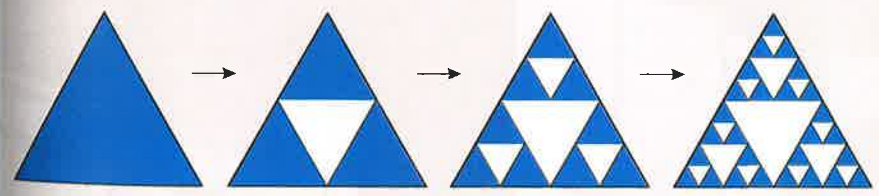
(a) What is each channel's market share after one year?

(b) Track the market shares of channels 1 and 2 in part **(a)** over a five-year period.

(c) If this trend continues, what is the market share of each station?

17. Sierpinski' s triangle is another fractal that can be constructed using transformations, starting with an equilateral triangle. Three transformations are required. The first three stages are shown.

At stage 1, we take a scale of ratio $\frac{1}{2}$ and create one triangle, which is positioned at the lower left corner of the original. Next, we create another triangle and translate it to fit the lower right corner, and similarly we fill the upper corner, thus cutting out the middle triangle as shown. The process is then repeated to each new shaded triangle, and so on.



(a) Count the number of shaded triangles at each stage (stage 0 has 1, stage 1 has 3). Predict the number of shaded triangles at stage 4 and stage 5.

(b) What is the number of triangles at stage n?

(c) Letting the area at stage 0 to be a, what is the total shaded area at each stage? As n becomes large without bound, what happens to the shaded area?

(d) Find the matrix transformations required to create the fractal.

18. A study in 2015 divided occupations in the United Kingdom into upper level, U, (executives and professionals), middle level, M, (supervisors and skilled manual workers), and lower level, L, (unskilled).

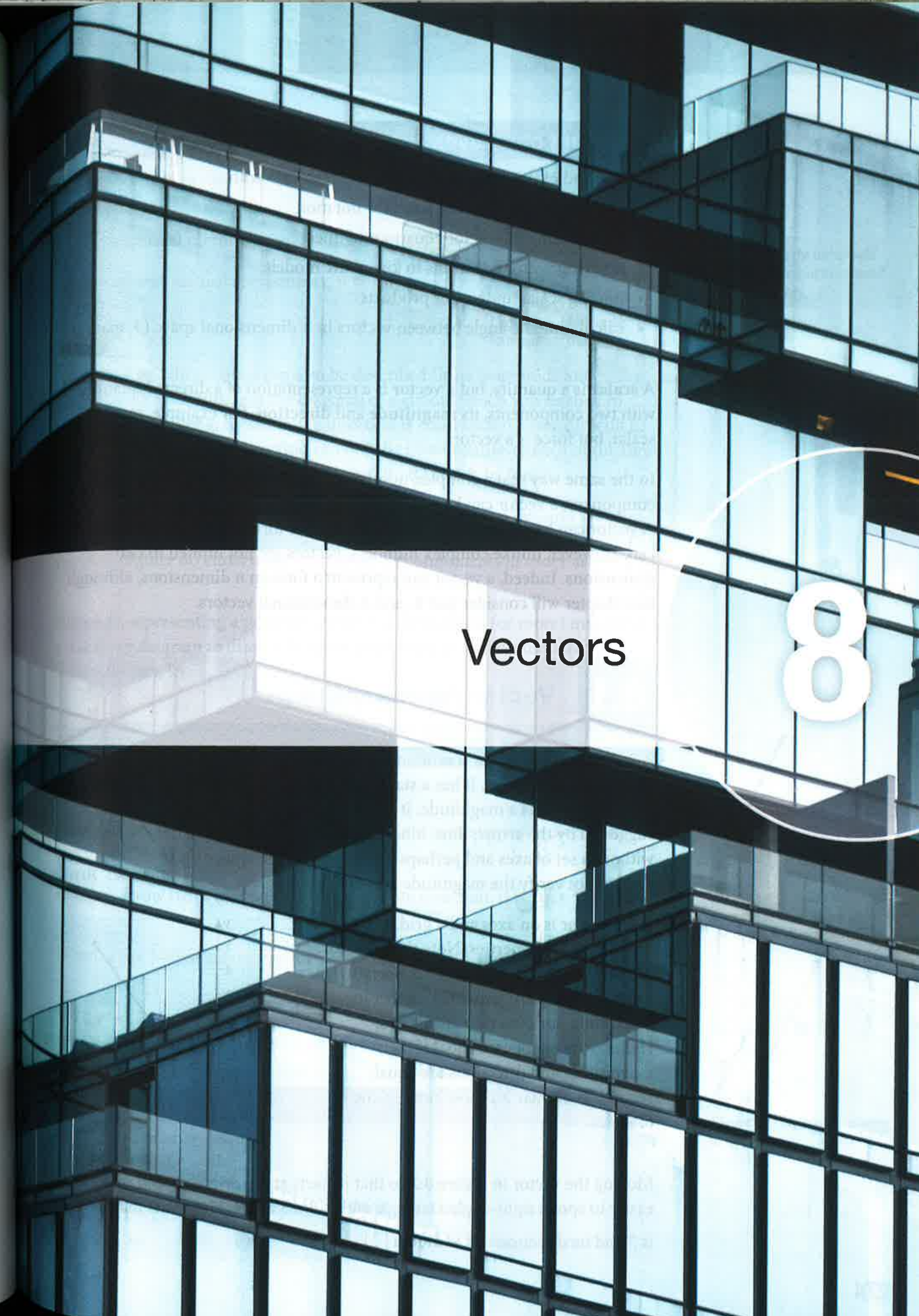
To determine the mobility across these levels in a generation, about two thousand men were asked, 'At which level are you, and at which level was your father when you were fourteen years old?' Here is a summary:

	Father's occupation			
	U	M	L	
Son's occupation	0.60	0.26	0.14	U
	0.29	0.37	0.34	M
	0.16	0.27	0.57	L

For example, a child of a middle-class worker has a 0.26 chance of moving into an upper class job.

With initial distribution of respondents' fathers given below, find the distributions for the next five generations.

Upper: 0.12, middle: 0.32, lower: 0.56



8 Vectors