18. A study in 2015 divided occupations in the United Kingdom into upper level, U, (executives and professionals), middle level, M, (supervisors and skilled manual workers), and lower level, L, (unskilled).

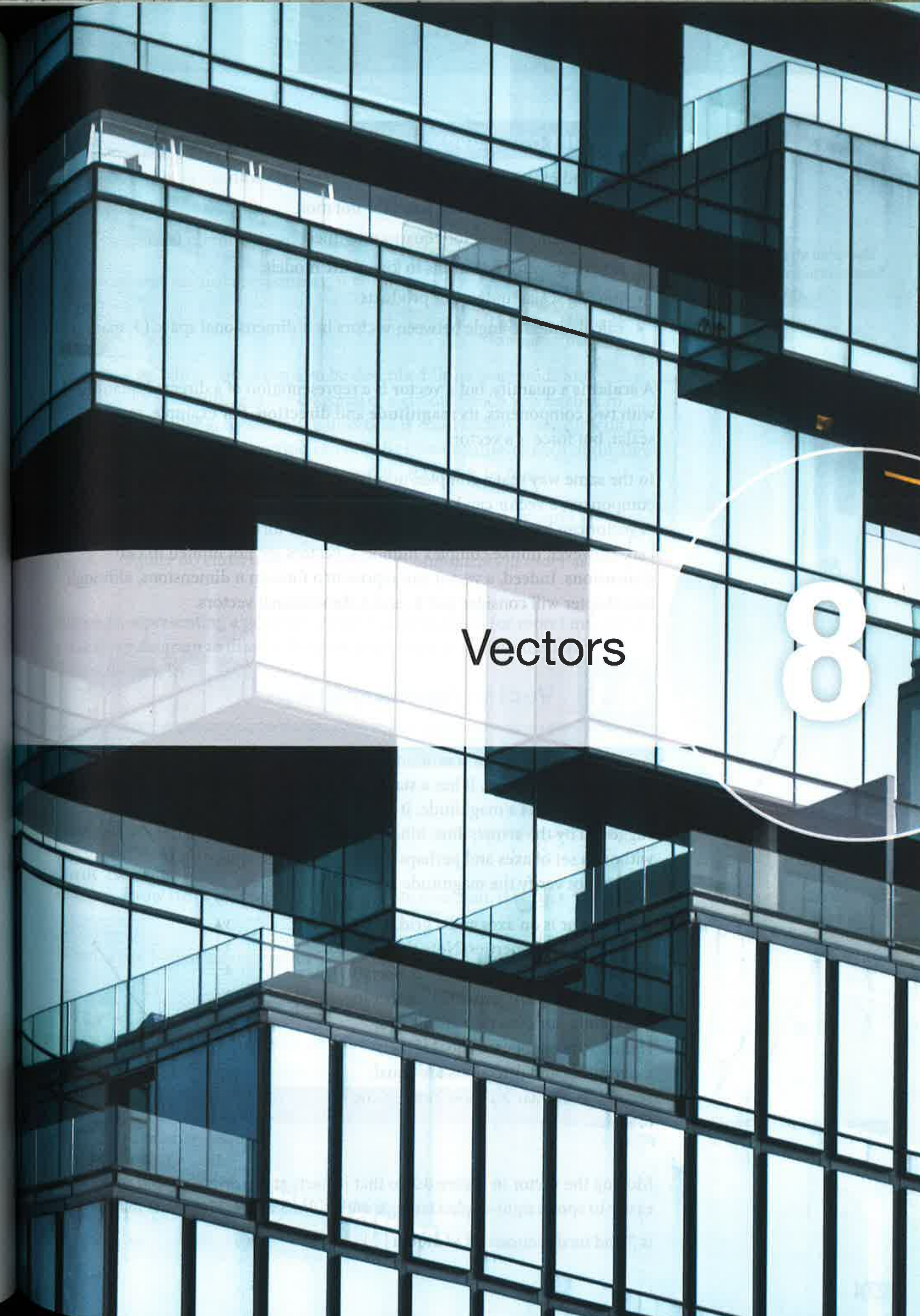
To determine the mobility across these levels in a generation, about two thousand men were asked, 'At which level are you, and at which level was your father when you were fourteen years old?' Here is a summary:

	Father's occupation			
	U	**M**	**L**	
Son's occupation	0.60	0.26	0.14	**U**
	0.29	0.37	0.34	**M**
	0.16	0.27	0.57	**L**

For example, a child of a middle-class worker has a 0.26 chance of moving into an upper class job.

With initial distribution of respondents' fathers given below, find the distributions for the next five generations.

Upper: 0.12, middle: 0.32, lower: 0.56



Learning objectives

By the end of this chapter, you should be familiar with…

- describing vectors using standard notation
- determining the vector equations of lines
- applying vector equations to kinematic models
- finding scalar and vector products
- calculating the angle between vectors in 3-dimensional space (3-space).

A **scalar** is a quantity, but a **vector** is a representation of a directed quantity with two components, its **magnitude** and **direction**. For example, mass is a scalar, but force is a vector.

In the same way that a complex number can have a real and an imaginary component, a vector can have a horizontal and a vertical component. A vector can be also expressed in trigonometric form, as a complex number can. However, unlike complex numbers, vectors are not limited to two dimensions. Indeed, a vector can represent a force in n dimensions, although this chapter will consider just 2- and 3-dimensional vectors.

8.1 Vector representation

A 2-dimensional vector $\boldsymbol{u}$ is often represented by an arrow in the Cartesian plane. It has a starting point and an ending point; thus, it has a magnitude. It has some specific heading suggested by the arrow; thus, it has a direction. However, without a set of axes and perhaps a grid as a background, you cannot verify the magnitude and the direction.

Figure 8.1 2D vector

If the vector is on axes and a grid, you can determine its properties. Note that a vector need not be drawn starting at the origin; it can lie anywhere, provided neither its magnitude nor direction are altered. Hence, two vectors are equal if their magnitudes and directions are equal, regardless of their location in the plane or space.

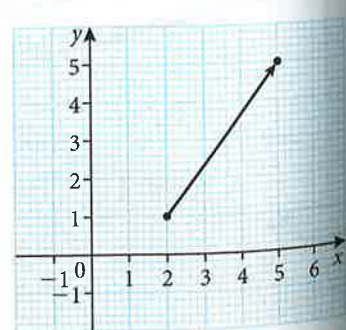


Figure 8.2 Vector on a grid

Moving the vector in Figure 8.2 so that it starts at the origin should make it easier to spot a right-angled triangle with sides 3 and 4. Hence, its magnitude is 5 and its direction is $\theta = \arctan\left(\frac{4}{3}\right)$

This should look familiar: complex numbers are described in trigonometric form, $r(\cos\theta + \mathrm{i}\sin\theta)$ and abbreviated as $r \operatorname{cis} \theta$, with a magnitude and direction.

A 2-dimensional vector can be described by its horizontal and vertical components, $\boldsymbol{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

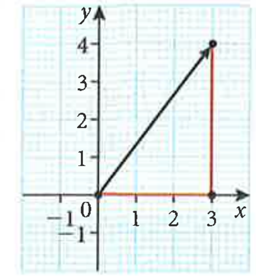


Figure 8.3 Moving the vector so it starts at the origin

Note the use of ***bold italic*** font to indicate vectors.

A vector $\boldsymbol{v}$ in 3 dimensions can also be described by its magnitude and direction. The magnitude is the diagonal of a cuboid which you learned to calculate in chapter 4; however, the direction is considerably more difficult to specify, since its correct direction can only be given relative to each of the three axes, and requires that all three directions, θ_x, θ_y, and θ_z be found.

Should $\boldsymbol{v}$ be described by its measurements relative to the x-, y-, and z-axes, it would require no effort other than to list its components in order as $\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Although representing a point or vector in 4 dimensions (or more) may present a daunting diagram to draw, adding another value to a column matrix is decidedly simpler.

Vector properties

Any vector $\boldsymbol{w}$ can be represented by an arrow.

A second arrow with the same magnitude points in the opposite direction, and is $-\boldsymbol{w}$.

Another arrow twice as long points in the same direction: it is $2\boldsymbol{w}$.

When $\boldsymbol{w}$ is a 3-dimensional vector defined as $\boldsymbol{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $-\boldsymbol{w} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$ and $2\boldsymbol{w} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

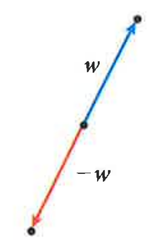


Figure 8.4 $-\boldsymbol{w}$ has the same magnitude as $\boldsymbol{w}$ but points in the opposite direction

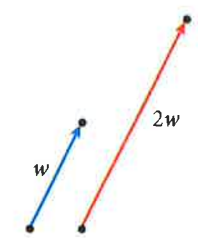


Figure 8.5 $2\boldsymbol{w}$ is twice as long as $\boldsymbol{w}$

Example 8.1

Let vector $\boldsymbol{u} = \begin{pmatrix} -2 \\ 14 \end{pmatrix}$

Determine the components of vector $\boldsymbol{v}$ when:

(a) $\boldsymbol{v} = 3\boldsymbol{u}$ (b) $\boldsymbol{v} = -2\boldsymbol{u}$ (c) $\boldsymbol{u} = 2\boldsymbol{v}$

Solution

(a) $v = 3\begin{pmatrix}-2\\14\end{pmatrix} = \begin{pmatrix}3\times-2\\3\times 14\end{pmatrix} = \begin{pmatrix}-6\\42\end{pmatrix}$ (b) $\begin{pmatrix}4\\-28\end{pmatrix}$ (c) $\begin{pmatrix}-1\\7\end{pmatrix}$

Vector addition and scalar multiplication

Just as complex numbers are added by summing the real and imaginary components separately, the measurements of vectors in the x-, y-, and z-directions are summed separately. Vectors with the same number of dimensions can be added and subtracted easily using their column matrix notation; vectors with different numbers of dimensions cannot. When vectors and their scalar multiples are combined, the result is called a **linear combination**.

Example 8.2

Let vector $u = \begin{pmatrix}-3\\5\\2\end{pmatrix}$ and $v = \begin{pmatrix}1\\0\\-2\end{pmatrix}$.

Determine the vector components of w, when:

(a) $w = u + v$ (b) $w = u - 2v$ (c) $w = 2u + 3v$

Solution

The linear combinations of u and v produce the following vectors.

(a) $w = \begin{pmatrix}-3\\5\\2\end{pmatrix} + \begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}-3+1\\5+0\\2-2\end{pmatrix} = \begin{pmatrix}-2\\5\\0\end{pmatrix}$ (b) $\begin{pmatrix}-5\\5\\6\end{pmatrix}$ (c) $\begin{pmatrix}-3\\10\\-2\end{pmatrix}$

Unit vectors

Unit vectors have a magnitude of 1. Assume that a vector v has components $\begin{pmatrix}a\\b\end{pmatrix}$.

Its magnitude, designated by the notation $|v|$, is equal to 1 only if $\sqrt{a^2 + b^2} = 1$

By dividing each component by $\sqrt{a^2 + b^2}$, its magnitude becomes equal to 1.

Example 8.3

Determine the components of vector u, the unit vector of $v = \begin{pmatrix}-6\\8\end{pmatrix}$, and find the components of the vector w in the same direction as v and with magnitude 7.

Solution

Vector $u = \frac{v}{|v|} = \frac{1}{\sqrt{6^2 + 8^2}}\begin{pmatrix}-6\\8\end{pmatrix} = \begin{pmatrix}-0.6\\0.8\end{pmatrix}$

Vector $w = 7u \Rightarrow w = 7\begin{pmatrix}-0.6\\0.8\end{pmatrix} = \begin{pmatrix}-4.2\\5.6\end{pmatrix}$

Unit vectors in three dimensions are found in the same manner.

Example 8.4

Given vector $v = \begin{pmatrix}2\\1\\-2\end{pmatrix}$, find the components of a vector w in the same direction as v with a magnitude of 9.

Solution

Vector $w = 9u \Rightarrow w = 9\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{pmatrix} = \begin{pmatrix}6\\3\\-6\end{pmatrix}$

In 2-dimensional space, the unit vectors along the x- and y-axes are called $i = \begin{pmatrix}1\\0\end{pmatrix}$ and $j = \begin{pmatrix}0\\1\end{pmatrix}$ respectively. Similarly, in 3-dimensional space, the unit vectors along the x-, y-, and z-axes are called $i = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $j = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $k = \begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively. These three vectors are also known as the **base vectors** in 3-dimensional space.

Vector operations illustrated

Consider the geometric view of vectors.

If $u = \begin{pmatrix}a\\b\end{pmatrix}$ and $v = \begin{pmatrix}c\\d\end{pmatrix}$, then $u + v = \begin{pmatrix}a+c\\b+d\end{pmatrix}$

Since $v + u = \begin{pmatrix}a+c\\b+d\end{pmatrix}$ as well, $u + v = v + u$

With two vectors starting at the same point, their sum is shown by the black vector (Figure 8.6) extending to the diagonally opposite corner of a parallelogram.

Online

Test your understanding of adding and subtracting vectors geometrically.

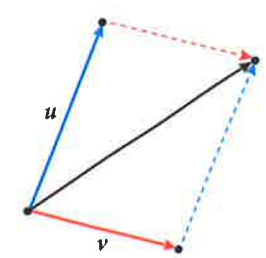


Figure 8.6 The sum is shown by the black vector

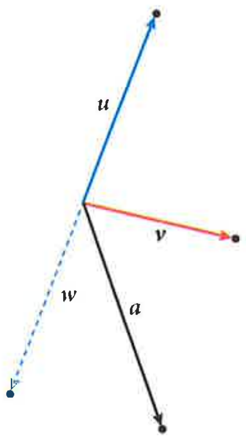


Figure 8.7 $w = -u$

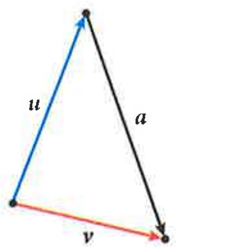


Figure 8.8 How can you tell that $a = v - u$?

Now, let vector $w = -u$ shown in Figure 8.7 as a dashed blue line in the opposite direction to u.

Therefore, the black vector, labelled $a = v + w = w + v$

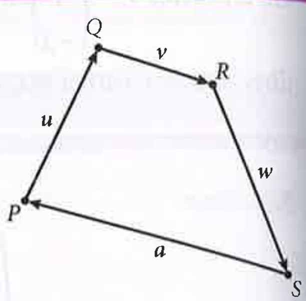
Since $w = -v$, $a = v - u = -u + v$

When vector a is placed in the original diagram with vectors u and v (Figure 8.8), it should become evident that it forms the other diagonal of the parallelogram. How can you tell from the diagram that $a = v - u$ and not $u - v$?

The tip of a is at the tip of v while its bottom is at the tip of u. Or to move from the bottom of a to its tip, move backwards through u, then forwards through v.

Example 8.5

Consider vectors u, v, w, and a drawn in an anti-clockwise sequence from point P.



(a) Describe each vector sum as a single vector.

(i) $u + v$

(ii) $u + v + w$

(b) How does the sum $u + v + w$ compare to vector a?

(c) What is the sum of all four vectors?

Solution

(a) (i) $u + v$ is a vector that starts at P and ends at R. The vector is $\overrightarrow{PR}$.

(ii) $u + v + w$ is a vector from P to S, and can simply be written as $\overrightarrow{PS}$.

(b) $\overrightarrow{PS}$ is the opposite vector to a.

(c) The sum of all four vectors is the zero vector, $\mathbf{0}$.

Exercise 8.1

1. $u = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$. Determine the components of vector v when:

(a) $v = 2u$ (b) $v = -3u$ (c) $u = \frac{v}{3}$

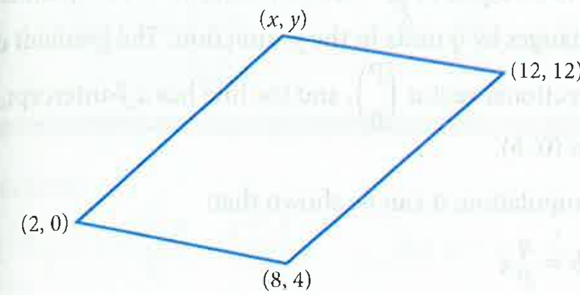
2. $u = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Determine the vector components of w when:

(a) $w = u - v$ (b) $w = u + 2v$ (c) $w = 3u - 4v$

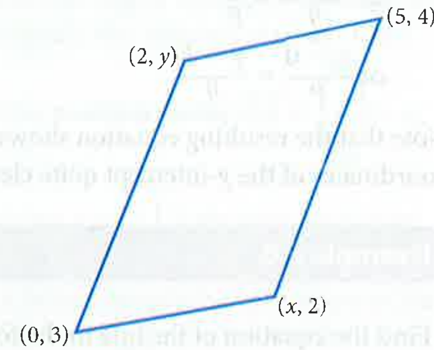
3. Find the unit vector of each given vector.

(a) $u = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ (b) $u = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ (c) $u = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

4. Find (x, y) so that the diagram is a parallelogram.



5. Find the values of x and y in the parallelogram.



6. Find the value of scalars r and s such that $\begin{pmatrix} 8 \\ 46 \end{pmatrix} = r\begin{pmatrix} 1 \\ 9 \end{pmatrix} + s\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

7. Write $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

8. Write $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

9. Write $\begin{pmatrix} -11 \\ 0 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

10. Find vectors, each with a magnitude of 10 units, parallel to these vectors.

(a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (b) $\begin{pmatrix} -5 \\ -12 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

8.2 Vector and parametric equations of lines

Lines require just two distinct points to be defined in both 2 and 3 dimensions. In 2 dimensions, a familiar description of a line may be in the form $y = mx + b$. Consider the gradient, m, to be equal to $\frac{q}{p}$, where for every non-zero p units in the x-direction, the line changes by q units in the y-direction. The gradient can then be described by a directional vector $\begin{pmatrix} p \\ q \end{pmatrix}$, and the line has a y-intercept, so it passes through the point $(0, b)$.

With simple algebraic manipulation, it can be shown that:

$$y = \frac{q}{p}x + b \Rightarrow y - b = \frac{q}{p}x$$

$$\Rightarrow \frac{y-b}{q} = \frac{x}{p}$$

$$\text{or } \frac{x-0}{p} = \frac{y-b}{q}$$

Note that the resulting equation shows the vector components and the coordinates of the y-intercept quite clearly.

Example 8.6

Find the equation of the line in the form $\frac{x-h}{p} = \frac{y-k}{q}$ that passes through the given point A with gradient m.

(a) $A(5, -2)$, $m = \frac{2}{3}$ (b) $A(1, 2)$, $m = -\frac{1}{12}$

Solution

(a) In this question, $h = 5$, $k = -2$, $p = 3$, and $q = 2$ hence, $\frac{x-5}{3} = \frac{y+2}{2}$

(b) $\frac{x-1}{12} = \frac{y-2}{-1}$

Now, consider a line in 3 dimensions that contains a point $A(x_0, y_0, z_0)$ with a gradient described by the vector $\begin{pmatrix} p \\ q \\ s \end{pmatrix}$

The addition of another dimension merely adds another displacement and another denominator $\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{s}$

This is commonly known as the Cartesian form of a line.

Example 8.7

Find the equation of the line in 3-space that passes through the given point A with the gradient described by the direction vector $\boldsymbol{u}$ with components $\begin{pmatrix} p \\ q \\ s \end{pmatrix}$

(a) $A(1, 2, 3)$, $\boldsymbol{u} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ (b) $A(6, -2, 4)$, $\boldsymbol{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Solution

(a) $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}$ (b) $x - 6 = \frac{y+2}{-2} = \frac{z-4}{3}$

Consider the point $A(x_0, y_0, z_0)$ in relation to the origin, $O(0, 0, 0)$. The segment $[OA]$ has a length, but no direction. Now, should this segment start at O and end at A, then it would be a vector, called the **position vector** $\overrightarrow{OA} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \boldsymbol{r}_0$

Now, with the point A and the direction vector $\boldsymbol{v} = \begin{pmatrix} p \\ q \\ s \end{pmatrix}$, both given as vectors, the equation of the line containing point A is then simply the vector sum of the position vector $\overrightarrow{OA} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ and the direction vector $\begin{pmatrix} p \\ q \\ s \end{pmatrix}$, the scalar multiple of which, $k\begin{pmatrix} p \\ q \\ s \end{pmatrix}$, would ensure it is a line, not a segment. The vector equation of a line is given in the key fact box on the right.

You may find it easier to work with the **parametric form** of this equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + k\begin{pmatrix} p \\ q \\ s \end{pmatrix} \text{ implies that } x = \begin{cases} x = x_0 + kp \\ y = y_0 + kq \\ z = z_0 + ks \end{cases} \text{ by matrix addition.}$$

Simplifying the equation will give us

$$x - x_0 = kp \Rightarrow \frac{x - x_0}{p} = k, \text{ and similarly, } \frac{y-y_0}{q} = k, \text{ as well as } \frac{z-z_0}{s} = k$$

Thus, you can see that the parametric form will lead to the Cartesian form of the equation of the line

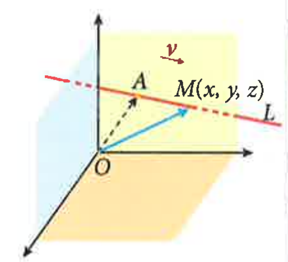
$$k = \frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{s}$$

In terms of notation, vector $\overrightarrow{OA}$ is also noted simply as $\boldsymbol{a}$, vector $\overrightarrow{OB}$ as $\boldsymbol{b}$, and so on.

The **vector equation** of the line $\boldsymbol{r} = \boldsymbol{r}_0 + k\boldsymbol{v}$

$$= \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + k\begin{pmatrix} p \\ q \\ s \end{pmatrix}$$

$\boldsymbol{r}$ is the position vector of any point on the line, while $\boldsymbol{r}_0$ is the position vector of a fixed point (A in this case) on the line and $\boldsymbol{v}$ is the vector parallel to the given line.



Example 8.8

Find the equation of the line in vector form which passes through the given point A with direction vector $\boldsymbol{u}$.

(a) $A(5, 2, -3)$, $\boldsymbol{u} = \begin{pmatrix} 4 \\ 7 \\ -6 \end{pmatrix}$ (b) $A(1, 0, -4)$, $\boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}$

Solution

(a) $\boldsymbol{r} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} + k\begin{pmatrix} 4 \\ 7 \\ -6 \end{pmatrix}$ (b) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + k\begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}$

Exercise 8.2

1. Find the equation of the line in the form $\frac{x-h}{p} = \frac{y-k}{q}$ that passes through the given point P with gradient m.

   (a) $A(1, -3)$, $m = \frac{3}{4}$ (b) $A(6, -1)$, $m = \frac{1}{4}$

   (c) $A(-9, 5)$, $m = -\frac{3}{2}$ (d) $A(7, 1)$, $m = -\frac{2}{5}$

2. Find the equation of the line in 3-space that passes through the given point A with the gradient described by the direction vector $\boldsymbol{u}$.

   (a) $A(6, 7, 0)$, $\boldsymbol{u} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ (b) $A(-3, -2, 9)$, $\boldsymbol{u} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$

3. Find the equation of the line in vector form which passes through the given point A with direction vector $\boldsymbol{u}$.

   (a) $A(1, 0, 2)$, $\boldsymbol{u} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ (b) $A(-2, 3, 0)$, $\boldsymbol{u} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

4. Name the coordinates of the point on the given line whose x-value is 6.

   (a) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + k\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (b) $\boldsymbol{r} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} + k\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

   (c) $\boldsymbol{r} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} + k\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ (d) $\boldsymbol{r} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + k\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$

5. Find the vector equation of each line given its Cartesian form.

   (a) $\frac{x-3}{2} = \frac{y+2}{-2} = \frac{z-1}{4}$ (b) $\frac{x}{3} = \frac{y-3}{-2} = z + 1$

   (c) $\frac{x+3}{\frac{1}{2}} = \frac{y}{5} = \frac{z+2}{-2}$ (d) $\frac{x-a}{m} = \frac{y-b}{n} = \frac{z-c}{p}$

6. Find the vector equation of the line passing through $P(3, -1, -5)$, parallel to each line.

   (a) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -6 \\ 0 \end{pmatrix} + k\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$ (b) $\boldsymbol{r} = \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} + k\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$

   (c) $\frac{x+6}{3} = \frac{y-8}{-1} = \frac{z+1}{5}$ (d) $\frac{x+9}{7} = y + 2 = \frac{z+11}{-2}$

7. A triangle ABC has coordinates $A(2, 4, 6)$, $B(4, -1, 7)$, and $C(-2, -3, 1)$. Determine the vector components of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ where $\boldsymbol{a}$ is $\overrightarrow{OA}$, $\boldsymbol{b}$ is $\overrightarrow{OB}$ and $\boldsymbol{c}$ is $\overrightarrow{OC}$.

8. Find the vector equation of the line passing through each pair of points.

   (a) $A(-3, 7)$ and $B(-1, 11)$ (b) $C(2, -5)$ and $D(-2, 1)$

   (c) $E(8, -2, 1)$ and $F(-3, 10, 7)$ (d) $G(0, -6, -3)$ and $H(7, -1, 0)$

   (e) $J(3, -4, 5)$ and $K(5, -2, 5)$ (f) $L(-7, -4, 2)$ and $M(2, -4, 12)$

9. Express each of the lines in question **8** in Cartesian form.

10. Find the point of intersection of the vector equations of the two lines:

$$\boldsymbol{r}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + k\begin{pmatrix} 3 \\ -2 \end{pmatrix} \text{ and } \boldsymbol{r}_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

11. Show that lines $\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-3}{1}$ and $\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-4}{2}$ intersect and find the coordinates of P, the point of intersection.

8.3 Kinematics

Objects in motion can be analysed with vectors, provided that the motion is constant. When rates of change are involved, the analyses require the use of differential calculus which will be presented in Chapter 13.

Constant motion directly towards or away from an observer is the simplest.

Example 8.9

A drone is hovering directly above a fixed point P. It then flies at a constant altitude in a direction described by the vector $\boldsymbol{v} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, moving 6 m to the east for every 8 m to the north, every second. Determine the distance to the drone from point P after:

(a) 1 second (b) 3.2 seconds

(c) 15 seconds (d) t seconds

Solution

The distance flown every second by the drone is the magnitude of the vector $|\boldsymbol{v}| = \sqrt{6^2 + 8^2} = 10\,\text{m}$. The diagram represents a plan view of the motion of the drone, rendered in 2D-space.

(a) $|\boldsymbol{v}| = 10\,\text{m}$ (b) $3.2|\boldsymbol{v}| = 32\,\text{m}$

(c) $15|\boldsymbol{v}| = 150\,\text{m}$ (d) $t|\boldsymbol{v}| = 10t\,\text{m}$

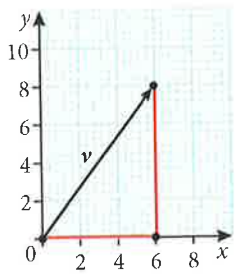


Figure 8.9 Diagram for solution to Example 8.9

Now, consider a situation when the drone does not fly directly towards or away from a fixed point.

Example 8.10

A drone starts from a position 10 m to the west of a fixed point O and flies along the vector $\boldsymbol{v} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

Find the distance of the drone from O after:

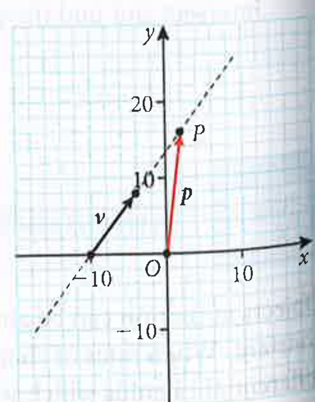
(a) 1 second (b) 2 seconds.

Solution

(a) Consider the point O to be at the origin (0, 0). The drone starts at $S(-10, 0)$. After t seconds, with the vector $\boldsymbol{v}$ moving the drone by $\begin{pmatrix} 6t \\ 8t \end{pmatrix}$, it will be at the point $P(-10 + 6t, 0 + 8t)$.

The position vector is $\overrightarrow{OP} = \begin{pmatrix} -10 + 6t \\ 8t \end{pmatrix}$

Its magnitude is the distance from the origin, and is $|\overrightarrow{OP}| = \sqrt{(-10 + 6t)^2 + (8t)^2}$



At $t = 1$

$$|\overrightarrow{OP}| = \sqrt{(-10 + 6)^2 + (8)^2} = \sqrt{80} = 4\sqrt{5}\,\text{m}$$

(b) At $t = 2$

$$|\overrightarrow{OP}| = \sqrt{(-10 + 12)^2 + (16)^2} = \sqrt{260} = 2\sqrt{65}\,\text{m}$$

Minimum distance between a point and an object in motion

The distance to the drone is not constant, as shown in Figure 8.10. Note that the expression under the square root sign is a quadratic expression with a positive leading coefficient; hence, an expression with a minimum.

Expand the brackets in the expression:

$$|\overrightarrow{OP}| = \sqrt{(-10 + 6t)^2 + (8t)^2} = \sqrt{100 - 120t + 36t^2 + 64t^2}$$

$$= \sqrt{100 - 120t + 100t^2}$$

Now, let $y = 100 - 120t + 100t^2$. As every quadratic function of the form $y = ax^2 + bx + c$ has a vertex at $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ and with $a > 0$, y has a minimum at $c - \frac{b^2}{4a} = 64$

So the minimum distance of the point P from O is $|\overrightarrow{OP}| = \sqrt{y} = \sqrt{64} = 8\,\text{m}$

Minimum distance between two objects in motion

Both the drone and the point of observation are moving along vectors.

While the drone flies along the path $\boldsymbol{d} = \begin{pmatrix} -10 + 6t \\ 8t \end{pmatrix}$, the observer travels from O(0, 0) along a road defined by the vector $\boldsymbol{u} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

The observer's path is defined as $\boldsymbol{u} = \begin{pmatrix} 4t \\ 10t \end{pmatrix}$

Disregarding the drone's altitude by using a 2-dimensional map, the distance to the drone can be expressed using vectors.

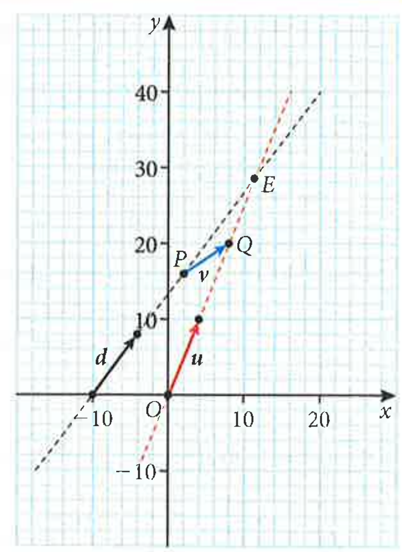


Figure 8.10 Vector from the drone to the observer

Note that although the paths cross at the point labelled E in the diagram, the observer and the drone reach it at different times.

The vector $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 4t \\ 10t \end{pmatrix} - \begin{pmatrix} -10 + 6t \\ 8t \end{pmatrix} = \begin{pmatrix} -2t + 10 \\ 2t \end{pmatrix}$ and the magnitude, similar to the earlier question, is $|\overrightarrow{PQ}| = \sqrt{(-2t + 10)^2 + (2t)^2}$

$= \sqrt{4t^2 - 40t + 100 + 4t^2}$

Let $y = 8t^2 - 40t + 100 \Rightarrow c - \frac{b^2}{4a} = 100 - \frac{(-40)^2}{4 \cdot 8} = 50$ m, and the minimum distance between the two moving points is $|\overrightarrow{PQ}| = \sqrt{50} = 5\sqrt{2}$m

Minimum distance between two moving objects in 3 dimensions

Finding the distance between objects in 3 dimensions requires the use of the same process as noted above. Vectors allow the flexibility to change the number of dimensions using the same process.

Example 8.11

A drone starts on the ground at the point $A(0, 0, 0)$ and flies at a constant velocity along the vector $\boldsymbol{u} = \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$ each second, while another drone, already hovering at the point $B(24, 12, 18)$ starts to descend at the same time in the direction $\boldsymbol{v} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$. Find the minimum distance between the two drones.

Solution

Since the first drone travels along the directional vector $\boldsymbol{u} = \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$ starting from its initial position $A(0, 0, 0)$, after t seconds, its position can be indicated by the vector

$$\boldsymbol{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ 8 \\ 2 \end{pmatrix}$$

Similarly, the second drone, starting from $B(24, 12, 18)$ with a directional $\boldsymbol{v} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$ will have a position that is indicated by $\boldsymbol{b} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}$

As in the example in 2 dimensions, $\boldsymbol{a} = \begin{pmatrix} 6t \\ 8t \\ 2t \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 24 - 2t \\ 12 + 4t \\ 18 - 4t \end{pmatrix}$

$$\Rightarrow \overrightarrow{AB} = \begin{pmatrix} 24 - 8t \\ 12 - 4t \\ 18 - 6t \end{pmatrix} \Rightarrow |\overrightarrow{AB}| = \sqrt{(24 - 8t)^2 + (12 - 4t)^2 + (18 - 6t)^2}$$

$$= \sqrt{116(9 - 6t + t^2)}$$

As the expression under the square root sign is a quadratic with a leading positive coefficient, it will have a minimum distance of

$c - \frac{b^2}{4a} = 9 - \frac{6^2}{4 \cdot 1} = 0$ when $t = -\frac{b}{2a} = \frac{6}{2} = 3$

The two drones will in fact crash at $t = 3$ seconds as the distance between them will be 0.

Did you notice that at $t = 4.5$ seconds, the second drone would have hit the ground?

Exercise 8.3

1. A robot vacuum cleaner, set in motion on an empty gymnasium floor, moves in the direction $\boldsymbol{u} = \begin{pmatrix} 1.2 \\ 2 \end{pmatrix}$, representing a path 1.2 m to the right for every 2 m forwards, every minute. Determine the distance travelled after:
   (a) 1 minute **(b)** 2 minutes **(c)** 10 minutes **(d)** t minutes

2. The vacuum cleaner in question **1** maintains movement along its set direction. An observer is 12 m to the right of the vacuum cleaner's starting position. Find the distance of the vacuum cleaner from the observer after:
   (a) 1 minute **(b)** 2 minutes.

3. Another robot vacuum cleaner starts at the same time as the first one in question **1**, but exactly 10 m to the right of the observer and in the direction of $\boldsymbol{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$
   Determine the closest distance between the two devices.

4. Consider a drone moving in the direction $\boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, given in m s^{-1}, when observed at the point $P(24, 16, 48)$.
   (a) How far is it from $O(0, 0, 0)$ when observed?
   (b) How far will it move in the next 10 seconds?
   (c) How far will it be from $O(0, 0, 0)$ after 10 seconds?

5. The drone in question **4** is being tracked by a sensor located at $Q(10, 0, 2)$.
   (a) Write an expression for the displacement from the sensor to the drone after t seconds.
   (b) Find the minimum distance between the drone and sensor to 3 significant figures.

6. Two projectiles travelling through space have directional vectors, given in m s^{-1}, of $\boldsymbol{u} = \begin{pmatrix} 20 \\ 20 \\ 20 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 30 \\ -20 \\ -10 \end{pmatrix}$

The first projectile, with directional vector $\boldsymbol{u}$, is at $P(1300, 2800, 1000)$ when the second, with directional vector $\boldsymbol{v}$, is at $Q(1000, 4000, 2000)$.

(a) Assume you are looking down on the projectiles from above. At what x- and y- coordinates do the paths of the projectiles intersect?

(b) Determine the time it would take each projectile to reach that (x, y) location.

(c) Hence, determine whether the two projectiles appear to collide when viewed from above.

(d) What is the closest that they will come to each other? Give an answer to 3 significant figures.

(e) At what time does this occur?

8.4 Scalar and vector products

The scalar product

The **scalar product** of two 2-dimensional vectors $\boldsymbol{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} d \\ e \end{pmatrix}$ is given by:

$$\boldsymbol{u} \cdot \boldsymbol{v} = ad + be$$

Similarly, the scalar product of two 3-dimensional vectors $\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ is given by:

$$\boldsymbol{u} \cdot \boldsymbol{v} = ad + be + cf$$

Note that due to the use of the dot symbol for this operation, the scalar product is also known as the **dot product**.

Example 8.12

Find the scalar product of each pair of vectors.

(a) $\boldsymbol{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 2 \\ 10 \\ -3 \end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Solution

(a) $\boldsymbol{u} \cdot \boldsymbol{v} = 2 \cdot (-1) + 3 \cdot 2 = 4$

(b) $\boldsymbol{u} \cdot \boldsymbol{v} = 4 \cdot 5 + 5 \cdot (-4) = 0$

(c) $\boldsymbol{u} \cdot \boldsymbol{v} = (-2) \cdot 2 + 1 \cdot 10 + 2 \cdot (-3) = 0$

(d) $\boldsymbol{u} \cdot \boldsymbol{v} = 7 \cdot 2 + 0 \cdot 1 + (-1) \cdot (-2) = 16$

Describing vectors by their directional components, for example, $\boldsymbol{v} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, facilitates the calculation of the scalar product. However, vectors can also be described by their magnitudes and directions. As the directions of vectors in 3-space require three separate measurements, θ_x, θ_y, and θ_z, calculating the angle between vectors by starting with their polar forms is not practical.

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be drawn from the same point as shown.

Then $|\boldsymbol{u} - \boldsymbol{v}|^2 = (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{u}^2 + \boldsymbol{v}^2 - 2\boldsymbol{u} \cdot \boldsymbol{v}$

$= |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2\boldsymbol{u} \cdot \boldsymbol{v}$

Also, using the law of cosines,

$$|\boldsymbol{u} - \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}| \cdot |\boldsymbol{v}| \cdot \cos\theta$$

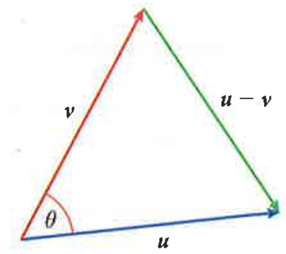


Figure 8.11 The difference of two vectors

By comparing the two results above:

$$|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2|\boldsymbol{u}| \cdot |\boldsymbol{v}| \cdot \cos\theta$$

$$-2\boldsymbol{u} \cdot \boldsymbol{v} = -2|\boldsymbol{u}| \cdot |\boldsymbol{v}| \cdot \cos\theta$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}| \cdot |\boldsymbol{v}| \cos\theta$$

When the magnitudes and the angle between vectors are known, you can calculate their scalar product as $\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}| \cdot |\boldsymbol{v}| \cos\theta$

Example 8.13

Find the scalar product for each pair of vectors, where θ is the angle between them.

(a) $|\boldsymbol{u}| = 4, |\boldsymbol{v}| = 8, \theta = 60°$ (b) $|\boldsymbol{u}| = 8\sqrt{3}, |\boldsymbol{v}| = 6, \theta = 30°$

(c) $|\boldsymbol{u}| = 10, |\boldsymbol{v}| = 4\sqrt{2}, \theta = 45°$ (d) $|\boldsymbol{u}| = 12, |\boldsymbol{v}| = 6, \theta = 120°$

Solution

(a) $\boldsymbol{u} \cdot \boldsymbol{v} = 4 \cdot 8 \cdot \cos 60° = 16$ (b) $\boldsymbol{u} \cdot \boldsymbol{v} = 8\sqrt{3} \cdot 6 \cdot \cos 30° = 72$

(c) $\boldsymbol{u} \cdot \boldsymbol{v} = 10 \cdot 4\sqrt{2} \cdot \cos 45° = 40$ (d) $\boldsymbol{u} \cdot \boldsymbol{v} = 12 \cdot 6 \cdot \cos 120° = -36$

The **work** done by any force is defined as the product of the force multiplied by the distance it moves a certain object in the direction of the force. In other words, it is the product of the force multiplied by the displacement of the object. As such, work is then the dot product between the vectors representing the force (F) and displacement (D), $W = F \cdot D$

The SI unit of work is the joule (J), defined as the work expended by a force of one newton through a displacement of one metre.

Example 8.14

Find the work done by the force $F = \begin{pmatrix} 400 \\ -50 \end{pmatrix}$ N in moving an object between points $M(2, 3)$ and $N(12, 43)$ given in metres. Your answer should be in joules (J).

Solution

$W = F \cdot D$ and $\overrightarrow{MN} = \begin{pmatrix}10\\40\end{pmatrix}$, so $W = \begin{pmatrix}400\\-50\end{pmatrix} \cdot \begin{pmatrix}10\\40\end{pmatrix} = 2000$ J

The vector product

The **vector product** of two vectors is another vector at right angles to the first two.

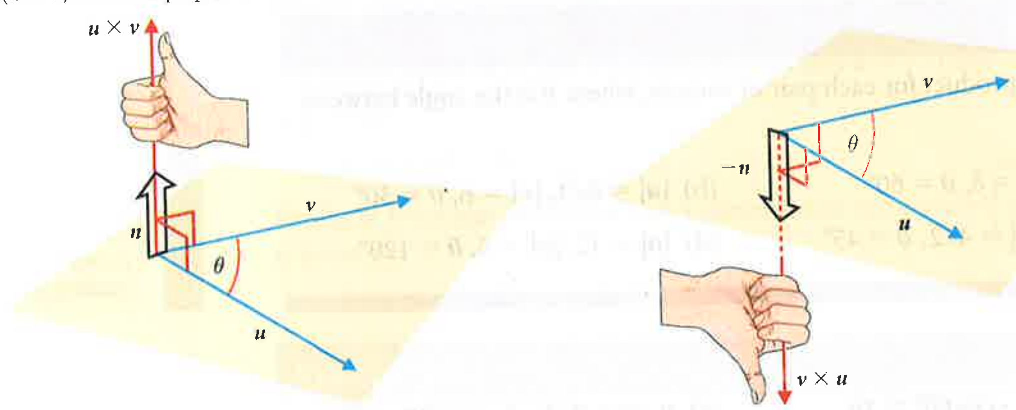
Using the properties of determinants, the vector (cross) product is equivalent to

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix}$$

Whereas **the scalar product** of two vectors yields a scalar quantity, the **vector product** of two vectors is **another** vector, at right angles to the two given vectors. The components of the vector product of two vectors $\boldsymbol{u} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix}d\\e\\f\end{pmatrix}$ are $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix}bf - ce\\cd - af\\ae - bd\end{pmatrix}$

Note that with the use of the symbol $\times$, the vector product is also known as the **cross product**. The resultant vector is orthogonal to the two original vectors and follows the **right hand rule**.

$(\boldsymbol{u} \times \boldsymbol{v})$ is a vector perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$ and obeying the right-hand rule, and has the magnitude: $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta$



Remember that $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$

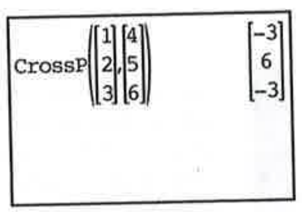


Figure 8.12 Using a GDC to find a vector product

The cross product $\boldsymbol{v} \times \boldsymbol{u}$ gives a vector in the opposite direction to $\boldsymbol{u} \times \boldsymbol{v}$

You can use a GDC to find the vector product. Look for a function with a reference to 'cross products'. The screenshot shows an example.

Example 8.15

Find the vector product $\boldsymbol{u} \times \boldsymbol{v}$ of each pair of vectors.

(a) $\boldsymbol{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix}-1\\2\\-2\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}3\\1\\2\end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$

Solution

(a) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix}0 \times 0 - 0 \times 1\\0 \times 0 - 1 \times 0\\1 \times 1 - 0 \times 0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ (b) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$

(c) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix}6\\-4\\-7\end{pmatrix}$ (d) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix}-2\\3\\2\end{pmatrix}$

Exercise 8.4

1. Find the scalar product of each pair of vectors.

(a) $\boldsymbol{u} = \begin{pmatrix}4\\-3\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}3\\3\end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix}6\\-2\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}-2\\-6\end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix}3\\-1\\4\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}-1\\5\\2\end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix}-7\\4\\3\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}-1\\-4\\3\end{pmatrix}$

2. Find the scalar product, accurate to 3 significant figures, of each pair of vectors where θ is the angle between them.

(a) $|\boldsymbol{u}| = 6$, $|\boldsymbol{v}| = 9$, $\theta = 30°$ (b) $|\boldsymbol{u}| = 12$, $|\boldsymbol{v}| = 8$, $\theta = 45°$

(c) $|\boldsymbol{u}| = 12$, $|\boldsymbol{v}| = 3$, $\theta = 23°$ (d) $|\boldsymbol{u}| = 10$, $|\boldsymbol{v}| = 13$, $\theta = 13°$

3. Find the work done by the force $F = \begin{pmatrix}30\\150\end{pmatrix}$ N in moving an object between points $A(0, 30)$ and $B(15, 70)$ given in metres. State your answer in joules.

4. Find the vector products $\boldsymbol{u} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{u}$ for each pair of vectors.

(a) $\boldsymbol{u} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix}3\\4\\-2\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}4\\-1\\4\end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix}-7\\2\\-3\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}-2\\2\\6\end{pmatrix}$

8.5 Angles between vectors

Using the scalar product, the angle between two vectors u and v can be given by $\theta = \arccos\left(\frac{u \cdot v}{|u||v|}\right)$

As there are two formulae for the scalar product, we can work out the angle between vectors, regardless of whether the vectors are in 2 or 3 dimensions. Since vectors are unchanged if their magnitudes and directions are unchanged, by considering any two vectors that share a common starting point, they can be drawn to be coplanar, and the angle between them identified.

Example 8.16

Find the angle, to the nearest degree, between each pair of vectors.

(a) $u = \begin{pmatrix}5\\12\end{pmatrix}$, $v = \begin{pmatrix}4\\-3\end{pmatrix}$ (b) $u = \begin{pmatrix}8\\6\end{pmatrix}$, $v = \begin{pmatrix}8\\15\end{pmatrix}$

(c) $u = \begin{pmatrix}2\\2\\-1\end{pmatrix}$, $v = \begin{pmatrix}1\\2\\-2\end{pmatrix}$ (d) $u = \begin{pmatrix}-2\\1\\2\end{pmatrix}$, $v = \begin{pmatrix}6\\-2\\-3\end{pmatrix}$

Solution

(a) $u \cdot v = ad + be$
$= 5 \times 4 + 12 \times (-3) = -16$
$|u| = 13, |v| = 5$
$u \cdot v = |u||v| \cos\theta$
$|u||v| \cos\theta = -16$
$\therefore \theta = \arccos\left(\frac{-16}{13 \cdot 5}\right) \approx 104°$

(b) $u \cdot v = 154, |u| = 10, |v| = 17$
$\therefore \theta = \arccos\left(\frac{154}{10 \cdot 17}\right) \approx 25°$

(c) $u \cdot v = 8, |u| = 3, |v| = 3$
$\therefore \theta = \arccos\left(\frac{8}{3 \cdot 3}\right) \approx 27°$

(d) $u \cdot v = -20, |u| = 3, |v| = 7$
$\therefore \theta = \arccos\left(\frac{-20}{3 \cdot 7}\right) \approx 162°$

Exercise 8.5

1. Find the angle between each pair of vectors, to the nearest degree.

(a) $u = \begin{pmatrix}-3\\1\end{pmatrix}$, $v = \begin{pmatrix}2\\6\end{pmatrix}$ (b) $u = \begin{pmatrix}-5\\4\end{pmatrix}$, $v = \begin{pmatrix}2\\3\end{pmatrix}$

(c) $u = \begin{pmatrix}-2\\1\\-2\end{pmatrix}$, $v = \begin{pmatrix}3\\-2\\-2\end{pmatrix}$ (d) $u = \begin{pmatrix}3\\-2\\2\end{pmatrix}$, $v = \begin{pmatrix}4\\-3\\9\end{pmatrix}$

2. Determine whether u is orthogonal, parallel to v, or neither.

(a) $u = \begin{pmatrix}-\frac{1}{2}\\2\end{pmatrix}$, $v = \begin{pmatrix}-2\\\frac{1}{2}\end{pmatrix}$ (b) $u = \begin{pmatrix}8\\4\end{pmatrix}$, $v = \begin{pmatrix}6\\-12\end{pmatrix}$

(c) $u = \begin{pmatrix}2\sqrt{3}\\2\end{pmatrix}$, $v = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$

3. Find the interior angles of the triangle ABC, the coordinates of whose vertices are given.

(a) $A(1, 2), B(3, 4), C(2, 5)$

(b) $A(3, 4), B(-1, -7), C(-8, -2)$

(c) $A(3, -5), B(1, -9), C(-7, -9)$

4. Find a vector perpendicular to $u = \begin{pmatrix}3\\5\end{pmatrix}$

In question 4, your answer will not be unique.

5. Determine if triangle ABC is right-angled by using the scalar product, where the coordinates of its vertices are $A(1, -3), B(2, 0)$, and $C(6, -2)$

6. For what non-zero value(s) of b are the vectors $\begin{pmatrix}-6\\b\end{pmatrix}$ and $\begin{pmatrix}b\\b^2\end{pmatrix}$ perpendicular?

7. Two vectors $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}x\\1\end{pmatrix}$ have an angle of 30° between them.

Find the possible values of x.

8. Use the scalar product to prove that the diagonals of a rhombus are perpendicular to each other.

9. Given the points A and B, use the scalar product to find the equation of a circle whose diameter is $[AB]$.

(a) $A(1, 2), B(3, 4)$ (b) $A(3, 4), B(-1, -7)$

Chapter 8 practice questions

1. Vector $u = \begin{pmatrix}8\\-2\end{pmatrix}$

Determine the components of vector v when:

(a) $v = -u$ (b) $v = \frac{u}{2}$ (c) $3u = 2v$

2. Vector $u = \begin{pmatrix}6\\-2\\7\end{pmatrix}$ and $v = \begin{pmatrix}-2\\8\\-3\end{pmatrix}$

Determine the vector components of w when:

(a) $w = -u + v$ (b) $w = u - v$ (c) $w = 5u - 2v$

3. Find the equation of each line in the form $\frac{x-h}{p} = \frac{y-k}{q}$ that passes through the given point P with gradient m.

(a) $P(-2, 2)$, $m = \frac{3}{5}$ (b) $P(4, -1)$, $m = -\frac{2}{3}$

(c) $P(0, 4)$, $m = \frac{3}{2}$ (d) $P(11, 7)$, $m = -\frac{4}{3}$

4. Find the equation of the line in 3-space that passes through the given point P with the gradient described by the direction vector $\boldsymbol{u}$.

(a) $P(3, 0, -2)$, $\boldsymbol{u} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix}$ (b) $P(-4, 4, 0)$, $\boldsymbol{u} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$

5. Find the equation of the line in vector form which passes through the given point P with direction vector $\boldsymbol{u}$.

(a) $P(-1, 3, -2)$, $\boldsymbol{u} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ (b) $P(-9, 3, -3)$, $\boldsymbol{u} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$

6. A radio-controlled boat in a pond moves in a direction described by the vector $\boldsymbol{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, moving 1 m to the west for every 2 m to the north, every minute. Determine, accurate to 3 significant figures, the distance covered by the boat in:

(a) 1 minute (b) 2.25 minutes (c) 10 minutes (d) t minutes

7. The boat in question **6** starts from a position 20 m to the east of the person controlling it and moves along the same vector $\boldsymbol{u}$. Find the distance from the container to the boat after:

(a) 1 minute (b) t minutes.

8. A second boat, starting at the same time and 25 m to the west of the boat in question 7, moves in a direction described by the vector $\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, moving 2 m to the east for every 1 m to the north, every minute.

(a) Determine the distance between the two boats after:

(i) 1 minute (ii) t minutes

(b) Would the two boats collide?

9. Find the scalar product for each pair of vectors.

(a) $\boldsymbol{u} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix} 8 \\ 2 \\ -7 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

10. Find the scalar product for each pair of vectors, where θ is the angle between them.

(a) $|\boldsymbol{u}| = 7$, $|\boldsymbol{v}| = 11$, $\theta = 60°$ (b) $|\boldsymbol{u}| = 11.2$, $|\boldsymbol{v}| = 5$, $\theta = 120°$

(c) $|\boldsymbol{u}| = 9$, $|\boldsymbol{v}| = 9$, $\theta = 45°$ (d) $|\boldsymbol{u}| = 13$, $|\boldsymbol{v}| = 6$, $\theta = 23°$

11. Find each vector product: $\boldsymbol{u} \times \boldsymbol{v}$

(a) $\boldsymbol{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$

12. Find the angle between each pair of vectors, correct to the nearest degree.

(a) $\boldsymbol{u} = \begin{pmatrix} -9 \\ 12 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (b) $\boldsymbol{u} = \begin{pmatrix} 13 \\ -12 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -15 \\ -8 \end{pmatrix}$

(c) $\boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ (d) $\boldsymbol{u} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$

13. Ryan and Jack have model aeroplanes which take off from level ground. Jack's aeroplane takes off after Ryan's.

The position of Ryan's aeroplane t seconds after it takes off is given by

$$\boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \text{m}$$

(a) Find the speed of Ryan's aeroplane.

(b) Find the height of Ryan's aeroplane after two seconds.

The position of Jack's aeroplane s seconds after it takes off is given by:

$$\boldsymbol{r} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix} \text{m}$$

(c) Show that the paths of the aeroplane are perpendicular.

The two aeroplane collide at the point $(-23, 20, 28)$.

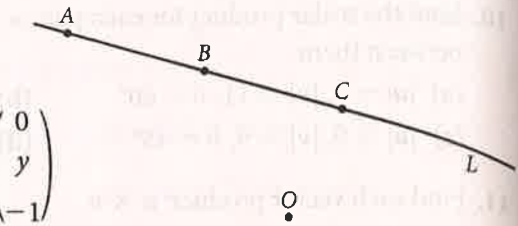
(d) How long after Ryan's aeroplane takes off does Jack's aeroplane take off?

14. A line L passes through points $A(-2, 4, 3)$ and $B(-1, 3, 1)$.

(a) (i) Show that $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ (ii) Find $|\overrightarrow{AB}|$

(b) Find a vector equation for L.

The diagram shows the line L and the origin O.



The point C also lies on L.

Point C has position vector $\begin{pmatrix} 0 \\ y \\ -1 \end{pmatrix}$

(c) Show that $y = 2$

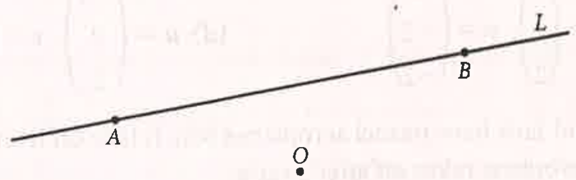
(d) **(i)** Find $\overrightarrow{OC} \cdot \overrightarrow{AB}$

(ii) Hence, write down the size of the angle between OC and L.

(e) Hence or otherwise, find the area of triangle OAB.

15. Let $\boldsymbol{u} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = m\boldsymbol{j} + n\boldsymbol{k}$, where $m, n \in \mathbb{R}$. Given that $\boldsymbol{v}$ is a unit vector perpendicular to $\boldsymbol{u}$, find the possible values of m and of n.

16. The points A and B lie on a line L, and have position vectors $\begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix}$ respectively. Let O be the origin.



(a) Find $\overrightarrow{AB}$

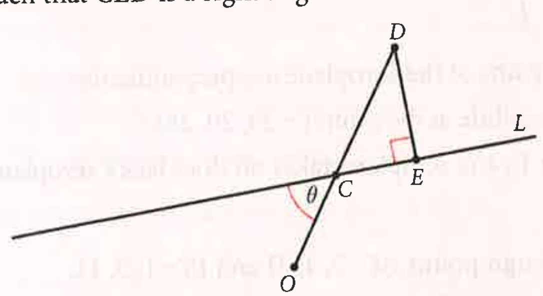
The point C also lies on L, such that $\overrightarrow{AC} = 2\overrightarrow{CB}$

(b) Show that $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

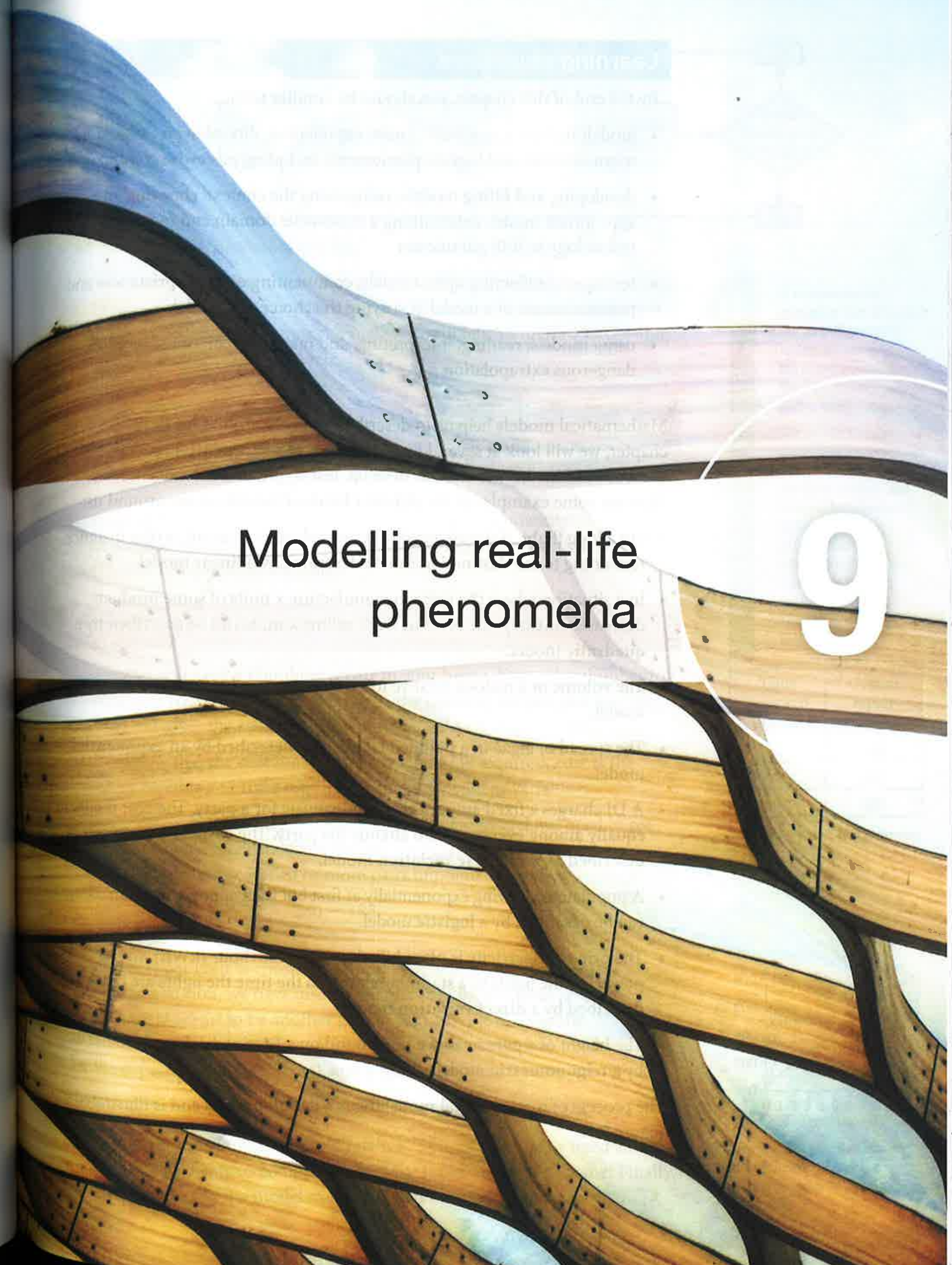
(c) Find θ.

Let D be a point such that $\overrightarrow{OD} = k\,\overrightarrow{OC}$, where $k > 1$. Let E be a point on L such that $C\hat{E}D$ is a right angle.



(d) **(i)** Show that $|\overrightarrow{DE}| = (k-1)|\overrightarrow{OC}|\sin\theta$

(ii) The distance from D to line L is less than 3 units. Find the possible values of k.



Modelling real-life phenomena

9