13 Approximating irregular spaces: integration

Concepts

- Space
- Approximation

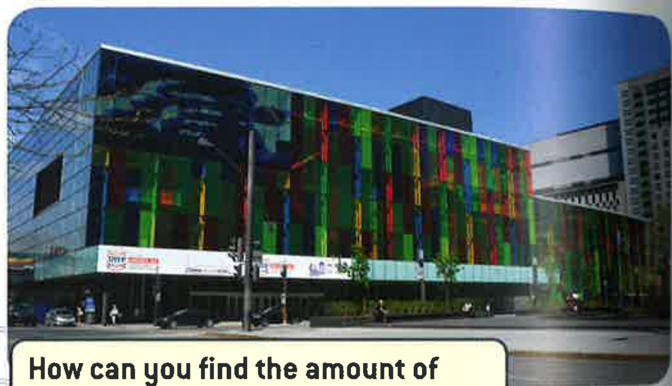
Microconcepts

- Lower limit
- Upper limit
- Definite integrals
- Area under a curve
- Trapezoidal rule
- Numerical integration
- Antiderivatives
- Indefinite integral
- Constant of integration

What do the graphs of functions that have the same derivative have in common? How do they differ?

This chapter explores integration, the reverse of differentiation. The area of an island and the amount of glass needed for the windows of a building can be represented by integrals. Integrals give you a way to estimate the values of areas that cannot be found using existing area formulae.

How can you estimate the area covered by oil spills out at sea?



How can you find the amount of glass in this building?

How can you estimate the area affected by a hurricane?

San Cristóbal is the easternmost island of the Galapagos. Here is a map of the island.

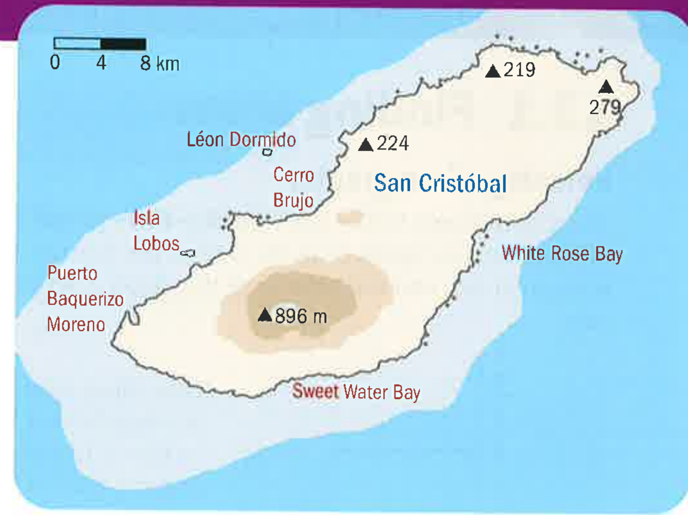
It is claimed that the total area of the island is 558 km^2. How can you test this value?

Use a rectangle to estimate the area of the island.

How did you use the map scale?

Does your result underestimate or overestimate the claimed area? Why?

How could you improve your estimate?



Developing inquiry skills

Write down any similar inquiry questions you might ask to model the area of something different; for example, the area of a national part, city of lake in your country.

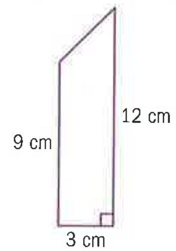
Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Use geometric formulae to find area

For example, the area of a trapezoid:



$$A = \frac{1}{2}(b_1 + b_2)h$$

$$= \frac{1}{2}(9 + 12) \times 3$$

$$= 31.5\text{ cm}^2$$

2 Find the derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$ where all the exponents of x are integers

For example, the derivative function of

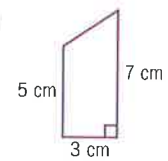
$f(x) = 3x^2 + \frac{x}{2} - 0.5$ is $f'(x) = 6x + \frac{1}{2}$

Skills check

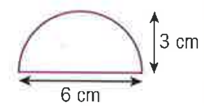
Click here for help with this skills check

1 Find the areas.

a



b



2 Find the derivative of each of these functions.

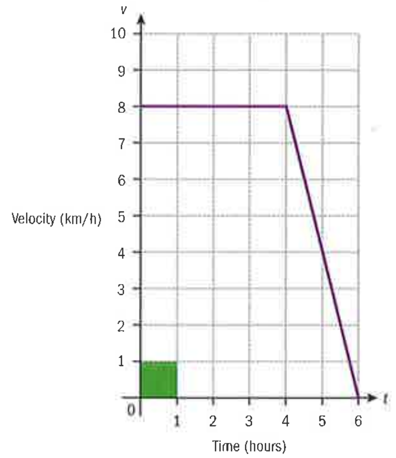
a $f(x) = x^3 + \frac{5x}{2} - 3$

b $g(x) = 4x^2 - x$

13.1 Finding areas

Velocity–time graphs

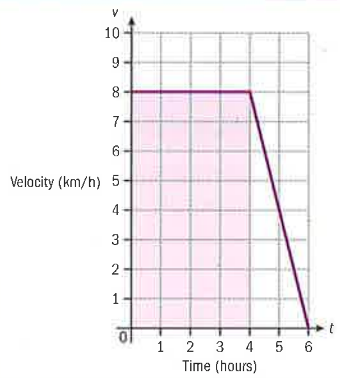
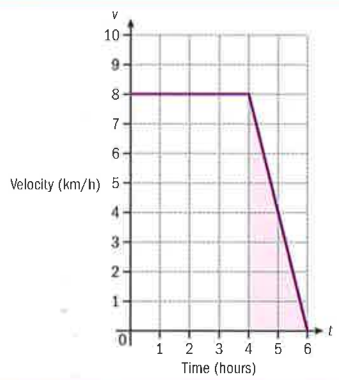
The graph shown below is a **velocity–time graph** for the journey of an object. The time taken from the start of the journey is represented on the horizontal axis and the velocity of the object is represented on the vertical axis.



This object travels at a constant velocity of 8 km/h for 4 hours and then it travels a further 2 hours with a steadily decreasing velocity until it reaches 0 km/h.

What distance did this object travel during the entire journey?

The graph is a piecewise linear function. To answer the question each part will be considered separately.

First 4 hours The velocity is 8 km/h. In 1 hour it travels 8 km therefore in 4 hours it travels $8 \times 4 = 32$ km. What is the relationship between 32 and **the area under the graph** in this first part? The green square shown in the graph above represents 1 km. Why?	
Last 2 hours How many green squares will fit under the second part of the piecewise function? Find the area under the graph. The shape is triangular. Applying the formula for the area of a triangle: $\frac{8\,\text{km/h} \times 2\,\text{h}}{2} = 8\,\text{km}$ Total distance travelled = 32 km + 8 km = 40 km	

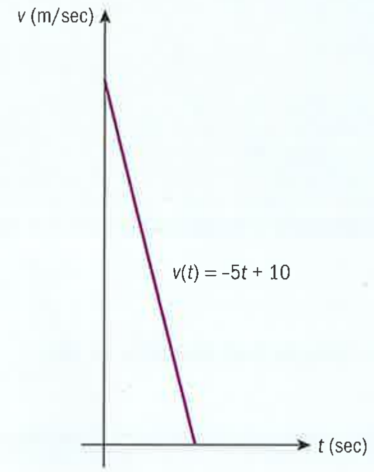
Reflect How does the area under a velocity–time graph relate to the distance travelled?

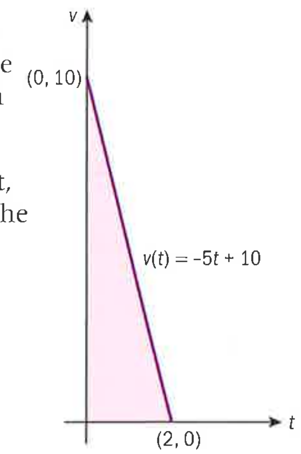
Now, if the velocity–time graph were curved, how would you calculate the area?

In this chapter you will study different methods to find or approximate areas between the graph of a function and the x-axis.

Example 1

For this velocity–time graph, find the distance travelled.

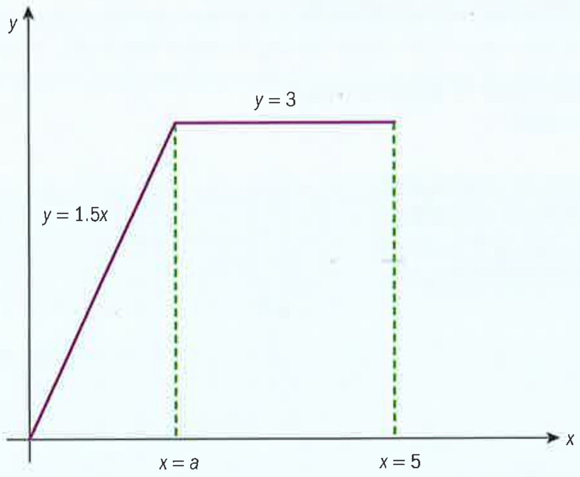


When $v = 0$: $-5t + 10 = 0$ then $t = 2$	The distance travelled is equal to the area under the velocity–time graph. To calculate the width of the triangle, find the point where the graph cuts the t-axis.
When $t = 0$: $v(0) = 10$	To calculate the height, find the point where the graph cuts the v-axis. At this point $t = 0$. 
Area $= \frac{1}{2} \times 10 \times 2 = 10$ Distance travelled is 10 m.	Substitute into the area formula using $b = 2$ and $h = 10$.

Calculus

Example 2

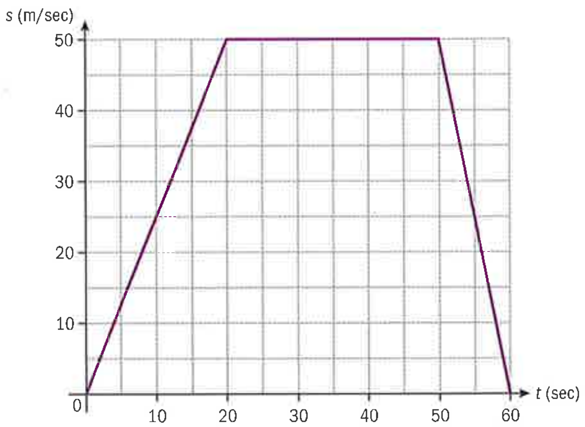
Find the area under the graph.



$1.5a = 3 \Rightarrow a = 2$	Find the value of a, where $y = 1.5x$ and $y = 3$ intersect.
Area of triangle $= \dfrac{2 \times 3}{2} = 3$	
Area of rectangle $= 3 \times 3 = 9$	Add the area of the triangle and the area of the rectangle.
Area under graph $= 3 + 9 = 12$	

Exercise 13A

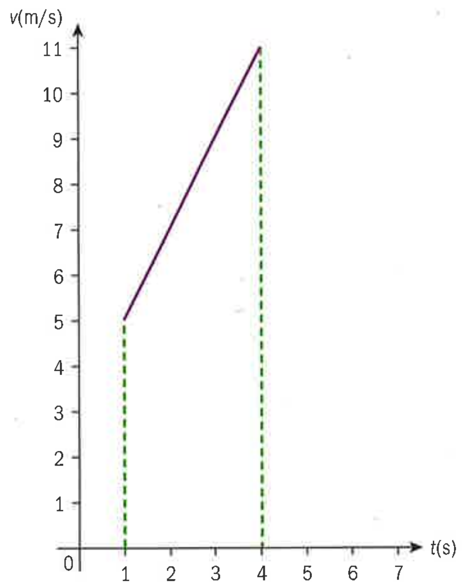
1 The graph below shows how the velocity of a car changes during the first 60 seconds of a journey.



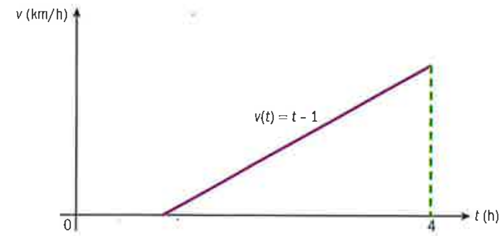
Find the distance travelled during the 60 seconds.

2 For each velocity–time graph, find the distance travelled.

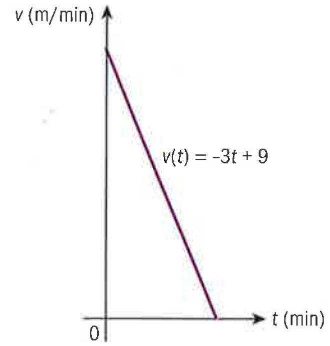
a



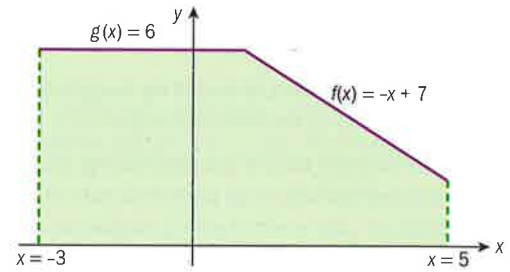
b



c

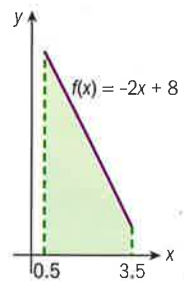


d

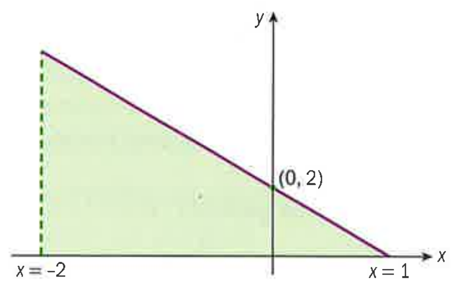


3 For each graph, find the shaded area.

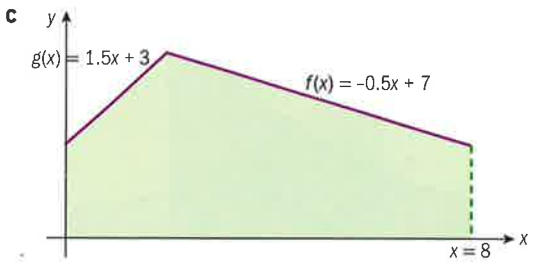
a



b



c



4 **a** Sketch the graph of the function $f(x) = 0.5x + 3$.

b Shade the region enclosed between the graph of f, the x-axis and the vertical lines $x = 1$ and $x = 6$.

c Find the area of the shaded region.

5 **a** Sketch the graph of the function $f(x) = -2x + 6$.

b Shade the region enclosed between the graph of f, the vertical line $x = 0$ and the x-axis.

c Find the area of the shaded region.

6 **a** Sketch the graph of the piecewise linear function

$$f(x) = \begin{cases} x, & 0 \le x \le 5 \\ 5, & 5 < x \le 9 \end{cases}$$

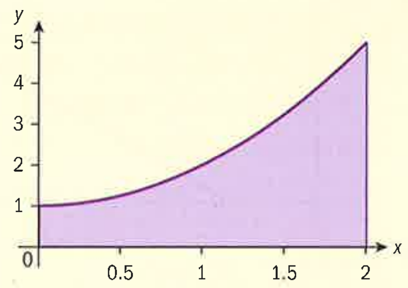
b Shade the region under the graph of $f(x)$ and above the x-axis.

c Find the area of the shaded region.

Investigation 1

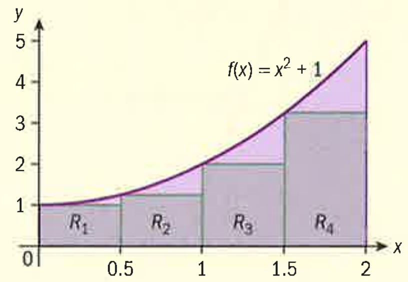
1 Consider the area bounded by the graph of the function $f(x) = x^2 + 1$, the vertical lines $x = 0, x = 2$ and the x-axis.

Estimate the area of the shaded region. Is your estimate an overestimate or an underestimate of the actual area? Discuss your method with a classmate.

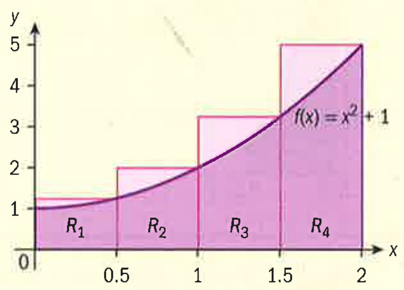


2 In this investigation you will be using rectangles or vertical strips to estimate this area. At the end of the investigation you can check how close your estimate was to the actual area.

- The graph shows four rectangles with the same width. The area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0, x = 2$ is also shown.
- What is the width of the rectangles? How do you calculate it?
- What is the relationship between the height of each rectangle and the graph of the function?
- Find the heights of each of these rectangles.
- Find the area of each of these rectangles and then find **the sum** of the areas of these rectangles.
- Is this an underestimate or an overestimate of the actual area? Why? The sum of these areas will be a **lower bound** of the area of the shaded region. This will give an **underestimate** of the area.

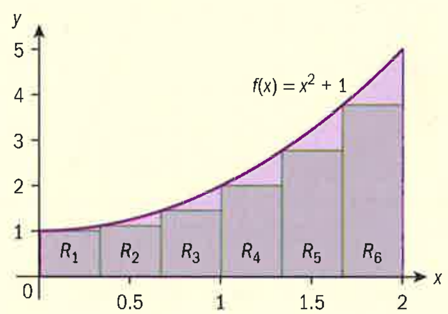


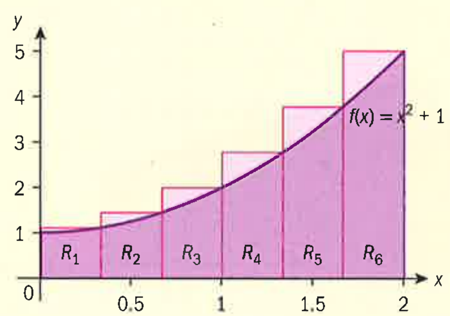
3 This graph also shows four rectangles with the same width. The area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0, x = 2$ is also shown.



- What is the width of each of these rectangles?
- Find the heights of each of these rectangles.
- Find the area of each of these rectangles and then find **the sum** of the areas of these rectangles.
- Is this an underestimate or an overestimate of the actual area? Why?
- If L_S represents the lower bound, A represents the actual area and U_S represents the upper bound, write an inequality relating L_S, U_S and A.

4 In each of the following graphs there are six rectangles. The area under the graph of $f(x) = x^2 + 1$ between the vertical lines $x = 0, x = 2$ and the x-axis is also shaded.





- Approximate the area under the curve by considering a lower bound and an upper bound. Why do you think that more rectangles are being used?
- Complete the following tables to organize the information. You can create a table with your GDC to calculate the heights of the rectangles. How would you calculate the widths? Remember that they are all equal.
- Lower bound with six rectangles:

Rectangle	Width	Height	Area
R_1			
R_2			
R_3			
R_4			
R_5			
R_6			
			$L_S =$

- Upper bound with six rectangles:

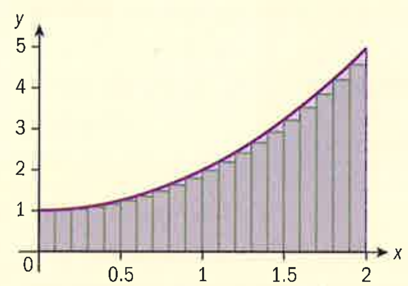
Rectangle	Width	Height	Area
R_1			
R_2			
R_3			
R_4			
R_5			
R_6			
			$U_S =$

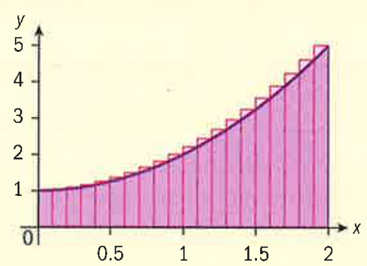
- What do you notice? Are the lower and upper bounds closer to each other than when you had four rectangles? How do you think these estimates can be improved?
- Write a new inequality relating L_S, U_S and A.

5 Look at the following graphs. The number of rectangles, n, has been increased in each case.

L_S and U_S are also given.

$n = 20,\ L_S = 4.47,\ U_S = 4.87$

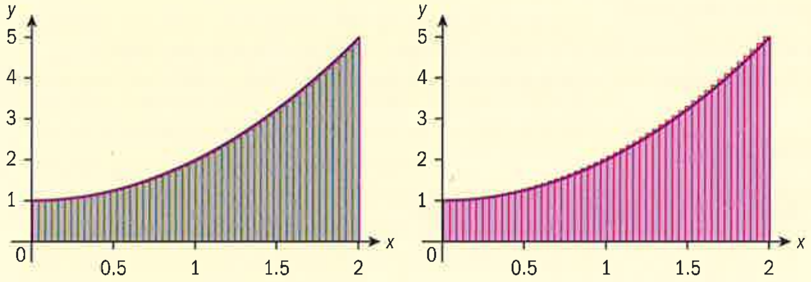




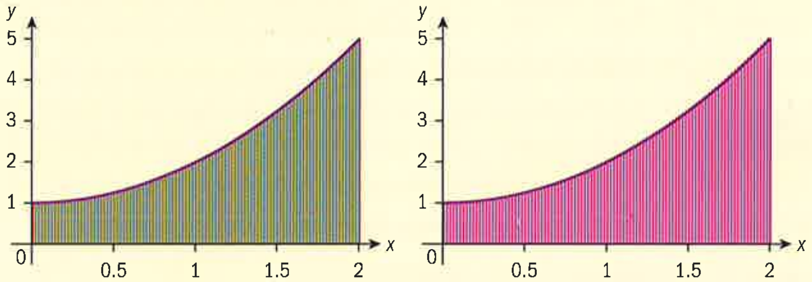
Calculus

Continued on next page

$n = 50,\ L_S = 4.5872,\ U_S = 4.7472$



$n = 100,\ L_S = 4.6268,\ U_S = 4.7068$



- What can you say about the values of L_S as n increases?
- What can you say about the values of U_S as n increases?
- Here are some more values for n, L_S and U_S.

n	L_S	U_S
500	4.65867	4.67467
1000	4.66267	4.67087
10000	4.66627	4.66707

- What happens when n tends to infinity (gets larger and larger)? What can you say about the value of A?

International-mindedness

A Riemann sum, named after 19th century German mathematician Bernhard Riemann, approximates the area of a region, obtained by adding up the areas of multiple simplified slices of the region.

TOK

We are trying to find a method to evaluate the area under a curve.

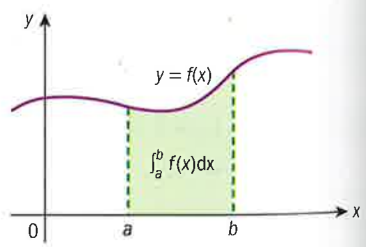
"The main reason knowledge is produced is to solve problems."

To what extent do you agree with this statement?

When f is a non-negative function for $a \le x \le b$, $\int_a^b f(x)\,dx$ gives the area under the curve from $x = a$ to $x = b$.

$\int_a^b f(x)\,dx$ is read as "the definite integral between $x = a$ and $x = b$."

The number a is called the **lower limit** of integration and b is called the **upper limit** of integration.



Investigation 1 (continued)

6 In the investigation, what is the lower limit of integration? What is the upper limit of integration? Is the function positive between the lower and the upper limits? How would you represent A using definite integral notation?

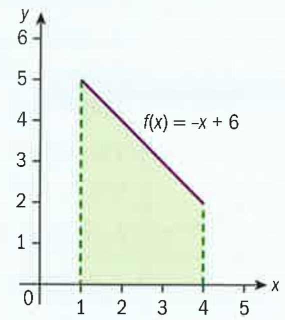
7 **Conceptual** How do areas under curves within a given interval relate to the definite integral and to lower and upper rectangle sums on the same interval?

Reflect What is a definite integral?

Example 3

a Write down a definite integral that gives the area of the shaded region.

b Find the value of the definite integral.



a $\int_1^4 (-x+6)\,dx$	The lower limit is $x = 1$. The upper limit is $x = 4$. The function is $f(x) = -x + 6$.
b $\int_1^4 (-x+6)\,dx = \frac{1}{2} \times (2+5) \times 3 = 10.5$	The shape is trapezoidal. Bases are: $b_1 = f(1) = -1 + 6 = 5$ $b_2 = f(4) = -4 + 6 = 2$ Height = 3 Substitute into the area of a trapezoid formula.

Example 4

For the definite integral $\int_{-1}^{3} (x+4)\,dx$:

a State clearly the function being integrated, the lower limit of integration and the upper limit of integration.

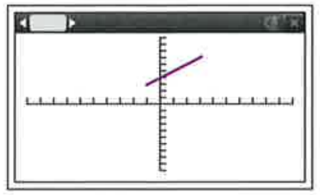
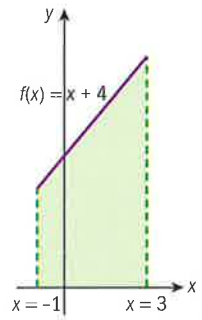
b Sketch the graph of the function. Shade the region whose area the definite integral represents.

c Find the value of the definite integral by using an area formula.

a The function is $f(x) = x + 4$. The lower limit is $x = -1$ and the upper limit is $x = 3$.	Given that this is a linear function, you should expect the graph to be a line segment in the interval $-1 \le x \le 3$.

Continued on next page

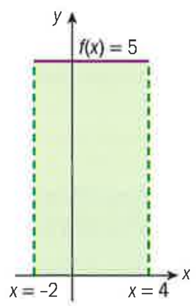
Calculus

	You can use technology to see the graph. 
b 	As the function is positive in the given interval, the definite integral represents the area between the graph and the x-axis. The shape is trapezoidal. The bases of the trapezoid are b_1 and b_2. $b_1 = f(-1) = -1 + 4 = 3$ $b_2 = f(3) = 3 + 4 = 7$ Height $= 3 - (-1) = 4$
b $\int_{-1}^{3}(x+4)\mathrm{d}x = \frac{1}{2}\times(3+7)\times 4 = 20$	Substitute into the area of a trapezoid formula.

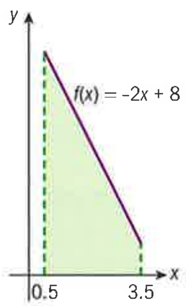
Exercise 13B

1 a Write down a definite integral that gives the area of each of the following regions.

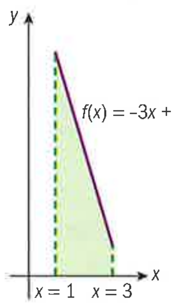
i



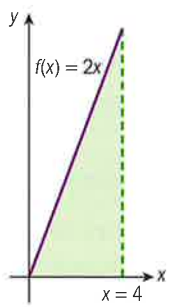
ii



iii



iv



b Calculate the definite integrals from part **a** using existing area formulae.

2 Find the value of the following definite integrals by using existing area formulae. In each case sketch the function in an appropriate interval and shade the region whose area the definite integral represents.

a $\int_{2}^{6}(x+1)\mathrm{d}x$

b $\int_{0}^{4}(-2x+8)\mathrm{d}x$

c $\int_{-2}^{0}(-0.5x+4)\mathrm{d}x$

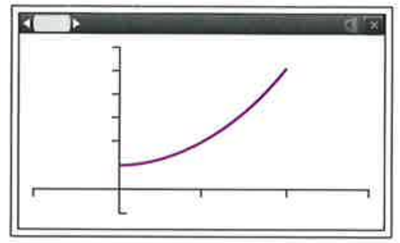
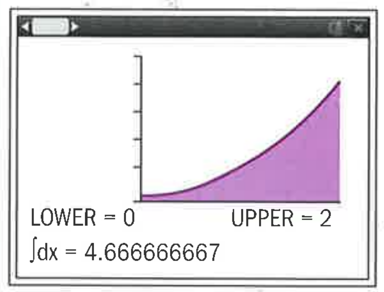
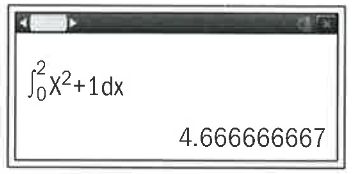
Example 5

Calculate the definite integral $\int_{0}^{2}(x^2+1)\mathrm{d}x$.

Compare your answer with the value found for A in Investigation 1.

TOK

Is imagination more important than knowledge?

$\int_{0}^{2}(x^2+1)\mathrm{d}x = \frac{14}{3}$ (4.67 to 3 sf). This value is the same as the one found for A, the area under the graph of the function $f(x) = x^2 + 1$ between the vertical lines $x = 0$ and $x = 2$, in Investigation 1.	**Method 1** Draw the graph of the function with your GDC.  Find the area. Then enter the lower and upper bounds.  The definite integral is equal to 4.67. **Method 2** Evaluate the definite integral. 

Example 6

Consider the region A enclosed between the curve $y = -x(x-3)$ and the x-axis.

a Write down the definite integral that represents the area of A.

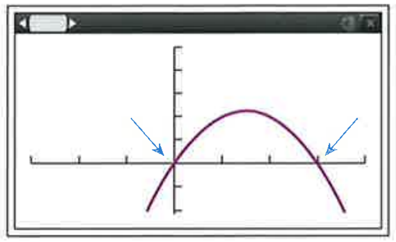
b Find the area of A.

Continued on next page

Calculus

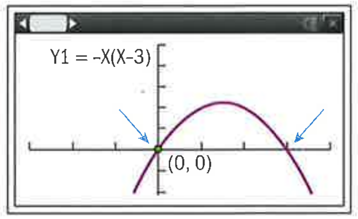
a $\int_0^3 -x(x-3)\,dx$

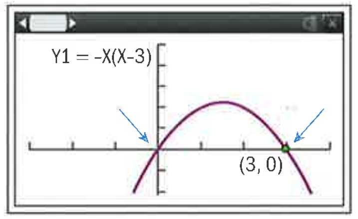
You first have to identify the region using your GDC.



In this case the lower and upper bounds are not given but from the graph it can be seen that these are the **roots** of the parabola.

Find the roots.





$x = 0$ is one of the roots, the lower bound of the definite integral.

$x = 3$ is the other root, the upper bound of the definite integral.

b $A = 4.5$

Once you have identified the region, write down the definite integral.

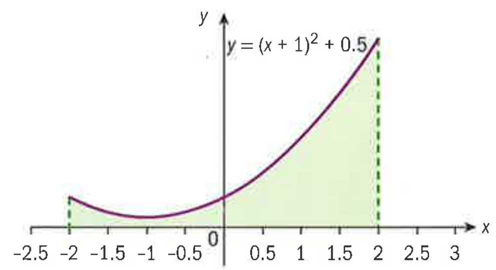
Exercise 13C

1 For each of the following diagrams:

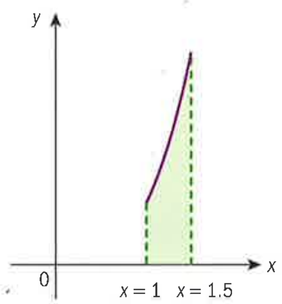
i Write down the definite integral that represents the area of the shaded region.

ii Find the area of the shaded region.

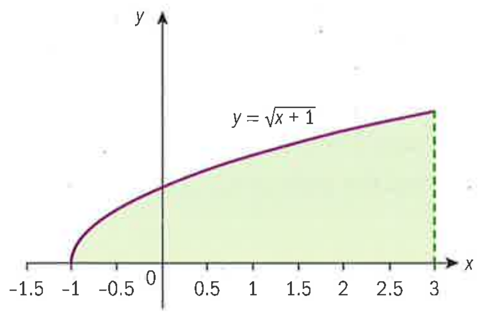
a



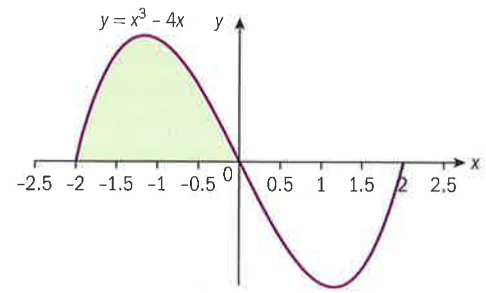
b



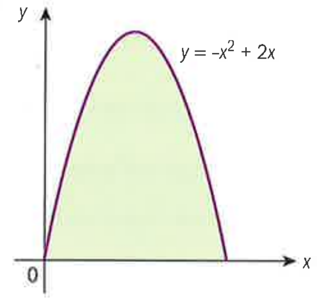
c



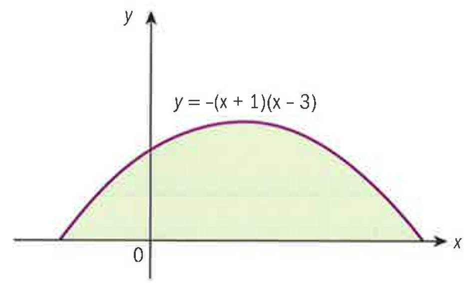
d



e



f



2 In each of the following:

i Write down the definite integral that represents the enclosed area.

ii Find the area.

a $y = x^2$, the x-axis and the vertical lines $x = 2$ and $x = 4$

b $y = 2x$, the x-axis and the vertical lines $x = -1$ and $x = 1$

c $y = \frac{1}{1+x^2}$ and the x-axis in the interval $-1 \le x \le 1$

d $y = \frac{1}{x}$ and the x-axis in the interval $0.5 \le x \le 3$

e $f(x) = -(x-3)(x+2)$, the vertical axis and the vertical line $x = 1$

f $f(x) = -(x-3)(x+2)$, the vertical axis and the horizontal axis

g $f(x) = -(x-3)(x+2)$ and the horizontal axis

h $f(x) = -x^2 + 2x + 15$ and the vertical lines $x = -2$ and $x = 4.5$

i $f(x) = -x^2 + 2x + 15$ and the line $y = 0$

j $f(x) = 3 - e^x$, the vertical line $x = -1$ and the x-axis

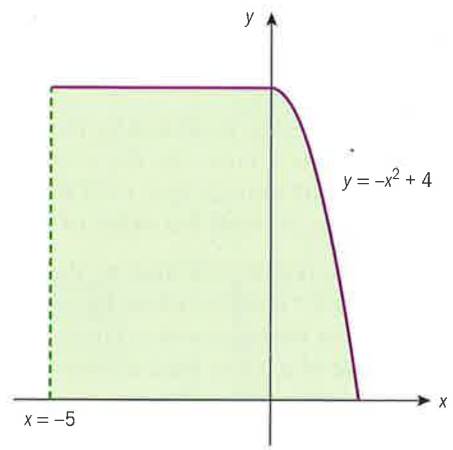
k $y = (x+2)^3 + 5$ and the coordinate axes.

3 Consider the curve $y = -x^2 + 4$.

a Find the zeros of this curve.

b Find the point where this curve cuts the y-axis.

Below is shown the graph of a piecewise function f made up by a horizontal line segment and part of the parabola $y = -x^2 + 4$. The area under the graph of f and above the x-axis has been shaded.



c Find the area under the graph of f in the interval $-5 \le x \le 0$.

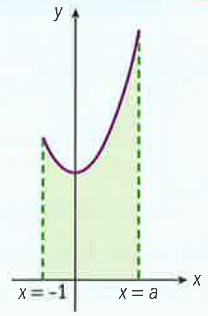
d **i** Write down an expression for the area under the graph of f and above the x-axis for $x > 0$.

ii Find the area.

e Find the area of the whole shaded region.

Example 7

The area of the region bounded by the graph of $f(x) = x^2 + 3$, the x-axis and the vertical lines $x = -1$ and $x = a$ with $a > -1$ is equal to 12. Find the value of a.



$a = 2$	The unknown, a, is the upper bound of this area. First write down the definite integral $\int_{-1}^{a}(x^2+3)\,dx = 12$. Use technology to find the value of a.

Exercise 13D

1 The area of the region bounded by the graph of $f(x) = -(x + 1)(x - 5)$, the x-axis, the y-axis and the vertical line $x = a$ where $a > 0$ is equal to 24. Find the value of a.

2 The area of the region bounded by the graph of $f(x) = 2^{-x}$ and the x-axis between $x = -3$ and $x = a$ where $a > -3$ is equal to 9. Find the value of a. Give your answer correct to four significant figures.

3 The area of the region bounded by the graph of $f(x) = x + \frac{1}{x}$ and the x-axis between $x = a$ and $x = 3$ where $0 < a < 3$ is equal to 6.

a Describe this region using a partially shaded diagram.

b Find the value of a. Give your answer correct to four significant figures.

4 Given that $\int_{-2}^{a} x^2\,dx = \frac{7}{3}$ where $a > -2$:

a Find the value of a.

b Describe the region whose area is defined by the definite integral on a partially shaded diagram on a set of axes.

5 Given that $\int_{-1}^{b}(1 + x^3)\,dx = 2$ where $b > -1$:

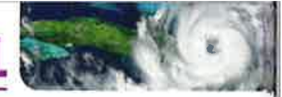
a Find the value of b.

b Describe the region whose area is defined by the definite integral on a partially shaded diagram on a set of axes.

6 Given that $\int_{-1}^{t}\sqrt{x+1}\,dx = \frac{16}{3}$ where $t > -1$:

a Find the value of t.

b Describe the region whose area is defined by the definite integral on a partially shaded diagram on a set of axes.



Numerical integration

You will now study a new **numerical method** to estimate areas between a curve and the x-axis in a given interval. Numerical integration is used, among other techniques, when we do not have a mathematical function to describe the area. Instead we have a set of points.

However, throughout the investigation we will use a function to illustrate this new method.

Investigation 2

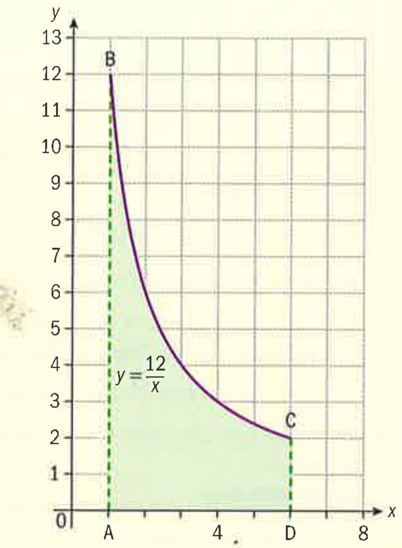
1 Consider the curve $y = \frac{12}{x}$, where $1 \le x \le 6$.

The area of the region enclosed between the graph of $f(x) = \frac{12}{x}$ and the x-axis in the interval $1 \le x \le 6$ will be called S.

Write down an expression for S and find its value. Give your answer correct to two decimal places.

2 Consider the trapezoid ABCD.

- What is the relationship between the base AB of the trapezoid ABCD and the graph of the function $f(x) = \frac{12}{x}$?
- What is the relationship between the base DC of the trapezoid ABCD and the graph of the function $f(x) = \frac{12}{x}$?
- Find the area of the trapezoid ABCD. Is this value an overestimate or an underestimate of S? What is the **error** in this approximation?

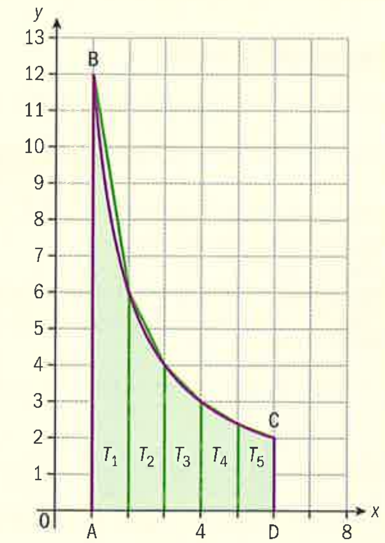


3 To find a better approximation to the value of S you can subdivide the shaded area into strips with **equal width**. The area of every strip can be approximated with the area of a trapezoid and then all these areas added.

The graph shows the shaded area subdivided into five strips. First, you will find the area of trapezoids T_1, T_2, T_3, T_4 and T_5.

You will approximate the value of S by adding up the areas of these five trapezoids.

- The height of each of these trapezoids (or the width of the strips) is equal to 1 unit. How is it found?
- Let the parallel sides of the trapezoids be y_0, y_1, y_2, y_3, y_4 and y_5 where y_0 is the length of AB and y_5 is the length of DC. How can you calculate the lengths of y_0, y_1, y_2, y_3, y_4 and y_5?
- The table on the next page will help you organize the calculations. Use a table on your GDC to complete it.



Continued on next page

Calculus

Trapezoid	Base 1 (b_1)	Base 2 (b_2)	h	$A = \frac{1}{2}(b_1 + b_2)h$
T_1	$y_0 = f(1) = \frac{12}{1} = 12$	$y_1 =$	1	
T_2	$y_1 =$	$y_2 =$	1	
T_3	$y_2 =$	$y_3 =$	1	
T_4	$y_3 =$	$y_4 =$	1	
T_5	$y_4 =$	$y_5 = f(6) = \frac{12}{6} = 2$	1	
		Sum of the areas of the five trapezoids		

- Is this estimation better than the estimate for S found in **1**? Why? How could you improve this value? Why?

4 Now, you will approximate the value of S by adding up the area of $n = 8$ trapezoids.

- What is the height of each of the trapezoids now? How did you find this value?
- Show that the sum of the area of the eight trapezoids is now equal to 21.87, correct to two decimal places. Draw a table similar to the one from part **3** to organize the calculations.
- What is the error made with this approximation?

5 What can you say about the error made in the approximation when the number of trapezoids, n, tends to infinity?

6 **Conceptual** How does the sum of the areas of trapezoids defined by a curve approximate the area under the curve within a given interval?

Trapezoidal rule

In the previous investigation you saw that the definite integral $\int_a^b f(x)\mathrm{d}x$, the area of the region bounded by the curve $y = f(x)$ and the x-axis over the interval $a \le x \le b$, can be approximated by the sum of the areas of trapezoids.

For example, when $n = 4$, the height of each of the trapezoids is $h = \frac{b-a}{4}$.

Recall that

$y_0 = f(a)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_3 = f(x_3)$ and $y_4 = f(b)$ so that

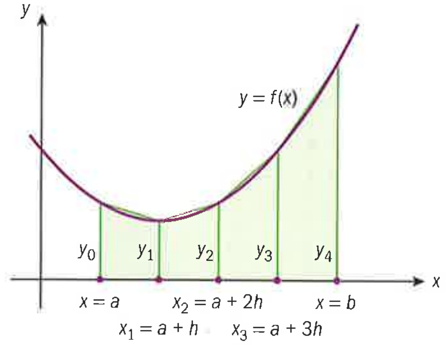
$$\int_a^b f(x)\mathrm{d}x \cong \underbrace{\frac{1}{2}(y_0 + y_1) \times \frac{b-a}{4}}_{\text{Area of } T_1} + \underbrace{\frac{1}{2}(y_1 + y_2) \times \frac{b-a}{4}}_{\text{Area of } T_2} + \underbrace{\frac{1}{2}(y_2 + y_3) \times \frac{b-a}{4}}_{\text{Area of } T_3} + \underbrace{\frac{1}{2}(y_3 + y_4) \times \frac{b-a}{4}}_{\text{Area of } T_4}$$

The right-hand side can be simplified because there is a common factor $\frac{1}{2} \times \frac{b-a}{4}$:

$$\int_a^b f(x)\mathrm{d}x \cong \frac{1}{2} \times \frac{b-a}{4}\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4)\right]$$

The sum inside the square brackets can be simplified to:

$$\int_a^b f(x)\mathrm{d}x \cong \frac{1}{2} \times \frac{b-a}{4}\{y_0 + 2(y_1 + y_2 + y_3) + y_4\}$$



More generally,

The trapezoid rule is $\int_a^b f(x)\mathrm{d}x \cong \frac{1}{2} \times \frac{b-a}{n} \times \{y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\}$

where the interval $a \le x \le b$ is divided into n intervals of equal width.

What is the value of x_i when $i = 0$?

What is the value of x_i when $i = n$?

What is the meaning of $\frac{b-a}{n}$ in this formula?

Reflect What geometric methods can be used to approximate integrals?

Example 8

Estimate the area under a curve over the interval $4 \le x \le 12$, with x- and y-values given in the following table.

x	4	6	8	10	12
y	5	13	10	3	4

Area of trapezoid 1 $= \frac{1}{2}(5 + 13) \times 2 = 18$

Area of trapezoid 2 $= \frac{1}{2}(13 + 10) \times 2 = 23$

In this example, there is no formula of the form $y = f(x)$. You need to find the area of each trapezoid and then sum these areas.

The table suggest that there are four trapezoids.

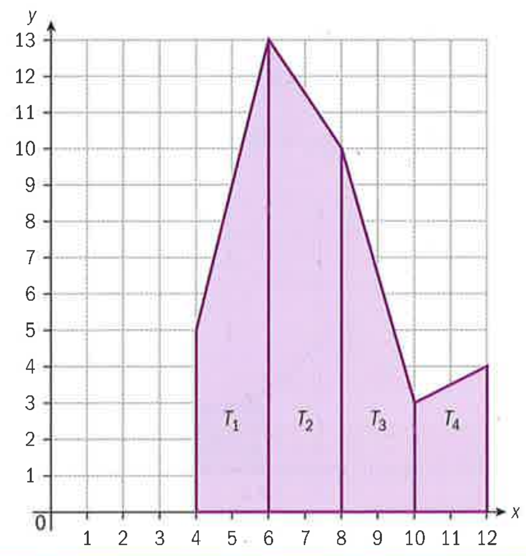
Continued on next page

Calculus

Area of trapezoid 3 $= \frac{1}{2}(10+3)\times 2 = 13$

Area of trapezoid 4 $= \frac{1}{2}(3+4)\times 2 = 7$

Area under the curve = 18 + 23 + 13 + 7 = 61



Example 9

Estimate the area under the graph of $f(x) = e^x$ over the interval $0 \le x \le 1$ using five trapezoids.

x	0	0.2	0.4	0.6	0.8	1
y	1	$e^{0.2}$	$e^{0.4}$	$e^{0.6}$	$e^{0.8}$	e

Draw a table of values.

$$\int_a^b e^x\,dx \cong \frac{1}{2}\times 0.2\times\left(1+2(e^{0.2}+e^{0.4}+e^{0.6}+e^{0.8})+e\right)$$

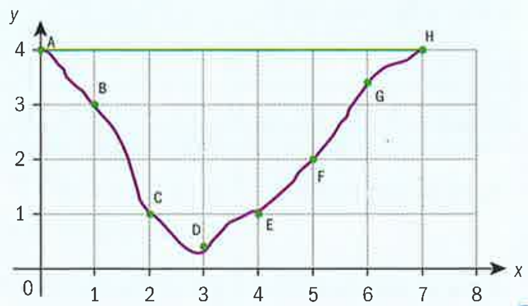
$\cong 1.7240\ldots$

$\cong 1.72$

Substitute into the trapezoid rule using $a = 0$, $b = 1$, $n = 5$, and with heights of trapezoids $= \frac{1-0}{5} = 0.2$.

Example 10

The cross-section of a river is shown here. If the water is flowing at 0.8 m/s use the trapezoidal rule, with seven trapezoids, to find an approximation for the volume of water passing this point in one minute. All lengths are in metres.



A	B	C	D	E	F	G	H
(0,4)	(1, 3)	(2, 1)	(3, 0.4)	(4, 1)	(5, 2)	(6, 3.4)	(7, 4)

Use the trapezoidal rule to find the area under the curve:

$$A \approx \frac{1}{2}\times 1\left(4+4+2(3+1+0.4+1+2+3.4)\right)$$

$= \frac{1}{2}\times 29.6 = 14.8\ \text{m}^2$

Cross-sectional area of river is:

$28 - 14.8 = 13.2\ \text{m}^2$

Volume of water per minute:

$= 13.2\times 0.8\times 60 \approx 634\ \text{m}^3$

The lengths of the parallel lines are given by the y-coordinates in the table.

The trapezoidal rule is applied to these values.

Volume of water is the amount that passes per second multiplied by 60.

Exercise 13E

1 Estimate the area under a curve over the interval $1 \le x \le 9$, with the x- and y-values given in the following table.

x	1	3	5	7	9
y	5	7	6	10	4

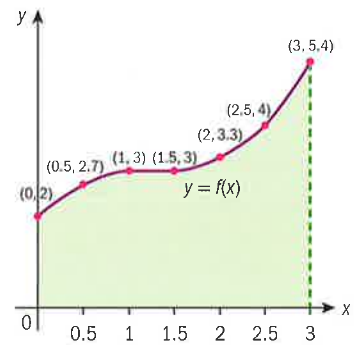
2 Estimate the area under a curve over the interval $0 \le x \le 6$, with the x- and y-values given in the following table.

x	0	1.5	3	4.5	6
y	1	4	2	5.5	0

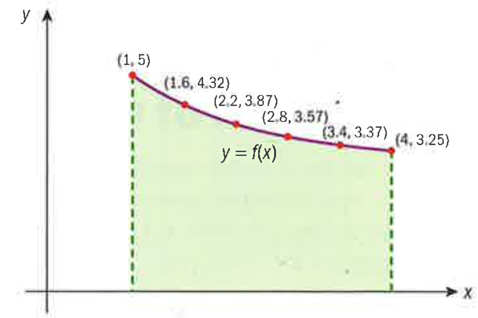
3 Estimate the area under a curve over the interval $-1 \le x \le 4$, with the x- and y-values given in the following table.

x	−1	0.25	1.5	2.75	4
y	5	7	3.5	6	8

4 Estimate the area under the graph of $y = f(x)$ using the data points given in the diagram.



5 Estimate the area under the graph of $y = f(x)$ using the data points given in the diagram.



6 Estimate the area between each curve and the x-axis over the given interval, using the specified number of trapezoids. Give your answers to four significant figures.

a $f(x) = \sqrt{x}$, interval $0 \le x \le 4$ with $n = 5$

b $f(x) = 2^x$, interval $-1 \le x \le 4$ with $n = 4$

c $f(x) = \frac{10}{x} + 1$, interval $2 \le x \le 5$ with $n = 6$

d $y = -0.5x(x-5)(x+1)$, interval $0 \le x \le 5$ with $n = 5$

7 Consider the region enclosed by the curve $y = -2(x-3)(x-6)$ and the x-axis.

a Sketch the curve and shade the region.

b **i** Write down a definite integral that represents the area of this region.

ii Find the area of this region.

Calculus

c Estimate the area of this region using six trapezoids.

d Find the percentage error made with the estimation made in part **c**.

8 Consider the region enclosed by the graph of the function $f(x) = 1 + e^x$, the x-axis and the vertical lines $x = 0$ and $x = 2$.

a Sketch the function f and shade the region.

b i Write down a definite integral that represents the area of this region.

ii Find the area of this region. Give your answer correct to four significant figures.

c Estimate the area of this region using five trapezoids. Give your answer correct to four significant figures.

d Find the percentage error made with the estimation found in part **c**.

Developing inquiry skills

In the opening scenario for this chapter you looked at how to estimate the area of an island.

How could you improve your estimation of the area of the island using what you have studied in this section?

How close is your estimate to the claimed area? Why is your answer an estimate?

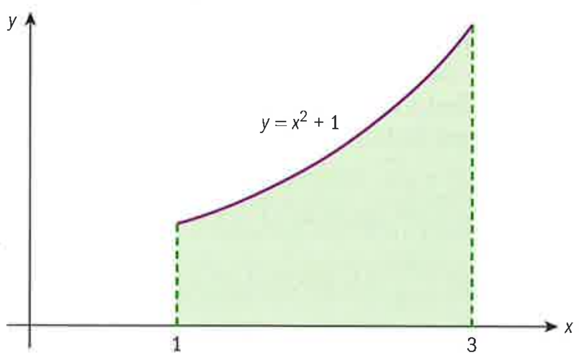
TOK

Galileo said that the universe is a grand book written in the language of mathematics.

Where does mathematics come from? Does it start in our brains or is it part of the universe?

13.2 Integration: the reverse process of differentiation

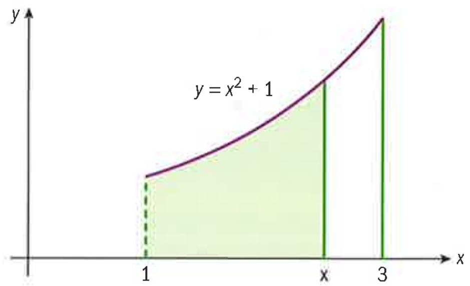
You have so far found areas under curves between two given values. For example, you have seen how to find the area enclosed between $y = x^2 + 1$, the x-axis and the vertical lines $x = 1$ and $x = 3$.



HINT

The area of the shaded region is $\int_1^3 (x^2+1)\,dx = \frac{32}{3}$.

You are now going to find expressions for areas when one of the limits is fixed and the other is variable.



HINT

As the value of x increases from 1 to 3, the area of the shaded region also increases.

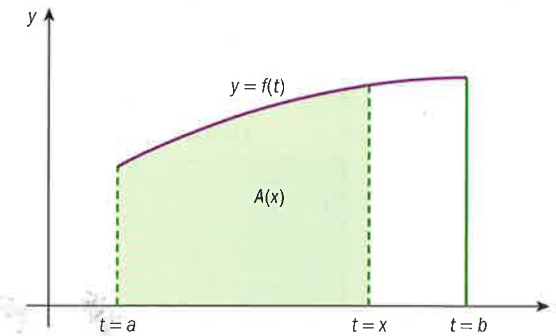
There is a function, the **area function,** which maps every value of x to the area of the shaded region.

What is the formula of the area function?

And what would the area function be in a different region?

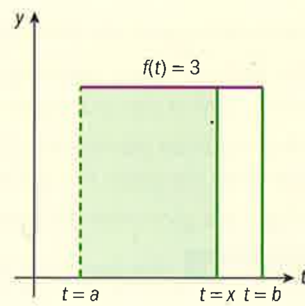
Consider a positive and continuous function $y = f(t)$ over the interval $a \le t \le b$. The area enclosed between the graph of $y = f(t)$, the t-axis and the vertical lines $t = a$ and $t = x$ where $a \le x \le b$ is defined by

$$A(x) = \int_a^x f(t)\,dt.$$



In this investigation you will find the relationship between the two major branches of calculus: integration and differentiation.

Investigation 3



1 Consider the area under the graph of $f(t) = 3$ between $t = a$ and $t = x$, where $a < x$.

Show that the area function is $A(x) = 3x - 3a$.

2 Consider the area under the graph of $f(t) = t$ between $t = a$ and $t = x$, where $a < x$. Show that the area function can be written as $A(x) = \frac{x^2}{2} - \frac{a^2}{2}$. Draw the graph of the function f and shade the area enclosed between this graph and the t-axis over the interval $a \le t \le x$.

3 Consider the area under the graph of $f(t) = 2t$ between $t = a$ and $t = x$, where $a < x$. Show that the area function can be written as $A(t) = x^2 - a^2$. Draw the graph of the function f and shade the area enclosed between this graph and the t-axis over the interval $a \le t \le x$.

4 The results from **1** to **3** are summarized in the first two columns from the table shown below. The last two columns will be completed later.

$f(t)$	$A(x)$	$F(x)$	$A(x)$
$f(t) = 3$	$A(x) = 3x - 3a$	$F(x) = 3x$	$F(x) - F(a)$
$f(t) = t$	$A(x) = \frac{x^2}{2} - \frac{a^2}{2}$		
$f(t) = 2t$	$A(x) = x^2 - a^2$		

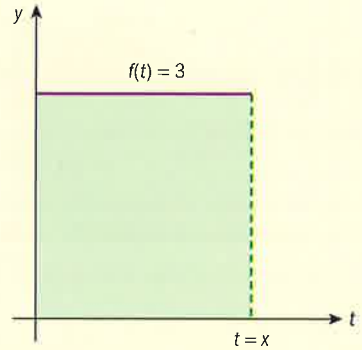
Continued on next page

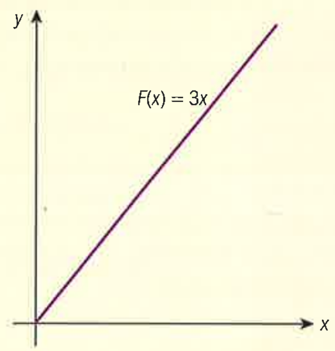
Calculus

- The expressions for $A(x)$ have been written as the difference between two terms. The first term is a function of x and the second term is constant. What do the two terms have in common?

Let the first term in each of the expressions for $A(x)$ be a new function $F(x)$.

- Complete the third column of the table with the corresponding expressions for $F(x)$. The first row has been completed for you.
- For which value of a are the expressions for $F(x)$ and $A(x)$ equal? What area would be represented by $F(x)$ when a takes this value? How would you represent this area using definite integrals?
- Complete the fourth column of the table by writing $A(x)$ in terms of $F(x)$. The first row has been completed for you.
- Below are shown the graphs of $f(t) = 3$ and $F(x) = 3x$.





- What is the gradient of the graph of F at every x? How does this relate to the graph of f?
- For each of the two remaining functions f, find an expression for the gradient of the function F at every x. What can you say about the relationship between the gradient function of F and the function f?
- Now consider the function $f(t) = 3t + 1$. What would be $F(x)$ in this case? What is the relationship between the graph of $F(x)$ and the graph of f?
- If you are given that the area function is $F(x) = x^3$, how would you find the formula of f?

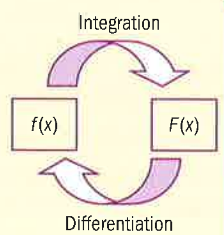
5 **Conceptual** How does integration relate to differentiation?

If $f(t) \geq 0$ is a continuous function, the area enclosed between the graph of f and the x-axis over the interval $a \leq t \leq x$ can be found with the definite integral $\int_a^x f(t)\,dt$.

Also $\int_a^x f(t)\,dt = F(x) - F(a)$ where $F'(x) = f(x)$.

If $F(x)$ is a function where $F'(x) = f(x)$, we say that $F(x)$ is an **antiderivative** of f.

The process of finding an antiderivative is called **antidifferentiation**.



International-mindedness

The **fundamental theorem of calculus** shows the relationship between the derivative and the integral and was developed in the 17th century by Gottfried Leibniz and Isaac Newton.

For example,

$F(x) = 3x$ is an antiderivative of $f(x) = 3$ because $F'(x) = 3$.

$F(x) = \frac{x^2}{2} - 3$ is an antiderivative of $f(x) = x$ because $F'(x) = x$.

$F(x) = x^2 + 1$ is an antiderivative of $f(x) = 2x$ because $F'(x) = 2x$.

Can you think of an antiderivative for $f(x) = 3x^2$?

Reflect What is an antiderivative of a function?

How is the antiderivative related to the definite integral?

TOK

How do "believing that" and "believing in" differ?

How does belief differ from knowledge?

Example 11

a Find an antiderivative of $y = 4 + 2x$.

b Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = 3x$.

a $F(x) = 4x + x^2$	An antiderivative of $y = 4 + 2x$ is a function F whose derivative is $4 + 2x$. The derivative of $4x$ is 4. The derivative of x^2 is $2x$. The derivative of $4x + x^2$ is $4 + 2x$.
b $g(x) = 1.5x^2$	If the gradient function of $y = g(x)$ is $3x$ then $g'(x) = 3x$. Function g is a function whose derivative is $3x$. The derivative of $1.5x^2$ is $3x$.

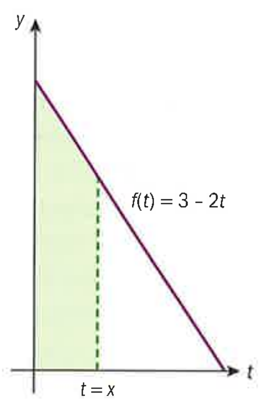
Example 12

Consider the definite integral $\int_0^x (3-2t)\,dt$ where $0 \leq x \leq 1.5$.

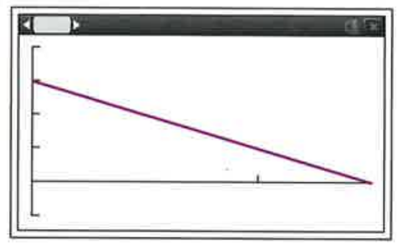
a Draw on a diagram the function $f(t) = 3 - 2t$ and shade the area represented by $\int_0^x (3-2t)\,dt$.

b By using antiderivatives find an expression for $\int_0^x (3-2t)\,dt$ in terms of x.

a



You can use your GDC to draw the graph.



HINT

Remember that $0 \leq x \leq 1.5$.

Shade the area under the graph of f between $t = 0$ and $t = x$.

Continued on next page

b $\int_0^x (3-2t)\,dt = 3x - x^2$

Find an antiderivative of $f(t) = 3 - 2t$, a function whose derivative is $f(t) = 3 - 2t$.

The derivative of $3t$ is 3.

The derivative of t^2 is $2t$.

The derivative of $3t - t^2$ is $3 - 2t$.

$F(x) = 3x - x^2$

$F(0) = 0$

$F(x) - F(0) = 3x - x^2$

HINT

Use the formula for the area of a trapezoid to check that this answer is correct.

Exercise 13F

1 **a** For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = 1$ and $F(x) = x$

ii $f(x) = 1$ and $F(x) = x + 3$

iii $f(x) = 1$ and $F(x) = x - 6$

b Find two further antiderivatives for $f(x) = 1$.

c Write down the general form of an antiderivative of $f(x) = 1$.

2 **a** For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = 2x$ and $F(x) = x^2$

ii $f(x) = 2x$ and $F(x) = x^2 + 1$

iii $f(x) = 2x$ and $F(x) = x^2 - 4$

b Find two further antiderivatives for $f(x) = 2x$.

c Write down the general form of an antiderivative of $f(x) = 2x$.

3 **a** For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = x$ and $F(x) = \frac{x^2}{2}$

ii $f(x) = x$ and $F(x) = \frac{x^2}{2} + 2.5$

iii $f(x) = x$ and $F(x) = \frac{x^2}{2} - 12$

b Write down the general form of an antiderivative of $f(x) = x$.

4 **a** For each of the following, show that $F(x)$ is an antiderivative of $f(x)$.

i $f(x) = 2x + 1$ and $F(x) = x^2 + x$

ii $f(x) = 2x + 1$ and $F(x) = x^2 + x - 3.2$

iii $f(x) = 2x + 1$ and $F(x) = x^2 + x + 4$

b Write down the general form of an antiderivative of $f(x) = 2x + 1$.

5 Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = 3$.

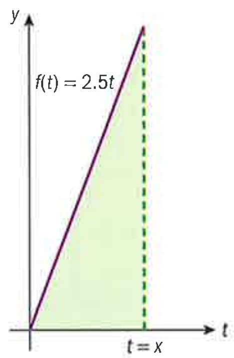
6 Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = -2$.

7 Find a function $y = g(x)$ when the gradient function is $\frac{dy}{dx} = \frac{x}{2}$.

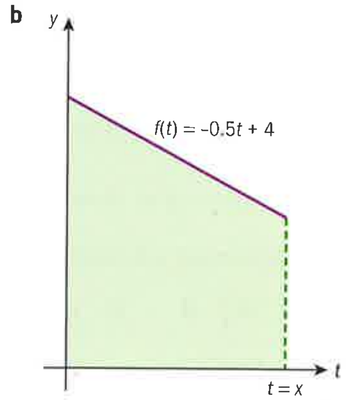
8 Find an antiderivative of $f(x) = x^2$.

9 Find the area function, $A(x)$, in each of the following situations.

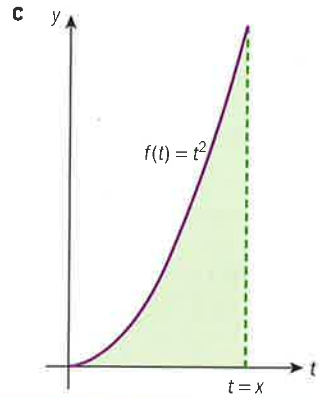
a



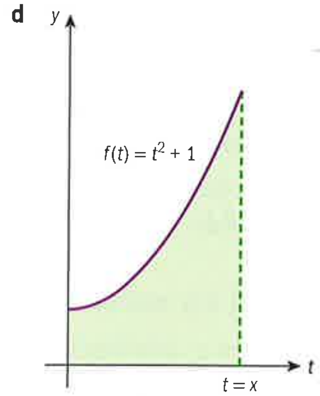
b



c



d



10 Consider the graph of the function $f(t) = 4$ over the interval $0 \le t \le x$. Find $\int_0^x 4\,dt$.

11 Consider the graph of the function $f(t) = 2t + 1$ over the interval $0 \le t \le x$. Find $\int_0^x (2t+1)\,dt$.

12 Consider the graph of the function $f(t) = 4t$ over the interval $2 \le t \le x$. Find $\int_2^x 4t\,dt$.

Investigation 4

1 On the same set of axes, sketch the graphs of

$y = x^2$ $\quad y = x^2 + 1$ $\quad y = x^2 - 2$ $\quad y = x^2 + 3$

over the interval $-4 \le x \le 4$.

- How can you describe their relative position?
- Find the gradient of each of these curves at $x = 1$. What do you notice?
- Now find the gradient of each of these curves at $x = -2$. What can you say about these answers?
- What is the gradient of each of these curves at any x?
- Find another curve for which the gradient at x is the same as the gradient of any of the curves from **1**.
- How would you write the formula of any curve with the gradient the same as the gradient of the curves from **1**?
- All these curves make up a family of functions that are antiderivatives of $2x$. Why is this?

The notation used to indicate this family of functions is $\int 2x\,dx = x^2 + c$, where c is a constant.

Continued on next page

This is read as "the integral of $2x$ with respect to x is $x^2 + c$".

- What is the value of c when the antiderivative is $y = x^2 + 1$? What is the constant of integration when the antiderivative is $y = x^2$?
- What is the derivative of $x^2 + c$ with respect to x?

2 What is the family of functions that are antiderivatives of $f(x) = 3$? What do their graphs have in common? Write this family of functions using integral notation.

3 What does the integral $\int x\,dx$ represent? Calculate it.

4 **Conceptual** How does an indefinite integral define a family of antiderivatives?

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$ where $c \in \mathbb{R}$.

The expression $\int f(x)\,dx$ is called an **indefinite integral** and $\int f(x)\,dx$ is read as "the integral of f with respect to x".

Note: c is called the **constant of integration**.

Reflect What does an indefinite integral represent?
How is an antiderivative related to an indefinite integral?

Example 13

a Find the family of antiderivatives of $f(x) = 5$.

b Find $\int 4x\,dx$.

a $F(x) = 5x + c$ where $c \in \mathbb{R}$	These are the functions F for which $F'(x) = 5$.
b $2x^2 + c$ where $c \in \mathbb{R}$	This represents the family of antiderivatives of $4x$, the functions whose derivatives are $4x$. This is $2x^2 + c$.

Exercise 13G

1 Find the family of antiderivatives of 2.

2 Find the family of antiderivatives of $x + 1$.

3 Find all the functions whose derivatives are equal to -1.

4 Find all the functions whose derivatives are equal to $2x$.

5 Calculate the following indefinite integrals.

a $\int 1\,dx$ **b** $\int 6\,dx$ **c** $\int \frac{1}{2}\,dx$

6 Calculate the following indefinite integrals.

a $\int x\,dx$ **b** $\int 2x\,dx$

c $\int 5x\,dx$ **d** $\int \frac{1}{2}x\,dx$

e $\int ax\,dx$ (a is a non-zero constant)

7 Calculate the following indefinite integrals.

a $\int x^2\,dx$ **b** $\int 3x^2\,dx$ **c** $\int 4x^2\,dx$

d $\int \frac{1}{2}x^2\,dx$ **e** $\int \frac{x^2}{3}\,dx$ **f** $\int \frac{3}{2}x^2\,dx$

g $\int ax^2\,dx$ (a is a non-zero constant)

8 Calculate the following indefinite integrals.

a $\int x^3\,dx$ **b** $\int 4x^3\,dx$ **c** $\int 2x^3\,dx$

d $\int \frac{4}{5}x^3\,dx$ **e** $\int \frac{x^3}{3}\,dx$ **f** $\int -x^3\,dx$

g $\int ax^3\,dx$ (a is a non-zero constant)

In questions **5** to **8** you may have noticed that all the functions were of the form ax^n.

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ where a and n are constants, $a \neq 0$ and $n \neq -1$. n is an integer.

This is an integration rule and is called the **power rule.**

Reflect Why is the condition $n \neq -1$ given in the power rule?

Example 14

a Find $\int \frac{x^5}{4}\,dx$. **b** Find $\int \left(\frac{x^2}{2} + 3x - 1\right)dx$. **c** Find $\int \frac{4}{x^2}\,dx$.

a $\int \frac{x^5}{4}\,dx = \frac{x^6}{24} + c$	Apply the power rule with $a = \frac{1}{4}$ and $n = 5$. **HINT** Do not forget the constant of integration. $\int \frac{x^5}{4}\,dx = \int \frac{1}{4}x^5\,dx = \frac{1}{4} \times \frac{x^{5+1}}{5+1} + c = \frac{1}{4} \times \frac{x^6}{6} + c$
b $\int \left(\frac{x^2}{2} + 3x - 1\right)dx = \frac{x^3}{6} + \frac{3x^2}{2} - x + c$	As the derivative of a sum is the sum of the derivatives, calculate an antiderivative for each term by applying the power rule and then add the three terms up. $\int \frac{x^2}{2}\,dx = \frac{1}{2} \times \frac{x^{2+1}}{2+1} + c_1 = \frac{1}{6}x^3 + c_1$

Continued on next page

Calculus

c $\int \frac{4}{x^2}dx = -\frac{4}{x} + c$	$\int 3x\,dx = 3 \times \frac{x^{1+1}}{1+1} + c_2 = \frac{3x^2}{2} + c_2$ $\int -1\,dx = -x + c_3$ Apply the power rule $a = 4; n = -2$ $\int \frac{4}{x^2}dx = \int 4x^{-2}dx = 4 \times \frac{x^{-2+1}}{-2+1} + c$

HINT
c_1, c_2, c_3 are three constants that added up give another constant c.

Reflect What is the integral of a sum of multiples of powers of x?
How can you tell that the indefinite integrals are correct?

Exercise 13H

1 Find the following indefinite integrals.
 a $\int 10\,dx$ **b** $\int 0.6x^2 dx$
 c $\int x^5 dx$ **d** $\int (7-2x)dx$
 e $\int (1+2x)dx$ **f** $\int (5 + x - \frac{1}{3}x^2)dx$
 g $\int (-x + \frac{3x^2}{4} + 0.5)dx$ **h** $\int (1 - x + \frac{x^3}{2})dx$
 i $\int (x^2 - \frac{1}{2}x + 4)dx$ **j** $\int \frac{1}{2}x^{-2}dx$
 k $\int \frac{2}{x^4}dx$ **l** $\int (4x + \frac{5}{x^3})dx$

2 For $f(x) = x^2 - \frac{x}{3} + 4$, find:
 a $f'(x)$ **b** $\int f(x)dx$

3 Find $\int (t - 3t^2)dt$.

4 Find $\int (4t^3 - 3t + 1)dt$.

5 Find **all** the functions F for which the gradient equals $3 + x - \frac{x^2}{4}$.

6 Find **all** the functions F for which the gradient is $y = -x + 0.5x^2$.

7 **a** Expand $(x+1)(x-2)$.
 b Hence, find all the functions $y = f(x)$ for which $\frac{dy}{dx} = (x+1)(x-2)$.

8 Find the indefinite integral of the function $g(x) = x^3 - \frac{2}{x^2} + 1$.

Example 15

a The curve $y = f(x)$ passes through the point (1, 3). The gradient of the curve is given by $f'(x) = 2 + \frac{x}{3}$. Find the equation of the curve.

b If $\frac{dy}{dx} = 2x - 4x^2$ and $y = -1$ when $x = 3$, find y in terms of x.

a $f(x) = \int (2 + \frac{x}{3})dx = 2x + \frac{1}{3} \times \frac{x^2}{2} + c$ $f(x) = 2x + \frac{x^2}{6} + c$	Apply the power rule to find an antiderivative of $f'(x)$.
$f(1) = 2 \times 1 + \frac{1^2}{6} + c$ $3 = 2 + \frac{1}{6} + c$ $c = \frac{5}{6}$ $f(x) = 2x + \frac{x^2}{6} + \frac{5}{6}$	If the curve passes through the point (1, 3) then $f(1) = 3$.
b $y = \int (2x - 4x^2)dx = x^2 - \frac{4x^3}{3} + c$	Integrate $\frac{dy}{dx}$ to find y in terms of x.
$-1 = 3^2 - \frac{4 \times 3^3}{3} + c$ $-1 = 9 - 36 + c$ $c = 26$ $y = x^2 - \frac{4x^3}{3} + 26$	Use the fact that $y = -1$ when $x = 3$ to find the value of the constant c.

Example 16

Find the cost function, $C(x)$, when the marginal cost is $M(x) = 1 + 2x$ and the fixed cost is US\$40.

$C(x) = x + x^2 + 40$	The cost function is an antiderivative of the marginal cost function. To find $C(x)$, integrate $M(x)$ $\int (1+2x)\,dx = x + x^2 + c$ The fixed cost is used to find the constant of integration. $C(0) = 40$ then $0 + 0^2 + c = 40$ and $c = 40$

Exercise 13I

1 The derivative of the function f is given by $f'(x) = 3x + 4x^2$. The point $(-1, 0)$ lies on the graph of f. Find an expression for f.

2 It is given that $\frac{dy}{dx} = x + \frac{x^2}{5} + 2$ and that $y = 3$ when $x = 4$. Find an expression for y in terms of x.

3 It is given that $f'(x) = 3 - x$ and $f(2) = 1$. Find $f(x)$.

4 It is given that $f'(x) = 3x^2 - \frac{x^4}{3}$ and $f(0) = 2$. Find $f(x)$.

5 It is given that $\frac{dy}{dx} = 0.2x + 3x^2 + 1$ and that $y = -1$ when $x = 1$. Find the value of y when $x = 0.5$.

6 The derivative of the function f is given by $f'(x) = -x + 3$. The point $(-2, 0)$ lies on the graph of f.

a Find an expression for f.

b Find $f(2)$.

7 A company's marginal cost function is $M(x) = x - \frac{2}{x^2} + 1$ and the fixed cost is US\$145. Find the cost function.

8 A company's marginal cost function is $M(x) = 3 - 4x + x^2$ and the fixed cost for the company is US\$1000. Find the cost function.

9 A refrigerator factory's marginal revenue function is $M(t) = t^2 - 80$, where t is the number of refrigerators produced by the factory and the revenue is given in US\$.

The factory earns US\$ 567,000 in revenue from selling 120 refrigerators.

a Find the revenue function, $R(t)$.

b Find the factory's total revenue from producing 150 refrigerators.

International-mindedness

Ibn Al Haytham, born in modern-day Iraq in the 10th century, was the first mathematician to calculate the integral of a function in order to find the volume of a paraboloid.

Developing your toolkit

Now do the Modelling and investigation activity on page 584.

TOK

Why do we study mathematics?

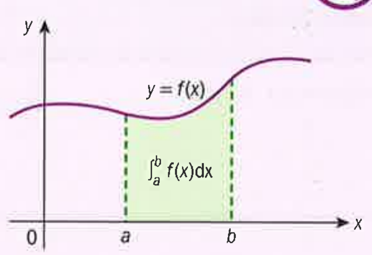
What's the point?

Can we do without it?

Chapter summary

- When f is a non-negative function for $a \le x \le b$, $\int_a^b f(x)dx$ gives the area under the curve from $x = a$ to $x = b$.
- If $v(t)$ is a velocity–time function and $v(t) \ge 0$ over the interval of time $a \le t \le b$ then

  distance travelled $= \int_a^b v(t)dt$.



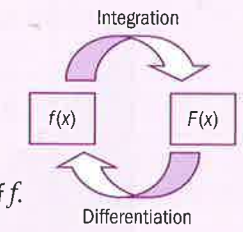
- The trapezoid rule is

  $$\int_a^b f(x)dx \approx \frac{1}{2} \times \frac{b-a}{n} \times \{y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

  where the interval $a \le x \le b$ is divided into n intervals of equal width.
- Consider a positive and continuous function $y = f(t)$ over the interval $a \le t \le b$. The area enclosed between the graph of $y = f(t)$, the t-axis and the vertical lines $t = a$ and $t = x$ where $a \le x \le b$ is defined by $A(x) = \int_a^x f(t)dt$.

- If $f(t) \ge 0$ is a continuous function, the area enclosed between the graph of f and the x-axis over the interval $a \le t \le x$ can be found with the definite integral $\int_a^x f(t)dt$. Also $\int_a^x f(t)dt = F(x) - F(a)$ where $F'(x) = f(x)$.

  If $F(x)$ is a function where $F'(x) = f(x)$, we say that $F(x)$ is an **antiderivative** of f. The process of finding an antiderivative is called **antidifferentiation.**



- If $F'(x) = f(x)$ then $\int f(x)dx = F(x) + c$ where $c \in \mathbb{R}$.

  The expression $\int f(x)dx$ is called an **indefinite integral** and $\int f(x)dx$ is read as "the integral of f with respect to x".

HINT

c is called the **constant of integration**.

- $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ where a and n are constants, $a \ne 0$ and $n \ne -1$.

  This is an integration rule and is called the **power rule.**

Developing inquiry skills

Write down any further inquiry questions you could ask and investigate how you could find the areas of irregular shapes and curved shapes.

Chapter review

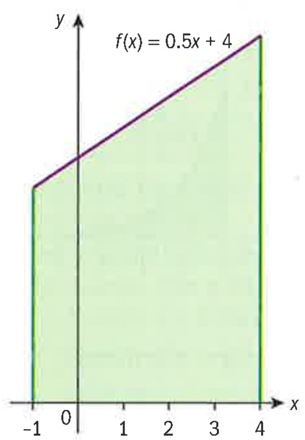
Click here for a mixed review exercise

1 For each of the following shaded regions:

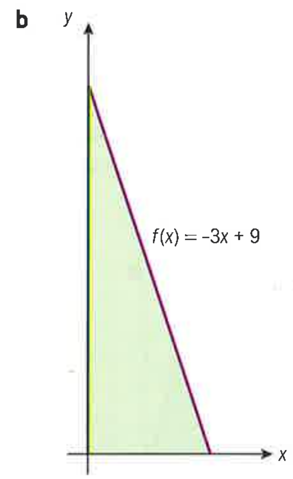
i Write down a definite integral that represents the area of the region.

ii Hence or otherwise, find the area of these shaded regions.

a



b



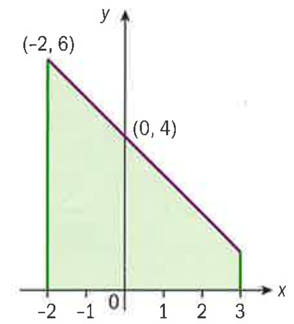
2 For each of the following regions:

i Write down a definite integral that represents the area of the region.

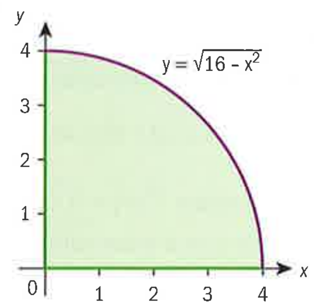
ii Hence or otherwise, find the area of these shaded regions.

Calculus

a



b

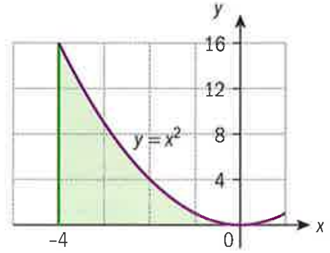


3 For each of the following regions:

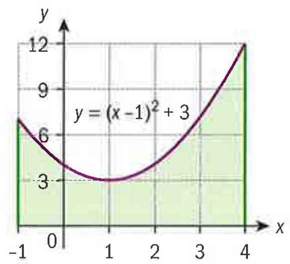
i Write down a definite integral that represents the area of the region.

ii Find the area of the region.

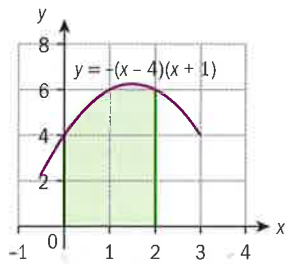
a



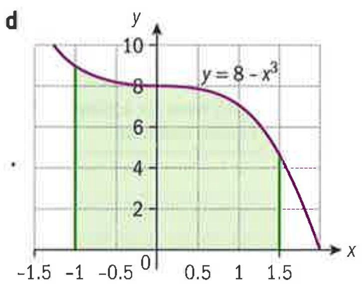
b



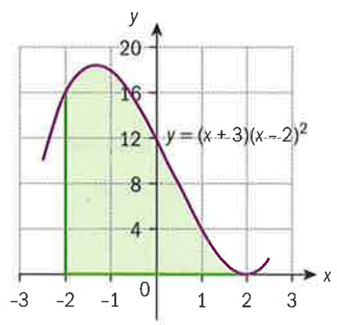
c



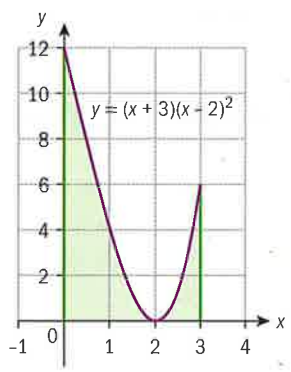
d



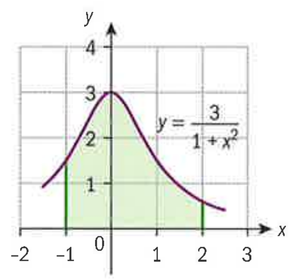
e



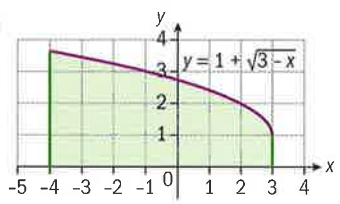
f



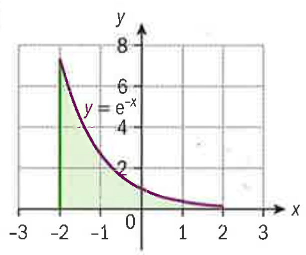
g



h



i



4 For each of the following definite integrals:

i On a diagram shade the region that they represent.

ii Find their value.

a $\int_0^4 \sqrt{x}\,dx$ **b** $\int_{-2}^{2} x^2 dx$

c $\int_{2.5}^{3} -(x-2)(x-4)\,dx$

d $\int_2^4 -(x-2)(x-4)\,dx$

e $\int_2^5 \frac{10}{x+1}dx$ **f** $\int_{-1}^{1} (3^x + 2)\,dx$

g $\int_{-2}^{3} (x^2 - 2x + 3)\,dx$

5 Consider the region A enclosed between the graph of $y = -(x+1)(x-4)$ and the x-axis.

a Write down a definite integral that represents the area of A.

b Find the value of this area.

6 Consider the curve $y = x(x-4)^2$. Let A be the region enclosed between this curve and the x-axis.

a Write down the x-intercepts of this curve.

b Write down a definite integral that represents the area of A.

c Find the value of this area.

7 Consider the curve $y = x^3$. Let A be the region enclosed between this curve, the x-axis and the vertical line $x = 2$.

a Write down the x-intercept of this curve.

b Sketch the curve and clearly label A.

c Write down a definite integral that represents the area of A.

d Find the value of this area.

8 Consider the graph of the function $f(x) = (x+2)^2 + 1$.
The region bounded by the graph of f, the x-axis, the y-axis and the vertical line $x = b$ with $b > 0$ has an area equal to 42.

a Sketch the region.

b Find the value of b.

9 The table below shows the coordinates (x, y) of five points that lie on a curve $y = f(x)$.

x	1	3.25	5.5	7.75	10
$y = f(x)$	3	9	7	12	5

Estimate the area under the curve over the interval $1 \le x \le 10$.

10 Estimate the area under the graph of $f(x) = \sqrt{x-2}$ over the interval $2 \le x \le 4$ using five trapezoids. Give your answer correct to four significant figures.

11 Consider the graph of the function $f(t) = -t^2 + 2t$ where $f(t) \ge 0$.

a Draw the graph of the function f and shade the area enclosed between this graph and the t-axis over the interval $0 \le t \le x$.

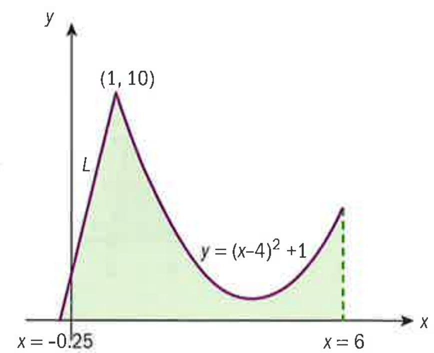
b Find an expression for the area under the graph of f over the interval $0 \le t \le x$.

12 Consider the graph of the function $f(t) = 4t$ over the interval $3 \le t \le x$. Find $\int_3^x 4t\,dt$.

13 Calculate $\int (2 + x)\,dx$.

14 Find $\int (1 + x - \frac{x^3}{4})\,dx$.

15 Line L passes through the points $(-0.25, 0)$ and $(1, 10)$. Consider the region bounded by the graph of line L for $-0.25 \le x \le 1$, the curve $y = (x-4)^2 + 1$ for $1 \le x \le 6$ and the x-axis. The region is shown below.



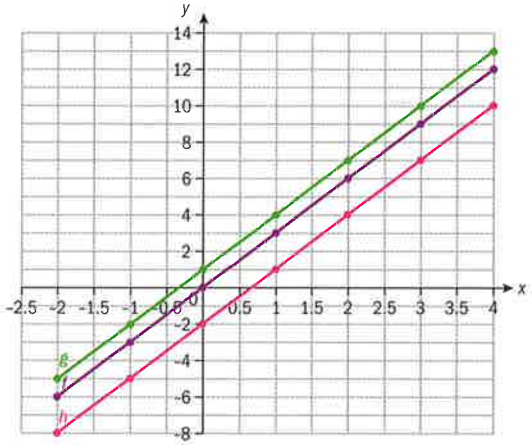
a Find the area under the graph of L for $-0.25 \le x \le 1$.

Calculus

b Write down an expression for the area under the curve $y = (x-4)^2 + 1$ for $1 \le x \le 6$.

c Hence, find the area of the shaded region.

16 The diagram shows the graph of three linear functions, g, f and h.



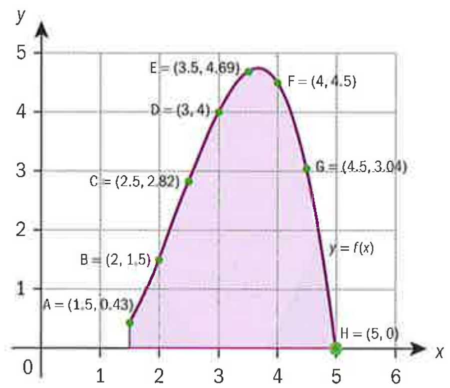
a Find the equation of each of these functions.

The three functions are antiderivatives of $y = t(x)$.

b Find the equation of $y = t(x)$.

c Find $\int t(x)\,dx$.

17 Estimate the area under the graph of $y = f(x)$ in the interval $1.5 \le x \le 5$ using the data points given in the diagram.



18 a Estimate the area A under the graph of $f(x) = e^{-x^2}$ over the interval $0 \le x \le 1$ using five trapezoids. Give your answer correct to four significant figures.

b i Write down a definite integral that represents A.

ii Hence, find the actual area. Give your answer correct to four significant figures.

c Find the percentage error made with the estimation found in part **a**.

Exam-style questions

19 P1: Find $\int (5 - 12x^2 + 4x^3)\,dx$, simplifying your answer as far as possible. (4 marks)

20 P1: The derivative of the function f is given by $f'(x) = \frac{3}{2}x^2 + x + 3$ and the curve $y = f(x)$ passes through the point $\left(-1, \frac{13}{2}\right)$.

Find an expression for f. (6 marks)

21 P2: a Find the coordinates of the points of intersection of the graphs of $y = 6x - x^2$ and $y = 10 - x$. (4 marks)

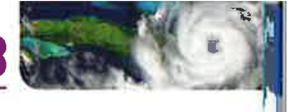
b On the same axes, sketch the graphs of $y = 6x - x^2$ and $y = 10 - x$. (2 marks)

c Find the exact value for the area bounded by the two curves. (7 marks)

22 P1: A particle P is travelling in a straight line. After t seconds, the particle has velocity $v = 0.6t^2 + 4t + 1$ m/s for $t \ge 0$.

a Find an expression for the displacement of the particle from the origin after t seconds. (2 marks)

b Hence, find the distance travelled by the particle during the third second of motion. (3 marks)



23 P1: Consider the curve $y = -\frac{x^2}{10}(x - 10)$.

The area A is defined as the region bounded by the curve and the x-axis.

a Sketch the curve, clearly showing the area defined as A. (3 marks)

b Write down a definite integral that represents A. (1 mark)

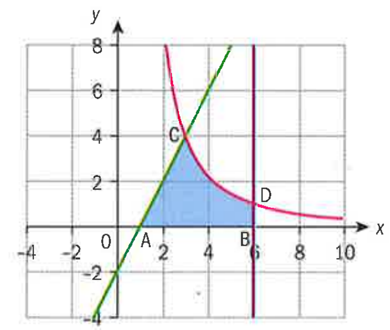
c Find the exact value of A. (5 marks)

24 P2: a By using technology, find the coordinates of the points of intersection of the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. (4 marks)

b On the same axes, sketch the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. (2 marks)

c Find an exact value for the total area bounded by the two curves. (6 marks)

25 P1: The diagram below shows an area bounded by the x-axis, the line $x = 6$, the line $y = 2x - 2$ and the curve $y = \frac{36}{x^2}$.



a Using technology or otherwise, find the coordinates of points A, B, C and D. (4 marks)

b Show that the shaded area is exactly 10 units². You must show all of your working. (6 marks)

26 P2: Consider the area enclosed by the curve $y = 5 - \frac{x^3}{25}$ and the positive x- and y-axis.

a Sketch the curve, shading the area described above. (3 marks)

b Using the trapezium rule with five strips, determine an approximation for the shaded area. (5 marks)

c Explain why your answer to part **b** will be an underestimate. (2 marks)

d Using integration, determine the exact value of the shaded area. (4 marks)

e Find the percentage error of your approximation, compared with the exact value. (2 marks)

Click here for further exam practice

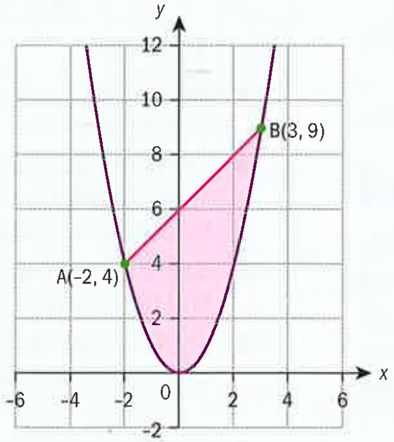
In the footsteps of Archimedes

Approaches to learning: Research, Critical thinking
Exploration criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)
IB topic: Integration, Proof, Coordinate geometry

The area of a parabolic segment

A parabolic segment is a region bounded by a parabola and a line.

Consider this shaded region which is the area bounded by the line $y = x + 6$ and the curve $y = x^2$:



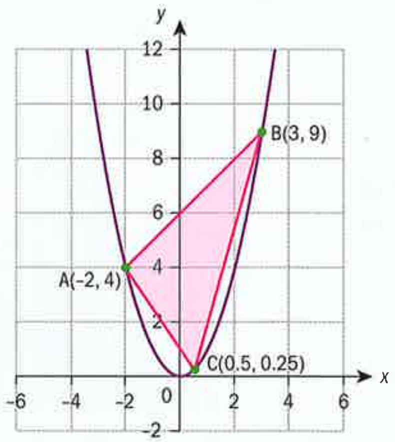
From this chapter you know that you can calculate the shaded area using integration.

On the diagram points A $(-2, 4)$ and B $(3, 9)$ are marked.

Point C is such that the x-value of C is halfway between the x-values of points A and B.

What are the coordinates of point C on the curve?

Triangle ABC is constructed as shown:



Archimedes showed that the area of the parabolic segment is $\frac{4}{3}$ of the area of triangle ABC.

Calculate the area of the triangle shown.

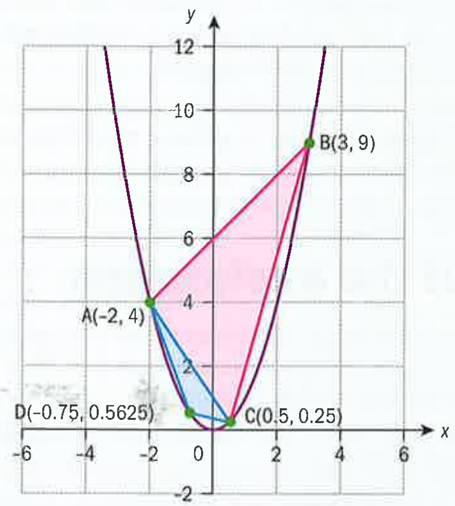
What methods are available to calculate the area of the triangle?

Use integration to calculate the area between the two curves.

Hence verify that Archimedes' result is correct for this parabolic segment.

You can show that this result is true for any parabola and for any starting points A and B on the parabola.

Consider another triangle by choosing point D on the parabola such that its x-value is halfway between the x-values of A and C, similar to before.



What are the coordinates of point D?

Calculate the area of triangle ACD.

Similarly, for line BC, find E such that its x-value is half-way between C and B.

What are the coordinates of point E?

Hence calculate the area of triangle BCE.

Calculate the ratio between the areas of the new triangles and original triangle ABC.

What do you notice?

You can see already that if you add the areas of triangles ABC, ACD and BCE, you have a reasonable approximation for the area of the parabolic segment.

You can improve this approximation by continuing the process and forming four more triangles from sides AD, CD, CE and BE.

If you add the areas of these seven triangles, you have an even better approximation.

How could the approximation be improved?

Generalize the problem

Let the area of the first triangle be X.

What is the total area of the next two, four and eight triangles in terms of X?

If you continued adding the areas of an *infinite* number of such triangles, you would have the *exact* area for the parabolic segment.

By summing the areas of all the triangles, you can show that they form a geometric series.

What is the common ratio?

What is the first term?

What is the sum of the series?

What has this shown?

Extension

This task demonstrates the part of the historical development of the topic of limits which has led to the development of the concept of calculus.

Look at another area of mathematics that you have studied on this course so far.

- What is the history of this particular area of mathematics?
- How does it fit into the development of the whole of mathematics?
- How significant is it?
- Who are the main contributors to this branch of mathematics?

13 Substituting (2, −1) gives (1 mark)

$-1 = 4a + 2b + 3$ (1 mark)

$4a + 2b = -4$

$2a + b = -2$

$\frac{dy}{dx} = 2ax + b$ (1 mark)

$8 = 4a + b$ (1 mark)

Solving simultaneously gives (1 mark)

$a = 5$ (1 mark)

$b = -12$ (1 mark)

14 **a** $y = 20 - x$

$xy = x(20 - x) = 20x - x^2$ (1 mark)

Differentiate and set to zero: (1 mark)

$20 - 2x = 0$ (1 mark)

$x = 10$ (1 mark)

So $(xy)_{MAX} = 100$ (1 mark)

b $x^2 + y^2 = x^2 + (20 - x)^2$ (1 mark)

$= x^2 + x^2 - 40x + 400$

$= 2x^2 - 40x + 400$

Differentiate and set to zero: (1 mark)

$4x - 40 = 0$ (1 mark)

$x = 10$ (1 mark)

So $(x^2 + y^2)_{MAX} = 10^2 + 10^2 = 200$ (1 mark)

c $\left.\frac{dy}{dx}\right|_{x=9.5} = 4\times9.5 - 40 = -2$ (1 mark)

$\left.\frac{dy}{dx}\right|_{x=10.5} = 4\times10.5 - 40 = +2$ (1 mark)

The derivative goes from negative to positive, therefore this is a maximum (1 mark)

Chapter 13

Skills check

1 **a** 66.6 km/h **b** 5 hours

2 **a** 25.5 **b** 9π

3 **a** $3x^2 + \frac{5}{2}$ **b** $8x - 1$

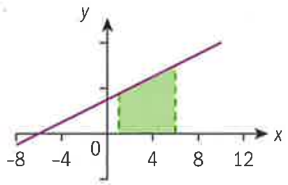
Exercise 13A

1 2250 m

2 **a** 24 m **b** 9 km **c** 13.5 m

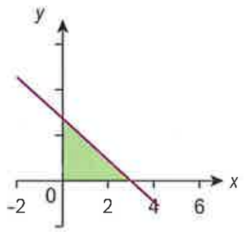
3 **a** 12 **b** 9 **c** 36 **d** 40

4 **a and b**



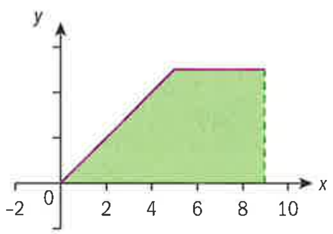
c 23.75

5 **a and b**



c 9

6 **a and b**



c 32.5

Exercise 13B

1 **a** **i** $\int_{-2}^{4} 5\,dx$

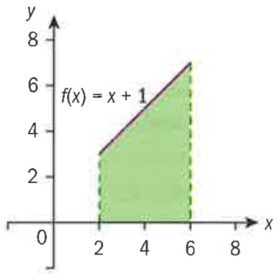
ii $\int_{0.5}^{3.5} (-2x + 8)\,dx$

iii $\int_{1}^{3} (-3x + 10)\,dx$

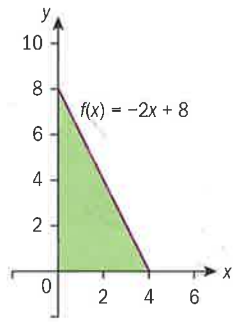
iv $\int_{1}^{3} 2x\,dx$

b **i** 30 **ii** 12

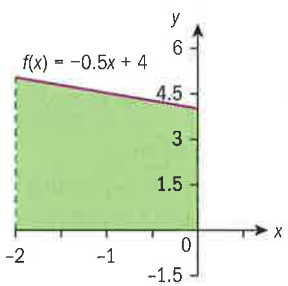
iii 8 **iv** 16



2 **a**



b



Exercise 13C

1 **a** **i** $\int_{-2}^{2} (x+1)^2 + 0.5\,dx$

ii $\frac{34}{3}$

b **i** $\int_{1}^{1.5} x^3\,dx$ **ii** $\frac{65}{64}$

c **i** $\int_{-1}^{3} \sqrt{x+1}\,dx$ **ii** 5.33

d **i** $\int_{-2}^{0} x^3 - 4x\,dx$ **ii** 4

e **i** $\int_{0}^{2} -x^2 + 2x\,dx$ **ii** $\frac{4}{3}$

f **i** $\int_{-1}^{3} -(x+1)(x+3)\,dx$

ii 16

2 **a** **i** $\int_{2}^{4} x^2\,dx$ **ii** $\frac{56}{3}$

b **i** $\int_{-1}^{1} 2^x\,dx$ **ii** 2.16

c **i** $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$ **ii** 1.57

d **i** $\int_{0.5}^{3} \frac{1}{x}\,dx$ **ii** 0.79

e **i** $\int_{0}^{1} -(x-3)(x+2)\,dx$

ii $\frac{37}{6}$

f **i** $\int_{-2}^{0} -(x-3)(x+2)\,dx$ or

$\int_{0}^{3} -(x-3)(x+2)\,dx$

ii $\frac{22}{3}$ or 13.5

g **i** $\int_{-2}^{3} -(x-3)(x+2)\,dx$

ii $\frac{125}{6}$

h **i** $\int_{-2}^{4.5} -x^2 + 2x + 15\,dx$

ii 80.7083

i **i** $\int_{-\sqrt{17}}^{\sqrt{17}} -x^2 + 2x + 15\,dx$

ii $\frac{68\sqrt{17}}{3}$

j **i** $\int_{-1}^{\ln(3)} 3 - e^x\,dx$

ii $\ln 27 + \frac{1}{e}$

k **i** $\int_{-2-5^{\frac{1}{3}}}^{0} (x+2)^3 + 5\,dx$

ii $14 + \frac{15\sqrt[3]{5}}{4}$

3 **a** −2, 2 **b** (0, 4)

c 20

d **i** $\int_{0}^{2} -x^2 + 4\,dx$

ii $\frac{16}{3}$

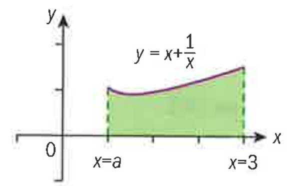
e $\frac{76}{3}$

Exercise 13D

1 3

2 −0.8169

3 **a**



b 0.5693

4 **a** −1

b Area between curve $y = x^2$ and the x-axis between $x = -2$ and $x = -1$

5 **a** $b = 1$

b Area between curve $y = (1 + x^3)$ and the x-axis, between $x = -1$ and $x = 1$

6 **a** $t = 3$

b Area between curve $y = \sqrt{x+1}$ and the x-axis, between $x = -1$ and $x = 3$

Exercise 13E

1 55

2 18

3 28.75

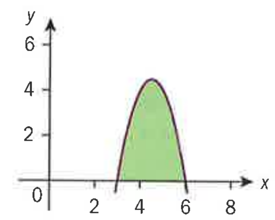
4 9.85

5 11.6 (3 s.f.)

6 **a** 5.198 **b** 23.74

c 12.21 **d** 35

7 **a**



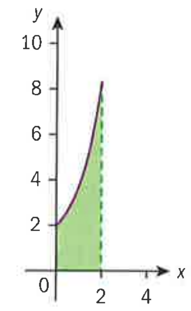
b **i** $\int_{3}^{6} -2(x-3)(x-6)\,dx$

ii 9

c 8.75

d 2.78% (3 s.f.)

8 **a**



b **i** $\int_{0}^{2} 1 + e^x\,dx$

ii 8.389

c 8.474

d 1.01% (3 s.f.)

Exercise 13F

1 **a** **i** If $F(x) = x$, $F'(x) = 1 = f(x)$, so $F(x) = x$ is an antiderivative of $f(x) = 1$

ii If $F(x) = x + 3$, $F'(x) = 1 = f(x)$, so $F(x) = x + 3$ is an antiderivative of $f(x) = 1$

iii If $F(x) = x - 6$, $F'(x) = 1 = f(x)$, so $F(x) = x - 6$ is an antiderivative of $f(x) = 1$

b For example, $F_1(x) = x - 10$, $F_2(x) = x + 20$

c $F(x) = x + c$, where c is any number.

2 **a** **i** If $F(x) = x^2$, $F'(x) = 2x = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

ii If $F(x) = x^2 + 1$, $F'(x) = 2x = f(x)$, so $F(x) = x^2 + 1$ is an antiderivative of $f(x) = 2x$

iii If $F(x) = x^2 - 4$, $F'(x) = 2x = f(x)$, so $F(x) = x^2 - 4$ is an antiderivative of $f(x) = 2x$

b For example, $F_1(x) = x^2 - 1$, $F_2(x) = x^2 + 2$

c $F(x) = x^2 + c$, where c is any number.

3 **a** **i** If $F(x)=\frac{x^2}{2}$, $F'(x) = x = f(x)$, so $F(x)=\frac{x^2}{2}$ is an antiderivative of $f(x) = x$

ii If $F(x)=\frac{x^2}{2}+2.5$, $F'(x) = x = f(x)$, so $F(x)=\frac{x^2}{2}+2.5$ is an antiderivative of $f(x) = x$

iii If $F(x)=\frac{x^2}{2}-12$, $F'(x) = x = f(x)$, so $F(x)=\frac{x^2}{2}-12$ is an antiderivative of $f(x) = x$

b $F(x)=\frac{x^2}{2}+c$, where c is any number.

4 **a** **i** If $F(x) = x^2 + x$, $F'(x) = 2x + 1 = f(x)$, so $F(x) = x^2 + x$ is an antiderivative of $f(x) = 2x + 1$

ii If $F(x) = x^2 + x - 3.2$, $F'(x) = 2x + 1 = f(x)$, so $F(x) = x^2 + x - 3.2$ is an antiderivative of $f(x) = 2x + 1$

iii If $F(x) = x^2 + x + 4$, $F'(x) = 2x + 1 = f(x)$, so $F(x) = x^2 + x + 4$ is an antiderivative of $f(x) = 2x + 1$

b $F(x) = x^2 + x + c$, where c is any number.

5 For example, $g(x) = 3x$

6 For example, $g(x) = -2x$

7 For example, $g(x)=\frac{x^2}{4}$

8 For example, $F(x)=\frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$

9 **a** $A(x) = 1.25x^2$

b $A(x) = -0.25x^2 + 4x$

c $A(x) = 1.25x^2$

d $A(x)=\frac{x^3}{3}+x$

10 4^x

11 $x(x + 1)$

12 $2(x + 2)(x - 2)$

Exercise 13G

1 $f(x) = 2x + c$, where c is any real number

2 $f(x)=\frac{x^2}{2}+x+c$, where c is any real number

3 $f(x) = -x + c$, where c is any real number

4 $f(x) = x^2 + c$, where c is any real number

5 **a** $x + c,\ c \in \mathbb{R}$

b $6x + c,\ c \in \mathbb{R}$

c $\frac{x}{2}+c, c \in \mathbb{R}$

6 **a** $\frac{x^2}{2}+c, c \in \mathbb{R}$

b $x^2 + c,\ c \in \mathbb{R}$

c $\frac{5x^2}{2}+c, c \in \mathbb{R}$

d $\frac{x^2}{4}+c, c \in \mathbb{R}$

e $\frac{ax^2}{2}+c, c \in \mathbb{R}$

7 **a** $\frac{x^3}{3}+c, c \in \mathbb{R}$

b $x^3 + c,\ c \in \mathbb{R}$

c $\frac{4x^3}{3}+c, c \in \mathbb{R}$

d $\frac{x^3}{6}+c, c \in \mathbb{R}$

e $\frac{x^3}{9}+c, c \in \mathbb{R}$

f $\frac{x^3}{2}+c, c \in \mathbb{R}$

g $\frac{ax^3}{3}+c, c \in \mathbb{R}$

8 **a** $\frac{x^4}{4}+c, c \in \mathbb{R}$

b $x^4 + c,\ c \in \mathbb{R}$

c $\frac{x^4}{2}+c, c \in \mathbb{R}$

d $\frac{x^4}{5}+c, c \in \mathbb{R}$

e $\frac{x^4}{12}+c, c \in \mathbb{R}$

f $-\frac{x^4}{4}+c, c \in \mathbb{R}$

g $\frac{ax^4}{4}+c, c \in \mathbb{R}$

Exercise 13H

1 **a** $10x + c,\ c \in \mathbb{R}$

b $0.2x^3 + c,\ c \in \mathbb{R}$

c $\frac{x^6}{6}+c, c \in \mathbb{R}$

d $\int 7\,dx - \int 2x\,dx = 7x + c_1 - x^2 - c_2 = 7x - x^2 + c\ (c = c_1 - c_2 \in \mathbb{R})$

e $\int 1\,dx + \int 2x\,dx = x + c_1 + x^2 + c_2 = x + x^2 + c\ (c = c_1 + c_2 \in \mathbb{R})$

f $\int 5dx+\int x\,dx+\int -\frac{x^2}{3}\,dx=5x+c_1+\frac{x^2}{2}+c_2-\frac{x^3}{6}+c_3=5x+\frac{x^2}{2}-\frac{x^3}{6}+c$ $(c= c_1+c_2 \in \mathbb{R})$

g $\int(-x+\frac{3x^2}{4}+0.5)\,dx = \int -x\,dx+\int\frac{3x^2}{4}\,dx+\int 0.5\,dx = -\frac{x^2}{2}+\frac{x^3}{4}+0.5x+c$

h $\int\left(1-x+\frac{x^3}{2}\right)dx=\int 1\,dx - \int x\,dx+\int\frac{x^3}{2}dx=x+c_1-\frac{x^2}{2}+c_2+\frac{x^4}{8}+c_3=x-\frac{x^2}{2}+\frac{x^4}{8}+c\ (c=c_1+c_2+c_3 \in \mathbb{R})$

i $\int(x^2-\frac{x}{2}+4)\,dx = \int x^2\,dx-\int\frac{x}{2}\,dx+\int 4\,dx = \frac{x^3}{3}+c_1-\frac{x^2}{4}+c_2+4x+c_3=\frac{x^3}{3}-\frac{x^2}{4}+4x+c$ $(c=c_1+c_2+c_3)$

j $-\frac{1}{2}x^{-1}+c$

k $-\frac{2}{3}x^{-3}+c$

l $2x^2-\frac{5}{4}x^{-4}+c$

2 **a** $2x-\frac{1}{3}$

b $\frac{x^3}{3}-\frac{x^2}{6}+4x+c$

3 $\frac{t^2}{2}-t^3+c$

4 $t^4-\frac{3t^2}{2}+t+c$

5 Any function F with gradient $3+x-\frac{x^2}{4}$ of the form $F(x)=\int\left(3+x-\frac{x^2}{4}\right)dx=3x+\frac{x^2}{2}-\frac{x^3}{12}+c\,(c \in \mathbb{R})$

6 Any function F with gradient $-x + 0.5x^2$ of the form $F(x)=\int(-x+0.5x^2)\,dx=-\frac{x^2}{2}+\frac{x^3}{6}+c\,(c\in\mathbb{R})$

7 **a** $x^2 - x + 2$

b Any function $y = f(x)$ with $\frac{dy}{dx}=(x+1)(x-2)$ of the form $y=f(x)=\int(x^2-x+2)\,dx=\frac{x^3}{3}-\frac{x^2}{2}+2x+c$, where c is any real number

8 $\frac{x^4}{4}+\frac{2}{x}+x+c$

Exercise 13I

1 $f(x)=\frac{3x^2}{2}+\frac{4x^3}{3}-\frac{1}{6}$

2 $y=\frac{x^2}{2}+\frac{x^3}{15}+2x-\frac{259}{15}$

3 $f(x)=3x-\frac{x^2}{2}-3$

4 $f(x)=x^3-\frac{x^5}{15}+2$

5 $y(0.5) = -2.45$

6 **a** $f(x)=-\frac{x^2}{2}+3x+8$

b $f(2)=-\frac{2^2}{2}+3\times 2+8=12$

7 $C(x)=\frac{x^2}{2}+\frac{2}{x}+x+145$

8 $C(x)=3x-2x^2+\frac{x^3}{3}+1000$

9 **a** $R(t)=\frac{t^3}{3}-80t+600$

b USD 1 113 600

Chapter review

1 **a** **i** $\int_{-1}^{4} 0.5x+4\,dx$

ii 23.75

b **i** $\int_{0}^{3}(-3x+9)\,dx$

ii 13.5

2 **a** **i** $\int_{-2}^{3}(4-x)\,dx$ **ii** 17.5

b **i** $\int_{0}^{4}\sqrt{16-x^2}\,dx$

ii 12.6 (3 s.f.)

3 **a** **i** $\int_{-4}^{0}x^2\,dx$ **ii** $\frac{64}{3}$

b **i** $\int_{-1}^{4}((x-1)^2+3)\,dx$

ii $\frac{80}{3}$

c **i** $\int_{0}^{2}(-(x-4)(x+1))\,dx$

ii $\frac{34}{3}$

d **i** $\int_{-1}^{1.5}(8-x^3)\,dx$

ii 14.75 (4 s.f.)

e **i** $\int_{-2}^{2}(x+3)(x-2)^2\,dx$

ii 42.7 (3 s.f.)

f **i** $\int_{0}^{4}(x+3)(x-2)^2\,dx$

ii 26.7 (3 s.f.)

g **i** $\int_{-1}^{2}\frac{3}{1+x^2}\,dx$

ii 5.68 (3 s.f.)

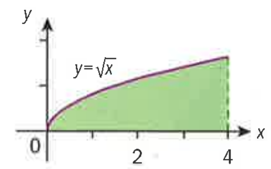
h **i** $\int_{-4}^{3}\left(1+\sqrt{3-x}\right)dx$

ii 19.3 (3 s.f.)

i **i** $\int_{-2}^{2}e^{-x}\,dx$

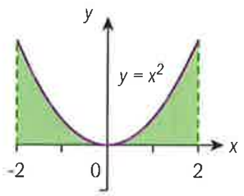
ii 7.25 (3 s.f.)

4 **a** **i**



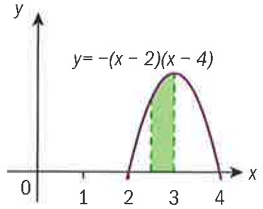
ii 5.33

b



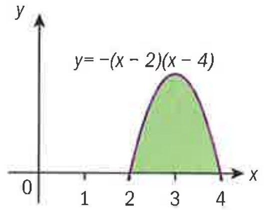
ii $\frac{16}{3}$

c i



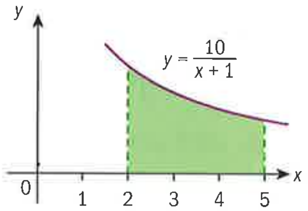
ii 0.458 (3 s.f.)

d i



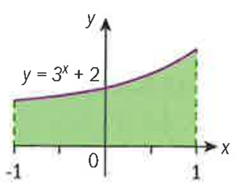
ii 1.33 (3 s.f.)

e i



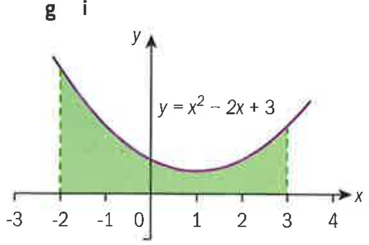
ii 6.93 (3 s.f.)

f i



ii 6.43 (3 s.f.)

g i



ii $\frac{65}{3}$

5 a $\int_{-1}^{4} -(x+1)(x-4)\,dx$

b 20.8 (3 s.f.)

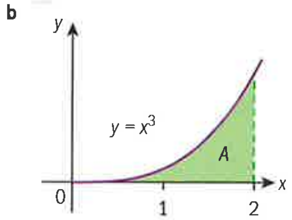
6 a $x = 0, 4$

b $\int_{0}^{4} x(x-4)^2\,dx$

c 21.3 (3 s.f.)

7 a 0

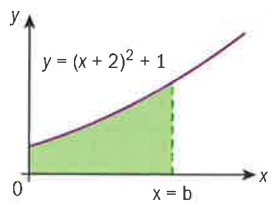
b



c $\int_{0}^{2} x^3\,dx$

d 4

8 a

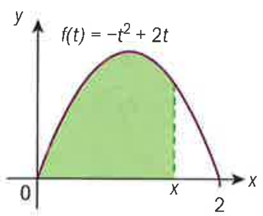


b 3

9 72

10 8.27 (3 s.f.)

11 a



b $\int_{0}^{2} (-t^2+2t)\,dt - \int_{2}^{x} (-t^2+2t)\,dt$

12 $2(x^2 - 9)$

13 $2x + \frac{x^2}{2} + c$, where $c \in \mathbb{R}$ is a constant of integration

14 $x + \frac{x^2}{2} - \frac{x^4}{16} + c$, where $c \in \mathbb{R}$ is a constant of integration

15 a 6.25

b $\int_{1}^{6} (x-4)^2 + 1\,dx$

c 22.9 (3 s.f.)

16 a Green line: $y = f_1(x) = 3x + 1$, purple line: $y = 3x$, pink line: $y = 3x - 2$

b $f_i(x)$ are anti-derivatives of $y = t(x)$ and hence $t(x) = f_i'(x) = 3$

c $3x + c$, where c is an arbitrary constant of integration

17 10.4 (3 s.f.)

18 a 0.744 (3 s.f.)

b i $\int_{0}^{1} e^{-x^2}\,dx$

ii 0.747 (3 s.f.)

c 0.329% (3 s.f.)

Exam-style questions

19 $5x - \frac{12x^3}{3} + \frac{4x^4}{4}(+c)$ (3 marks)

$= 5x - 4x^3 + x^4 + c$ (1 mark)

20 $\int\left(\frac{3}{2}x^2 + x + 3\right)$ (3 marks)

$dx = \frac{x^3}{2} + \frac{x^2}{2} + 3x + c$

Substituting $x = -1$ and equating to $\frac{13}{2}$ (1 mark)

$-\frac{1}{2} + \frac{1}{2} - \frac{3}{2} + c = \frac{13}{2}$ (1 mark)

$c = 8$ (1 mark)

$f(x) = \frac{x^3}{2} + \frac{x^2}{2} + 3x + 8$

21 a $6x - x^2 = 10 - x$ (1 mark)

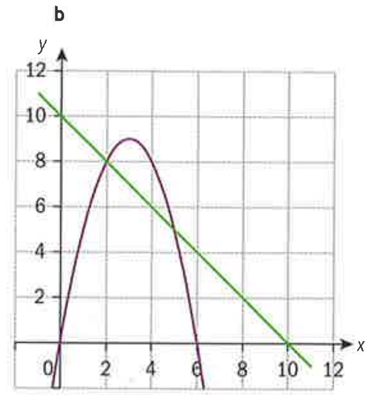
$x^2 - 7x + 10 = 0$

$(x - 2)(x - 5) = 0$ (1 mark)

Coordinates are (2,8) (1 mark)

and (5,5) (1 mark)

b



c Area under curve

$= \int_{2}^{5} 6x - x^2\,dx$ (1 mark)

$= \left[3x^2 - \frac{x^3}{3}\right]_2^5$ (1 mark)

$= \frac{100}{3} - \frac{28}{3} = \frac{72}{3}$ (1 mark)

$= 24$ (1 mark)

Area under line

$= \frac{1}{2} \times (5+8) \times 3 = \frac{39}{2}$ (1 mark)

Required area $= 24 - \frac{39}{2}$ (1 mark)

$= \frac{9}{2}$ units² (1 mark)

[Note: Alternatively you can find $\int_{2}^{5} (6x - x^2) - (10 - x)\,dx$ to give the same answer]

22 Displacement $= \int 0.6t^2 + 4t + 1\,dt$ (1 mark)

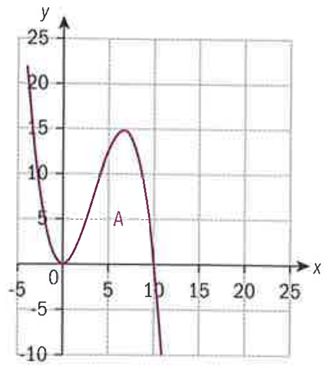
$= 0.2t^3 + 2t^2 + t + k$ (1 mark)

Distance travelled in 3rd second $= \int_{2}^{3} (0.6t^2 + 4t + 1)\,dt$ (1 mark)

$= [0.2 \times 3^3 + 2 \times 3^2 + 3] - [0.2 \times 2^3 + 2 \times 2^2 + 2]$ (1 mark)

$= 14.8$ m (1 mark)

23 a



(3 marks)

b $\int_{0}^{10} -\frac{x^2}{10}(x - 10)\,dx$ (1 mark)

c $-\frac{x^2}{10}(x - 10) = x^2 - \frac{x^3}{10}$

$\int_{0}^{10} \left(x^2 - \frac{x^3}{10}\right) dx$ (1 mark)

$= \left[\frac{x^3}{3} - \frac{x^4}{40}\right]_0^{10}$ (2 marks)

$= \frac{1000}{3} - \frac{10\,000}{40}$ (1 mark)

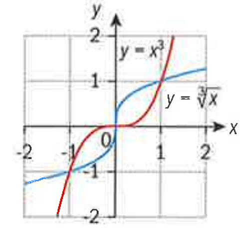
$= \frac{1000}{3} - \frac{1000}{4}$

$= \frac{1000}{12}\left(= \frac{250}{3}\right)$ units² (1 mark)

24 a Use of GDC (1 mark)

$(-1, -1), (0, 0), (1, 1)$ (3 marks)

b



(2 marks)

c $\int_{0}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$ (2 marks)

$\int_{0}^{1} x^{\frac{1}{3}}\,dx = \left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^1 = \frac{3}{4}$ (1 mark)

Positive area is therefore

$\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ (1 mark)

Therefore total area is

$2 \times \frac{1}{2} = 1$ unit² (2 marks)

25 a $A(1,0)$ (1 mark)

$B(6,0)$ (1 mark)

$C(3,4)$ (1 mark)

$D(6, 1)$ (1 mark)

b Area $= \frac{1}{2} \times 2 \times 4 + \int_{3}^{6} \frac{36}{x^2}\,dx$ (3 marks)

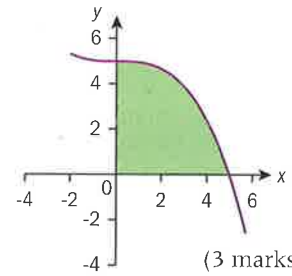
$= 4 + 36\int_{3}^{6} x^{-2}\,dx$

$= 4 + 36\left[-\frac{1}{x}\right]_3^6$ (1 mark)

$= 4 + 36\left[-\frac{1}{6} - \left(-\frac{1}{3}\right)\right]$ (1 mark)

$= 4 + 36 \times \frac{1}{6} = 10$ units² (1 mark)

26 a



(3 marks)

b

x	0	1	2	3	4	5
y	5	4.96	4.68	3.92	2.44	0

(2 marks)

Area

$= \int_{0}^{5} \left(5 - \frac{x^3}{25}\right) dx \approx \frac{1}{2}\left(\frac{5-0}{5}\right)$

$\left[5 + 0 + 2\left(\begin{matrix}4.96 + 4.68\\ +3.92 + 2.44\end{matrix}\right)\right]$ (2 marks)

$= 18.5$ units² (1 mark)

c The curve is concave down in the interval $0 \le x \le 5$, so each trapezium will be an underestimate. (1 mark)

Therefore the sum of the trapezia will also be an underestimate. (1 mark)

d $\int_0^5 \left(5-\frac{x^3}{25}\right)dx = \left[5x-\frac{x^4}{100}\right]_0^5$ (2 marks)

$=25-\frac{625}{100}=18.75 \text{ units}^2$ (2 marks)

e Percentage error

$=\frac{18.75-18.5}{18.5}\times100=1.35\%$ (2 marks)

Paper 1

Exam-style questions

1 a $P\left(1+\frac{6}{100\times12}\right)^{12\times10}$

$=1\,500\,000$ (1 mark)

$P = 824449.10$ (1 mark)

Ans: COP 824449.10

b $V_0 = 1500000 - 824449.10$
$= 675550.90$ (1 mark)

$V_{10} = 675550.90 \times (0.9)^{10}$
$= 235550.03$ (2 marks)

2 a $d = 40$ (metres) (1 mark)

b EITHER

$200 + 40 \times 13 = 720$ (2 marks)

OR

$360 + 40 \times 9 = 720$ (2 marks)

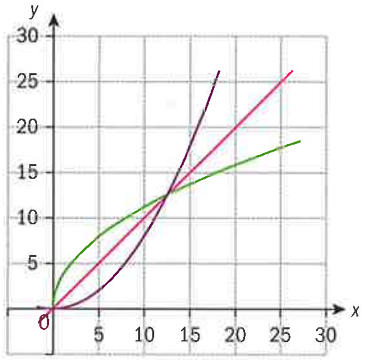
c $S_{21}=\frac{2\times200+40\times20}{2}\times21$ (2 marks)

$= 12\,600$ (1 mark)

3 a i For each given area A, there is only one possible value for the radius, which is $\sqrt{\frac{A}{\pi}}$. Therefore, since $C = 2\pi r$, C is also unique for that value of A. (1 mark)

ii For each given circumference C, there is only one possible value for the radius, which is $\frac{C}{2\pi}$. Therefore, $A = \pi r^2$, A is unique for that value of C, so C^{-1} exists. (1 mark)

b



(2 marks)

c $A = C^{-1}(25) \Rightarrow C(A) = 25 \Rightarrow$
$A = 49.7$ (2 marks)

4 a i $T(0) = 86 \Rightarrow 22 + a = 86$
$\Rightarrow a = 64$ (2 marks)

ii $T(0.5) = 28 \Rightarrow 22 + 64 \times 2^{0.5b} = 28$ (1 mark)

$b = -6.83$ (1 mark)

b $T = 22$ (1 mark)

c The temperature of the hot chocolate approaches 22 °C as t gets very large, indicating that the temperature of the room is 22°C. (1 mark)

5 a Discrete (1 mark)

Number of walks can be counted (1 mark)

b i 3 (1 mark)

ii 2.68 (3 s.f.) (2 marks)

6 a $\frac{1}{9}+\frac{1}{6}+\frac{1}{12}+\frac{2}{9}+\frac{1}{6}+p=1$

(2 marks)

$p=\frac{1}{4}$ (2 marks)

b $\frac{1}{9}\times2+\frac{1}{6}\times3+\frac{1}{12}\times5+\frac{2}{9}\times1+\frac{1}{6}\times3+\frac{1}{4}\times4=\frac{103}{36}$

(2.86, 3 sf) (2 marks)

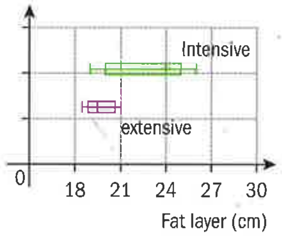
7 a $\frac{8}{14}\left(=\frac{4}{7}\right)$ or 0.571 (3 s.f.)

(1 mark)

b $\binom{5}{2}\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)^3=0.257$

(3 s.f.) (3 marks)

8 a



Intensive

Scale and labelled axis (1 mark)

Correct diagrams (2 marks)

extensive

b The average layer of fat was thicker on the cows from the Intensive programme. (1 mark)

The interquartile range and range of fat layer sizes was also much greater for cows from the Intensive programme. (2 marks)

9 a $L = 10\log_{10}(10^{-6}\times10^{12})$
$= 60\text{dB}$ (2 marks)

Yes as the value is between 0 and 140. (1 mark)

b $10\log_{10}(S\times10^{12}) = 80$ (1 mark)

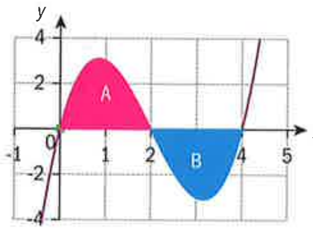
$S = 0.0001 \text{ Wm}^{-2}$ (1 mark)

10 a H_0:Favourite colour is independent of gender. (1 mark)

b 0.0276 (1 mark)

c As 0.0276 < 0.05 the null hypothesis is rejected at 5% level of significance. (2 marks)

11 a Labelled axes (1 mark)
Each region (2 marks)



b Use of definite integration (1 mark)

i $\text{area}(A)=\int_0^2 f(x)dx=4$ (1 mark)

ii $\text{area}(B)=\int_2^4 |f(x)|dx=4$ (1 mark)

iii area(A) + area(B) = 8 (1 mark)

12 a i $H_0: \mu_1 = \mu_2$ (1 mark)

ii $H_1: \mu_1 \neq \mu_2$ (1 mark)

b H_0: $p = 0.209$ (3 sf) (2 marks)

c As $0.209 > 0.1$ there is no evidence to reject the null hypothesis at the 10% level of significance. (2 marks)

13 a $AE=\sqrt{(2-4)^2+(9-6)^2}$

$= 3.61$ (3 s.f.) (2 marks)

b Attempt to find perpendicular bisector of BE. (1 mark)

One technique is shown here:

$PB = PE \Rightarrow \sqrt{(x-2)^2+(y-3)^2}$
$=\sqrt{(x-4)^2+(y-6)^2}$ (1 mark)

Attempt to expand (1 mark)

$x^2 - 4x + 4 + y^2 - 6y + 9 =$
$x^2 - 8x + 16 + y^2 - 12y + 36$ (1 mark)

$4x + 6y - 39 = 0$ (1 mark)

c The cell corresponds to the region of the park that has E as the closest well. (1 mark)

14 a $AD^2 = 100^2 - 65^2 = 5775$ or $AD = 76.0$ 3 s.f. (1 mark)

Area

$=\frac{\pi}{2}\left((50+AD)^2-AD^2\right)$

$= 15\,864 \text{ m}^2$ (3 marks)

b $C\hat{A}D$

$=\sin^{-1}\left(\frac{65}{100}\right)=40.54...$"

$= 40.5°$ (2 marks)

c $B\hat{A}C = 180 - C\hat{A}D = 139.46$

Cosine rule

$\Rightarrow (BC)^2 = (AB)^2 + (AC)^2$

$-2AB\times AC\times\cos B\hat{A}C$ (1 mark)

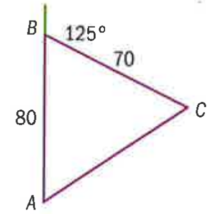
$BC=\sqrt{50^2+100^2-2\times50\times100\cos(139.46...)}$ (1 mark)

$BC = 142$ m (3 s.f.) (1 mark)

Paper 2

Exam-style questions

1 a



(2 marks)

b $A\hat{B}C = 180 - 125 = 55$ (1 mark)

$AC^2 = 80^2 + 70^2 - 2 \times 70 \times 80 \times \cos 55$ (1 mark)

$AC = 69.8279... = 69.8\text{m}$ (3 s.f.) (1 mark)

c Area $=\frac{1}{2}\times80\times70\times\sin55$ (1 mark)

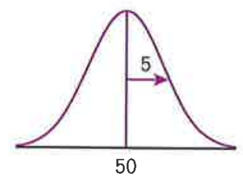
$= 2293.62...$
$= 2.29 \times 10^3 \text{m}$ (3 s.f.) (1 mark)

d $\frac{69.8279}{\sin55}=\frac{80}{\sin C}\Rightarrow C=69.8$ (2 marks)

Bearing is 360 − 55 − 69.8 (2 marks)

$= 235°$ (3 s.f.) (1 mark)

2 a



b $P(45 < t < 55) = 0.683$ (3 s.f.) (2 marks)

c $P(t < 40) = 0.0228$ (3 s.f.) (2 marks)

d $P(t < M) = 0.75 \Rightarrow$
$M = 53.4$ (3 s.f.) mins (2 marks)

e $P(t<40|\text{gained medal})$

$=\frac{P(t<40\cap\text{gained medal})}{P(\text{gained medal})}$

$=\frac{0.02275...}{0.75}=0.303$ (3 s.f.) (3 marks)

f $10\,000 \times P(t < 33) = 3.37...$ so 3 competitors (2 marks)

3 a 11 m (1 mark)

b $\frac{360}{30}=12$ hours (2 marks)

c since $-1 \le \sin \le +1$

i 16 m at 03:00 and 15:00 (2 marks)

ii 6 m at 09:00 and 21:00 (2 marks)