1 Number and algebra 1

CHAPTER OBJECTIVES:

1.1 Natural numbers, $\mathbb{N}$; integers, $\mathbb{Z}$; rational numbers, $\mathbb{Q}$; real numbers, $\mathbb{R}$

1.2 Approximation: decimal places, significant figures; estimation; percentage errors

1.3 Expressing numbers in standard form; operations with numbers in standard form

1.4 SI and other basic units of measurement

Before you start

You should know how to:

1 Substitute into formulae, e.g.
G and F are linked through the formula $G=\dfrac{F-1}{\sqrt{F+2}}$. Find the value of G when $F = 98$. $G=\dfrac{98-1}{\sqrt{98+2}}=9.7$

2 Solve simple equations in one variable, e.g.

a $2x - 8 = 10$
$2x = 18$
$x = 9$

b $x^2 = 25$
$x = 5$ or $x = -5$

3 Calculate percentages, e.g.
Calculate 5% of 240. $\dfrac{5}{100}\times 240 = 12$

4 Solve inequalities and represent the solution on the number line, e.g.
$2x + 7 \le 10$
$2x \le 3$
$x \le 1.5$

(Number line marked −1, 0, 1.5, 2, arrow pointing left from 1.5)

5 Calculate the absolute value of a number, e.g. $|2.5| = 2.5$, $|-1.3| = 1.3$, $|0| = 0$, $|5 - 10| = 5$.

Skills check

1 Find the value of y when $x = -0.1$ if x and y are linked through the formula

a $y = 3x^2(x - 1)$ **b** $y=\dfrac{(x-1)^2}{x}$

c $y = (1 - x)(2x + 1)$.

2 Solve for x.

a $3x - 7 = 14$ **b** $2(x - 6) = 4$

c $\frac{1}{2}(1-x)=0$ **d** $x^2 = 16$

3 Calculate

a 8% of 1200 **b** 0.1% of 234.

4 Solve the following inequalities. Represent their solutions on the number line.

a $10 - x \le 1$ **b** $3x - 6 > 12$

c $2x \le 0$

5 Calculate

a $|-5|$ **b** $\left|\frac{1}{2}\right|$

c $|5 - 7|$ **d** $\left|\dfrac{12-8}{8}\right|\times 100$

▲ This is the biggest snow castle in the world, in northern Finland. First built in 1996, it has been rebuilt every winter when there has been enough snow.

- The castle is 100 km south of the Arctic Circle.
- It takes approximately 6 weeks to build.
- The temperature has to be no more than −8 °C to prevent it melting.
- The castle's area varies each year. So far it has ranged from 13 000 to 20 000 m².
- Approximately 300 000 people from around the world visited the castle when it was first open.
- The castles have had towers taller than 20 m and walls longer than 1000 m.

These facts and figures about the snow castle use different types of number and different types of unit. Some are approximate values.

This chapter will help you to classify numbers, round numbers and make approximations, as well as showing you how to write very large or very small numbers in standard form, and convert between different units of measurement.

1.1 The number sets

These expressions use several different types of number.

- In Finland the lowest temperature in winter is around −45 °C.
- In 2010 unemployment in Ireland was more than 13%.
- Approximately $\frac{4}{5}$ of the world's population has a mobile or cell phone.
- Usain Bolt won the men's 100 metres at the 2008 Olympic Games with a world record time of 9.69 seconds.
- The area of a circle with a radius of 1 cm is π cm².

The numbers 60, −45, $\frac{1}{3}$, 9.69 and π belong to different **number sets**, which are described over the next few pages.

At the end of this section you will be able to classify them as elements of these sets.

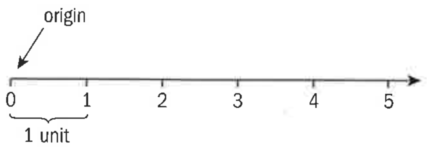
Natural numbers, ℕ

→ The set of **natural numbers** ℕ is 0, 1, 2, 3, 4, ...

We use these numbers

- *to count*: for example '205 nations are expected to take part in the 2012 Olympic Games'
- *to order*: for example 'The Congo rainforest is the 2nd largest in the world'

You can represent the natural numbers on the number line by setting an **origin** and a **unit**.



We write ℕ = {0, 1, 2, 3, 4, 5, ...}
The curly brackets enclose the elements of a set.

Example 1

a Find the value of these expressions when $a = 5$ and $b = 7$.
 i $a + b$ **ii** $a \times b$ **iii** $a - b$ **iv** $b - a$

b State whether your answers to part **a** are natural numbers or not.

Answers

a **i** $5 + 7 = 12$ **ii** $5 \times 7 = 35$ **iii** $5 - 7 = -2$ **iv** $7 - 5 = 2$

b **i** natural **ii** natural **iii** not natural **iv** natural

There are as many natural numbers as even numbers.

Remember that the negative numbers are not in ℕ.

Exercise 1A

a Find the value of these expressions when $a = 2$ and $b = 4$.
 i $2a + b$ **ii** $2(a + b)$ **iii** $a^2 - b^2$ **iv** $(a - b)^2$

b State whether your answers to part **a** are natural numbers or not.

Investigation – natural numbers

State whether each statement is true or false. If it is false, give an example to show why.

a True or false? *Whenever you add two natural numbers the* ***sum*** *will be a natural number.*

If $a + b = c$, we say that c is the sum of a and b.

b True or false? *Whenever you multiply two natural numbers, the* ***product*** *will be a natural number.*

If $a \times b = c$, we say that c is the product of a and b.

c True or false? *Whenever you subtract two natural numbers the* ***difference*** *will be a natural number.*

If $a - b = c$, we say that c is the difference of a and b.

The set of integers, ℤ

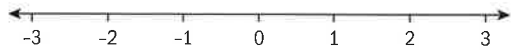
In Example 1 you saw that the difference of two natural numbers is *not always* a natural number. So we need a new set as there are quantities that cannot be represented with natural numbers. The new set is ℤ, the set of integers.

ℤ is an extension of ℕ.

→ The set of **integers** ℤ is {..., −4, −3, −2, −1, 0, 1, 2, 3, 4, ...}

Any natural number is also an integer but not all integers are natural numbers.

You can represent ℤ on the number line like this:



On this number line
- positive integers are placed to the right of zero
- negative integers are placed to the left of zero.
- Zero is neither positive nor negative.

Example 2

Solve each equation for x. State whether the solution to the equation is an integer or not.

a $x + 5 = 11$ **b** $-3x = 10$

Answers

a $x + 5 = 11$
 $x = 6$ x is an integer.

b $-3x = 10$
 $x = \frac{-10}{3}$ x is not an integer.

We use negative numbers to represent many everyday situations.
List at least three.

Example 3

a Find the value of the following expressions when $j = 4$ and $k = -2$.
 i $\frac{5k - j}{k + j}$ **ii** $\frac{j^2 - k}{j^2 + 2k}$

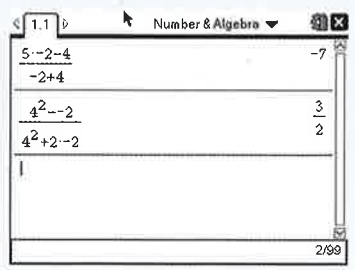
b State whether your answers to part **a** are integers or not.

Answers

a **i** $\frac{5(-2) - 4}{-2 + 4} = \frac{-14}{2} = -7$

Write the expressions, substituting the numbers for the letters.

ii $\frac{4^2 - (-2)}{4^2 + 2(-2)} = 1.5$



You can use your GDC to evaluate this.

When using your GDC to input fractional expressions, remember to use brackets to indicate clearly the numerator and the denominator, or use the fraction template.

b **i** integer **ii** not an integer

Brahmagupta lived from 589 to 669 CE in India. He is credited with writing the first book that included zero and negative numbers.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 1B

1 **a** Solve the equation $4x + 2 = 0$.
  b State whether or not your solution to part **a** is an integer.

2 **a** Solve the equation $x^2 = 4$.
  b State whether or not your solutions to part **a** are integers.

3 **a** Find the value of these expressions when $a = -2$ and $b = 4$.
  i $\frac{a-b}{a+b}$ **ii** $3a^2 - \frac{9}{b}$
  b State whether or not your answers to part **a** are integers.

Investigation – integers

State whether each of these statements is true or false. If false, give an example to show why.

a *The* ***sum*** *of two integers is always an integer.*
b *The* ***difference*** *of two integers is always an integer.*
c *The* ***quotient*** *of two integers is always an integer.*
d *The* ***product*** *of two integers is always an integer.*

If $\frac{a}{b} = c$ then we say that c is the **quotient** of a and b. *Quotient* means **ratio.**

The set of rational numbers, $\mathbb{Q}$

In the investigation you should have found that the quotient of two integers is not always an integer. So we need a new set as there are quantities that cannot be represented with integers. This set is $\mathbb{Q}$, the set of rational numbers.

$\mathbb{Q}$ is an extension of $\mathbb{Z}$.

→ The set of **rational numbers** $\mathbb{Q}$ is
$\left\{\frac{p}{q}\text{ where } p \text{ and } q \text{ are integers and } q \neq 0\right\}$

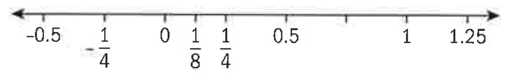
Note that $q \neq 0$ as division by 0 is not defined.

This definition means that a number is rational if it can be written as the quotient of two integers. Here are examples of rational numbers.

- 7 is a rational number as it can be written as $\frac{7}{1}$, and both 7 and 1 are integers.
- -3 is a rational number as it can be written as $\frac{-3}{1}$, and both -3 and 1 are integers.
- 0 is a rational number as it can be written as $\frac{0}{4}$, and both 0 and 4 are integers.
- -1.5 is a rational number as it can be written as $\frac{-3}{2}$, and both -3 and 2 are integers.
- $0.\dot{6} = 0.666...$ is a rational number as it can be written as $\frac{6}{9}$, and both 6 and 9 are integers.

The decimal expansion of a rational number may have a finite number of decimal places (for example -1.5) or may recur (for example $0.\dot{6}$). A number with recurring digits has a **period**, which is the digit or group of digits that is repeated after the decimal point. For example, the period of 0.66666... is 6 and the period of 0.767676... is 76.

From these examples we can see that any integer is also a rational number but not all rational numbers are integers. You can represent some of the rational numbers on the number line like this:



Find out more about the history of rational numbers on pages 40–41.

Example 4

a Express $1.\dot{3}$ as a fraction.
b Hence calculate $1.\dot{3} + \frac{4}{5}$
Give your answer as a fraction.

Answers

a Let $a = 1.\dot{3}$ then
$a = 1.3333...$
$10a = 13.333...$ — *Multiply by 10 to obtain another number with the same period.*
$10a - a = 13.333... - 1.3333...$ — *Subtract a from 10a.*
$= 12$
$9a = 12$ — *Divide both sides by 9.*
$a = \frac{12}{9} = \frac{4}{3}$ — *Simplify to its simplest form.*

b $1.\dot{3} + \frac{4}{5} = \frac{4}{3} + \frac{4}{5} = \frac{32}{15}$ — *Use a common denominator of 15 or your GDC.*

'Hence' is a command term that is frequently used in exams. If you read 'hence' then try to use the preceding work to find the required result.

Exercise 1C

1 **a** Find the decimal expansion of these fractions.
  $\frac{2}{3}$ $-\frac{5}{4}$ $\frac{2}{9}$ $\frac{4}{7}$ $\frac{-11}{5}$
  b For each fraction in **a**, state whether its decimal expansion
  i is finite **ii** recurs.

$\frac{2}{3} \longrightarrow 2 \div 3$, use your GDC.

2 **a** Express $0.\dot{5}$ as a fraction. **b** Express $1.\dot{8}$ as a fraction.
  c Hence calculate $0.\dot{5} + 1.\dot{8}$. Give your answer as a fraction.

3 **a** Write down a rational number whose decimal expansion is finite.
  b Write down a rational number whose decimal expansion recurs.
  c Write down a rational number whose decimal expansion has a period that starts in the fourth digit after the decimal point.

For any pair of rational numbers, you can always find a rational number that lies between them on the number line. For example, the **arithmetic mean** of two numbers is halfway between those numbers.

Express $1.\dot{9}$ as a fraction. What do you notice? Is it true that $1.\dot{9} = 2$?

Example 5

a Write down a rational number that lies on the number line between $\frac{2}{3}$ and 1.

b Write down a second rational number that lies on the number line between $\frac{2}{3}$ and 1.

c Write down a third rational number that lies on the number line between $\frac{2}{3}$ and 1.

Answers

a $\frac{\frac{2}{3}+1}{2}=\frac{5}{6}$

b $\frac{\frac{2}{3}+\frac{5}{6}}{2}=\frac{3}{4}$

c $\frac{\frac{2}{3}+\frac{3}{4}}{2}=\frac{17}{24}$

Find the arithmetic mean of $\frac{2}{3}$ and 1. Use your GDC to simplify it.

'**Write down**' is a command term that means you don't need to show much or any working.

How many rational numbers are there between two rational numbers?

→ A number is rational if
- it can be written as a quotient of two integers, or
- its decimal expansion is finite, or
- its decimal expansion is non-terminating but has a recurring digit or pattern of digits.

'Non-terminating' is the opposite of 'finite'.

Example 6

For each of the expressions **a** $(x+y)^2$ **b** $\sqrt{\frac{x+5}{y}}$

i Calculate the value when $x=-4$ and $y=\frac{1}{2}$.

ii State whether your answers to **i** are rational numbers or not. Justify your answer.

Answers

a i $\left(-4+\frac{1}{2}\right)^2=\left(-\frac{7}{2}\right)^2=\frac{49}{4}$

ii It is a rational number as it can be written as the quotient of two integers.

To justify your answer, explain how you know it is rational.

b i $\sqrt{\frac{-4+5}{\frac{1}{2}}}=\sqrt{\frac{1}{\frac{1}{2}}}=\sqrt{2}$

ii It is not a rational number. Its decimal expansion is 1.4142135... It does not have a finite number of decimal places and does not recur.

Exercise 1D

1 Write down three rational numbers that lie on the number line between 2 and $\frac{9}{4}$.

2 a Calculate the value of the expression $\sqrt{2(y-x)}$ when $y=3$ and $x=-\frac{1}{8}$.

b State whether your answer to part **a** is a rational number or not.

3 a Write down three rational numbers between $\frac{9}{5}$ and $\frac{11}{6}$.

b i Write down three rational numbers between $-\frac{28}{13}$ and -2.

ii How many rational numbers are there between $-\frac{28}{13}$ and -2?

Investigation – rational numbers

State whether each of these statements is true or false. If a statement is false, explain why by giving an example.

a *The **difference** of two rational numbers is always a rational number.*

b *The **square** of a rational number is always a rational number.*

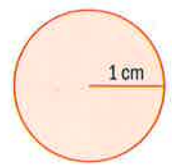
c *The **quotient** of two rational numbers is sometimes a rational number.*

d *The **square root** of a rational number is always a rational number.*

The set of real numbers, $\mathbb{R}$

In the investigation you should have found that the square root of a rational number is not always a rational number. So we need a new set, as there are quantities that cannot be represented with rational numbers. For example, we could think of a circle with radius 1 cm.

What is the area, A, of this circle?



$A=\pi\times r^2$

$A=\pi\times(1\text{ cm})^2$

$A=\pi\text{ cm}^2$

Is π a rational number? The decimal expansion of π from the GDC is 3.141592654 – but these are just the first nine digits after the decimal point.

The decimal expansion of π has an infinite number of digits after the decimal point, and no **period** (no repeating pattern).

You can find the first ten thousand digits of π from this website: http://www.joyofpi.com/pi.html.

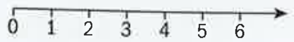
→ Any number that has a decimal expansion with an infinite number of digits after the decimal point and no period is an **irrational number**.

Irrational numbers include, for example, π, $\sqrt{2}$, $\sqrt{3}$.

→ The set of rational numbers together with the set of irrational numbers complete the number line and form the set of **real numbers**, $\mathbb{R}$.

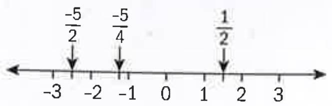
How many real numbers are there? Can we count them?

Natural numbers $\mathbb{N}$

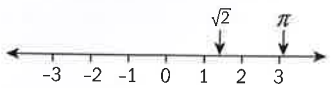


Integers $\mathbb{Z}$

Rationals $\mathbb{Q}$



Real numbers $\mathbb{R}$ complete the number line



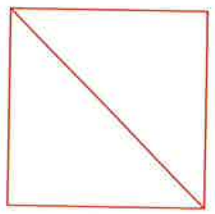
On March 14 (or, in month/day format, 3/14) a lot of people around the world celebrate **Pi Day**, as 3, 1 and 4 are the three most significant digits of π. Also March 14 is Albert Einstein's birthday so sometimes both events are celebrated together. **Pi Approximation Day** is July 22, or in the day/month format 22/7, which is an approximation to the value of π.

Example 7

Calculate each of these measurements and state whether it is rational or irrational.

a The length l of a diagonal of a square with side length of 1 cm.

b The area A of a circle with radius $\frac{1}{\sqrt{\pi}}$ cm.



Answers

a $l^2 = 1^2 + 1^2$ — *Use Pythagoras' theorem.*

$l^2 = 2$

$l = \sqrt{2}$ — *$\sqrt{2} = 1.4142...$ It is not finite, not recurring.*

$\sqrt{2}$ is an irrational number

b $A = \pi r^2$ — *Use the formula for the area of a circle.*

$A = \pi \times \left(\frac{1}{\sqrt{\pi}}\right)^2 = \pi \times \frac{1}{\pi}$

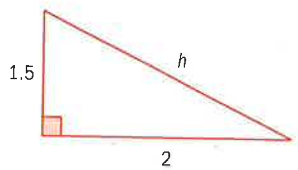
$A = 1\text{ cm}^2$

1 is a rational number

Exercise 1E

1 a Calculate the length, h, of the hypotenuse of a right-angled triangle with sides 2 cm and 1.5 cm.

b State whether h is rational or irrational.



2 a Calculate the area, A, of a circle with diameter 10 cm.

b State whether A is rational or irrational.

Example 8

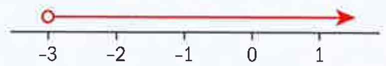
a Solve this inequality and represent the solution on the number line.
$8 + x > 5$

b State whether $p = -\pi$ is a solution to the inequality given in part **a**.

Answers

a $8 + x > 5$

$x > -3$



b $-\pi = -3.142...$, so $-\pi < -3$

p is not a solution of the inequality.

Do we all use the same notation in mathematics? We are using an empty dot to indicate that $x = -3$ is *not* included. Different countries have different notations to represent the same thing. Furthermore, different teachers from the same country use different notations!

Exercise 1F

1 a Solve these inequalities.

i $0.5 < \frac{x}{2} \le 1.5$ **ii** $3 - x \ge 1$

b Represent the solution to part **a** on the number line.

c State whether the numbers $q = 1.5$ and $t = \sqrt{5}$ are solutions to the inequalities given in part **a**.

2 a Solve these inequalities.

i $2x + 1 > -1$ **ii** $4 \le x + 1 \le 8$ **iii** $2 - x > -1$

b Represent the solutions to part **a** on the number line.

c Copy and complete this table. Put a ✓ if the number p is a solution to the inequality given.

p \ **Inequality**	$2x + 1 > -1$	$4 \le x + 1 \le 8$	$2 - x > -1$
$\frac{-2}{3}$			
$\sqrt{10}$			
2π			

1.2 Approximations and error

It is important that you understand the difference between an **exact value** and an **approximate value.**

Sometimes, as in the following examples, we approximate a quantity because the exact values are not known (maybe because the instrument we use to take the measurements only reaches a certain accuracy).

- The approximate area of Ecuador is 283 561 km².
- The present height of the Great Pyramid of Giza is approximately 138.8 m.
- The weight of an apple is approximately 250 g.

Sometimes we approximate a quantity because we don't need the exact value, as in the following examples.

- India's population is about 1 800 000 000.
- I run for about 3 hours every Sunday.
- China's economy grew at an average rate of 10% per year during the period 1990–2004.

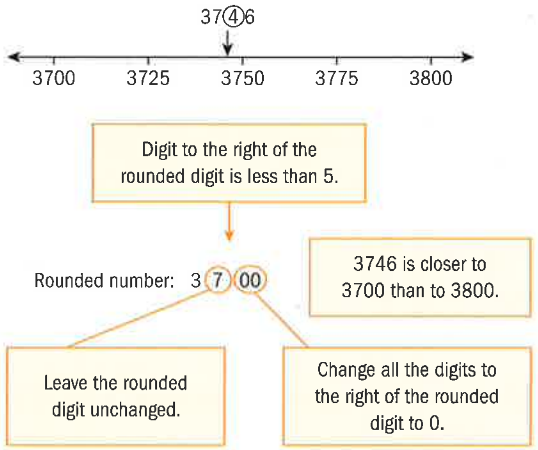
Rounding a number is the process of approximating this number to a given degree of accuracy.

Rounding numbers to the nearest unit, nearest 10, nearest 100, nearest 1000, etc.

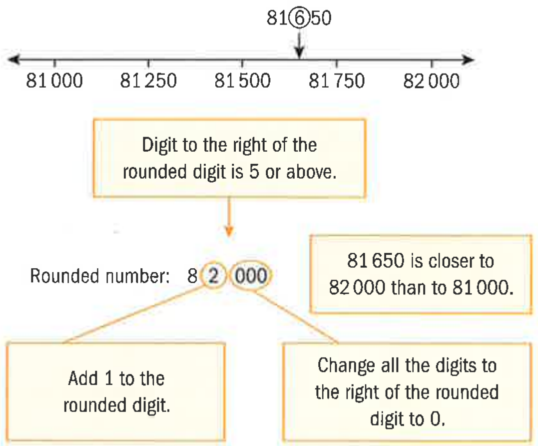
→ Rounding a number to the **nearest 10** is the same as rounding it to the **nearest multiple of 10**.

Rounding a number to the **nearest 100** is the same as rounding it to the **nearest multiple of 100**.

To round 3746 to the nearest hundred:



To round 81 650 to the nearest thousand:



→ **Rules for rounding**

If the digit after the one that is being rounded is less than 5 then keep the rounded digit unchanged and change all the remaining digits to the right of this to 0.

If the digit after the one that is being rounded is 5 or more then add 1 to the rounded digit and change all remaining digits to the right of this to 0.

Example 9

a Write down 247 correct to the nearest ten.
b Write down 1050 correct to the nearest hundred.

Answers

a 250	*Both 240 and 250 are multiples of 10 but 250 is closer to 247.*
b 1100	*Both 1000 and 1100 are multiples of 100 and 1050 is exactly in the middle. Because the digit after the one being rounded is 5, round up.*

Exercise 1G

1 Write these numbers correct to the nearest unit.
a 358.4 **b** 24.5 **c** 108.9 **d** 10 016.01

2 Write these numbers correct to the nearest 10.
a 246.25 **b** 109 **c** 1015.03 **d** 269

3 Write these numbers correct to the nearest 100.
a 140 **b** 150 **c** 1240 **d** 3062

4 Write these numbers correct to the nearest 1000.
a 105 607 **b** 1500 **c** 9640 **d** 952

5 Write down a number that correct to the nearest 100 is 200.

6 Write down a number that correct to the nearest 1000 is 3000.

7 Write down a number that correct to the nearest unit is 6.

Rounding numbers to a given number of decimal places (dp)

This is rounding numbers to the nearest tenth, to the nearest hundredth, etc.

→ Rounding a number **correct to one decimal place** is the same as rounding it to the **nearest tenth**.

Rounding a number **correct to two decimal places** is the same as rounding it to the **nearest hundredth**.

Rounding a number **correct to three decimal places** is the same as rounding it to the **nearest thousandth**.

To write 3.021 correct to 1 dp:

		Rounded digit	First digit to the right is less than 5	
NUMBER	3	• 0	2	1
ROUNDED NUMBER	3	• 0	......	......
		Rounded digit remains unchanged	Digits to the right of rounded digit are deleted	Digits to the right of rounded digit are deleted

3.021 = 3.0 (1 dp)

To write 10.583 correct to 2 dp:

NUMBER	1	0	•	5	8	3
ROUNDED NUMBER	1	0	•	5	8	......
					Rounded digit remains unchanged	Digits to the right of rounded digit are deleted

10.583 = 10.58 (2 dp)

To write 4.371 to 1 dp:

		Rounded digit	First digit to the right is more than 5	
NUMBER	4	• 3	7	1
ROUNDED NUMBER	4	• 4	......	......
		Rounded digit is changed to 1 more	Digits to the right of rounded digit are deleted	Digits to the right of rounded digit are deleted

4.371 = 4.4 (1dp)

→ **Rounding rules for decimals**

- If the digit after the one that is being rounded is less than 5 keep the rounded digit unchanged and delete all the following digits.
- If the digit after the one that is being rounded is 5 or more then add 1 to the rounded digit and delete all the following digits.

Example 10

a Write down 10.045 correct to 2 dp.
b Write down 1.06 correct to 1 dp.

Answers

a 10.045 = 10.05 (2 dp)

10.045 Next digit is 5, so round up: 10.05

b 1.06 = 1.1 (1 dp)

1.06 Next digit is 6, so round up: 1.1

Exercise 1H

1 Write these numbers correct to 1 dp.
a 45.67 **b** 301.065 **c** 2.401 **d** 0.09

2 Write these numbers correct to 2 dp.
a 0.0047 **b** 201.305 **c** 9.6201 **d** 28.0751

3 Write these numbers correct to the nearest thousandth.
a 10.0485 **b** 3.9002 **c** 201.7805 **d** 0.00841

4 Calculate $\frac{\sqrt{1.8}}{3.08 \times 0.012^2}$; use your GDC.
Give your answer correct to
a 1 dp **b** 2 dp **c** 3 dp **d** nearest 100 **e** nearest 1000.

5 Given that $p = 3.15$ and $q = 0.8$, find the value of $\frac{(p+q)^3}{p+q}$ giving your answer correct to
a 2 dp **b** 3 dp **c** nearest unit **d** nearest ten.

6 Write down a number that correct to 2 dp is 2.37.

7 Write down a number that correct to 1 dp is 4.1.

Rounding numbers to a given number of significant figures (sf)

→ The number of **significant figures** in a result is the number of figures that are known with some degree of reliability.

This sometimes depends on the measurement that is being taken. For example, if the length of a pencil is measured with a ruler whose smallest division is 1 mm, then the measurement is only accurate to the nearest millimetre.

You can say: *The length of this pencil is 14.6 cm.*
But you cannot say: *The length of this pencil is 14.63 cm.*
The length of the pencil can be given correct to 3 sf but cannot be given correct to 4 sf.

Rules for significant figures:	
• All non-zero digits are significant.	2578 kg has 4 sf
• Zeros between non-zero digits are significant.	20 004 km has 5 sf
• Zeros to the left of the first non-zero digit are **not** significant.	0.023 g has 2 sf
• Zeros placed after other digits but to the right of the decimal point are significant.	0.100 ml has 3 sf

Make sure you understand when a digit is significant.

The rules for rounding to a given number of significant figures are similar to the ones for rounding to the nearest 10, 1000, etc. or to a number of decimal places.

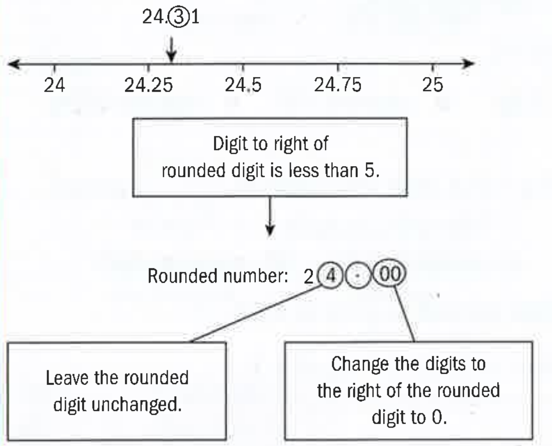
This example shows you the method.

Example 11

a Write down 24.31 correct to 2 sf.
b Write down 1005 correct to 3 sf.
c Write down 0.2981 correct to 2 sf.

Answers

a 24.31 = 24 (2 sf)



b 1005 = 1010 (3 sf)

Digit to right of rounded digit is equal to 5. Add 1 to the rounded digit. Change all digits to the right of the rounded digit to 0.

c 0.2981 = 0.30 (2 sf)

Digit to right of rounded digit is greater than 5. Add 1 to the rounded digit. Change all digits to the right of the rounded digit to 0.

9 + 1 = 10 Replace the rounded digit with 0. Add 1 to the digit to the left of the rounded digit.

→ Rounding rules for significant figures

- If the $(n+1)$th figure is less than 5 then keep the nth figure unchanged.
- If the $(n+1)$th figure is 5 or more then add 1 to this figure.
- In both cases all the figures to the right of figure n should be deleted if they are to the right of the decimal point and should be replaced by zeros if they are to the left of the decimal point.

Example 12

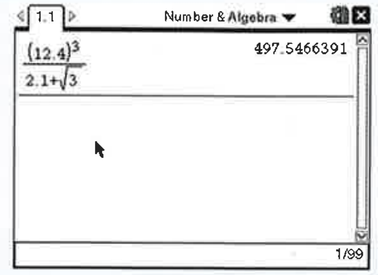
Let $t = \dfrac{12.4^3}{2.1 + \sqrt{3}}$.

a Write down the value of t giving the full calculator display.
b Write the answer to part **a** correct to
  i 3 significant figures **ii** 2 significant figures.

▶ Continued on next page

Answers

a 497.5466391



b i 498 *497.54 3 = 498 (3 sf)*

ii 500 *49 7.54 2 = 500 (2 sf)*

Exercise 1I

1 Write the number of significant figures of each of these numbers.
a 106 **b** 200 **c** 0.02 **d** 1290 **e** 1209

2 Write these numbers correct to 1 sf.
a 280 **b** 0.072 **c** 390.8 **d** 0.001 32

3 Write these numbers correct to 2 sf.
a 355 **b** 0.0801 **c** 1.075 **d** 1560.03

4 Write these numbers correct to 3 sf.
a 2971 **b** 0.3259 **c** 10 410 **d** 0.5006

5 Calculate $\dfrac{\sqrt{8.7 + 2 \times 1.6}}{0.3^4}$.
Give your answer correct to
a 1 sf **b** 3 sf **c** 1 dp **d** nearest hundredth.

6 Write the value of π correct to
a nearest unit **b** 2 dp **c** 2 sf **d** 3 dp.

7 Write down these numbers to the accuracy stated.
a 238 (1 sf) **b** 4609 (3 sf) **c** 2.7002 (3 sf)

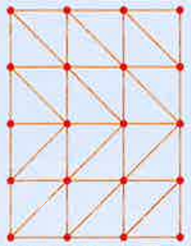
8 a Calculate $\dfrac{\sqrt[3]{3.375}}{1.5^2 + 1.8}$. Write down the full calculator display.
b Give your answer to part **a** correct to
  i 2 sf **ii** 3 sf **iii** 4 sf.

Often in exams you need to do multi-step calculations.
In those situations, *keep at least one more significant digit in intermediate results* than needed in your final answer.
For instance, if the final answer needs to be given correct to 3 sf, then carry at least 4 sf in the intermediate calculations, or store the unrounded values in your GDC.

The general rule in Mathematical Studies is *Unless otherwise stated in the question answers must be given exactly or correct to three significant figures.*

Example 13

The diagram represents a window grille made of wire, to keep pigeons out of the house.
The small triangles are right-angled triangles and are all **congruent**. Their hypotenuse is 15 cm long. The other two sides are equal lengths.
Find the total length of the wire, L. Give your answer correct to 3 significant figures.



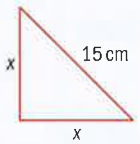
'Congruent' means exactly the same shape and size.

Answers

Let x be the side length of the triangles.

$x^2 + x^2 = 15^2$
$2x^2 = 225$
$x^2 = 112.5$
$x = \sqrt{112.5}$

First find the length of the shorter sides using Pythagoras.



$x = 10.6066\ldots$

Keep this value either exact or with more than three significant figures as this is just an intermediate value.

$L = 31 \times x + 12 \times 15$
$L = 31 \times 10.6066\ldots + 12 \times 15$
$L = 508.804\ldots$
$L = 509$ cm (3 sf)

In the grille there are 31 sides of triangles with length x and 12 sides with length 15.

Do not forget to write down the units in your answers.

Exercise 1J

EXAM-STYLE QUESTIONS

1 The area of a circle is 10.5 cm^2.
  a Find the length of its radius. Give your answer correct to four significant figures.
  b Find the length of its circumference. Give your answer correct to two significant figures.

2 Let the numbers $p = \sqrt{2}$ and $q = \sqrt{10}$.
  a Find the arithmetic mean of p and q. Give your answer correct to 4 sf.
  b Find the value of $(p + q)^2$. Give your answer correct to 3 sf.
  c Find the area of a rectangle whose sides are p cm and q cm long. Give your answer correct to 2 sf.

Estimation

An **estimate of a quantity** is an approximation that is usually used to check the reasonableness of an answer.

→ To estimate the answer to a calculation, round all the numbers involved to 1 sf.

Example 14

A theater has 98 rows; each row has 23 seats. Estimate the number of seats in the theater.

Answer

$100 \times 20 = 2000$ seats

Round 98 to 1 sf → 100
Round 23 to 1 sf → 20

Exact answer is $98 \times 23 = 2254$ seats.

Example 15

Estimate the average speed of a car that travels 527 km in 6 hours.

Answer

Average speed $= \dfrac{\text{distance traveled}}{\text{time taken}}$

$\dfrac{500}{5} = 100$ km h^{-1}

527 → 500 (1 sf)
Round 6 down to 5 to make the division calculation easier.

Exact answer is $\dfrac{527}{6} = 87.8$ km h^{-1} (3 sf)

Exercise 1K

1 Estimate the answers to these calculations.
  a 298×10.75 **b** 3.8^2 **c** $\dfrac{147}{11.02}$ **d** $\sqrt{103}$

2 A lorry is carrying 210 containers with pipes. There are 18 pipes in each container. Estimate the number of pipes that the lorry is carrying.

3 Japan covers an area of approximately 377 835 km^2 and in March 2009 Japan's population was 127 076 183. Estimate Japan's population density in 2009.

Population density $= \dfrac{\text{total population}}{\text{land area}}$

4 A tree yields on average 9000 copy pages. Estimate the number of reams that can be made from one tree.

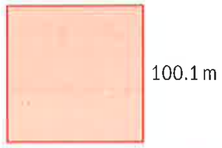
A ream has 500 pages.

5 Mizuki runs 33 km in 1.8 hours. Estimate Mizuki's average speed.

Average speed $= \dfrac{\text{distance traveled}}{\text{time taken}}$

6 The Badaling Section and the Ming Mausoleums Scenic Area of the Great Wall are limited to 53 000 visitors per day. Estimate the number of visitors per year.

7 Peter calculates the area of this square as 1020.01 m². Use estimation to decide whether Peter is correct.



▲ The Great Wall of China

Percentage errors

Sometimes you need to know the difference between an estimated value and the exact value.

→ The difference between an estimated or **approximated value** and the **exact value** is called the **error**:
Error = $v_A - v_E$
where v_A is the approximated value and v_E is the exact value

Why do errors arise?
What kind of errors do you know?
Do 'error' and 'mistake' have the same meaning?

Example 16

Olivia and Ramesh each went to a different concert.
In the concert that Olivia attended there were 1450 people and Olivia estimated that there were 1300.
In the concert that Ramesh attended there were 1950 people and Ramesh estimated that there were 1800.
Calculate the errors Olivia and Ramesh made in their estimations.

Answer

Olivia: Error = 1450 – 1300
Error = 150 people

Ramesh: Error = 1950 – 1800
Error = 150 people

$v_A - v_E$ is negative, so use $v_E - v_A$ instead.

$|v_A - v_E|$ is the modulus, or positive value, of $v_A - v_E$.

In Example 16, Olivia and Ramesh both made the same error, 150. However, Ramesh's estimate is more accurate as 150 out of 1950 is a smaller proportion than 150 out of 1450.

Using **percentages**:

$$\frac{150}{1450} \times 100\% = 10.3\% \text{ (3 sf)} \quad \text{and} \quad \frac{150}{1950} \times 100\% = 7.69\% \text{ (3 sf)}$$

Olivia's error represents 10.3% of the total.
Ramesh's error represents 7.69% of the total.
These percentages help us to have a better idea of the accuracy of the estimations. They are called **percentage errors**.

→ Percentage error = $\left|\frac{v_A - v_E}{v_E}\right| \times 100\%$

where v_A represents **approximated value** or **estimated value** and v_E represents the **exact value**.

Sometimes we don't have the exact value. In these cases we replace V_E with the **accepted value**.

Example 17

The size of angle M is 125.7°. Salomon measures M with a protractor as 126°. Find the percentage error he made in measuring angle M.

Answer

Percentage error

$= \left|\frac{126 - 125.7}{125.7}\right| \times 100\%$

Percentage error
= 0.239% (3 sf)

Percentage error

$= \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$

with $v_A = 126$, $v_E = 125.7$
Use your GDC. Round to 3 sf.

Exercise 1L

EXAM-STYLE QUESTIONS

1 Let $a = 5.2$ and $b = 4.7$.
a Find the exact value of $3a + b^3$.
Xena estimates that the answer to part **a** is 140.
b Find the percentage error made by Xena in her estimation.

2 Ezequiel's Biology marks are 8.3, 6.8 and 9.4 out of 10. His final grade in Biology is the mean of these three marks.
a Calculate Ezequiel's final grade in Biology.
Ezequiel wrote the three marks correct to the nearest unit to find his final grade in Biology.
b Calculate the final grade that Ezequiel found.
c Calculate the percentage error made by Ezequiel when finding his final grade in Biology.

3 The measurements of the length and width of a rectangular kitchen are 5.34 m and 3.48 m respectively.
a Calculate in m² the exact area of the kitchen.
b Write down both the length and the width correct to 1 dp.
c Calculate the percentage error made if the area was calculated using the length and the width correct to 1 dp.

4 The area of a circular garden is 89 m².
a Find the radius of the garden. Give your answer correct to three decimal places.
b Find the perimeter of the garden.
José estimates that the perimeter of the garden is 30 m.
c Use your answer to part **b** to find the percentage error made by José. Give your answer correct to two significant figures.

1.3 Standard form

- The number of internet users in the world up to June 2010 was $\mathbf{2 \times 10^9}$.
- The mass of the Earth is about $\mathbf{5.97 \times 10^{24}}$ kg.
- An estimate for the average mass of a human cell is about $\mathbf{10^{-9}}$ g.

If we did not use standard form, we would write the mass of the Earth as 5 970 000 000 000 000 000 000 000 kg

These numbers are either very large or very small.

They are written in **standard form**: a way of writing very large or very small numbers without writing a lot of zeros.

When numbers are written in standard form it is easier to
- compare them
- calculate with them.

→ A number is written in standard form if it is in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.

A **googol** is the number 1 followed by 100 zeros. In standard form it is 10^{100}. The name googol was invented by a 9-year-old, who was asked by his uncle, the American mathematician Edward Kasner, to think up a name for a very large number.

The name of the company Google comes from a misspelling of the word googol and is related to the amount of information that the company handles.

Example 18

These numbers are written in standard form ($a \times 10^k$).
For each of them state the value of a and of k.
a 2×10^9 **b** 5.97×10^{24} **c** 10^{-9}

Answers
a $a = 2$; $k = 9$
b $a = 5.97$; $k = 24$
c $a = 1$; $k = -9$

Compare with $a \times 10^k$

Example 19

Decide which of these numbers are *not* written in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer. Justify your decisions.

a 2.06×10^{-5} **b** 13×10^{-1} **c** $6.13 \times 10^{\frac{1}{3}}$
d 7.05 **e** 0.12×10^6

Answers
b 13×10^{-1} is not written in standard form as 13 is greater than 10.
c $6.13 \times 10^{\frac{1}{3}}$ is not written in standard form as $\frac{1}{3}$ is not an integer.
e 0.12×10^6 is not written in standard form as 0.12 is smaller than 1.

Compare with $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$

Abu Kamil Shuja (c. 850–c. 930), also know as al-Hasib al-Misri, meaning 'the calculator from Egypt', was one of the first to introduce symbols for indices, such as $x^m x^n = x^{m+n}$, in algebra.

Example 20

Write these numbers in standard form, showing your working.
a 257 000 000 **b** 0.000 43

Answers
a 257 000 000 so $k = 8$
$257\,000\,000 = 2.57 \times 10^8$

First significant figure of 257 000 000 is 2. Place the decimal point immediately after 2.
Moving the decimal point 8 places to the right is the same as multiplying by 10^8.

b 0.000 43 so $k = -4$
$0.000\,43 = 4.3 \times 10^{-4}$

First significant figure of 0.000 43 is 4. Place the decimal point immediately after 4.
Moving the decimal point 4 places to the left is the same as multiplying by 10^{-4}.

Tips to write a number in standard form:
1 Write down a: write down all the significant figures of the number and place the decimal point immediately after the first significant figure.
2 Find k.

Exercise 1M

1 Which of these numbers are written in standard form?

2.5×10^{-3} 12×10^5 10^{10} $3.15 \times 10^{\frac{1}{2}}$ 0.81×10^2

2 Write these numbers in standard form.
a 135 600 **b** 0.002 45 **c** 16 000 000 000
d 0.000 108 **e** 0.23×10^3

3 Write these numbers in ascending order.
2.3×10^6 3.4×10^5 0.21×10^7 215×10^4

4 Write these numbers in descending order.
3.621×10^4 31.62×10^2 0.3621×10^4 3.261×10^3

Change them to decimal numbers, e.g $2.3 \times 10^6 = 2\,300\,000$.
A decimal number is a 'normal' number written to base 10. It doesn't necessarily have a decimal point or decimal places.

Example 21

Let $x = \dfrac{-5 + \sqrt{121}}{(7-1)^2}$.
a Calculate the value of x. Write down the full calculator display.
b Write your answer to part **a** correct to 3 sf.
c Write your answer to part **b** in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

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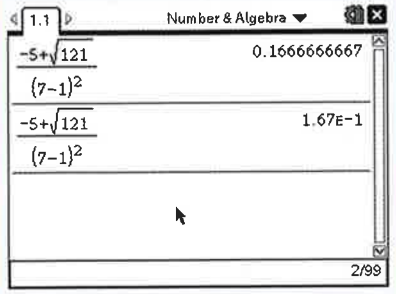
Answers

a 0.1666666667

Use your GDC.



b 0.167 — *0.166¦666... 3 sf, round up*

c 1.67×10^{-1}

> **Careful!**
> 1.67 E-1 is calculator notation and is *not* accepted as an answer. You must interpret it as 1.67×10^{-1}.

Calculations with numbers expressed in standard form

You can use your GDC for calculations in standard form.

Example 22

Let $x = 2.4 \times 10^4$ and $y = 5.10 \times 10^5$.

a Find the value of $3x + y$.

b Write your answer to part **a** correct to 2 sf.

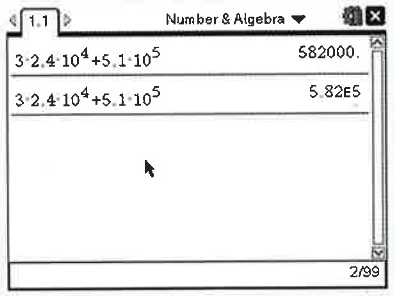
c Write your answer to part **b** in the form $a \times 10^k$ where $1 \le a < 10$ and k is an integer.

Answers

a $3 \times 2.4 \times 10^4 + 5.10 \times 10^5$
$= 582\,000$

b 580 000

c 5.8×10^5



> Always use a GDC in this type of question, but show the working as shown in **a**.

Exercise 1N

1 Given that $x = 6.3 \times 10^6$ and $y = 2.8 \times 10^{10}$, calculate the following. Give your answers in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

a $x \times y$ **b** $\frac{x}{y}$ **c** $\sqrt{\frac{x}{y}}$

2 Let $x = 2.5 \times 10^6$ and $y = 3.48 \times 10^6$.

a Find the arithmetic mean of x and y. Give your answer in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

b Give your answer to part **a** correct to the nearest million.

EXAM-STYLE QUESTIONS

3 Let $t = 22.05 \times 10^8$ and $q = 3.15 \times 10^6$

a Write down t in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

b Calculate $\frac{t}{q}$.

c Write your answer to part **b** in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

4 Let $x = 225 \times 10^8$.

a Write x in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

b State whether the following statement is true: $x^2 > 10^{20}$
Justify your answer.

c i Calculate $\frac{x}{\sqrt{x}}$.

ii Give your answer to part **i** in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

1.4 SI units of measurement

Ariel is baking a tuna pie.
He needs a tuna can whose net weight is 180 **g**.
Another ingredient is 240 **ml** *of milk.*
He bakes the pie in a preheated oven to 200°**C** *for 20* **minutes**.
Ariel recycles material. He has decided to use the metal from the can so he needs to take some measurements:
The height of the tuna can is 4 **cm**.
The total area of metal used to make the tuna can was 219 **cm²**.
The volume of the tuna can is 314 **cm³**.

Here we have seen how in an everyday situation we deal with different kinds of units such as g, ml, °C, minutes, cm, cm², cm³. These units are internationally accepted and have the same meaning in any part of the world.

> The 11th General Conference on Weights and Measures, CGPM, held in 1960, adopted the name Système International d'Unités. The CGPM is made up of representatives from 54 member states and 31 associate states and economies.

SI is the international abbreviation for the *International System of Units* (in French, *Système international d'unités*). There are seven **base units** (see table). Each unit is accurately defined and the definition is independent from the other six units.

The seven base units and their respective quantities are given in the following table.

Base quantity	Base unit name	Base unit symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Intensity of light	candela	cd

One metre is defined in the SI as the distance travelled by light in a vacuum in $\frac{1}{299\,792\,458}$ of a second.

In the **SI** there are other units, the **derived units**. These units are expressed in terms of the base units. Some of these units along with their quantities are listed below:

- The **square metre** (m^2) for area
- The **cubic metre** (m^3) for volume
- The **metre per second** ($m\,s^{-1}$) for speed or velocity
- The **kilogram per cubic metre** ($kg\,m^{-3}$) for density or mass density

Derived units are products of powers of **base units**.

→ In Mathematical Studies, the most common **SI base units** used are m, kg and s, and **derived units** are m^2 (area), m^3 (volume), $km\,h^{-1}$ (velocity), $kg\,m^{-3}$ (density).

Example 23

Write down the symbol used for the quantities in bold:

a The **velocity** of an object that travels 1000 km in 3 hours.

b The **density** of an object with a mass of 550 g and a volume of 400 cm^3.

Answers

a $km\,h^{-1}$

b $g\,cm^{-3}$

Velocity is kilometres per hour.
Density is grams per cubic centimetre.

SI prefixes

To avoid writing very small or very large quantities we use prefix names and prefix symbols. Some of these are shown in this table.

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^3	kilo	k	10^{-3}	milli	m
10^2	hecto	h	10^{-2}	centi	c
10^1	deca	da	10^{-1}	deci	d

The kilogram is the only SI base unit with a prefix as part of its name.

Investigation – SI units

a How many prefix names and symbols are there nowadays?

b Six prefix names and their symbols are listed in the table. Find the others.

c Choose at least two of them and describe situations where they are used.

Does the use of SI notation help us to think of mathematics as a 'universal language'?

Example 24

Convert each measurement to the stated unit.

a 1 dm to m **b** 1 das to s **c** 1 hg to g

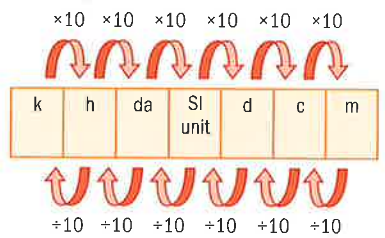
Answers

a $1\,dm = 10^{-1}\,m$

b $1\,das = 10^1\,s$

c $1\,hg = 10^2\,g$

Use the information on prefixes given in the table on the previous page.
dm *reads decimetre*
das *reads decasecond*
hg *reads hectogram*



This diagram will help you to convert between SI units.

Example 25

Convert each measurement to the stated unit. Give your answers in standard form.

a 2.8 m to hm **b** 3200 s to ms **c** 0.5 kg to dg

Answers

a $1\,m = 10^{-2}\,hm$

$2.8\,m = 2.8 \times 10^{-2}\,hm$

In this example replace 'SI unit' in the diagram with m.
To convert from m to hm divide by 10 twice therefore 1 m = 10^{-2} hm.

b $1\,s = 10^3\,ms$

$3200\,s = 3200 \times 10^3\,ms$

$= 3.2 \times 10^6\,ms$

In this example replace 'SI unit' in the diagram with s.
To convert from s to ms multiply by 10 three times therefore 1 s = 10^3 ms.

c $1\,kg = 10^4\,dg$

$0.5\,kg = 0.5 \times 10^4\,dg$

$= 5 \times 10^3\,dg$

In this example replace 'SI unit' in the diagram with g.
To convert from kg to dg multiply by 10 four times therefore 1 kg = 10^4 dg.

Exercise 10

1 Write down the symbol used for the quantities in bold.

a The **acceleration** of an object that has units measured in kilometres per hour squared.

b The **density** of an object with a mass of 23 kg and a volume of $1.5\,\text{m}^3$.

c The average **speed** of an object that travels 500 m in 70 seconds.

2 Write down these units in words.

a dag **b** cs **c** mm **d** dm

3 Convert each of these to the stated unit.

a 32 km to m **b** 0.87 m to dam **c** 128 cm to m

4 Convert each of these to the stated unit.

a 500 g to kg **b** 357 kg to dag **c** 1080 dg to hg

5 Convert each of these to the stated unit.

a 0.080 s to ms **b** 1200 s to das **c** 0.8 hs to ds

6 **a** Convert 67 800 000 mg to kg. Give your answer correct to the nearest kg.

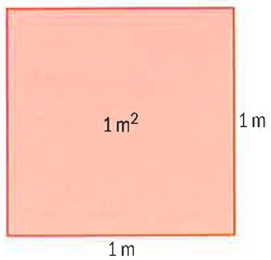
b Convert 35 802 m to km. Give your answer correct to the nearest km.

c Convert 0.654 g to mg. Give your answer in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

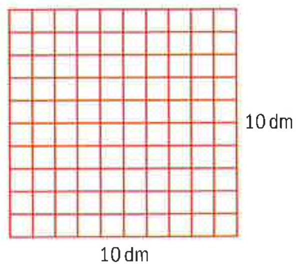
Area and volume SI units

Area

The diagrams show two different ways of representing $1\,\text{m}^2$.



▲ A square metre is equal to the area of a square with sides of length 1 m.

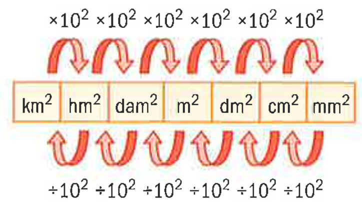


▲ $1\,\text{m}^2 = 100\,\text{dm}^2$

$1\,\text{m}^2 = 1\,\text{m} \times 1\,\text{m} = 10\,\text{dm} \times 10\,\text{dm} = 100\,\text{dm}^2$

To convert from m^2 to dm^2 we multiply by **100** or 10^2.
You can use the same method to convert from

- km^2 to hm^2
- hm^2 to dam^2
- dam^2 to m^2
- dm^2 to cm^2
- cm^2 to mm^2



Example 26

Convert each measurement to the stated unit.
Give your answers in full.

a $1.5\,\text{m}^2$ to cm^2

b $3240\,\text{m}^2$ to km^2

Answers

a $1\,\text{m}^2 = 10^4\,\text{cm}^2$
Therefore
$1.5\,\text{m}^2 = 1.5 \times 10^4\,\text{cm}^2$
$= 15\,000\,\text{cm}^2$

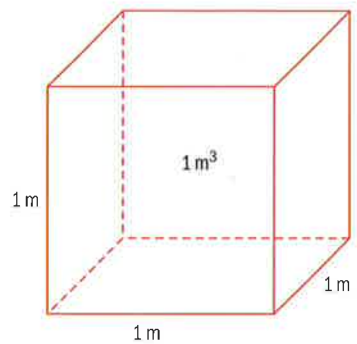
To convert from m^2 to cm^2 multiply by 10^2 twice; this is multiply by 10^4.
$(10^2)^2 = 10^4$

b $1\,\text{m}^2 = 10^{-6}\,\text{km}^2$
Therefore
$3240\,\text{m}^2 = 3240 \times 10^{-6}\,\text{km}^2$
$= 0.003\,240\,\text{km}^2$

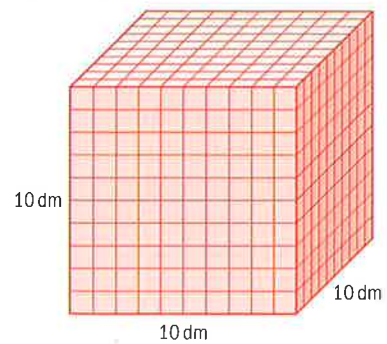
To convert from m^2 to km^2 divide by 10^2 three times; this is divide by 10^6 or multiply by 10^{-6}.
$(10^2)^3 = 10^6$

Volume

The diagrams show two different ways of representing $1\,\text{m}^3$.



▲ A cubic metre is equal to the volume of a cube with sides of length 1 m.

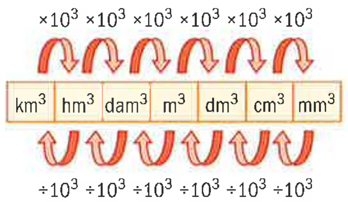


▲ $1\,\text{m}^3 = 1000\,\text{dm}^3$

$1\,\text{m}^3 = 1\,\text{m} \times 1\,\text{m} \times 1\,\text{m} = 10\,\text{dm} \times 10\,\text{dm} \times 10\,\text{dm} = 1000\,\text{dm}^3$

To convert from m^3 to dm^3 we multiply by **1000** or 10^3.
You can use the same method to convert from

- km^3 to hm^3
- hm^3 to dam^3
- dam^3 to m^3
- dm^3 to cm^3
- cm^3 to mm^3



Example 27

Convert each measurement to the stated unit.
Give your answers in standard form.

a $0.8\,m^3$ to cm^3
b $15\,900\,cm^3$ to dam^3

Answers

a $1\,m^3 = 10^6\,cm^3$ Therefore $0.8\,m^3 = 0.8 \times 10^6\,cm^3$ $= 8 \times 10^5\,cm^3$	*To convert from m^3 to cm^3 multiply by 10^3 twice; this is multiply by 10^6. $(10^3)^2 = 10^6$*
b $1\,cm^3 = 10^{-9}\,dam^3$ Therefore $15\,900\,cm^3$ $= 15\,900 \times 10^{-9}\,dam^3$ $= 1.59 \times 10^{-5}\,dam^3$	*To convert from cm^3 to dam^3 divide by 10^3 three times; this is multiply by 10^{-9}.*

Exercise 1P

1 Convert these measurements to the stated unit.
Give your answers in full.
a $2.36\,m^2$ to cm^2 **b** $1.5\,dm^2$ to dam^2
c $5400\,mm^2$ to cm^2 **d** $0.06\,m^2$ to mm^2
e $0.8\,km^2$ to hm^2 **f** $35\,000\,m^2$ to km^2

2 Convert these measurements to the stated unit.
Give your answers in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.
a $5\,m^3$ to cm^3 **b** $0.1\,dam^3$ to m^3
c $3\,500\,000\,mm^3$ to dm^3 **d** $255\,m^3$ to mm^3
e $12\,000\,m^3$ to dam^3 **f** $0.7802\,hm^3$ to dam^3

3 The side length of a square is 13 cm. Find its area in
a cm^2 **b** m^2


13 cm

4 The side length of a cube is 0.85 m. Find the volume of the cube in
a m^3 **b** cm^3

5 Write these measurements in order of size starting from the smallest.
$0.081\,dam^2$, $8\,000\,000\,mm^2$, $82\,dm^2$, $7560\,cm^2$, $0.8\,m^2$

Convert all to the same unit.

6 Write these measurements in order of size starting from the smallest.
$11.2\,m^3$, $1200\,dm^3$, $0.01\,dam^3$, $11\,020\,000\,000\,mm^3$, $10\,900\,000\,cm^3$

Convert all to the same unit.

Extension material on CD: *Worksheet 1 - Calculations with measures*

Non-SI units accepted in the SI

→ There are some units that are **non-SI** units but are accepted for use with the SI because they are widely used in everyday life, for example, min, h, ℓ.

Each of these non-SI units has an exact definition in terms of an SI unit. The table below shows some of these units along with their equivalents in SI units.

Quantity	Name of unit	Symbol	Equivalents in SI units
time	minute	min	1 min = 60 s
	hour	h	1 h = 60 min = 3600 s
	day	d	1 d = 24 h = 86 400 s
area	hectare	ha	$1\,ha = 1\,hm^2 = 10^4\,m^2$
volume	litre	L, ℓ	$1\,\ell = 1\,dm^3$
mass	tonne	t	$1\,t = 10^3\,kg$

The SI prefixes are used with ℓ, but *not* used with min, h and d.

Example 28

a Convert 3 d 15 h 6 min to seconds.
b Convert the average speed of $12\,km\,h^{-1}$ to $m\,s^{-1}$.

Answers

a 1 d = 86 400 s ⇒ 3 d = 259 200 s 1 h = 3600 s ⇒ 15 h = 54 000 s 1 min = 60 s ⇒ 6 min = 360 s Therefore 3 d 15 h 6 min = 259 200 s + 54 000 s + 360 s = 313 560 s	*1 day = 24 hours* *= 24 × 60 min* *= 24 × 60 × 60 s*
b Average speed = $12\,km\,h^{-1}$ ⇒ in 1 h the object moved 12 km ⇒ in 3600 s it moved 12 000 m Average speed = $\frac{12\,000\text{ m}}{3600\text{ s}}$ $= 3.33\,m\,s^{-1}$ (3 sf)	*1 h = 60 min* *= 60 × 60 s* *12 km = 12 000 m*

'⇒' means 'therefore' or 'implies'.

Average speed $= \frac{\text{distance traveled}}{\text{time taken}}$

Example 29

Convert

a 120 hl to cl

b 5400 ℓ to m^3

Answers

a $120\,\text{hl} = 120 \times 10^4\,\text{cl}$
$= 1\,200\,000\ \text{cl}$

To convert from hl to cl, multiply by 10 four times, i.e. multiply by 10^4.

b $1\,\ell = 1\,\text{dm}^3$
$\Rightarrow 5400\,\ell = 5400\,\text{dm}^3$
$5400\,\text{dm}^3 = 5400 \times 10^{-3}\,\text{m}^3$
$= 5.4\,\text{m}^3$

To convert from dm^3 to m^3 we divide by 10^3 once; this is multiply by 10^{-3}.

Exercise 1Q

1 **a** Convert 1 d 2 h 23 m to seconds.
b Give your answer to part **a** correct to the nearest 100.

2 **a** Convert 2 d 5 m to seconds.
b Give your answer to part **a** in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

3 Convert these measurements to the stated unit.
Give your answers in full.
a 5 ℓ to ml **b** 0.56 ml to hl **c** 4500 dal to cl

4 Convert these measurements to the stated unit. Give your answers in the form $a \times 10^k$ where $1 \leq a < 10$ and $k \in \mathbb{Z}$.
a 500 ℓ to cm^3 **b** 145.8 dl to dm^3 **c** 8 hl to cm^3

5 Convert these measurements to the stated unit.
Give your answers to the nearest unit.
a 12.5 dm^3 to ℓ **b** 0.368 m^3 to hl **c** 809 cm^3 to cl

6 A particle travels 3000 m at an average speed of $40\,\text{m}\,\text{min}^{-1}$.
a Find in minutes the time travelled by the particle.
b Give your answer to part **a** in seconds.

EXAM-STYLE QUESTION

7 A cubic container has sides that are 1.5 m long.
a Find the volume of the container. Give your answer in m^3.
b Give your answer to part **a** in dm^3.
c Decide whether 4000 ℓ of water can be poured in the container.
Justify your answer.

EXAM-STYLE QUESTIONS

8 The volume of a tea cup is 220 cm^3. Mercedes always serves a tea cup to $\frac{4}{5}$ of its capacity to avoid spilling any.
a Find, in ℓ, the amount of tea that Mercedes serves in a tea cup.

The volume of Mercedes' teapot is 1.5 ℓ.
b Find the maximum number of tea cups that Mercedes can serve from one teapot.

9 The distance by air from Buenos Aires to Cape Town is 6900 km. An airplane flies at an average speed of $800\,\text{km}\,\text{h}^{-1}$.
a Find the time it takes for this airplane to fly from Buenos Aires to Cape Town.

Abouo takes this flight and then flies to Johannesburg, which is 1393 km from Cape Town. The flight is 2 hours long.
b Find the average speed of this second airplane.

Abouo leaves Buenos Aires at 10:00 a.m. When he arrives at Cape Town he waits 1.5 hours until the next flight.
c Find the time at which he arrives at Johannesburg.

Temperature

→ There are three temperature scales:
- **kelvin** (K)
- **Celsius** (°C)
- **Fahrenheit** (°F)

Fahrenheit 451 is the name of a book written by Ray Bradbury. The title refers to the temperature at which paper combusts. This temperature is also known as the flashpoint of paper.

The kelvin (K) is the only SI base unit of temperature and is mainly used by scientists. The °C is an SI derived unit. The Celsius scale is used in most countries but not in the United States, where the Fahrenheit scale is used. In the following table the freezing and boiling points of water for each of the three scales are shown.

Scale	Freezing point of water	Boiling point of water
Fahrenheit (°F)	32	212
Celsius (°C)	0	100
kelvin (K)	273.15	373.15

The formula used to convert from °C to °F is

$$t_F = \frac{9}{5} \times t_C + 32$$

In this formula t_C represents temperature in °C and t_F represents temperature in °F.

The formula used to convert from K to °C is

$$t_C = t_K - 273.15$$

In this formula t_C represents temperature in °C and t_K represents temperature in K.

Example 30

Convert		
a 25 °C to °F	**b** 300 K to °C	**c** 200 °F to °C

Answers

a $\frac{9}{5} \times 25 + 32 = 77$ °F — *Use the formula* $t_F = \frac{9}{5} \times t_C + 32$

b $300 - 273.15 = 26.85$ °C — *Use the formula* $t_C = t_K - 273.15$

c $200 = \frac{9}{5} \times t_C + 32$ — *Rearrange to make* t_C *the subject of the formula.*

$t_C = (200 - 32) \times \frac{5}{9}$

$t_C = 93.3$ °C (3 sf)

You will derive formulae like this to model real-life situtations in chapter 6.

Exercise 1R

1 Convert into °C. Give your answer correct to one tenth of a degree.
a 280 K **b** 80 °F

2 Convert into °F. Give your answer correct to the nearest degree.
a 21 °C **b** 2 °C

3 a Convert 290 K to °C.
b Hence convert 290 K to °F.

4 a The formula to convert from K to °C is $t_C = t_K - 273.15$.
Find the formula used to convert from °C to K.

b The formula to convert from °C to °F is $t_F = \frac{9}{5} \times t_C + 32$.
Find the formula used to convert from °F to °C.

Review exercise

Paper 1 style questions

EXAM-STYLE QUESTION

1 Consider the numbers $5, \frac{\pi}{2}, -3, \frac{5}{4}, 2.\dot{3}$ and the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$.
Complete the following table by placing a tick (✓) in the appropriate box if the number is an element of the set.

	5	$\frac{\pi}{2}$	−3	$\frac{5}{4}$	$2.\dot{3}$
$\mathbb{N}$					
$\mathbb{Z}$					
$\mathbb{Q}$					
$\mathbb{R}$					

EXAM-STYLE QUESTIONS

2 Given the numbers
14.1×10^{-1} 1.4×10^2 $\sqrt{2}$ 0.00139×10^2 1414×10^{-2}
a state which of these numbers is irrational
b write down $\sqrt{2}$ correct to five significant figures
c write down these numbers in ascending order.

3 The mass of a container is 2690 kg.
a Write down this weight in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.
Nelson estimates that the mass of the container is 2.7×10^3 kg.
b i Write down this mass in full.
ii Find the percentage error made by Nelson in his estimation.

4 Light travels in empty space at a speed of 299 792 458 m s^{-1}.
a Write this value correct to three significant figures.
b Use your answer to part **a** to find in km the distance that the light travels in 1 second.
c Use your answer to part **b** to find in km h^{-1} the speed at which the light travels in empty space. Give your answer in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

5 The total mass of 90 identical books is 52 200 g.
a Calculate the exact mass of one book in kg.
b Write down your answer to part **a** correct to one significant figure.

Matilda estimates that the mass of any of these books is 0.4 kg. She uses the answer to part **b** to find the percentage error made in her estimation.
c Find this percentage error.

6 The volume, V, of a cubic jar is 1560 cm^3.
a Write down V in dm^3.

Sean works in the school cafeteria making juice. He pours the juice in these jars. He always fills the jars up to $\frac{3}{4}$ of their height.
b Find in ℓ the amount of juice that Sean pours in each jar.
He makes 25 ℓ of juice per day.
c i Find the number of jars that Sean pours per day.
ii Write down the amount of juice left.

7 Let $x = \frac{30y^2}{\sqrt{y+1}}$.
a Find the exact value of x when $y = 1.25$.
b Write down the value of x correct to three significant figures.
c Write down your answer to part **b** in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

EXAM-STYLE QUESTIONS

8 The side length of a square field is x m.

a Write down in terms of x an expression for the area of the field.

The area of the field is $2.56\,\text{km}^2$.

b **i** Find the value of x.

ii Find, in **metres,** the perimeter of the field.

9 The formula to convert from the kelvin scale to the Fahrenheit scale is

$$t_F = \frac{9}{5} \times t_K - 459.67$$

where t_K represents temperature in K and t_F represents temperature in °F.

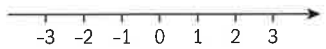
a Find the temperature in °F for 300 K.

b Find the temperature in K for 100 °F. Give your answer to the nearest unit.

10 Consider the inequality $2x + 5 > x + 6$.

a Solve the inequality.

b Represent the solution to part **a** on a copy of the number line.



c Decide which of these numbers are solutions to the inequality given in part **a**.

$1 \qquad \frac{\pi}{4} \qquad -5 \qquad \sqrt{3} \qquad 2.0\dot{6} \qquad \frac{101}{100} \qquad 1.2 \times 10^{-3}$

11 The size of an A4 sheet is $210\,\text{mm} \times 297\,\text{mm}$.

a Find the area of an A4 sheet. Give your answer in mm^2.

b Give your answer to part **a** in m^2.

One ream has 500 pages. It weighs $75\,\text{g}\,\text{m}^{-2}$.

c Find the mass of one page.

d Find the mass of one ream. Give your answer in kg.

Paper 2 style questions

EXAM-STYLE QUESTION

1 The figure shows a rectangular field. The field is 1260 m wide and 2500 m long.

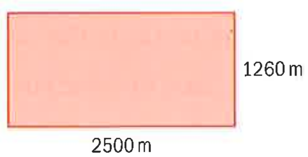


Figure not to scale

a Calculate the perimeter of the field. Give your answer in km.

The owner of the field, Enrico, wants to fence it. The cost of fencing is \$327.64 per km.

b Calculate the cost of fencing the field. Give your answer correct to two decimal places.

Enrico estimates that the perimeter of the field is 7.6 km. He uses this estimation to calculate the cost of fencing the field.

c Calculate the percentage error made by Enrico when using his estimation of the perimeter of the field to calculate the cost of fencing.

d Calculate the area of the field. Give your answer in square kilometres (km^2).

EXAM-STYLE QUESTIONS

2 A running track is made of a rectangular shape 800 m by 400 m with semicircles at each end as shown in the figure below.

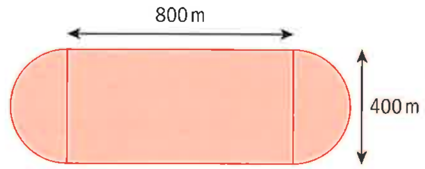


Figure not to scale

a Find the perimeter of the running track. Give your answer correct to the nearest metre.

Elger runs 14 200 m around the track.

b Find the number of complete laps that Elger runs around the running track.

Elger runs at an average speed of $19\,\text{km}\,\text{h}^{-1}$.

c Find the time it takes Elger to complete **one** lap. Give your answer in hours.

d Find the time **in minutes** it takes Elger to run 14 200 m. Give your answer correct to 5 sf.

Elger estimates that it takes him 44 minutes to run 14 200 m.

e Find the percentage error made by Elger in his estimation.

3 A chocolate shop makes spherical chocolates with a diameter of 2.5 cm.

a Calculate the volume of each of these chocolates in cm^3. Give your answer correct to two decimal places.

The chocolates are sold in cylindrical boxes which have a radius of 12.5 mm and a height of 15 cm.

b Calculate the volume of each of these cylindrical boxes in cm^3. Give your answer correct to two decimal places.

c Show that the maximum number of chocolates that fit in each of these boxes is 6.

The boxes are filled with 6 chocolates.

d Find the volume of the box that is **not** occupied by the chocolates.

e Give your answer to part **d** in mm^3.

f Give your answer to part **d** in the form $a \times 10^k$ where $1 \le a < 10$ and $k \in \mathbb{Z}$.

CHAPTER 1 SUMMARY

The number sets

- The set of **natural numbers** $\mathbb{N}$ is $\{0, 1, 2, 3, 4, 5, \ldots\}$
- The set of **integers** $\mathbb{Z}$ is $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.
- The set of **rational numbers** $\mathbb{Q}$ is $\left\{\frac{p}{q} \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\right\}$.
  A number is rational if
  - it can be written as a quotient of two integers, or
  - its decimal expansion is finite, or
  - its decimal expansion has a recurring digit or pattern of digits.
- Any number that has a decimal expansion with an infinite number of digits after the decimal point and with no period is an **irrational number**.
- The set of rational numbers together with the set of irrational numbers complete the number line and form the set of **real numbers**, $\mathbb{R}$.

Approximations and error

- Rounding a number to the **nearest 10** is the same as rounding it to the **nearest multiple of 10**.
- Rounding a number to the **nearest 100** is the same as rounding it to the **nearest multiple of 100**.
- **Rules for rounding**
  - If the digit after the one that is being rounded is less than 5 then keep the rounded digit unchanged and change all the remaining digits to the right of this to 0.
  - If the digit after the one that is being rounded is 5 or more then add 1 to the rounded digit and change all remaining digits to the right of this to 0.
- Rounding a number **correct to one decimal place** is the same as rounding it to the **nearest tenth**.
- Rounding a number **correct to two decimal places** is the same as rounding it to the **nearest hundredth**.
- Rounding a number **correct to three decimal places** is the same as rounding it to the **nearest thousandth**.
- **Rounding rules for decimals**
  - If the digit after the one that is being rounded is less than 5 keep the rounded digit unchanged and delete all the following digits.
  - If the digit after the one that is being rounded is 5 or more then add 1 to the rounded digit and delete all the following digits.
- The number of **significant figures** in a result is the number of figures that are known with some degree of reliability.

Continued on next page

- **Rules for significant figures:**
  - All non-zero digits are significant.
  - Zeros between non-zero digits are significant.
  - Zeros to the left of the first non-zero digit are *not* significant.
  - Zeros placed after other digits but to the right of the decimal point are significant.
- **Rounding rules for significant figures**
  - If the $(n+1)$th figure is less than 5 then keep the nth figure unchanged.
  - If the $(n+1)$th figure is 5 or more then add 1 to this figure.
  - In both cases all the figures to the right of figure n should be deleted if they are to the right of the decimal point and should be replaced by zeros if they are to the left of the decimal point.
- To **estimate** the answer to a calculation, round all the numbers involved to 1 sf.
- The difference between an **estimated** or **approximated value** and the **exact value** is called the **error**:

  $$\text{Error} = v_A - v_E$$

  where v_A is the approximated value and v_E is the exact value.
- Percentage error $= \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$

  where v_A represents **approximated value** or **estimated value** and v_E represents the **exact value**.

Standard form

- A number is written in **standard form** if it is in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer.

SI units of measurement

- In Mathematical Studies the most common **SI base units** used are m, kg and s, and **derived units** are m^2 (area), m^3 (volume), $km\,h^{-1}$ (velocity), $kg\,m^{-3}$ (density).
- To avoid writing very small or very large quantities we use **prefix** names and prefix symbols. Some of these are shown in this table.

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^3	kilo	k	10^{-3}	milli	m
10^2	hecto	h	10^{-2}	centi	c
10^1	deca	da	10^{-1}	deci	d

- There are some units that are **non-SI** units but are accepted for use with the SI because they are widely used in everyday life, for example, min, h, *l*.
- There are three temperature scales: **kelvin** (K), **Celsius** (°C) and **Fahrenheit** (°F).

A rational explanation

The Pythagorean School, around 2500 years ago, believed that all numbers were rational. This idea was expressed in terms of sticks of different lengths which could be measured exactly by a third, shorter stick.

For example these sticks:

can both be measured by this one:

like this:

- What is the ratio of the shorter stick to the longer one?
- What fraction of the longer stick is the shorter stick?
- What fraction of the shorter stick is the longer stick?

Because a smaller stick exists which divides exactly into both larger sticks, the two sticks are said to be 'commensurable'. The early Pythagoreans believed that all numbers could be represented by a series of commensurable lines.

The Pythagorean School had very strict rules and was a school of philosophy as well as of mathematics. Find out more about its principles and beliefs.

Brahmi		−	=	≡	+	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Hindu	०	१	२	३	४	५	६	७	८	९
Arabic	٠	١	٢	٣	٤	٥	٦	٧	٨	٩
Medieval	o	ı	2	3	ꝛ	ϥ	6	ʌ	8	9
Modern	0	1	2	3	4	5	6	7	8	9

- Where do our numerals for zero to ten come from?
- When was zero discovered – or was it invented?

Hippasus demonstrates an irrational number

According to legend one of the Pythagoreans, Hippasus, first demonstrated that $\sqrt{2}$ was not rational. It is likely that Hippasus used the idea that sticks of length $\sqrt{2}$ and 1 could not both be measured by the same stick, no matter how small.

First, a few things Hippasus did know…

1 Pythagoras' theorem: this square with side length 1 has a diagonal of length $\sqrt{2}$.

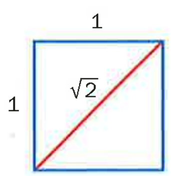


Figure 1

2 If a stick could measure two larger sticks it could measure the difference between them. In the example above, the difference is 2 times the measuring stick.

So Hippasus reasoned that if there was a stick that could measure both the side and the diagonal of a square then it could measure the difference, shown in green in Figure 2.

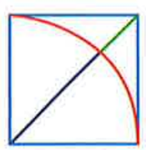


Figure 2

William Dunham in his book *Journey through Genius* hints at how Hippasus might have done this.

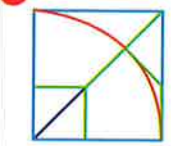


Figure 3

He knew enough circle theorems to deduce that all the green line segments (Figure 3) were the same length and so could be measured by the original stick. So could the part colored purple.

So he started again with the small square and diagonal and made the same picture within it… and again within that …


Figure 4

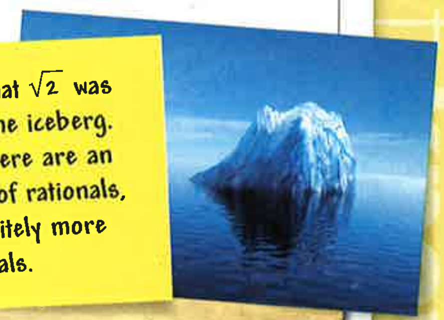

Figure 5

He argued that because the square was getting smaller every time, the measuring stick must be even smaller, in the end vanishing, because the reduction could be repeated indefinitely. Because the stick got so small it would vanish, it must therefore not exist in the first place.

His colleagues were convinced, but they most certainly weren't happy, and they threw him off a ship leaving him to drown. No doubt the story has gained a few details over the years, but the discovery of irrational numbers did have a profound effect on the Greek mathematicians, who for several centuries left the study of Number and concentrated on the 'safe' topic of Geometry.

- Were irrational numbers created or discovered?
- Do irrational numbers exist?

Now we know that $\sqrt{2}$ was just the tip of the iceberg. Even though there are an infinite number of rationals, there are infinitely more irrationals.



Cantor's proof

Georg Cantor described infinitely big sets as either countably infinite or uncountably infinite. Countably infinite sets described a set where each member could be counted with the natural numbers: 1, 2, 3, 4, … . The process would go on forever, but because the members of the set had been put in some sort of order you could move forward counting them without leaving any out. He showed that rational numbers could be ordered in this way, but that it was impossible to do this with irrational numbers. Whatever order you devised, there would always be irrational numbers missing from the list.

Cantor's theories are today a standard (if slightly uncomfortable) part of mathematics, but at the time they caused even more controversy than Hippasus did in his day. Cantor was seen as undermining mathematics and his ideas were rejected by almost all contemporary mathematicians of that time. He suffered from severe depression and finished his life in a sanatorium.

Cantor lived in Vienna during World War I, when the Austro-Hungarian Empire was collapsing. His fellow citizens were fearful of the change they saw around them. Perhaps Cantor 'changing' the concept of number was a step too far?

- Can mathematics develop 'in a bubble'?
- Can mathematicians ever be free from external influences?

2 Descriptive statistics

CHAPTER OBJECTIVES:

4.1–4.3 Discrete and continuous data: frequency tables; mid-interval values; upper and lower boundaries. Frequency histograms

4.4 Cumulative frequency tables; cumulative frequency curves; median and quartiles. Box and whisker diagrams

4.5 Measures of central tendency: mean; median; mode; estimate of a mean; modal class

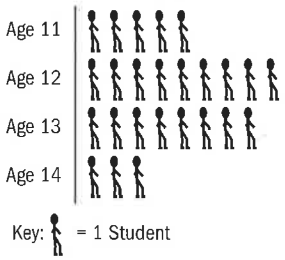
4.6 Measures of dispersion: range, interquartile range, standard deviation

Before you start

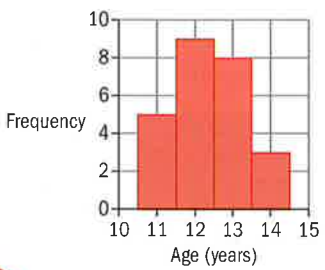
You should know how to:

1 Collect and represent data using

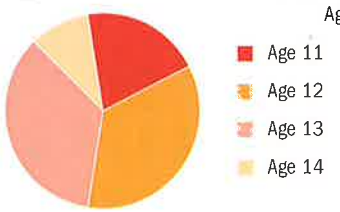
a a pictogram



b a bar chart



c a pie chart



2 Set up axes on graphs using given scales.

Skills check

1 Maerwen wants to find out information about the numbers of men, women, boys and girls using a library. Design a suitable data-collection sheet to collect the information.

2 These data show the number of different colored sweets in a packet.

Color	blue	green	red	orange	yellow
Frequency	5	7	8	4	6

a Draw a pictogram to represent these data.

b Draw a bar chart to represent these data.

c Draw a pie chart to represent these data.

3 On graph paper, draw a set of axes such that 1 cm represents 2 units on the x-axis and 1 cm represents 10 units on the y-axis.

Every country needs basic information on its population so that it can plan and develop the services it needs. For example, to plan a road network you need to know the size of the population so you can estimate the amount of traffic in an area.

To collect information on a population, governments often organize a census. A census is a survey of the **whole population** of a country.

The information collected includes data on age, gender, health, housing, employment and transport. The data are then analyzed and presented in tables, charts and spreadsheets. All data should be processed so that information on individuals is protected. The United Nations recommends that population censuses should be taken at least every 10 years.

In what other areas of society is mathematics used in a practical way?

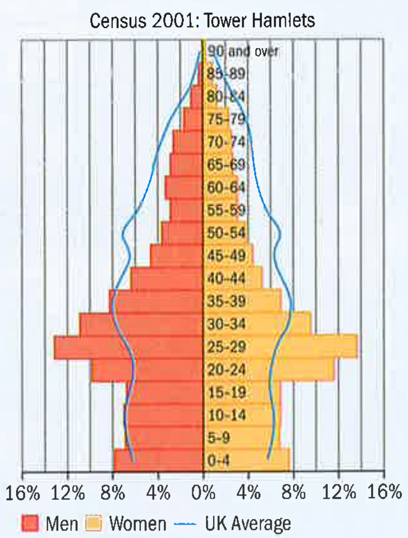
What are the benefits of sharing and analyzing data from different countries?

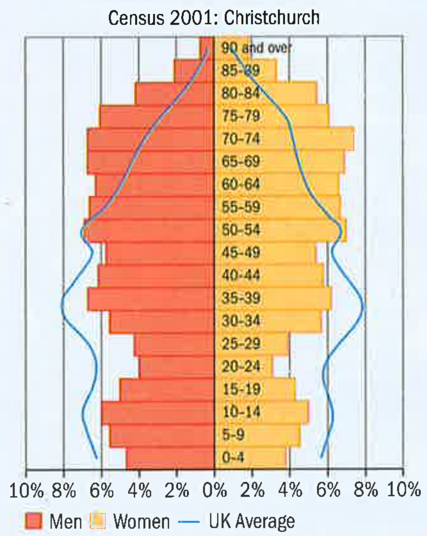
When was the last census in your country? Is the census information published in the public domain? How has technology changed the way census data is collected and presented?

Investigation – population distribution

In the United Kingdom, there is a census every 10 years.

These population pyramids are based on information collected in the 2001 census. They show the distribution of age ranges in Tower Hamlets, London, and Christchurch, Dorset.





Compare the population pyramids for Tower Hamlets and Christchurch.

Simply based on these data, make a number of conjectures about these two areas.

Fully research the areas to test your conjectures. How accurate were you?

All information from the 2001 census can be found at www.ons.gov.uk by searching for '2001 census data'.

In this chapter you will organize data in frequency tables, graph data in a variety of diagrams, and analyze data using a range of measures.

How is education data used to investigate the link between the level of education and patterns of creating families and fertility?

2.1 Classification of data

There are two main types of data: **qualitative** and **quantitative**. Qualitative data are data that are not given numerically, for example, favorite color. Quantitative data are numerical.

Quantitative data can be further classified as **discrete** or **continuous**.

→ **Discrete data** are either data that can be **counted** or data that can only take **specific values**.

Examples of data that can be counted include the number of sweets in a packet, the number of people who prefer tea to coffee, and the number of pairs of shoes that a person owns.

Examples of data that can only take specific values include shoe size, hat size and dress size.

→ **Continuous data** can be **measured**. They can take any value within a range.

Examples of continuous data include weight, height and time.

Continuous data can be expressed to a required number of significant figures. The greater the accuracy required, the more significant figures the data will have.

The weighing scale was invented at a time when countries began trading materials and a standard measurement was required to ensure fair trading.

Time is a continuous measure because it can take any numerical value in a particular range. For example, the time taken for world-class sprinters to run 100 m can be recorded as any fraction of a second.

Is the number of grains of salt in a salt cellar discrete?

The number of shoes and shoe size are examples of discrete data.

Population and sample

When conducting a statistical investigation, the whole of a group from which we may collect data is known as the **population**. It is not always possible, or even necessary, to access data for a whole population.

You can make conclusions about a population by collecting data from a sample. It is usually cheaper and quicker to collect data from a sample.

A **sample** is a small group chosen from the population.

A **random sample** is one where each element has the same chance of being included.

A **biased sample** is one that is not random.

It is important that a sample is random and not biased – it must be **representative** of the elements being investigated. To ensure that the different members of the population have an equal probability of being selected you could choose people by picking names out of a hat. Or you could assign a number to each member of the population and then choose numbers at random using the random number function on a GDC.

A weighing scale gives us continuous data.

Can the wording of a survey question and the way the data are presented introduce bias?

Sampling will not be examined. However, if you use sampling when writing your Mathematical Studies project, you will need to discuss how you picked your sample and convince the moderator that it is indeed a random sample.

Are exit polls a good way of predicting the results of an election?

Example 1

Kiki wants to find out if there is any connection between eating breakfast and grades among students in her school. However, there are too many students in the school to ask everyone. She needs to pick a sample.
How can she make sure that the sample she picks is a random sample?

Answer
Kiki can use her GDC to generate random numbers and use the students who have those numbers on the school register.

Does each student have the same chance of being included in her sample? If yes, it is a random sample.

In market research, a sample of the population is interviewed in order to collect data about customers. Many research methods have been developed since companies began to carry out formal market research in the 1920s.

Example 2

Ayako is conducting a survey to find out how much money women who live in London spend on fashion in a month. She only interviews women coming out of Harrods (a very exclusive store). Is this a random sample?

Answer
No, because the sample will not come from the total population of women in London and some of the women she interviews may not even belong to the population.

Is Ayako only asking 'women who live in London'?
Do all women who live in London shop at Harrods?

Exercise 2A

1 State whether these data are discrete or continuous.
  - **a** The number of sweets in a packet
  - **b** The heights of students in Grade 8
  - **c** The dress sizes of a girls' pipe band
  - **d** The number of red cars in a parking lot
  - **e** The weights of kittens
  - **f** The marks obtained by Grade 7 in a science test
  - **g** The times taken for students to write their World Literature paper
  - **h** The weights of apples in a 5 kg bag
  - **i** The number of cm of rain each day during the month of April
  - **j** The number of heads when a coin is tossed 60 times
  - **k** The times taken for athletes to run a marathon
  - **l** The number of visitors to the Blue Mosque each day.

2 State whether the following samples are random or biased.
  - **a** When researching if people eat breakfast, only interview the people in the canteen.
  - **b** When researching spending habits, interview every third person you meet.
  - **c** When researching spending habits on cars, Josh interviews men exiting a garage.
  - **d** When comparing GDP to child mortality, Eizo chooses the countries from a numbered list, by generating random numbers on his GDC.
  - **e** When researching the sleeping habits of children, Adham distributes a questionnaire to the students in his school.

GDP, gross domestic product, is the total value of goods produced and services provided in a country in a year.

2.2 Simple discrete data

When there is a large amount of data, it is easier to interpret if the data are organized in a **frequency table** or displayed as a graph.

Example 3

The numbers of sweets in 24 packets are shown below.

22 23 22 22 23 21 22 22 20 22 24 21
22 21 22 23 22 22 24 20 22 23 22 22

Organize this information in a frequency table.

Answer

Number of sweets	Tally	Frequency
20	||	2
21	|||	3
22	卌 卌 |||	13
23	||||	4
24	||	2
	TOTAL	**24**

Draw a chart with three columns.

Write the possible data values in the 'Number of sweets' column.

Use tally marks to record each value in the 'Tally' column.

For each row, count up the tally marks and write the total in the 'Frequency' column.

Add up the values in the 'Frequency' column to work out the total frequency.

Now you can see how many packets have each number of sweets.

Exercise 2B

1 The numbers of goals scored by Ajax football team during their last 25 games were:

1 3 0 2 1 1 2 3 0 1 2 2 5 0 2 1 4 3 2 1 0 1 2 3 5

Organize this information in a frequency table.

2 The numbers of heads obtained when twelve coins were tossed 50 times are recorded below.

8 3 5 7 1 9 2 10 5 12 7 6 6 8 12 4 10 2 6 6 8 4 5 11 3
4 6 8 6 7 5 3 11 2 10 5 6 7 5 8 9 2 10 11 0 12 3 6 6 5

Organize this information in a frequency table.

3 The ages of the girls in a hockey club are:

10 11 12 10 9 11 15 13 12 16 11 13 14 12 10 10 11 9 9 10
10 12 15 16 12 11 13 10 15 13 12 11 15 16 11 12 10 9 10 11

Organize this information in a frequency table.

4 It is stated that there are 90 crisps in a box.
Viktoras checked 30 boxes and the numbers of crisps in them are recorded below.

90 90 91 90 89 89 90 90 92 90 90 88 89 90 90
91 90 89 90 88 89 90 91 90 92 88 89 90 90 90

Organize this information in a frequency table.

5 Sean threw a dice 50 times. The numbers that appeared are shown below.

1 1 3 2 6 6 5 6 4 4 3 6 2 1 3 5 6 3 2 1 4 5 6 3 2
1 5 3 4 6 2 5 5 4 2 1 3 6 4 2 3 1 6 3 2 5 3 3 2 6

Organize this information in a frequency table.

EXAM-STYLE QUESTION

6 The numbers of games played in matches at a badminton tournament are recorded below.

8 8 10 11 9 7 8 7 11 12 7 8 10 10 11 9 9 8 11 7 9 8

The raw data have been organized in the frequency table.

Games	Frequency
7	4
8	m
9	4
10	n
11	4
12	1

Write down the values of m and n.

2.3 Grouped discrete or continuous data

When there are a lot of data values spread over a wide range it is useful to **group** the data. Depending on the number of data values, there should be between 5 and 15 groups, or classes, of equal width.

The classes must cover the range of the values and they must not overlap – each data value must belong to only one class.

You can organize both discrete and continuous data in **grouped frequency tables**.

Example 4

Loni made 30 telephone calls one week. The times of the calls, in minutes, were recorded.

3.1	12.2	9.6	8.1	2.2	1.2	15.0	4.8	21.2	13.6
17.3	22.3	1.5	4.6	31.2	26.7	7.8	18.2	35.4	1.6
2.9	5.5	12.8	28.3	16.9	1.3	5.6	7.8	2.3	6.9

Organize this information in a grouped frequency table.

The frequency table gives a much clearer picture of the data.

Answer

Time (t)	Frequency
$0 \le t < 5$	10
$5 \le t < 10$	7
$10 \le t < 15$	3
$15 \le t < 20$	4
$20 \le t < 25$	2
$25 \le t < 30$	2
$30 \le t < 35$	1
$35 \le t < 40$	1

First decide on the size and the number of classes:

Smallest number = 1.2 so classes start at 0.

Largest number = 35.4 so classes finish at 40.

Using a class width of 5, there will be (40 ÷ 5 =) 8 classes in total.

Exercise 2C

1 Organize each of these sets of data in a grouped frequency table.

a 2 5 12 21 7 9 25 31 17 19 22 23 15 24 5
34 45 32 13 43 7 11 32 6 18 40 23 32 22 8

b 10 24 31 29 42 19 55 65 46 72 35 48 68 56 92
12 33 77 56 45 82 76 56 34 12 78 89 45 59 32
26 97 67 54 34 18 77 59 34 27 13 19 63 65 22

c 1 3 8 12 4 2 6 3 9 10 11 9 7 5 14 2 3 16
9 5 13 14 4 8 17 3 15 19 5 3 9 10 11 14 15

Upper and lower boundaries

To find the **upper** and **lower boundaries** of a class, calculate the mean of the upper value from one class and the lower value from the following class.

Example 5

This table shows the heights of flowers in a garden.
Write down
a the upper boundary of the first class
b the lower boundary of the third class.

Height (x cm)	Frequency
$0 \le x < 10$	5
$10 \le x < 20$	12
$20 \le x < 30$	21
$30 \le x < 40$	15
$40 \le x < 50$	6

Answers

a $\frac{10+10}{2} = 10$

Upper value of the first class is 10. Lower value of the second class is 10. The upper boundary of the first class is the mean of these two values.

b $\frac{20+20}{2} = 20$

Upper value of the second class is 20. Lower value of the third class is 20. The lower boundary of the third class is the mean of these two values.

Example 6

The table shows the numbers of pairs of shoes of each size sold in a shop one day.
Write down
a the upper boundary of the first class and the last class
b the lower boundary of the first class and the fourth class.

Shoe size	Frequency
15–19	3
20–24	9
25–29	12
30–34	22
35–39	45
40–44	31

These are European shoe sizes. What are the equivalent shoe sizes in your country?

Answers

a Upper boundary of the first class: $\frac{19+20}{2} = 19.5$

Upper boundary of the last class: $\frac{44+45}{2} = 44.5$

Upper value of the first class is 19. Lower value of the second class is 20. The upper boundary of the first class is the mean of these two numbers. Similarly for last class.

b Lower boundary of the first class: $\frac{14+15}{2} = 14.5$

Lower boundary of the fourth class: $\frac{29+30}{2} = 29.5$

Upper value of the previous class would be 14. Lower value of the first class is 15. The lower boundary of the first class is the mean of these two numbers. Similarly for fourth class.

How could the shoe shop manager use this data?

Exercise 2D

1 Copy these tables and fill in the missing lower and upper boundary values.

a

Class	Lower boundary	Upper boundary
9–12		12.5
13–16		
17–20	16.5	
21–24		

b

Time (t seconds)	Lower boundary	Upper boundary
$2.0 \le t < 2.2$		
$2.2 \le t < 2.4$		
$2.4 \le t < 2.6$		

Frequency histograms

A **frequency histogram** is a useful way to represent data visually.

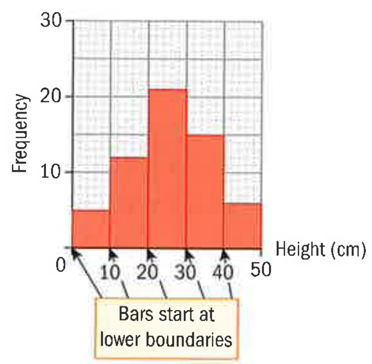
→ To draw a frequency histogram, find the lower and upper boundaries of the classes and draw the bars between these boundaries. There should be no spaces between the bars.

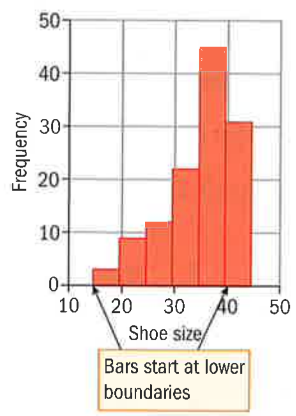
In the Mathematical Studies course you will only deal with frequency histograms that have equal class intervals.

The class boundaries are plotted on the x-axis and the frequency values on the y-axis.

Here are the frequency histograms for Examples 5 and 6.

English statistician Karl Pearson (1857–1936) first used the term 'histogram' in 1895.





Exercise 2E

1 The costs, in euros, of 80 dinners are shown in the table.
Draw a histogram to display this information.

Cost of dinner in euros (c)	Frequency
$10 \le c < 15$	2
$15 \le c < 20$	8
$20 \le c < 25$	11
$25 \le c < 30$	25
$30 \le c < 35$	14
$35 \le c < 40$	11
$40 \le c < 45$	6
$45 \le c < 50$	3

2 The table shows the age distribution of teachers at Genius Academy.

a Write down the lower and upper boundaries of each class.

b Draw a histogram to represent the information.

Age (x)	Frequency
$20 \le x < 30$	4
$30 \le x < 40$	8
$40 \le x < 50$	10
$50 \le x < 60$	9
$60 \le x < 70$	3

3 The masses of 150 melons are recorded in the table.

a Write down the lower and upper boundaries of the third class.

b Draw a histogram to represent the information.

Mass (x kg)	Frequency
$0.4 \le x < 0.6$	21
$0.6 \le x < 0.8$	36
$0.8 \le x < 1.0$	34
$1.0 \le x < 1.2$	29
$1.2 \le x < 1.4$	18
$1.4 \le x < 1.6$	12

4 The lengths of 100 worms (to the nearest cm) are given in the table.

Length (cm)	4	5	6	7	8	9	10
Frequency	18	20	26	15	8	6	7

a Write down the lower and upper boundaries of each class.

b Draw a histogram to represent the information.

5 50 people were asked how often they traveled by train each month. The results were:

8	7	10	5	23	4	16	9	62	28
14	53	29	11	34	33	68	75	12	79
22	54	67	55	13	32	41	58	36	2
26	80	65	38	52	71	2	16	36	40
18	24	52	64	76	16	6	18	28	40

a Organize this information in a grouped frequency table.

b Draw a histogram to represent the information graphically.

6 Yuri decided to count the number of weeds in one square metre of grass. He chose 80 plots of one square metre. The results for each square metre are:

22	24	21	12	8	14	34	62	54	6	28	42	35	22	14	18	9	24	12	18
31	47	17	9	35	24	41	52	38	19	5	23	31	65	32	46	15	13	74	22
9	13	22	55	47	52	14	13	21	19	52	33	71	12	22	17	58	42	31	16
2	15	31	73	45	31	12	8	4	33	42	57	61	48	43	27	14	5	14	26

a Organize this information in a grouped frequency table.

b Draw a histogram to represent the information graphically.

7 Simi recorded the numbers of vans per five minutes that drove down her street over a period of eight hours. Her results were:

Number of vans (x)	Frequency
$1 \le x \le 5$	12
$6 \le x \le 10$	23
$11 \le x \le 15$	31
$16 \le x \le 20$	13
$21 \le x \le 25$	9
$26 \le x \le 30$	5
$31 \le x \le 35$	2
$36 \le x \le 40$	1

a Write down the lower and upper boundaries of the fourth class.

b Draw a histogram to represent the information.

EXAM-STYLE QUESTION

8 The numbers of visitors per hour to the Taj Mahal are recorded in the table.

Time (t)	Number of visitors
$09{:}00 \le t < 10{:}00$	324
$10{:}00 \le t < 11{:}00$	356
$11{:}00 \le t < 12{:}00$	388
$12{:}00 \le t < 13{:}00$	435
$13{:}00 \le t < 14{:}00$	498
$14{:}00 \le t < 15{:}00$	563
$15{:}00 \le t < 16{:}00$	436
$16{:}00 \le t < 17{:}00$	250
$17{:}00 \le t < 18{:}00$	232

Draw a histogram to represent this information.

2.4 Measures of central tendency

Data can be summarized by using a measure of central tendency such as the mode, median or mean.

→ The **mode** of a data set is the value that occurs most frequently.

The **median** of a data set is the value that lies in the middle when the data are arranged in size order.

The **mean** of a data set is the sum of all the values divided by the number of values.

When there are two 'middle' values, the median is the midpoint between the two middle values. To find the midpoint, add the two middle values and divide by two.

How do you know which measure of central tendency is the best to use?

Can you mislead people by quoting statistics? For example, the numbers 1, 1, 100 have mode = 1, median = 1 and mean = 34.

You have to be aware that there may be outliers (isolated points outside the normal range of values) that skew the statistics.

What are the ethical implications of using statistics to mislead people?

Example 7

Here is a set of data: 5 4 8 4 4 7 8 9 11 1 5
Find the mode, median and mean.

Answer

5 (4) 8 (4) (4) 7 8 9 11 1 5
Mode = 4

The value '4' occurs three times.

1 4 4 4 5 (5) 7 8 8 9 11
Median = 5

First arrange the data in size order. There are 11 entries, so the median is the $\frac{11+1}{2}$ *= 6th value.*

Mean = $\frac{1+4+4+4+5+5+7+8+8+9+11}{11}$

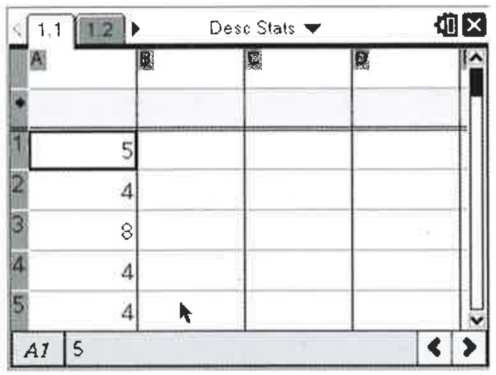
$= \frac{66}{11}$

Mean = 6

The mean is the $\frac{\text{sum of all the values}}{\text{number of values}}$

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

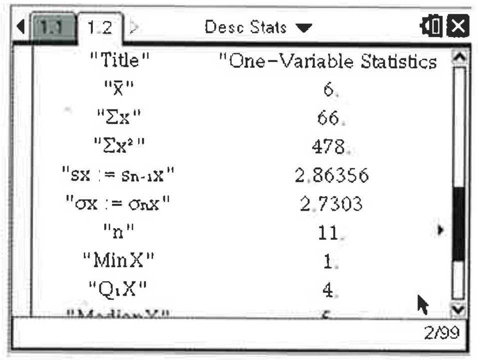
You can also use a GDC to calculate the median and mean.
Enter the data values:



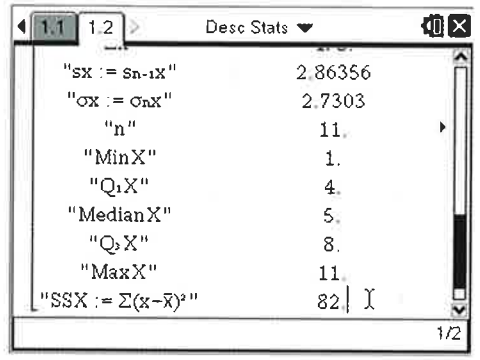
For help with entering data values, see Chapter 12, Section 2.1.

The GDC screen is too small to display all of the values in the list. Scroll down to see the remaining values.

The value of the mean is given by $\bar{x}$ (pronounced 'x-bar'):



The value of the median is shown as 'MedianX':



Exercise 2F

1 Calculate the mode, median and mean for each data set.

a 7 3 8 9 1 10 1

b 3 4 8 2 5 6 11 13 3 5 6 5

2 Calculate the values of **a**, **b**, **c**, **d** and **e** in this table.

Data	Median	Mode	Mean
Height (m): 1.52, 1.74, 1.83, 1.52, 1.67, 1.91	**a**	**b**	1.70
Age (years): 21, 34, 17, 22, 56, 38	28	none	**c**
Weight (kg): 54.7, 48.6, 63.2, 55.1, 77.9, 48.6	**d**	48.6	**e**

The German psychologist Gustav Fechner (1801–1887) popularized the use of the median, although French mathematician and astronomer Pierre-Simon Laplace (1749–1827) had used it previously.

3 The weights of eight pumpkins are
26.3 kg, 12.6 kg, 33.5 kg, 8.9 kg, 18.7 kg, 22.6 kg, 31.8 kg and 45.3 kg.

a Find the median weight.

b Calculate the mean weight.

EXAM-STYLE QUESTIONS

4 For these data the mode is 5, the median is 6 and the mean is 6.5.

1 1 2 3 s 5 5 7 8 9 10 t 12 12

Given that $s < t$, find the values of s and t.

5 Jin obtained marks of 76, 54 and 65 in his Physics, Biology and History examinations respectively.

a Calculate his mean mark for the three examinations.

b Find the mark that Jin must achieve in Mathematics so that the mean mark for the four examinations is exactly 68.

EXAM-STYLE QUESTION

6 Zoe and Shun compared their test scores. Zoe had a mean of 81 after taking five tests and Shun had a mean of 78 after taking three tests. Each of them took one more test and ended up with the same mean score of 80.

a Find the grade that Zoe gained on her sixth test.

b Find the grade that Shun gained on his fourth test.

Mean, median and mode from a frequency table

→ For data in a frequency table, the **mode** is the entry that has the largest frequency.

The **median** is the middle entry as the entries in the table are already in order. For n pieces of data, the median is the $\frac{n+1}{2}$th value.

The next example shows how to calculate the mean from a frequency table.

Example 8

Calculate the mode, median and mean of these data.

Number of sweets	Frequency
20	2
21	3
22	13
23	4
24	2
TOTAL	24

Answer

Mode = 22

'22' has the highest frequency (13).

Median = 22

Median is the $\frac{24+1}{2}$ *= 12.5th entry, so it is between the 12th and 13th entry. Both the 12th and 13th entries are 22, so the median = 22.*

To calculate the mean: Label the first column x_i. Label the second column f_i. Add a third column and label it f_ix_i.

Work out $f_i \times x_i$ for each row:

$2 \times 20 = 40$

$3 \times 21 = 63$

$13 \times 22 = 286$

$4 \times 23 = 92$

$2 \times 24 = 48$

Work out the total of the f_i column and the total of the f_ix_i column.

Number of sweets, x_i	Frequency, f_i	f_ix_i
20	2	40
21	3	63
22	13	286
23	4	92
24	2	48
TOTAL	24	529

Mean = $\frac{529}{24}$ = 22.0 (to 3 sf)

Mean = $\frac{\text{total of } f_ix_i}{\text{total of } f_i}$

Sometimes a question asks for the 'modal value'. This means 'the mode'.

→ The **mean** from a frequency table is:

$$\text{mean} = \frac{\text{total of } f_i \times x_i}{\text{total frequency}}$$

where f_i is the frequency of each data value x_i and $i = 1, \ldots, k$, where k is the number of data values.

The IB formula for the mean is:

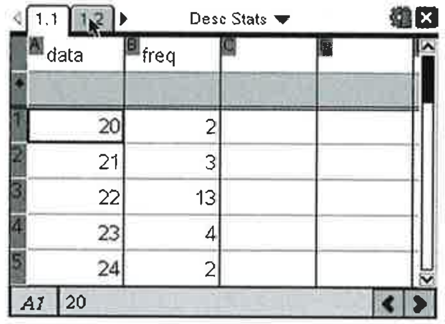
$\bar{x} = \frac{\sum_{i=1}^{k} f_ix_i}{n}$, where $n = \sum_{i=1}^{k} f_i$

The Σ notation simply means 'sum'.

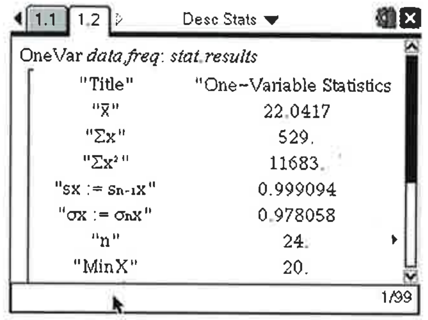
This formula is given in the Formula booklet.

You can also use your GDC to calculate the mean and median from a frequency table.

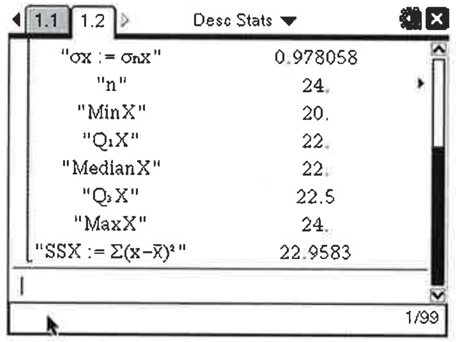
Enter the data values:



The value of the mean is given by $\bar{x}$:



The value of the median is shown as 'MedianX':



Exercise 2G

1 A dice is thrown 29 times and the score noted. The results are shown in the table.

a Write down the modal score.

b Write down the median score.

c Calculate the mean score.

Score	Frequency
1	4
2	7
3	3
4	8
5	5
6	2

EXAM-STYLE QUESTION

2 The table shows the frequency of the number of visits to the doctor per year for a group of children.

a How many children are in the group?

b Write down the modal number of visits.

c Calculate the mean number of visits.

Number of visits	0	1	2	3	4	5
Frequency	4	3	8	5	4	1

EXAM-STYLE QUESTIONS

3 A bag contains six balls numbered 1 to 6. A ball is drawn at random and its number noted. The ball is then returned to the bag.
The numbers for the first 30 draws are:

Number	1	2	3	4	5	6
Frequency	4	5	3	n	6	5

a Write down the value of n.
b Calculate the mean number.
c Write down the modal number.

4 The table gives the frequency of grades achieved by students in an IB school.
a Calculate the mean grade.
b What percentage of students achieved a grade 4 or 5?
c Write down the modal grade.

Grade	Frequency
1	1
2	6
3	19
4	34
5	32
6	18
7	10

Mean, median and mode for grouped data

For grouped data, you can find the modal class and an **estimate of the mean**.

→ For grouped data, the **modal class** is the group or class interval that has the largest frequency.

The next example shows how to calculate an estimate of the mean.

Example 9

The times, in seconds, taken to complete 200 bouts of sumo wrestling are shown in the table.

Time (t seconds)	Frequency
$0 \le t < 20$	37
$20 \le t < 40$	62
$40 \le t < 60$	46
$60 \le t < 80$	25
$80 \le t < 100$	11
$100 \le t < 120$	9
$120 \le t < 140$	6
$140 \le t < 160$	4
TOTAL	200

You do not know the exact data values for each group. Use the midpoint of each class interval as an estimate of the values in each group. You may also find the midpoint referred to as the 'mid-interval value'.

To find the midpoint of a class, find the mean of the class limits.
$\text{midpoint} = \frac{\text{lower boundary} + \text{upper boundary}}{2}$

Calculate **a** the modal class and **b** an estimate of the mean.

▶ Continued on next page

Answer

a Modal class = $20 \le t < 40$

This class interval has the largest frequency (62).

b

Time (t seconds)	Frequency, f_i	Midpoint, x_i	$f_i\, x_i$
$0 \le t < 20$	37	10	370
$20 \le t < 40$	62	30	1860
$40 \le t < 60$	46	50	2300
$60 \le t < 80$	25	70	1750
$80 \le t < 100$	11	90	990
$100 \le t < 120$	9	110	990
$120 \le t < 140$	6	130	780
$140 \le t < 160$	4	150	600
TOTAL	200		9640

Mean = $\frac{9640}{200}$ = 48.2 (to 3 sf)

*To work out an estimate of the mean, you must first work out the **midpoint** of each class interval. Add a third column and label it 'Midpoint, x_i'. Work out each midpoint:*

Midpoint of $0 \le t < 20$: $\frac{0+20}{2} = 10$

Midpoint of $20 \le t < 40$: $\frac{20+40}{2} = 30$

Midpoint of $40 \le t < 60$: $\frac{40+60}{2} = 50$

Next add a fourth column and label it '$f_i x_i$'. Then work out $f_i \times x_i$ for each row:
$9 \times 110 = 990$
$6 \times 130 = 780$

Work out the total of the f_i column and the total of the $f_i\, x_i$ column.

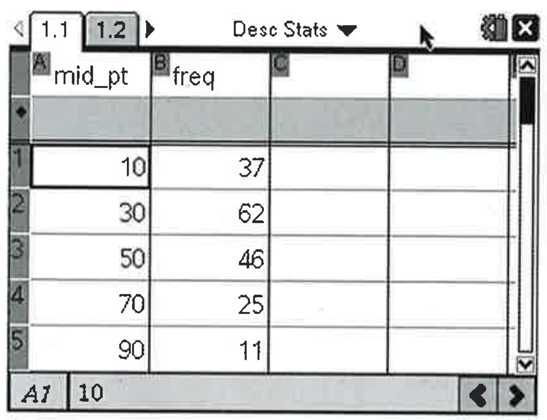
Mean = $\frac{\text{total of } f_i x_i}{\text{total } f_i}$

→ To calculate an estimate of the **mean** from a grouped frequency table, use $\frac{\text{total of } f_i \times x_i}{\text{total frequency}}$ where f_i is the frequency and x_i is the corresponding midpoint of each class.

Why does this give an estimate of the mean and not an exact value?

You can also use a GDC to calculate an estimate of the mean from a grouped frequency table.

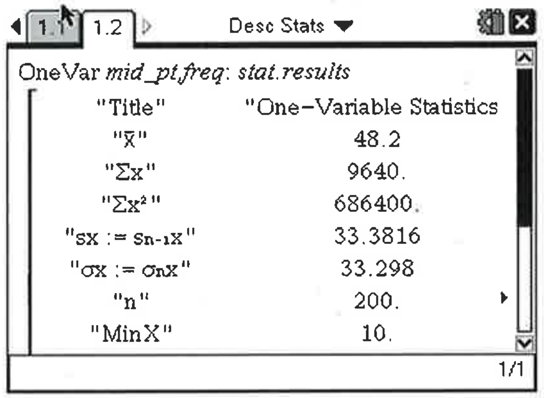
Enter the data values:



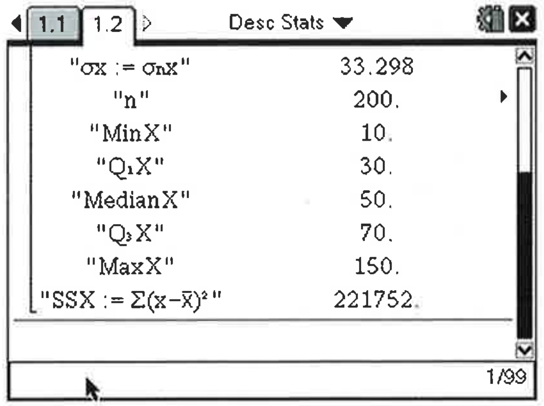
For help with entering data values, see Chapter 12, Section 2.2.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

The value of the mean is given by $\bar{x}$:



You were not asked for the median but the GDC works it out as part of the calculation screen (this too is only an estimate as we do not have all the individual values):



Exercise 2H

EXAM-STYLE QUESTIONS

1 The table shows the times taken for 25 cheetahs to cover a distance of 50 km.

a Write down the modal class.

b Calculate an estimate of the mean time taken.

Time taken (t minutes)	Frequency
$20 \le t < 22$	2
$22 \le t < 24$	5
$24 \le t < 26$	8
$26 \le t < 28$	4
$28 \le t < 30$	3
$30 \le t < 32$	2
$32 \le t < 34$	1

2 The speeds of vehicles passing under a bridge on a road are recorded in the table.

a Write down the modal class.

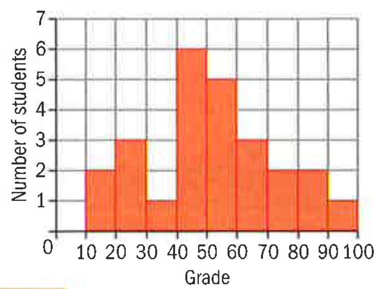
b Calculate an estimate of the mean speed of the vehicles.

Speed (s km h^{-1})	Frequency
$60 \le s < 70$	8
$70 \le s < 80$	15
$80 \le s < 90$	12
$90 \le s < 100$	10
$100 \le s < 110$	8
$110 \le s < 120$	3
$120 \le s < 130$	4

3 The results of a Geography test for 25 students are given in the diagram.

a Write down the modal class.

b Calculate an estimate of the mean grade.



2.5 Cumulative frequency curves

→ The **cumulative frequency** is the sum of all of the frequencies up to and including the new value. To draw a **cumulative frequency curve** you need to construct a cumulative frequency table, with the upper boundary of each class interval in one column and the corresponding cumulative frequency in another. Then plot the upper class boundary on the x-axis and the cumulative frequency on the y-axis.

Example 10

A supermarket is open 24 hours a day and has a free car park. The number of parked cars each hour is monitored over a period of several days. The results are shown in the table.
Organize this information in a cumulative frequency table.
Draw a graph of the cumulative frequency.

Number of parked cars per hour	Frequency
0–49	6
50–99	23
100–149	41
150–199	42
200–249	30
250–299	24
300–349	9
350–399	5

Answer

Number of parked cars per hour	Frequency	Upper boundary	Cumulative frequency
0–49	6	49.5	6
50–99	23	99.5	29
100–149	41	149.5	70
150–199	42	199.5	112
200–249	30	249.5	142
250–299	24	299.5	166
300–349	9	349 5	175
350–399	5	399.5	180

Add a third column and label it 'Upper boundary'.
Work out the upper boundary of each class:

Upper boundary $= \frac{49+50}{2} = 49.5$

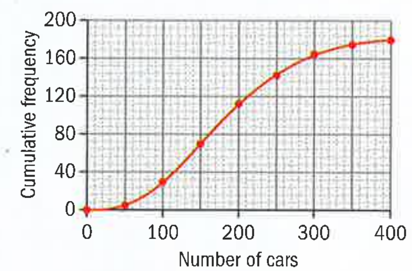
Upper boundary $= \frac{99+100}{2} = 99.5$

Upper boundary $= \frac{149+150}{2} = 149.5$

Now add a fourth column and label it 'Cumulative frequency'.
Work out the cumulative frequency for each row:

$6 + 23 = 29$
$29 + 41 = 70$
$166 + 9 = 175$
$175 + 5 = 180$

The final cumulative frequency value should equal the total frequency value. Cumulative frequency is always plotted on the **vertical** axis.



To draw the graph of the cumulative frequency, plot the value of the upper boundary against the cumulative frequency value.

Interpreting cumulative frequency graphs

We can use the cumulative frequency curve to find estimates of the **percentiles** and **quartiles**.

Percentiles separate large ordered sets of data into hundredths. Quartiles separate large ordered sets of data into quarters.

When the data are arranged in order, the lower quartile is the 25th percentile, the median is the 50th percentile (middle value) and the upper quartile is the 75th percentile.

→ To find the **lower quartile**, Q_1, read the value on the curve corresponding to $\frac{n+1}{4}$ on the cumulative frequency axis, where n is the total frequency.

To find the median, read the value on the curve corresponding to $\frac{n+1}{2}$ on the cumulative frequency axis.

To find the **upper quartile**, Q_3, read the value on the curve corresponding to $\frac{3(n+1)}{4}$ on the cumulative frequency axis.

To find the **percentiles**, p%, read the value on the curve corresponding to $\frac{p(n+1)}{100}$ on the cumulative frequency axis.

To find the **interquartile range** subtract the lower quartile from the upper quartile: $\text{IQR} = Q_3 - Q_1$.

The cumulative frequency curve is sometimes called an ogive.

Per cent means out of 100.

$\frac{1}{4} = 25\%$

$\frac{1}{2} = 50\%$

$\frac{3}{4} = 75\%$

There are no universally agreed formulae for the quartiles. For large n and grouped data: n rather than $n + 1$ may be used.

The IQR shows the spread of the middle 50% of the data

For any set of data:

- 25% or one-quarter of the values are between the smallest value and the lower quartile
- 25% are between the lower quartile and the median
- 25% are between the median and the upper quartile
- 25% are between the upper quartile and the largest value
- 50% of the data lie between the lower and upper quartiles.

In this cumulative frequency diagram (from the data in Example 10), $n = 180$.

Lower quartile ≈ 120 (blue)

This is the value corresponding to $\frac{180+1}{4} = 45.25$.

Median ≈ 173 (green)

This is the value corresponding to $\frac{180+1}{2} = 90.5$.

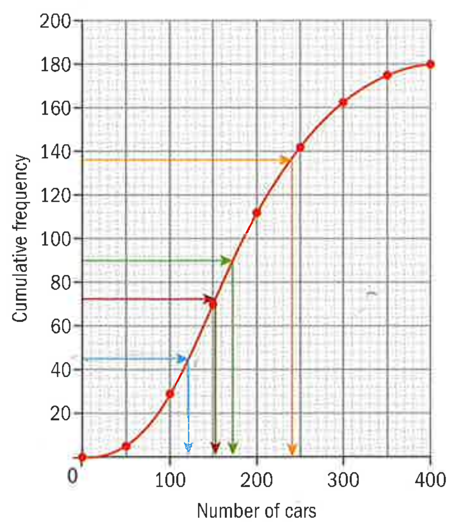
Upper quartile ≈ 238 (orange)

This is the value corresponding to $\frac{3(180+1)}{4} = 135.75$.

40th percentile ≈ 153 (brown)

This is the value corresponding to $\frac{40(180+1)}{100} = 72.4$.

The interquartile range ≈ 238 – 120 = 118



Example 11

50 contestants play the game of Oware. In total they have to play 49 games to arrive at a champion. The average times for the 49 games are given in the table.

Time (t minutes)	Frequency
$3 \le t < 4$	4
$4 \le t < 5$	12
$5 \le t < 6$	18
$6 \le t < 7$	9
$7 \le t < 8$	3
$8 \le t < 9$	2
$9 \le t < 10$	1

The game of Oware is played all over the world and there is even an Oware Society.
Why must 50 contestants play 49 games to arrive at a champion?
Can you prove this?

a Construct a cumulative frequency table for these data.

b Draw a cumulative frequency graph for these data.

c Use your graph to estimate

i the lower quartile **ii** the median

iii the upper quartile **iv** the interquartile range

v the 30th percentile.

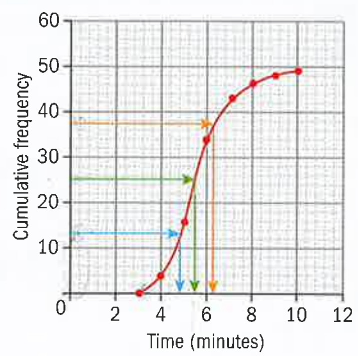
Answers

a

Time (t minutes)	Frequency	Upper boundary	Cumulative frequency
$3 \le t < 4$	4	4	4
$4 \le t < 5$	12	5	16
$5 \le t < 6$	18	6	34
$6 \le t < 7$	9	7	43
$7 \le t < 8$	3	8	46
$8 \le t < 9$	2	9	48
$9 \le t < 10$	1	10	49

Check:
Total frequency: 4 + 12 + 18 + 9 + 3 + 2 + 1 = 49
Final cumulative frequency value = 49

b



Plot each cumulative frequency at the upper boundary.

Continued on next page

c **i** $n = 49$

$\frac{n+1}{4} = \frac{50}{4} = 12.5$

Lower quartile ≈ 4.7 minutes

25% of games last 4.7 minutes or less.

Read across from 12.5 on the vertical axis, then down to the horizontal axis.

ii $\frac{n+1}{2} = \frac{49+1}{2} = 25$

Median ≈ 5.5 minutes

50% of games last 5.5 minutes or less.

This is the value on the horizontal axis corresponding to 25 on the vertical axis.

iii $\frac{3(n+1)}{4} = \frac{3(49+1)}{4} = 37.5$

Upper quartile ≈ 6.4 minutes

75% of games last 6.4 minutes or less.

This is the value on the horizontal axis corresponding to 37.5 on the vertical axis.

iv Interquartile range = 6.4 − 4.7 = 1.7 minutes

The 'middle' 50% of games last between 4.7 and 6.4 minutes.

v $\frac{30(n+1)}{100} = \frac{30(49+1)}{100} = 15$

30th percentile ≈ 4.9 minutes

30% of games last 4.9 minutes or less.

This is the value on the horizontal axis corresponding to 15 on the vertical axis.

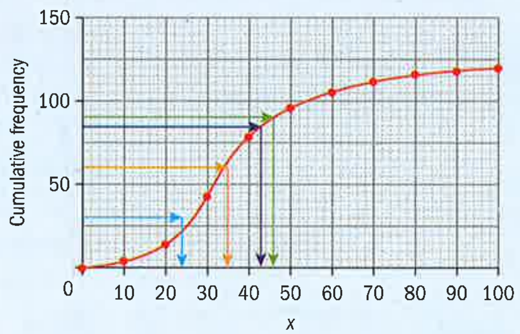
Example 12

From this cumulative frequency graph find

i the median

ii the interquartile range

iii the 70th percentile.



Answers

i $n = 120$

$\frac{n+1}{2} = \frac{121}{2} = 60.5$

Median ≈ 35

n = 120 from graph

Median is the value on the horizontal axis corresponding to 60.5 on the vertical axis.

ii Lower quartile $\frac{120+1}{4} = 30.25$th value

Lower quartile = 26

Upper quartile $\frac{3(120+1)}{4} = 90.75$th value

Upper quartile = 46

Interquartile range ≈ 46 − 26 = 20

Interquartile range = upper quartile − lower quartile

Upper quartile is the value corresponding to 90.75 on the vertical axis.

Lower quartile is the value corresponding to 30.25 on the vertical axis.

iii $\frac{70(120+1)}{100} = 84.7$th value

70th percentile ≈ 43

This is the value corresponding to 84.7 on the vertical axis.

Exercise 2I

EXAM-STYLE QUESTIONS

1 A dice is tossed 50 times. The number shown is recorded each time and the results are given in the table.

a Write down the value of N.

b Find the values of a, b and c.

Number	Frequency	Cumulative frequency
1	6	6
2	a	14
3	10	24
4	b	c
5	5	43
6	7	50
	N	

2 The table shows the percentages scored by candidates in a test.

Marks (%)	0–9	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89	90–100
Frequency	1	5	7	11	19	43	36	15	2	1

Here is the cumulative frequency table for the marks.

a Calculate the values of s and of t.

b Draw a cumulative frequency graph for these data.

c Use your graph to estimate

i the median mark

ii the lower quartile

iii the pass mark, if 40% of the candidates passed.

Marks (%)	Cumulative frequency
< 9.5	1
< 19.5	6
< 29.5	s
< 39.5	24
< 49.5	43
< 59.5	86
< 69.5	t
< 79.5	137
< 89.5	139
≤ 100	140

3 A safari park is open to visitors every day of the year. The numbers of cars that pass through the park each day for a whole year were recorded and are shown in the table.

a Draw a cumulative frequency graph to represent this information.

b Find the median and the interquartile range.

c On what percentage of days were there more than 800 cars in the park?

Number of cars (n)	Frequency
$0 < n \le 150$	25
$150 < n \le 300$	36
$300 < n \le 450$	68
$450 < n \le 600$	102
$600 < n \le 750$	64
$750 < n \le 900$	41
$900 < n \le 1050$	19
$1050 < n \le 1200$	10

4 Sofia studied an article in the *Helsingborgs Dagblad*. She recorded the numbers of words in each sentence in the frequency table.

a Draw a cumulative frequency graph to represent this information.

b Work out the lower quartile, the median and the upper quartile of the data.

Number of words	Frequency
1–4	4
5–8	19
9–12	38
13–16	23
17–20	8
21–24	4
25–28	2
29–32	1
33–36	1

EXAM-STYLE QUESTIONS

5 A salmon farmer records the lengths of 100 salmon, measured to the nearest cm. The results are given in the table.

a Construct a cumulative frequency table for the data in the table.

b Draw a cumulative frequency curve.

c Use the cumulative frequency curve to find

i the median length of salmon

ii the interquartile range of salmon length.

Length of salmon (x cm)	Number of salmon
$25 < x \leq 28$	3
$28 < x \leq 31$	4
$31 < x \leq 34$	11
$34 < x \leq 37$	23
$37 < x \leq 40$	28
$40 < x \leq 43$	15
$43 < x \leq 46$	12
$46 < x \leq 49$	4
TOTAL	100

6 The table shows the times taken by 100 students to complete a puzzle.

Time (t minutes)	11–15	16–20	21–25	26–30	31–35	36–40
Number of students	6	13	27	31	15	8

a Construct a cumulative frequency table.

b Draw a cumulative frequency graph.

c Use your graph to estimate

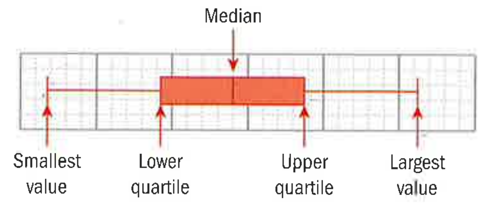
i the median time

ii the interquartile range of the time

iii the time within which 75% of the students completed the puzzle.

2.6 Box and whisker graphs

Another useful way to represent data is a **box and whisker graph** (or box and whisker plot).
A box and whisker graph looks something like this.



→ To draw a box and whisker graph, five pieces of information are needed: calculate the lower quartile, median and upper quartile for the data. Find the smallest and largest values.

Draw the box and whisker graph to scale on graph paper.

Note:
An **outlier** is a value that is much smaller or much larger than the other values.
Normally we consider an outlier to be a point with a value:

- less than 'the lower quartile − 1.5 × the interquartile range' or
- greater than 'the upper quartile + 1.5 × the interquartile range'.

Outliers will not be examined but they may be useful for projects.

Example 13

A yacht club hosts an annual race. The numbers of people in each yacht are recorded in the table.

a Find the median number of people in a yacht.

b Find the upper and lower quartiles.

c Draw a box and whisker graph to represent the information.

Number of people	Frequency
4	1
5	8
6	16
7	25
8	28
9	16
10	5
TOTAL	99

▶ Continued on next page

Answers

a $n = 99$, so the median is the number of people in the $\frac{99+1}{2} = \frac{100}{2} = 50$th yacht.

The 50th yacht is in the group highlighted red.

Number of people	Frequency	Cumulative frequency
4	1	1
5	8	9
6	16	25
7	25	50
8	28	78
9	16	94
10	5	99

The median number of people is 7.

b The lower quartile is the number of people in the $\frac{99+1}{4} = 25$th yacht.

The 25th yacht is in the group highlighted green.

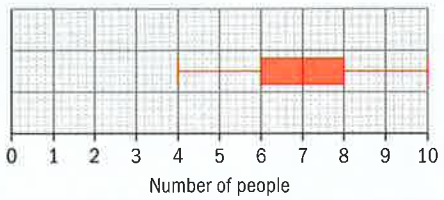
The lower quartile is 6.

The upper quartile is the number of people in the $\frac{3(99+1)}{4} = 75$th yacht.

The 75th yacht is in the group highlighted blue.

The upper quartile is 8.

c

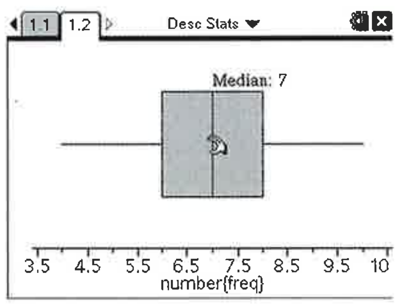


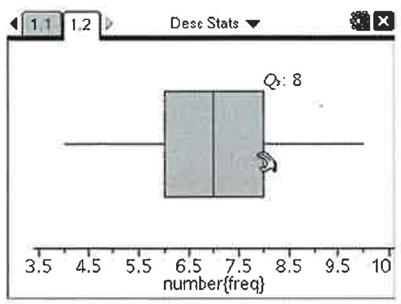
Need five pieces of information to draw a box and whisker graph:
Smallest number of people = 4
*Lower quartile = 6 (from part **b**)*
*Median = 7 (from part **a**)*
*Upper quartile = 8 (from part **b**)*
Largest number = 10

You can also find all the data for the box and whisker graph using your GDC.

Enter the 'Number of people' and 'Frequency' into lists named 'Number' and 'Freq' in a Lists & Spreadsheets page. Add a Data & Statistics page and press MENU 2: Plot Properties | 5: Add X Variable with Frequency and select the two lists. To read the values use the touchpad to move the arrow over them.

These GDC screenshots show the median and the upper quartile (Q_3).





GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Example 14

The weights, in kilograms, of 25 koala bears are:
4.3, 7.2, 5.6, 4.8, 10.7, 9.7, 5.6, 7.8, 8.2, 11.4, 7.9, 12.6, 13.1, 5.7, 9.9, 11.3, 13.4, 8.8, 7.5, 5.8, 9.2, 10.3, 12.1, 6.5, 8.6
Draw a box and whisker graph to represent the information.

Answer

First arrange the data in ascending order:
4.3, 4.8, 5.6, 5.6, 5.7, 5.8, 6.5, 7.2, 7.5, 7.8, 7.9, 8.2, 8.6, 8.8, 9.2, 9.7, 9.9, 10.3, 10.7, 11.3, 11.4, 12.1, 12.6, 13.1, 13.4
$n = 25$

Lowest value = 4.3

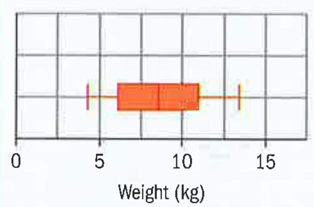
Lower quartile: $\frac{25+1}{4} = 6.5$,
so between 6th and 7th value
6th value = 5.8,
7th value = 6.5,
6.5th value = $\frac{5.8+6.5}{2} = 6.15$

To find the 6.5th value, calculate the mean of the 6th and 7th values.

Median = 8.6
$\left(\text{the } \frac{25+1}{2} = 13\text{th value}\right)$

Upper quartile: $\frac{3\times 26}{4} = 19.5$,
so between 19th and 20th value
19th value = 10.7,
20th value = 11.3,
19.5th value = $\frac{10.7+11.3}{2} = 11$

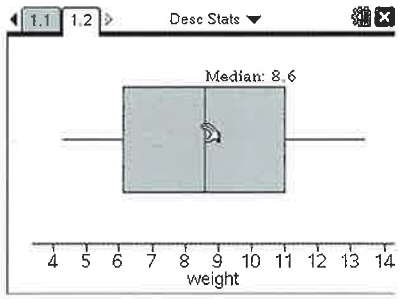
Largest value = 13.4

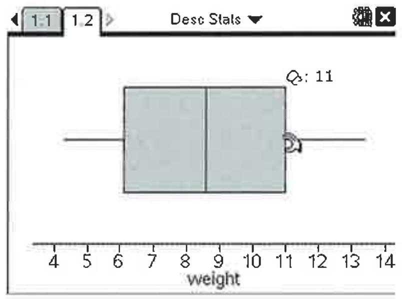


Need five pieces of information to plot a box and whisker graph.

Using a GDC:
Enter the data into a list. You do not need to put it in order.
These GDC screenshots show the median and the upper quartile (Q_3).

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*





You cannot use a GDC to draw box and whisker graphs for grouped frequency tables.

Exercise 2J

1 The numbers of sweets in 45 bags are:

34 33 35 33 32 33 34 34 32 35 33 32 36 31 33 34
33 34 33 32 35 31 33 32 32 34 33 36 33 30 33 32
34 35 32 33 33 32 33 31 34 33 32 33 34

a Construct a frequency table to represent the information.
b Find the median, the lower quartile and the upper quartile.
c Draw a box and whisker graph to represent this information.
Use a GDC to check your answer.

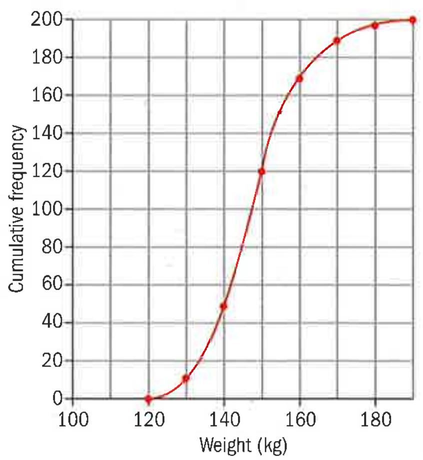
2 An experiment was performed 60 times. The scores from the experiment were recorded in the table.
a Find the median, the lower quartile and the upper quartile.
b Draw a box and whisker graph to represent this information.
Use a GDC to check your answer.

Score	Frequency
1	6
2	12
3	13
4	15
5	8
6	6

EXAM-STYLE QUESTION

3 The cumulative frequency graph shows the weights, in kg, of 200 sumo wrestlers.
a Write down
i the median
ii the lower quartile
iii the upper quartile.

The lightest wrestler weighs 125 kg and the heaviest weighs 188 kg.
b Draw a box and whisker graph to represent the information.



EXAM-STYLE QUESTIONS

4 The heights, in cm, of 180 students are given in the cumulative frequency table.
a Draw a cumulative frequency diagram to represent this information.
b Write down
i the median
ii the lower quartile and the upper quartile.
c The smallest student is 146 cm and the tallest is 183 cm. Represent this information on a box and whisker graph.

Height (x cm)	Cumulative frequency
$x \le 145$	0
$x \le 150$	26
$x \le 155$	81
$x \le 160$	119
$x \le 165$	142
$x \le 170$	154
$x \le 175$	167
$x \le 180$	174
$x \le 185$	180

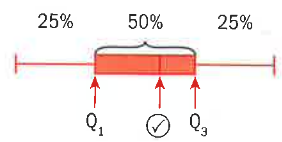
5 The table shows the heights, in cm, of 50 kangaroos.
a Construct a cumulative frequency table and use it to draw the cumulative frequency curve.
b Write down the median.
c Find the lower quartile and the upper quartile.

The smallest kangaroo is 205 cm and the tallest is 258 cm.
d Draw a box and whisker graph to represent the information.

Height (x cm)	Frequency
$200 \le x < 210$	4
$210 \le x < 220$	6
$220 \le x < 230$	11
$230 \le x < 240$	22
$240 \le x < 250$	5
$250 \le x < 260$	2

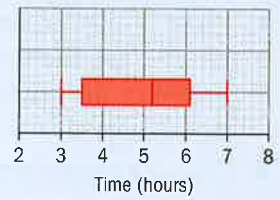
Interpreting box and whisker graphs

For any set of data:
- 25% or one-quarter of the values are between the smallest value and the lower quartile
- 25% are between the lower quartile and the median
- 25% are between the median and the upper quartile
- 25% are between the upper quartile and the largest value
- 50% of the data lie between the lower and upper quartiles.



Example 15

The box and whisker graph shows the times, in hours, that it takes to build an igloo.
a Write down the median time.
b Find the interquartile range.
c Write down the percentage of people who took less than 5.2 hours to build an igloo.
d x% of the people took more than 6.1 hours to build an igloo. Write down the value of x.



▶ Continued on next page

Answers

a The median time is 5.2 hours.

b The interquartile range = 6.1 − 3.5 = 2.6 hours. — *From the graph, upper quartile = 6.1, lower quartile = 3.5*

c 50% of the people took less than 5.2 hours to build an igloo. — *5.2 hours = median (from part **a**) 50% of data are at or below this value*

d 25% of the people took more than 6.1 hours to build an igloo. — *Upper quartile = 6.1 75% of data are at or below this value*

Exercise 2K

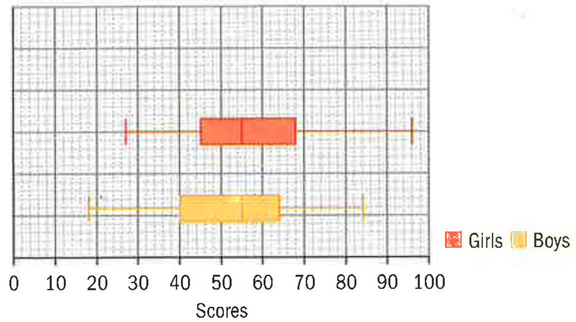
1 The box and whisker graphs represent the scores on a Psychology test for 40 boys and 40 girls.

a Find the median score for the boys and the girls.

b Write down the interquartile range for the boys' scores and the girls' scores.

c Write down the percentage of boys that scored more than 55.

d Write down the percentage of girls that scored more than 68.



2 The box and whisker graph represents the number of faults made by horses in a jumping competition.
Write down

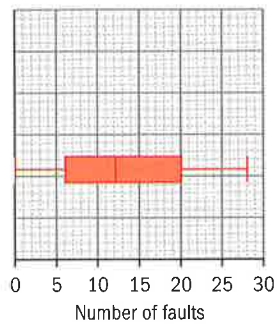
a the lowest number of faults

b the median

c the interquartile range

d the largest number of faults

e the percentage of horses that had fewer than six faults.



EXAM-STYLE QUESTION

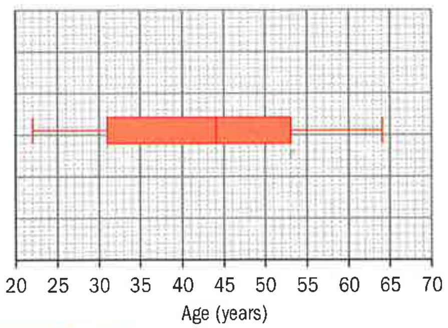
3 The box and whisker graph represents the ages of the teachers at Myschool High.

a Write down the age of the youngest teacher.

b Write down the median age.

c If 25% of the teachers are older than x, write down the value of x.

d Find the interquartile range of the ages.



Extension material on CD: *Worksheet 2 - Standard deviation, standardization and outliers*

2.7 Measures of dispersion

Measures of dispersion measure how spread out a set of data is.
The simplest measure of dispersion is the **range**.

→ The **range** is found by subtracting the smallest value from the largest value.

Example 16

The numbers of piglets in the litters of 10 pigs are:
10 12 12 13 15 16 9 10 14 11
Find the range.

Answer
Range = 16 − 9 = 7 — *Identify the largest value (16) and the smallest value (9).*

The **interquartile range** is found by subtracting the lower quartile, Q_1, from the upper quartile, Q_3: IQR = $Q_3 - Q_1$.

To work out the lower and upper quartiles the values must be arranged in size order

Example 17

Find the interquartile range of this data set.
4 5 6 6 7 8 10 10 11 14 15

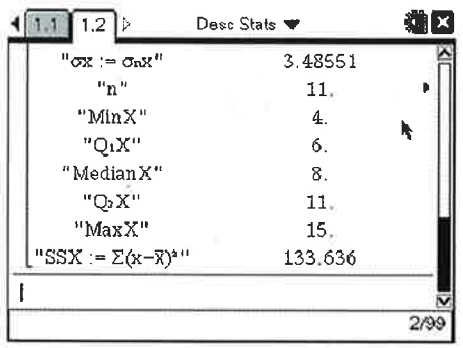
Answer — *There are 11 numbers so $n = 11$.*

Q_1 is the $\frac{11+1}{4}$ = 3rd number,
so $Q_1 = 6$.

Q_3 is the $\frac{3(11+1)}{4}$ = 9th number,
so $Q_3 = 11$.

IQR = 11 − 6 = 5

Using a GDC:
Enter the data into a list. Then use One Variable Statistics.
Scroll down to find the quartiles.
The value of Q_1 is given as 'Q_1X' and Q_3 as 'Q_3X'.



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can use a GDC for drawing graphs for frequency tables but not for grouped frequency tables.

For finding the interquartile range from a cumulative frequency graph see page 62. For finding the interquartile range from a box and whisker graph see page 71.

Exercise 2L

1 For each set of data calculate

i the range **ii** the interquartile range.

a 6 3 8 5 2 9 11 21 15 8

b 5 3 6 8 9 12 10 9 8 13 16 12 9 11 8

c

Price of main course in euros	Frequency
18	6
19	4
20	5
21	8
22	3
23	2
24	5
25	4

Standard deviation

The **standard deviation** is a measure of dispersion that gives an idea of how the data values are related to the mean.

When is the standard deviation of a set of data small?
Can the standard deviation equal zero?

Why do we take the square root to find the standard deviation?

Why is the standard deviation sometimes called the root-mean-square deviation?

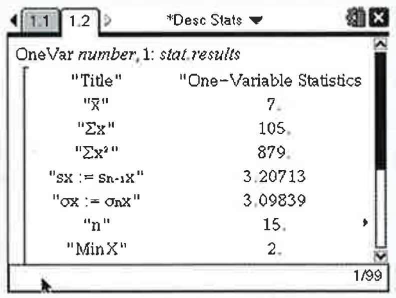
You are expected to use a GDC to calculate standard deviations.

Example 18

Find the mean and standard deviation of this data set.
4 5 6 8 12 13 2 5 6 9 10 9 8 3 5

Answer
Mean = 7
Standard deviation = 3.10 (to 3 sf)

Using a GDC:
Enter the data.
Mean is indicated by $\bar{x}$.
Standard deviation is indicated by σ_x



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Example 19

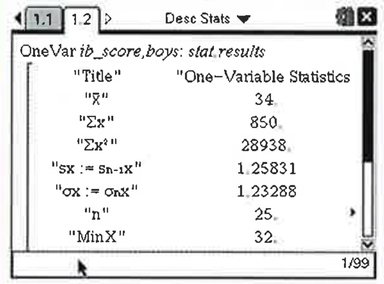
50 students were asked the total number of points that they received on their IB Diploma. The results are shown in the table.

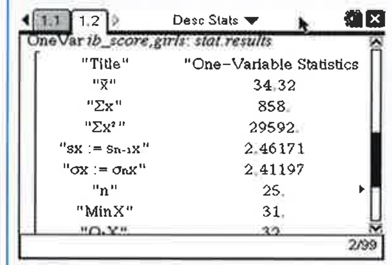
Score on IB Diploma	Boys	Girls
31	0	3
32	2	4
33	6	3
34	11	5
35	4	3
36	1	2
37	0	1
38	1	2
39	0	2

Use your GDC to calculate the mean and standard deviation for the boys and girls separately and comment on your answer.

Answer
Boys' mean = 34
Boys' standard deviation = 1.23 (to 3 sf)
Girls' mean = 34.3 (to 3 sf)
Girls' standard deviation = 2.41 (to 3 sf)
Both the boys and the girls have a mean of about 34 points. The standard deviation for the boys is small, which implies that most boys achieved close to 34 points. However, the standard deviation for the girls is larger which implies that some girls will have much less than 34 points and some will have much more.

Using a GDC:





To make a comment, compare the mean to the corresponding standard deviation.

Is standard deviation a mathematical discovery or an invention?

It is often impossible to find the mean and standard deviation for a whole population. This could be due to time restrictions, financial constraints or other reasons.

If we have, say, a random sample of 12 babies' heights from the UK, then the standard deviation of those 12 babies' heights is given as 'σ_x' on a GDC. This is the one we use for Mathematical Studies.

If we wanted to estimate the standard deviation of all the babies' heights in the UK, based on our random sample, then we would use 's_x' on the GDC.

The IB notation for standard deviation is s_n. When you use your GDC, choose σ_x

Exercise 2M

1 For each set of data calculate the standard deviation.

a 5 3 6 8 9 12 10 9 8 13 16 12 9 11 8

b

Price of main course in euros	Frequency
18	6
19	4
20	5
21	8
22	3
23	2
24	5
25	4

2 Calculate the mean and standard deviation for these data.

6 3 8 5 2 9 11 21 15 8

3 An experiment was performed 50 times. The scores from the experiment were recorded in the table.

a Write down the range.
b Find the interquartile range.
c Find the mean and standard deviation.

Score	Frequency
1	4
2	12
3	11
4	15
5	6
6	2

4 A boat club hosts an annual race. The numbers of people in each boat are recorded in the table.

a Write down the range.
b Find the interquartile range.
c Find the mean and standard deviation.

Number of people	Frequency
4	2
5	7
6	25
7	15
8	30
9	16
10	5

5 The numbers of telephone calls to a call center were monitored every hour for a month. The data collected are shown in the table.

Use your GDC to find

a the mean number of calls per hour
b the standard deviation
c the range
d the interquartile range.

Number of calls per hour	Frequency
60	18
62	45
64	40
66	55
68	31
70	32
72	15
74	13
76	14
78	16

EXAM-STYLE QUESTIONS

6 The mean of these numbers is 33.

16 41 24 x 62 18 25

a Find the value of x.
b Calculate the standard deviation.
c Find the range.
d Find the interquartile range.

7 80 plants were measured and their heights (correct to the nearest cm) recorded in the table.

a Write down the value of m.
b Find the mean height.
c Find the standard deviation of the heights.
d Find the interquartile range of the heights.

Height (cm)	Frequency
10	7
11	m
12	21
13	22
14	11
15	7
16	3

8 The 60 IBDP students at Golden Globe Academy complete a questionnaire about the number of pairs of shoes that they own. The results are shown in the table.

a Find the range and interquartile range.
b Find the mean and standard deviation.

Pairs of shoes	Frequency
5	6
6	8
7	15
8	10
9	5
10	12
11	1
12	3

EXAM-STYLE QUESTIONS

9 The times taken for 50 students to complete a crossword puzzle are shown in the table.

Time (*m* minutes)	Frequency
$15 \le m < 20$	3
$20 \le m < 25$	7
$25 \le m < 30$	10
$30 \le m < 35$	11
$35 \le m < 40$	12
$40 \le m < 45$	5
$45 \le m < 50$	2

Use the midpoint of each class to estimate the mean and the standard deviation of grouped data.

Find an approximation for the mean and standard deviation.

10 The percentage marks obtained for an ITGS (Information Technology for a Global Society) test by the 25 boys and 25 girls at Bright High are shown in the table.

a Calculate an estimated value for the mean and standard deviation for the girls and the boys separately.

b Comment on your findings.

Girls' frequency	Percentage mark	Boys' frequency
0	$0 \le x < 10$	2
0	$10 \le x < 20$	1
0	$20 \le x < 30$	1
3	$30 \le x < 40$	1
5	$40 \le x < 50$	5
7	$50 \le x < 60$	9
8	$60 \le x < 70$	2
2	$70 \le x < 80$	0
0	$80 \le x < 90$	2
0	$90 \le x < 100$	2

Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

1 The mean of the twelve numbers listed is 6.

3 4 *a* 8 3 5 9 5 8 6 7 5

a Find the value of *a*.

b Find the median of these numbers.

2 The mean of the ten numbers listed is 5.

4 3 *a* 6 8 4 6 6 7 5

a Find the value of *a*.

b Find the median of these numbers.

EXAM-STYLE QUESTIONS

3 For the set of numbers

3 4 1 7 6 2 9 11 13 6 8 10 6

a calculate the mean

b find the mode

c find the median.

4 The lengths of nine snakes, in metres, are:

6.5 4.6 7.2 5.0 2.4 3.9 12.9 10.3 6.1

a **i** Find the mean length of the snakes.

ii Find the standard deviation of the length of the snakes.

b Find the median length of the snakes.

5 A survey was conducted of the number of bathrooms in 150 randomly chosen houses. The results are shown in the table.

Number of bathrooms	1	2	3	4	5	6
Number of houses	79	31	22	10	5	13

a State whether the data are discrete or continuous.

b Write down the mean number of bathrooms per house.

c Write down the standard deviation of the number of bathrooms per house.

6 The table shows the age distribution of members of a chess club.

Age (years)	Number of members
$20 \le x < 30$	15
$30 \le x < 40$	23
$40 \le x < 50$	34
$50 \le x < 60$	42
$60 \le x < 70$	13

a Calculate an estimate of the mean age.

b Draw a histogram to represent these data.

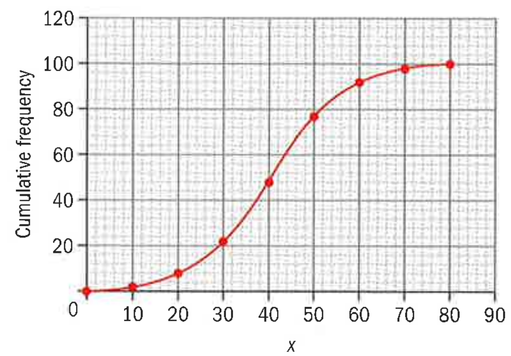
7 Using the cumulative frequency graph, write down the value of

a the median

b the lower quartile

c the upper quartile

d the interquartile range.



EXAM-STYLE QUESTION

8 The numbers of horses counted in 35 fields are represented in the table.
Draw a box and whisker graph to represent this information.

Number of horses	Frequency
8	4
10	9
12	7
15	12
21	3

Paper 2 style questions

EXAM-STYLE QUESTIONS

1 Nineteen students carried out an experiment to measure gravitational acceleration in cm s^{-2}.
The results are given to the nearest whole number.

96 97 101 99 100 98 99 94 96 100
97 98 101 98 99 96 96 100 97

a Use these results to find an estimate for
 i the mean value for the acceleration
 ii the modal value for the acceleration.
b **i** Construct a frequency table for the results.
 ii Use the table to find the median value and the interquartile range.

2 A gardener wanted to estimate the number of weeds on the sports field.
He selected at random 100 sample spots, each of area $100\,cm^2$, and counted the number of weeds in each spot.

The table shows the results of his survey.

Number of weeds	Frequency
0–4	18
5–9	25
10–14	32
15–19	14
20–24	7
25–29	4

a **i** Construct a cumulative frequency table and use it to draw the cumulative frequency curve.
 ii Write down the median number of weeds.
 iii Find the percentage of spots that have more than 19 weeds.
b **i** Estimate the mean number of weeds per spot.
 ii Estimate the standard deviation of the number of weeds per spot.

The area of the field is $8000\,m^2$.
 iii Estimate the total number of weeds on the field.

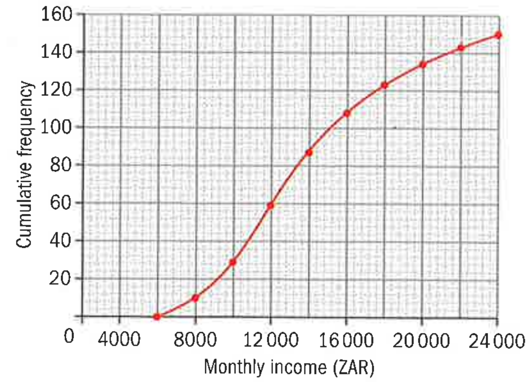
3 The marks for a test are given in the frequency table.
a Complete a cumulative frequency table and use it to draw the cumulative frequency curve.
b Find the median mark.
c Find the interquartile range.

60% of the candidates passed the examination.
d Find the pass mark.
e Given that the lowest mark was 9 and the highest was 98, draw a box and whisker graph to represent the information.

Mark, x	Frequency
$0 \le x < 10$	3
$10 \le x < 20$	14
$20 \le x < 30$	21
$30 \le x < 40$	35
$40 \le x < 50$	42
$50 \le x < 60$	55
$60 \le x < 70$	43
$70 \le x < 80$	32
$80 \le x < 90$	15
$90 \le x < 100$	10

EXAM-STYLE QUESTIONS

4 The cumulative frequency graph shows the monthly incomes, in South African Rand, ZAR, of 150 people.

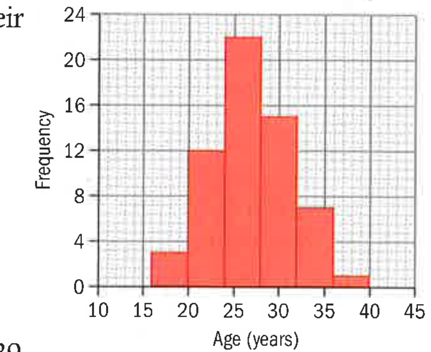


a Write down the median and find the interquartile range.
b Given that the lowest monthly income is 6000 ZAR and the highest is 23 500 ZAR, draw a box and whisker graph to represent this information.
c Draw a frequency table for the monthly incomes.
d Use your GDC to find an estimate of the mean and standard deviation of the monthly incomes.

5 The weights of 200 female athletes are recorded in the table.
a Write down the modal group.
b Calculate an estimate of the mean and the standard deviation.
c Construct a cumulative frequency table and use it to draw the cumulative frequency graph.
d Write down the median, the lower quartile and the upper quartile.
e The lowest weight is 47 kg and the heaviest is 76 kg. Use this information to draw a box and whisker graph.

Weight (w kg)	Frequency
$45 \le w < 50$	4
$50 \le w < 55$	16
$55 \le w < 60$	45
$60 \le w < 65$	58
$65 \le w < 70$	43
$70 \le w < 75$	28
$75 \le w < 80$	6

6 A group of 60 women were asked at what age they had their first child. The information is shown in the histogram.



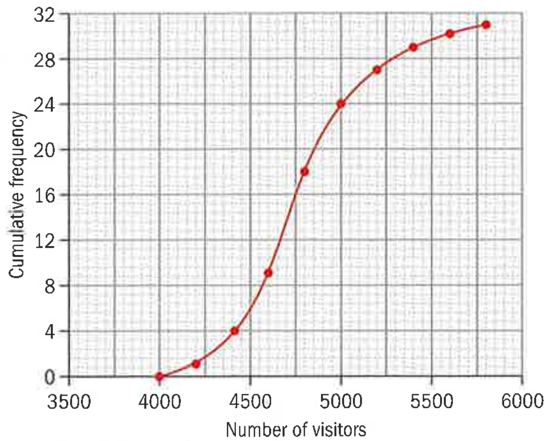
a Calculate an approximation for the mean and standard deviation.
b Write down the modal class.
c Construct a cumulative frequency table for the data and draw the cumulative frequency curve.
d Use your graph to find the median and interquartile range.
e Given that the youngest age was 16 and the oldest was 39, draw a box and whisker graph to represent the information.

7 The average times, to the nearest second, that 100 participants waited for an elevator are shown in the table.
a Write down the modal class.
b Calculate an estimate of the mean time and the standard deviation.
c Construct a cumulative frequency table and use it to draw the cumulative frequency graph.
d Write down the median and interquartile range.

Time (t seconds)	Frequency
$0 \le t < 10$	5
$10 \le t < 20$	19
$20 \le t < 30$	18
$30 \le t < 40$	22
$40 \le t < 50$	16
$50 \le t < 60$	12
$60 \le t < 70$	8

EXAM-STYLE QUESTIONS

8 The cumulative frequency graph shows the daily number of visitors to the Mausoleum on Tiananmen Square in the month of January.



a Write down the median, the lower quartile and the upper quartile.

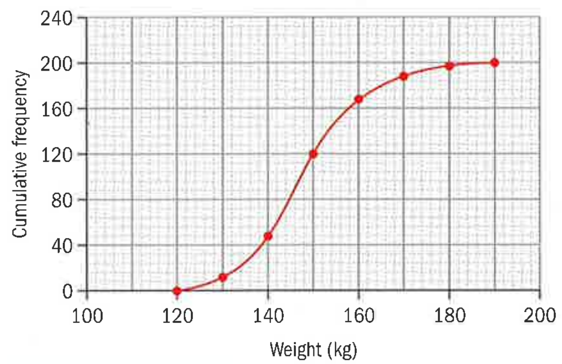
b Given that the least number of visitors was 4000 and the most was 5700, draw a box and whisker graph to represent the information.

c Construct a frequency table for this information.

d Write down the modal class.

e Calculate an estimate of the mean and the standard deviation.

9 The cumulative frequency graph shows the weights, in kg, of 200 professional wrestlers.



a Construct a grouped frequency table for this information.

b Write down the modal class.

c Calculate an estimate of the mean weight.

CHAPTER 2 SUMMARY

Classification of data

- **Discrete data** are either data that can be counted or data that can only take specific values.
- **Continuous data** can be measured. They can take any value within a range.

Grouped discrete or continuous data

- To draw a **frequency histogram**, find the lower and upper boundaries of the classes and draw the bar between these boundaries. There should be no spaces between the bars.

Measures of central tendency

- The **mode** of a data set is the value that occurs most frequently.
- The **median** of a data set is the value that lies in the middle when the data are arranged in size order.
- The **mean** of a data set is the sum of all the values divided by the number of values.
- For data in a frequency table, the **mode** is the entry that has the largest frequency.

Continued on next page

- The **median** is the middle entry as the entries in the table are already in order. For n pieces of data, the median is the $\frac{n+1}{2}$th value.
- The **mean** from a frequency table is:

$$\text{mean} = \frac{\text{total of } f_i \times x_i}{\text{total frequency}}$$

where f_i is the frequency of each data value x_i and $i = 1, \ldots, k$, where k is the number of data values.
- For grouped data, the **modal class** is the group or class interval that has the largest frequency.
- To calculate the **mean** from a grouped frequency table, an estimate of the mean is

$$\frac{\text{total of } f_i \times x_i}{\text{total frequency}}$$

where f_i is the frequency and x_i is the corresponding midpoint of each class.

Cumulative frequency curves

- The **cumulative frequency** is the sum of all of the frequencies up to and including the new value. To draw a **cumulative frequency curve** you need to construct a cumulative frequency table, with the upper boundary of each class interval in one column and the corresponding cumulative frequency in another. Then plot the upper class boundary on the x-axis and the cumulative frequency on the y-axis.
- To find the **lower quartile**, Q_1, read the value on the curve corresponding to $\frac{n+1}{4}$ on the cumulative frequency axis, where n is the total frequency.
- To find the median, read the value on the curve corresponding to $\frac{n+1}{2}$ on the cumulative frequency axis.
- To find the **upper quartile**, Q_3, read the value on the curve corresponding to $\frac{3(n+1)}{4}$ on the cumulative frequency axis.
- To find the **percentiles**, $p\%$, read the value on the curve corresponding to $\frac{p(n+1)}{100}$ on the cumulative frequency axis.
- To find the **interquartile range** subtract the lower quartile from the upper quartile: $\text{IQR} = Q_3 - Q_1$.

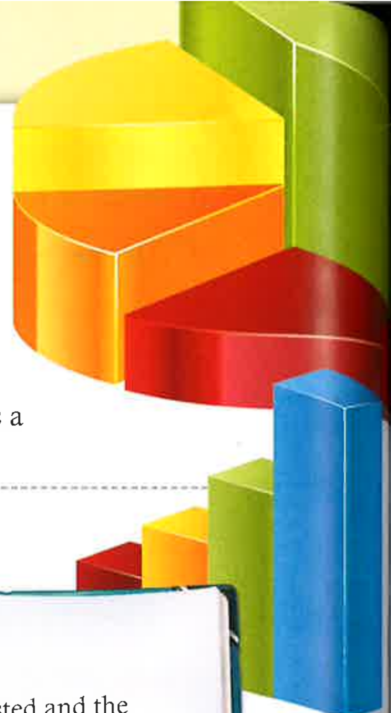
Box and whisker graphs

- To draw a box and whisker graph, five pieces of information are needed: calculate the lower quartile, median and upper quartile for the data. Find the smallest and largest values.

Measures of dispersion

- The **range** is found by subtracting the smallest value from the largest value.
- The **interquartile range** is found by subtracting the lower quartile, Q_1, from the upper quartile, Q_3: $\text{IQR} = Q_3 - Q_1$.
- The standard deviation is often referred to as the 'root-mean-square deviation' because we find the **deviation** of each entry from the mean, then we **square** these values and find the **mean** of the squared values, and, finally, we take the square **root** of this answer.

Statistically speaking



Descriptive statistics describe the basic features of a data set.

Descriptive statistics reduce lists of data into a simple summary such as a single average (a number) or a visual form such as a graph or diagram.

Morals and statistics

Case study 1

A company has 3 employees and a boss.

The employees earn 2500 euros a month and the boss earns 25 000 euros a month.

A report in the local newspaper states that the average salary in the company is 8125 euros a month.

- Which average has the newspaper used?
- Does this average give a fair representation of the average salary?
- Which would be the most appropriate average to use? Why?

Case study 2

Ten appliances were tested and the number of faults each one had recorded below.

0 0 0 0 0 0 15 19 25 31

The company advertises that the average number of faults is 0.

- Which average has the company used?
- Is the company misleading people?
- Is it morally acceptable for the company to advertise the 'facts' in this manner?

"Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write."

H. G. Wells (1866–1946)

- What do you think H. G. Wells meant?
- Do you agree with him?

- How accurate are these visual representations:
  - X-rays
  - Snapshots
  - Paintings?

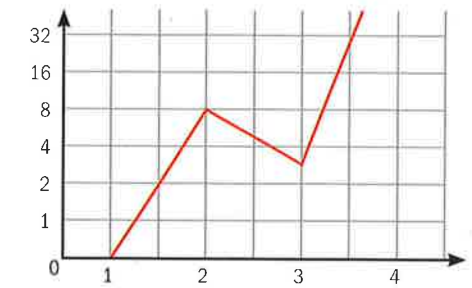
"There are three kinds of lies: lies, damned lies, and statistics."

Benjamin Disraeli (1084–1881)
Popularized by
Mark Twain (1835–1910)

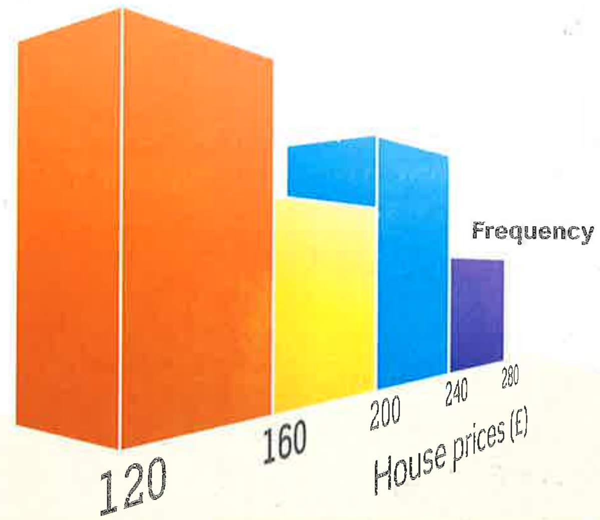
- Do statistics 'lie'?
- Are all statistics 'accurate'?

Misleading graphs

▼ What is wrong with this graph?



▼ What is wrong with this 3D histogram?



3 Geometry and trigonometry 1

CHAPTER OBJECTIVES:

5.1 Gradient; intercept; equation of a line in two dimensions; point of intersection of two lines; parallel lines; perpendicular lines

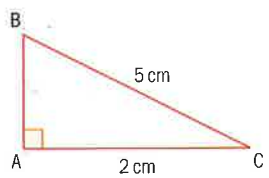
5.2 Use of sine, cosine and tangent ratios to find the sides and angles of a right-angled triangle; angles of depression and elevation

5.3 Use of the sine rule and the cosine rule; use of area of a triangle; construction of labeled diagrams from verbal statements

Before you start

You should know how to:

1 Use Pythagoras' theorem, e.g.
Find the length of side AC if AB = 2 cm and BC = 5 cm.



$$AB^2 + AC^2 = BC^2$$
$$2^2 + AC^2 = 5^2$$
$$AC^2 = 25 - 4$$
$$AC = \sqrt{21}\text{ cm}$$
$$= 4.58\text{ cm (3 sf)}$$

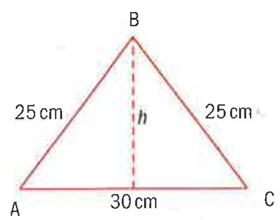
2 Find the midpoint of a line and the distance between two given points, e.g.
If A is (–3, 4) and B is (1, 2):

a Midpoint of AB is $\left(\frac{-3+1}{2}, \frac{4+2}{2}\right)$
$= (-1, 3)$

b Distance AB is
$$\sqrt{(1-(-3))^2 + (2-4)^2} = \sqrt{4^2 + 2^2}$$
$$= \sqrt{20}$$
$$= 4.47\text{ (3 sf)}$$

Skills check

1 a Find the height h of triangle ABC.



b Find the side length of a square if the length of its diagonal is 10 cm.

2 a A is the point (–3, 5) and B is the point (3, 7).
i Find the midpoint of AB.
ii Find the distance AB.

b The midpoint between C(2, p) and D(q, –4) is M(2.5, 1).
Find the values of p and q.

▲ *Les Eclaireurs Lighthouse*, in Tierra del Fuego, Argentina, is near Ushuaia, the southernmost city in the world. It has been guiding sailors since 1920.

When a lighthouse is designed, distances and angles are involved. The lighthouse needs to be tall enough for the light to be seen from a distance. Also, if a boat comes close into shore, could it still see the light?

In a manned lighthouse, if the keeper lowers his eyes and looks down to a boat, he can use this angle and the height of the lighthouse to calculate how far out the boat is. Problems like this can be solved using **trigonometry** – the part of mathematics that links the angles and lengths of a triangle. Using trigonometry you can calculate lengths that cannot be measured directly, such as the distance from a boat to the base of the lighthouse, the height of a tree or a building, the width of a river, etc.

This chapter will show you how to draw diagrams to represent these types of problem, and use trigonometry to solve them.

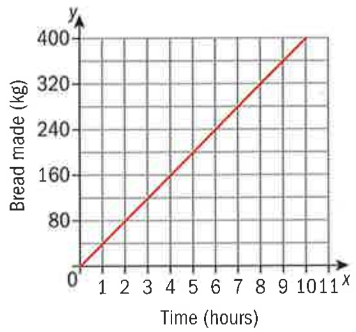
> **Geometry** came before trigonometry. In Egypt, after the flood seasons, nobody would know the borders of their lands so geo-metry, the art of 'earth-measuring', was invented. Geometry and trigonometry complement each other and are used extensively in a number of fields such as astronomy, physics, engineering, mechanics and navigation.

> This lighthouse is sometimes called *the Lighthouse at the End of the World*, as in the Jules Verne novel. However, the writer was inspired by the lighthouse *San Juan de Salvamento*, on another island nearby.

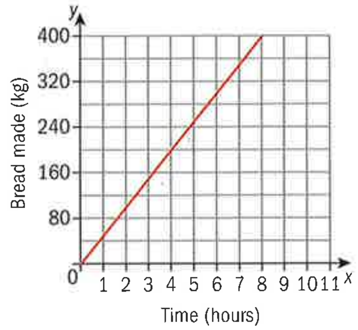
3.1 Gradient of a line

A bread factory has two bread-making machines, A and B.
Both machines make 400 kg of bread per day **at a constant rate**.
Machine A makes 400 kg in 10 hours.
Machine B makes the 400 kg in 8 hours.
For each machine, these graphs show the number of kilograms of bread made, y, in x hours. For example, in 2 hours machine A makes 80 kilograms of bread and machine B makes 100 kilograms of bread.

Machine A



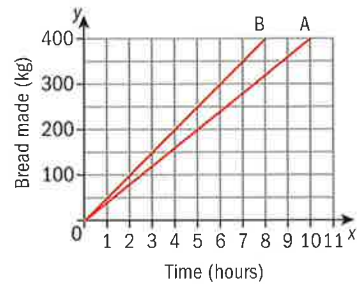
Machine B



This graph shows that machine A makes **40** kg of bread per hour.

This graph shows that machine B makes **50** kg of bread per hour.

The next graph shows the number of kilograms of bread made by both machines.

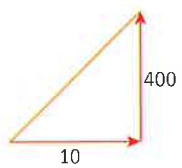


The line for machine B is steeper than the line for machine A. The **gradient** of a line tells you how steep it is. The gradient of line B is greater than the gradient of line A.

The gradient of a line = $\frac{\text{vertical step}}{\text{horizontal step}}$

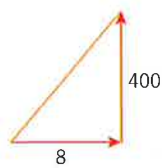
Gradient of line A = $\frac{\text{vertical step}}{\text{horizontal step}}$

$= \frac{400}{10} = 40$



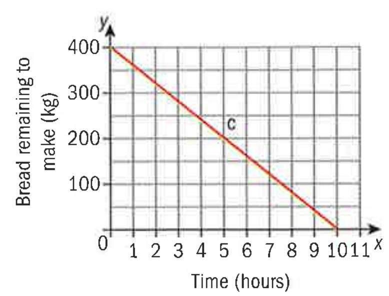
Gradient of line B = $\frac{\text{vertical step}}{\text{horizontal step}}$

$= \frac{400}{8} = 50$



The gradient tells you the rate at which the machine is working:
A's rate = 40 kg per hour and
B's rate = 50 kg per hour

This graph shows the number of kilograms of bread still to be made by machine A. At the beginning of the day the machine has 400 kg to make, after 1 hour the machine has 360 kg to make, and so on.

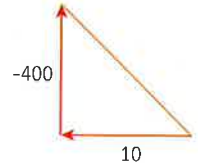


Line C has a negative gradient; it slopes downwards from left to right.

Gradient of line C = $\frac{\text{vertical step}}{\text{horizontal step}}$

$= \frac{-400}{10}$

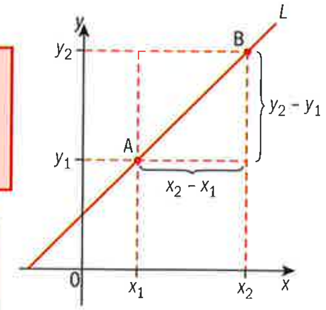
$= -40$



Each hour there is 40 kg less bread to be made.

→ If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points that lie on line L, the gradient of L is $m = \frac{y_2 - y_1}{x_2 - x_1}$

Note that the suffix order 2, then 1 in the gradient formula is the same in both the numerator and denominator.



Example 1

Find the gradient of the line L that passes through the points

a $A(1, 5)$ and $B(2, 8)$
b $A(0, 4)$ and $B(3, -2)$
c $A(2, 6)$ and $B(-1, 6)$
d $A(1, 5)$ and $B(1, -2)$

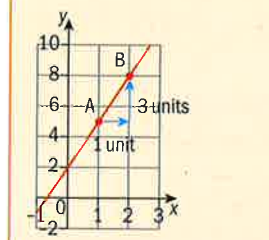
Answers

a $\left.\begin{array}{l} x_1 = 1 \\ y_1 = 5 \\ x_2 = 2 \\ y_2 = 8 \end{array}\right\} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{8-5}{2-1} = 3$

Substitute into the gradient formula.

Gradient = 3
For each 1 unit that x increases, y increases 3 units.

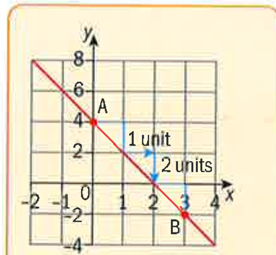


b $\left.\begin{array}{l} x_1 = 0 \\ y_2 = 4 \\ x_2 = 3 \\ y_2 = -2 \end{array}\right\} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-2-4}{3-0} = -2$

Substitute into the gradient formula.

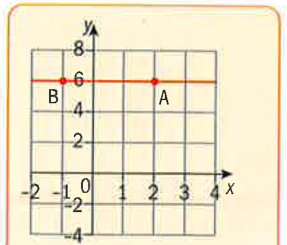
Gradient = −2
For each 1 unit that x ***increases****, y* ***decreases*** *by 2 units.*



c $\left.\begin{array}{l} x_1 = 2 \\ y_1 = 6 \\ x_2 = -1 \\ y_2 = 6 \end{array}\right\} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{6-6}{-1-2} = 0$

Substitute into the gradient formula.
Gradient = 0
For each 1 unit that x ***increases****, y* ***remains constant****. The line is horizontal.*



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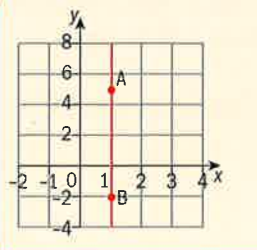
d $\left.\begin{aligned} x_1 &= 1 \\ y_1 &= 5 \\ x_2 &= 1 \\ y_2 &= -2 \end{aligned}\right\} \Rightarrow m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{-2-5}{1-1} = \dfrac{-7}{0}$	*Substitute into the gradient formula.* ***Remember** that division by zero is not defined therefore **the gradient of this line is not defined**. The line is vertical.*



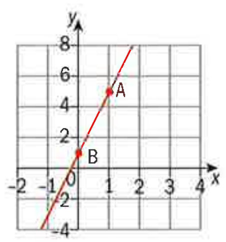
Exercise 3A

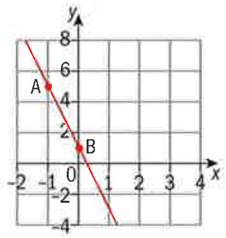
1 Plot the points A(2, 7), B(0, 9), C(0, −9) and D(2, −7) on a graph. Find the gradients of the lines

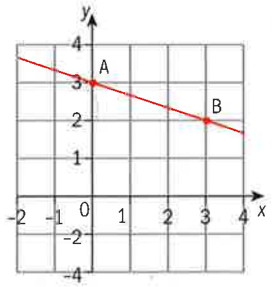
a AB **b** AC
c BD **d** CD

2 For each of these lines
  i write down the coordinates of the points A and B
  ii find the gradient.

The scales on the *x*-axis and *y*-axis are not always the same.

a


b


c


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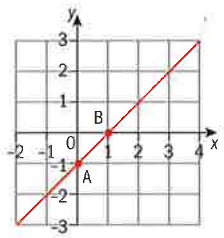

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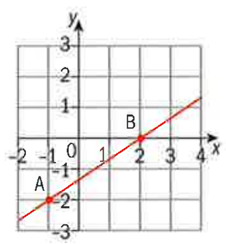

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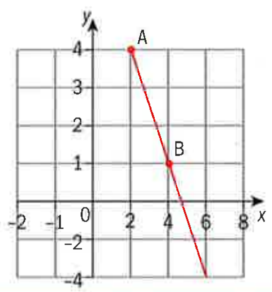
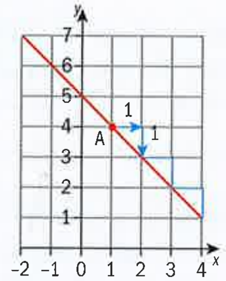


Example 2

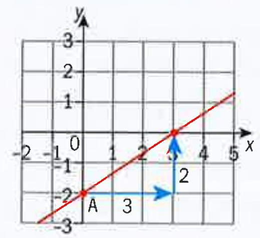
a Draw a line that passes through the point A(1, 4) with gradient −1.
b Draw a line that passes through the point A(0, −2) with gradient $\frac{2}{3}$.

Road gradients are often given as percentages or ratios. How do road signs show gradient in your country?

Answers

a


Plot the point A(1, 4).
The gradient is −1 so,
$m = -1 = \frac{-1}{1} = \frac{y\text{-step}}{x\text{-step}}$
so every time that x increases by 1 unit, y decreases by 1 unit.

b


Plot the point A(0, −2).
The gradient is $\frac{2}{3}$ so,
$m = \frac{2}{3} = \frac{y\text{-step}}{x\text{-step}}$
so every time that x increases by 3 units, y increases by 2 units.

Exercise 3B

1 **a** Draw a line with gradient $\frac{1}{2}$ that passes through the point A(0, 3).
b Draw a line with gradient −3 that passes through the point B(1, 2).
c Draw a line with gradient 2 that passes through the point C(3, −1).

2 For each of these lines, points A, B and C lie on the same line.
  i find the gradient of line AB.
  ii find the second coordinate of point C:

a *A*(2, 5), *B*(3, 7) and *C*(4, *p*)
b *A*(0, 2), *B*(1, 6) and *C*(2, *t*)
c *A*(0, 0), *B*(1, −5) and *C*(2, *q*)
d *A*(0, −1), *B*(1, 0) and *C*(4, *s*)
e *A*(−5, 1), *B*(−6, 4) and *C*(−4, *r*)

You may use a graph or the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

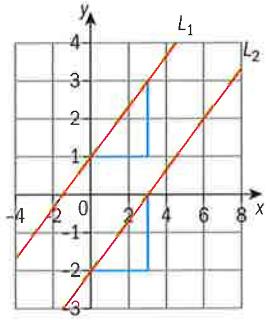
EXAM-STYLE QUESTIONS

3 The gradient of the line passing through points P(−1, 5) and Q(*a*, 10) is 4.
a Write down an expression in terms of *a* for the gradient of PQ.
b Find the value of *a*.

4 In line MN, every time that *x* increases by 1 unit, *y* increases by 0.5 units. Point M is (2, 6) and point N is (−3, *t*).
a Write down the gradient of MN.
b Write down an expression for the gradient of MN in terms of *t*.
c Find the value of *t*.

Parallel lines

→ **Parallel lines** have the **same gradient**. This means that
- if two lines are parallel then they have the same gradient
- if two lines have the same gradient then they are parallel.



The symbols, $L_1 \parallel L_2$ mean 'L_1 is parallel to L_2'.

Note that, although the gradient of a vertical line is not defined, two vertical lines are parallel.

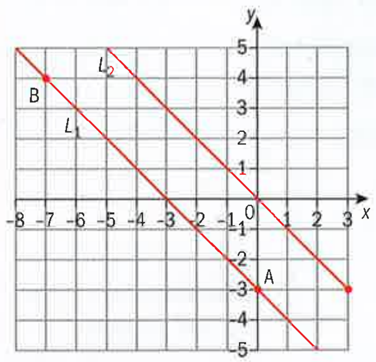
Example 3

Line L_1 passes through the points A(0, –3) and B(–7, 4).
a Find the gradient of L_1. **b** Draw and label L_1.
c Draw and label a second line L_2 passing through the origin and parallel to L_1.

Answers

a $m = \frac{4-(-3)}{-7-0} = -1$ *Substitute into the gradient formula.*

b and **c** *For L_1, plot A and B and join them. For L_2, draw a line through the origin parallel to L_1.*



Remember that the ***origin*** is the point O(0, 0), the point where the *x*-axis and the *y*-axis meet.

Exercise 3C

1 Line L_1 passes through the points A(2, 5) and B(0, –4).
a Find the gradient of L_1. **b** Draw and label L_1.
c Draw and label a second line L_2 passing through the point C(0, 2) and parallel to L_1.

2 Decide whether each line is parallel to the *y*-axis, the *x*-axis or neither:
a the line passing through the points P(1, 7) and Q(12, 7)
b the line passing through the points P(1, 7) and T(1, –3)
c the line passing through the points P(1, 7) and M(2, 5).

3 Complete these statements to make them true.
a Any horizontal line is parallel to the ______-axis.
b Any vertical line is parallel to the ______-axis.
c Any horizontal line has gradient equal to ______.

4 PQ is parallel to the *x*-axis. The coordinates of P and Q are (5, 3) and (8, *a*) respectively. Write down the value of *a*.

5 MN is parallel to the *y*-axis. The coordinates of M and N are respectively (*m*, 24) and (–5, 2). Write down the value of *m*.

Perpendicular lines

→ Two lines are **perpendicular** if, and only if, they make an angle of 90°.
This means that
- if two lines are perpendicular then they make an angle of 90°
- if two lines make an angle of 90° then they are perpendicular.

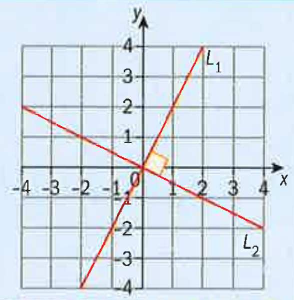
The *x*-axis and the *y*-axis are perpendicular.

The next example shows you the **numerical relationship** between the gradients of two perpendicular lines that are not horizontal and vertical.

Any vertical line is perpendicular to any horizontal line.

Example 4

The diagram shows two perpendicular lines L_1 and L_2.
a Find the gradients of L_1 and L_2.
b Show that the product of their gradients is equal to –1.

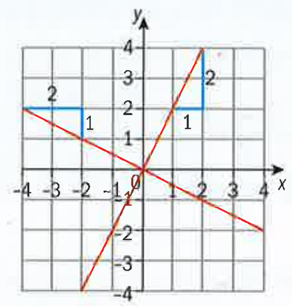


Note that the gradient of L_1 is positive and the gradient of L_2 is negative.

Answers

a Let m_1 be the gradient of L_1 and m_2 the gradient of L_2.
$m_1 = 2$ and $m_2 = -\frac{1}{2}$

Use the diagram to find m_1 and m_2.



b $2 \times -\frac{1}{2} = -1$

In general, if the gradient of a line is *m*, the gradient of a perpendicular line is $-\frac{1}{m}$

→ Two lines are **perpendicular** if the product of their gradients is –1.

a and *b* are reciprocal if $a \times b = 1$ or $a = \frac{1}{b}$
For example:
2 and $\frac{1}{2}$, $\frac{4}{3}$ and $\frac{3}{4}$

Exercise 3D

1 Which of these pairs of numbers are negative reciprocals?

a 2 and $-\frac{1}{2}$ **b** $-\frac{4}{3}$ and $\frac{3}{4}$ **c** 3 and $\frac{1}{3}$ **d** −1 and 1

2 Which of these pairs of gradients are of perpendicular lines?

a $\frac{2}{5}$ and $\frac{5}{2}$ **b** $\frac{4}{3}$ and $-\frac{3}{4}$ **c** −3 and $-\frac{1}{3}$ **d** 1 and −1

3 Find the gradient of lines that are perpendicular to a line with gradient

a −3 **b** $\frac{2}{3}$ **c** $-\frac{1}{4}$ **d** 1 **e** −1

4 Find the gradient of any line perpendicular to the line passing through the points

a A(−2, 6) and B(1, −1) **b** A(5, 10) and B(0, −2)

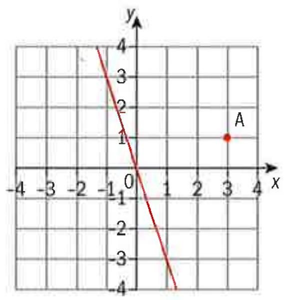
5 Each diagram shows a line and a point A.

i Write down the gradient of the line.

ii Write down the gradient of any line that is perpendicular to this line.

iii Copy the diagram and draw a line perpendicular to the red line passing through the point A.

a



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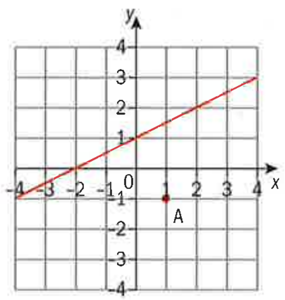

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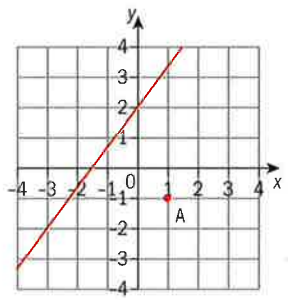


EXAM-STYLE QUESTIONS

6 Line L_1 passes through the points P(0, 3) and Q(−2, *a*).

a Find an expression for the gradient of L_1 in terms of *a*.

L_1 is perpendicular to line L_2. The gradient of L_2 is 2.

b Write down the gradient of L_1.

c Find the value of *a*.

7 The points A(3, 5) and B(5, −8) lie on the line L_1.

a Find the gradient of L_1.

A second line, L_2, is perpendicular to L_1.

b Write down the gradient of L_2.

L_2 passes through the points P(5, 0) and Q(*t*, 2).

c Find the value of *t*.

3.2 Equations of lines

The coordinates *x* and *y* of **any** point on a line *L* are linked by an equation, called the **equation of the line**.

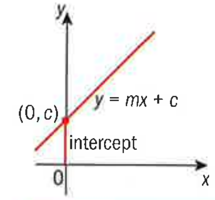
This means that:

- If a point *Q* lies on a line *L* then the coordinates of *Q* satisfy the equation of *L*.
- If the coordinates of any point *Q* satisfy the equation of a line *L*, then the point *Q* lies on *L*.

→ The equation of a straight line can be written in the form $y = mx + c$, where
- *m* is the **gradient**
- *c* is the ***y*-intercept** (*y*-coordinate of the point where the line crosses the *y*-axis).

$y = mx + c$ is the **gradient-intercept** form of the straight line equation.

Values for variables *x* and *y* are said to **satisfy** an equation if, when the variables are replaced by the respective values, the two sides of the equation are equal.



The equation $y = mx + c$ is in the Formula booklet. You will revisit this equation again in Chapter 4.

As well as $y = mx + c$, some people express the equation of a line as $y = ax + b$ or $y = mx + b$

Example 5

The line *L* passes through the point A(1, 7) and has gradient 5.
Find the equation of *L*.
Give your answer in the form $y = mx + c$.

Answer

Let P(*x*, *y*) be **any** point on *L*.
The gradient of *L* is 5

$\left.\begin{matrix} x_1 = 1 \\ y_1 = 7 \\ x_2 = x \\ y_2 = y \end{matrix}\right\} \Rightarrow \frac{y-7}{x-1} = 5$ *Use the gradient formula with A and P, and equate to 5.*

$y - 7 = 5(x - 1)$ *Multiply both sides by (x − 1)*
$y - 7 = 5x - 5$ *Expand brackets.*
$y = 5x + 2$ *Add 7 to both sides.*
y = mx + c where m = 5 and c = 2

Use A(7, 1) to check:
$7 = 5 \times 1 + 2$

Check that
- *the coordinates of the point A(1, 7) satisfy the equation of the line.*

Note that in the equation $y = 5x + 2$
- 5 multiplies *x*, and the gradient of the line is $m = 5$
- Putting $x = 0$ in the equation of *L*, $y = 5 \times 0 + 2 = 2$

Therefore the point (0, 2) lies on *L*.

Example 6

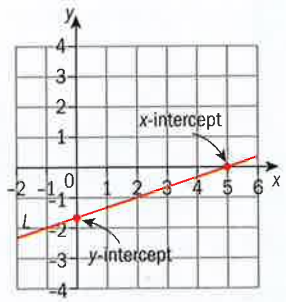
The line L has gradient $\frac{1}{3}$ and passes through A(2, –1).

a Find the equation of L. Give your answer in the form $y = mx + c$.
b Write down the point of intersection of L with the y-axis.
c Find the point of intersection of L with the x-axis.
d Draw the line L showing clearly the information found in **b** and **c**.

Answers

a $y = \frac{1}{3}x + c$	*Substitute $m = \frac{1}{3}$ in the equation $y = mx + c$.*
$-1 = \frac{1}{3} \times 2 + c$	*Substitute the coordinates of point A(2,–1) in the equation of the line.*
$-1 = \frac{2}{3} + c$	
$c = -\frac{5}{3}$	*Make c the subject of the equation.*
$y = \frac{1}{3}x - \frac{5}{3}$	*Substitute c in the equation of the line.*
b $\left(0, -\frac{5}{3}\right)$	*The line crosses the y-axis at the point (0, c).*
c $0 = \frac{1}{3}x - \frac{5}{3}$	*Any point on the x-axis has the form (k, 0).* *Substitute y = 0 in the equation of L.*
$\frac{1}{3}x = \frac{5}{3}$	
$x = 5$	
Therefore L intersects the x-axis at the point (5, 0).	

d



Note that you could find the equation of L using the same method as in Example 5.

Exercise 3E

1 Find the equation of a line with
a gradient 3 that passes through the point A(1, 4)
b gradient $\frac{5}{3}$ that passes through the point A(4, 8)
c gradient –2 that passes through the point A(–3, 0)
Give your answers in the form $y = mx + c$.

2 For each of these lines write down
i the gradient
ii the point of intersection with the y-axis
iii the point of intersection with the x-axis.

a $y = 2x + 1$ **b** $y = -3x + 2$ **c** $y = -x + 3$ **d** $y = -\frac{2}{5}x - 1$

EXAM-STYLE QUESTIONS

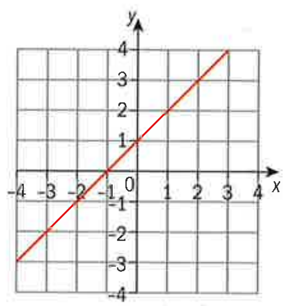
3 A line has equation $y = \frac{3(x-6)}{2}$.
a Write the equation in the form $y = mx + c$.
b Write down the gradient of the line.
c Write down the y-intercept.
d Find the point of intersection of the line with the x-axis.

4 The line AB joins the points A(2, –4) and B(1, 1).
a Find the gradient of AB.
b Find the equation of AB in the form $y = mx + c$

5 The line PQ joins the points P(1, 3) and Q(2, 5).
a Find the gradient of PQ.
b Find the equation of PQ in the form $y = mx + c$
c Find the gradient of all lines perpendicular to PQ.
d Find the equation of a line perpendicular to PQ that passes through A(0, 2).

6 Line L_1 has gradient 3 and is perpendicular to line L_2.
a Write down the gradient of L_2.
Line L_2 passes through the point P(5, 1).
b Find the equation of L_2. Give your answer in the form $y = mx + c$
c Find the x-coordinate of the point where L_2 meets the x-axis.

7 Find the equations of these lines, in the form $y = mx + c$

a


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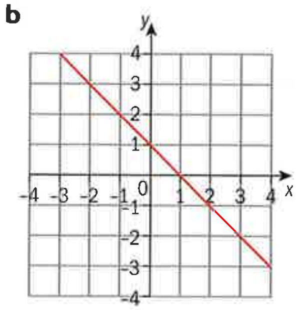

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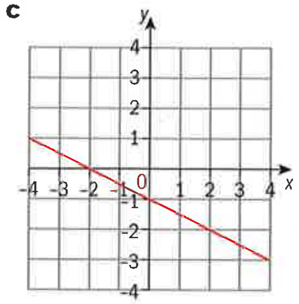

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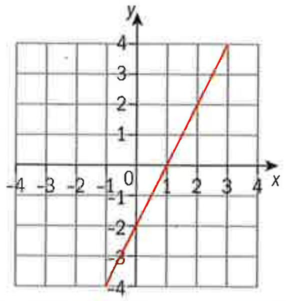
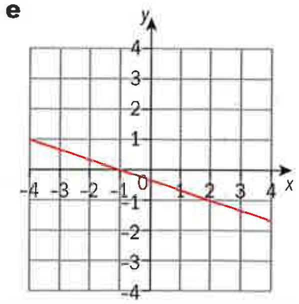


e


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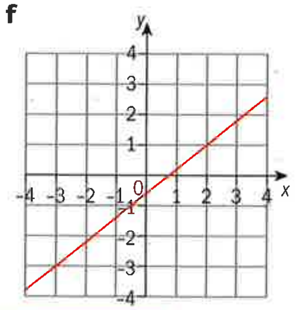


Example 7

a Line L joins the points A(–3, 5) and B(1, 2).
Find the equation of line L.
Give your answer in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$

b The point $Q\left(\frac{5}{3}, t\right)$ lies on L. Find the value of t.

Answers

a The gradient of L is
$m = \frac{2-5}{1-(-3)} = -\frac{3}{4}$ — *Use the gradient formula with the coordinates of A and B.*

Let P(x, y) be **any** point on L.
The gradient of L is also — *Use the gradient formula with A and P (or B and P).*

$$\left.\begin{array}{l} x_1 = -3 \\ y_1 = 5 \\ x_2 = x \\ y_2 = y \end{array}\right\} \Rightarrow m = \frac{y-5}{x-(-3)}$$

$\frac{y-5}{x-(-3)} = -\frac{3}{4}$ — *Equate gradients.*

$4(y - 5) = -3(x + 3)$ — *Cross multiply.* $\frac{a}{b} = \frac{c}{d} \Leftrightarrow a \times d = b \times c$

$4y - 20 = -3x - 9$ — *Expand brackets.*

$3x + 4y - 11 = 0$ — *Rearrange equation to form $ax + by + d = 0$*

$a = 3$, $b = 4$, $d = -11$

***Check that** both points A and B satisfy the equation of the line.*

b The point $Q\left(\frac{5}{3}, t\right)$ lies on L so its coordinates must satisfy the equation of L.

$3x + 4y - 11 = 0$

$3 \times \frac{5}{3} + 4 \times t - 11 = 0$ — *Substitute the coordinates of Q in the equation of L.*

$5 + 4t - 11 = 0$ — *Solve for t.*

$4t - 6 = 0$

$4t = 6$

$t = 1.5$

The equation $ax + by + d = 0$ is called **the general form** and is also in the Formula booklet.

Note that any multiple of this equation would also be correct as long as $a, b, d \in \mathbb{Z}$, e.g. $-3x - 4y + 11 = 0$ or $6x + 8y - 22 = 0$

Discuss: How many points do we need to determine a line?
Investigate: the meaning of the word 'collinear'. When do we say that three or more points are collinear?

→ The equation of a straight line can be written in the form
$ax + by + c = 0$
where a, b and $c \in \mathbb{Z}$.

Exercise 3F

1 Find the equations of these lines. Give your answers in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$.
- **a** A line with gradient –4 that passes through the point A(5, 0).
- **b** A line with gradient $\frac{1}{2}$ that passes through the point A(2, 3).
- **c** The line joining the points A(3, –2) and B(–1, 3).
- **d** The line joining the points A(0, 5) and B(–5, 0).

2 Rewrite each of these equations in the form $y = mx + c$.
- **a** $3x + y = 0$
- **b** $x + y + 1 = 0$
- **c** $2x + y - 1 = 0$
- **d** $2x - 4y = 0$
- **e** $6x + 3y - 9 = 0$

Make y the subject of the formula

3 The line L has equation $3x - 6y + 6 = 0$.
- **a** Write down the equation of L in the form $y = mx + c$.
- **b** Write down the x-intercept.
- **c** Write down the y-intercept.

4 The equation of a line is $y = 2x - 6$
- **a** Which of these points lie on this line?
  A(3, 0), B(0, 3), C(1, –4), D(4, 2), E(10, 12), F(5, 4)
- **b** The point (a, 7) lies on this line. Find the value of a.
- **c** The point (7, t) lies on this line. Find the value of t.

5 The equation of a line is $-6x + 2y - 2 = 0$
- **a** Which of these points lie on this line?
  A(1, 4), B(0, 1), C(1, 0), D(2, 6), E($-\frac{1}{3}$, 0), F(–1, 2)
- **b** The point (a, 3) lies on this line. Find the value of a.
- **c** The point (10, t) lies on this line. Find the value of t.

6 The table has four equations and four pairs of conditions. Match each equation with the pair of conditions that satisfies that line.

	Equation		Conditions
A	$6x - 3y + 15 = 0$	E	The x-intercept is 2.5 and the y-intercept is 5
B	$y = 2x - 5$	F	The gradient is –2 and the line passes through the point (1, –7)
C	$10x + 5y + 25 = 0$	G	The line passes through the points (0, –5) and (2.5, 0)
D	$y = -2x + 5$	H	The y-intercept is (0, 5) and the gradient is 2

EXAM-STYLE QUESTION

7 The line L_1 has equation $2x - y + 6 = 0$
- **a** Write down the gradient of L_1.
- **b** Write down the y-intercept of L_1.
- **c** The point A(c, 1.5) lies on L_1. Find the value of c.
- **d** The point B(5, t) lies on L_1. Find the value of t.

Line L_2 is parallel to L_1.
- **e** Write down the gradient of L_2.
- **f** Find the equation of L_2 if it passes through C(0, 4).

EXAM-STYLE QUESTION

8 The line L_1 joins the points A(1, 2) and B(–1, 6).

a Find the equation of L_1.

C is the point (10, –16).

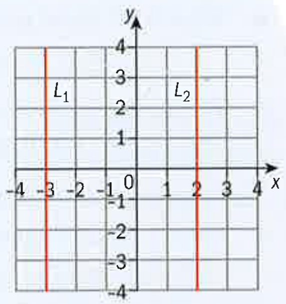
b Decide whether A, B and C are collinear, giving a reason for your answer.

Vertical and horizontal lines

Vertical lines are parallel to the y-axis.
Horizontal lines are parallel to the x-axis.

Investigation – vertical and horizontal lines

The diagram shows two **vertical** lines, L_1 and L_2.

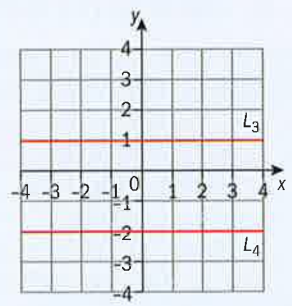


1 **a** Write down the coordinates of at least five points lying on L_1.
b What do you notice about the coordinates of the points from **a**? What do their coordinates have in common?
c What is the condition for a point to lie on L_1? Write down this condition in the form $x = k$ where k takes a particular value.

2 **a** Write down the coordinates of at least five points lying on L_2.
b What do you notice about the coordinates of the points from **a**? What do their coordinates have in common?
c What is the condition for a point to lie on L_2? Write down this condition in the form $x = k$ where k takes a particular value.

3 What is the equation of a vertical line passing through the point (1, –3)?

The diagram shows two **horizontal** lines, L_3 and L_4.



4 **a** Write down the coordinates of at least five points lying on L_3.
b What do you notice about the coordinates of the points from **a**? What do their coordinates have in common?
c What is the condition for a point to lie on L_3? Write down this condition in the form $y = k$ where k takes a particular value.

5 **a** Write down the coordinates of at least five points lying on L_4.
b What do you notice about the coordinates of the points from **a**? What do their coordinates have in common?
c What is the condition for a point to lie on L_4? Write down this condition in the form $y = k$ where k takes a particular value.

6 What is the equation of a horizontal line passing through the point (1, –3)?

→ • The equation of any vertical line is of the form $x = k$ where k is a constant.
• The equation of any horizontal line is of the form $y = k$ where k is a constant.

Intersection of lines in two dimensions

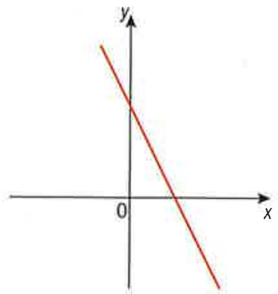
→ If two lines are parallel then they have the same gradient and do not intersect.

Parallel lines L_1 and L_2 can be:

- *Coincident lines (the same line)*

  e.g. $2x + y = 3$ and $6x + 3y = 9$

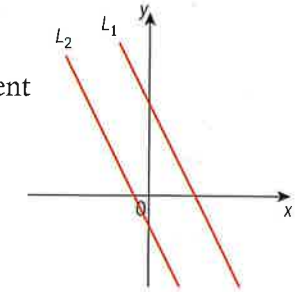
  $L_1 = L_2$ therefore they have the same gradient and the same y-intercept. There is an infinite number of points of intersection.



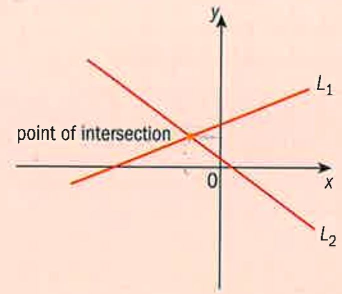
- *Different lines*

  e.g. $2x + y = 3$ and $2x + y = -1$

  L_1 and L_2 have the same gradient but different y-intercepts. There is no point of intersection.



→ If two lines L_1 and L_2 are not parallel then they intersect at just one point.



To find the point of intersection write $m_1x_1 + c_1 = m_2x_2 + c_2$ and solve for x.

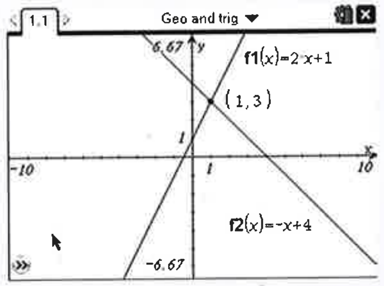
Example 8

Find the point of intersection of the lines $y = 2x + 1$ and $-x - y + 4 = 0$.

Answer	
Algebraically	
$y = 2x + 1$ and $y = -x + 4$	*Write both equations in the gradient–intercept form.*
$2x + 1 = -x + 4$	*Equate expressions for y.*
$3x = 3$	*Solve for x.*
$x = 1$	
so $y = 2 \times 1 + 1$	*Substitute for x in one of the equations to find y.*
$= 3$	
The point of intersection is (1, 3).	

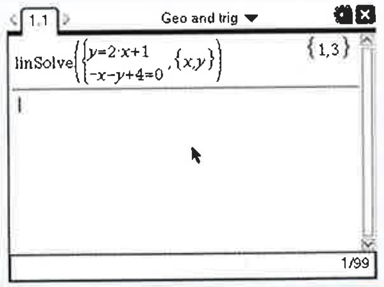
▶ Continued on next page

Using GDC Method 1



Rearrange both equations into the gradient–intercept form.

Using GDC Method 2



Solve the pair of simultaneous equations

$$\begin{cases} -2x + y = 1 \\ -x - y = -4 \end{cases}$$

For help with drawing graphs on your GDC, see Chapter 12, Section 3.4, Example 18.

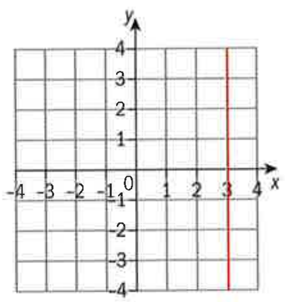
For help with solving simultaneous equations on your GDC, see Chapter 12, Section 1.1, Example 1.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

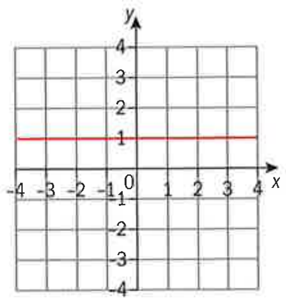
Exercise 3G

1 Write down the equations of these lines.

a



b



2 Find the point of intersection of each pair of lines.

a $y = 3x - 6$ and $y = -x + 2$

b $-x + 5y = 0$ and $\frac{1}{5}x + y - 2 = 0$

c $y = 3$ and $x = -7$

d $y = 1.5x + 4$ and $y = 1$

e $-x + 2y + 6 = 0$ and $x + y - 3 = 0$

f y-axis and $y = 4$

3 Show that the lines L_1 with equation $-5x + y + 1 = 0$ and L_2 with equation $10x - 2y + 4 = 0$ are parallel.

4 State, with reasons, whether each pair of lines meet at

i only one point **ii** an infinite number of points

iii no point.

a $y = 3(x - 5)$ and $x - \frac{1}{3}y + 6 = 0$

b $\frac{y+1}{x-2} = -1$ and $y = -x + 1$

c $y = 4x - 8$ and $4x - 2y = 0$

d $x - y + 3 = 0$ and $3x - 3y + 9 = 0$

EXAM-STYLE QUESTION

5 Line L_1 has gradient 5 and intersects line L_2 at the point A(1, 0).

a Find the equation of L_1.

Line L_2 is perpendicular to L_1.

b Find the equation of L_2.

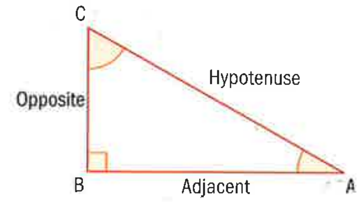
Point *A* lies on both lines.

3.3 The sine, cosine and tangent ratios

Trigonometry is the study of lengths and angles in triangles. This section looks at trigonometry in *right-angled* triangles.
In a **right-angled triangle** the side opposite the right angle is the **hypotenuse**, which is the **longest side**.

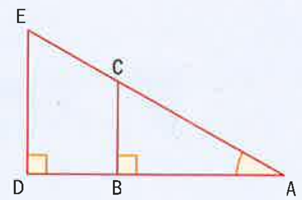
- AC is the hypotenuse
- AB is adjacent to angle A (Â)
- BC is opposite Â

Some textbooks use 'right triangle' instead of right-angled triangle.



Investigation – right-angled triangles

Draw a diagram of two triangles like this.



1 Measure the angles at E and C. What do you notice?

2 Measure the lengths AB and AD. Calculate the ratio $\frac{AD}{AB}$

3 Measure the lengths AE and AC. Calculate the ratio $\frac{AE}{AC}$

4 Measure the lengths DE and BC. Calculate the ratio $\frac{DE}{BC}$

What do you notice about your answers to questions 2 to 4?

In the diagram the right-angled triangles ABC and ADE have the same angles, and corresponding sides are in the same ratio.
The ratios $\frac{AB}{AC}, \frac{BC}{AC}$ and $\frac{BC}{AB}$ in triangle ABC are respectively equal to the ratios $\frac{AD}{AE}, \frac{DE}{AE}$ and $\frac{DE}{AD}$ in triangle ADE.

Two triangles with the same angles and corresponding sides in the same ratio are called **similar triangles**.

Therefore

$\frac{AB}{AC} = \frac{AD}{AE} = \frac{\text{Adjacent to } \hat{A}}{\text{Hypotenuse}}$ Note that both AB and AD are adjacent to Â, and AC and AE are the hypotenuses.

$\frac{BC}{AC} = \frac{DE}{AE} = \frac{\text{Opposite } \hat{A}}{\text{Hypotenuse}}$ Note that both BC and DE are opposite Â, and both AC and AE are the hypotenuses.

$\frac{BC}{AB} = \frac{DE}{AD} = \frac{\text{Opposite } \hat{A}}{\text{Adjacent to } \hat{A}}$ Note that both BC and DE are opposite Â, and both AB and AD are adjacent to Â.

In any triangle **similar** to triangle ABC these ratios will remain the same.

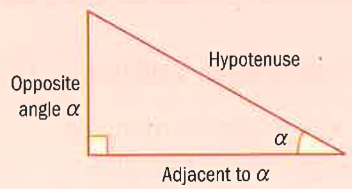
Some textbooks call the two shorter sides of a right-angled triangle the 'legs' of the triangle.

→ Three trigonometric ratios in a right-angled triangle are defined as

$\sin\alpha = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

$\cos\alpha = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$

$\tan\alpha = \frac{\text{Opposite side}}{\text{Adjacent side}}$



α is the Greek letter 'alpha'.
- 'sin α' is read 'sine of α'
- 'cos α' is read 'cosine of α'
- 'tan α' is read 'tangent of α'

You can use the acronym **SOHCAHTOA** to help you remember which ratio is which.

SOH as **S**in $\alpha = \frac{O}{H}$

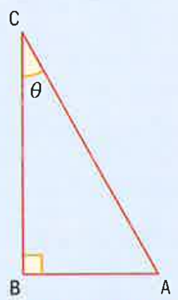
CAH as **C**os $\alpha = \frac{A}{H}$

TOA as **T**an $\alpha = \frac{O}{A}$

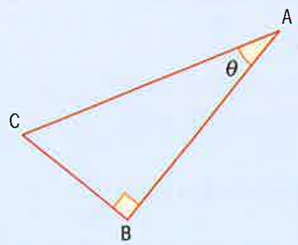
Example 9

For each triangle, write down the three trigonometric ratios for the angle θ in terms of the sides of the triangle.

a



b



Answers

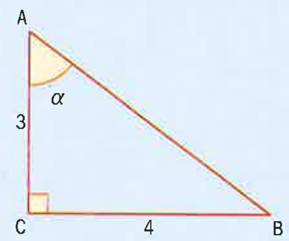
a $\sin\theta = \frac{AB}{AC}$, $\cos\theta = \frac{BC}{AC}$, $\tan\theta = \frac{AB}{BC}$

b $\sin\theta = \frac{BC}{AC}$, $\cos\theta = \frac{AB}{AC}$, $\tan\theta = \frac{BC}{AB}$

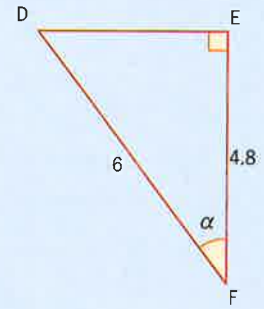
Example 10

For each of these right-angled triangles find the value of
i $\sin\alpha$ **ii** $\cos\alpha$ **iii** $\tan\alpha$.

a



b



Answers

a $AB^2 = 3^2 + 4^2$ *First find the hypotenuse. Use Pythagoras.*

$AB = 5$

Now

i $\sin\alpha = \frac{BC}{AB}$ *$\sin\alpha = \frac{opp}{hyp}$*

$\sin\alpha = \frac{4}{5}$

ii $\cos\alpha = \frac{AC}{AB}$ *$\cos\alpha = \frac{adj}{hyp}$*

$\cos\alpha = \frac{3}{5}$

iii $\tan\alpha = \frac{BC}{AC}$ *$\tan\alpha = \frac{opp}{adj}$*

$\tan\alpha = \frac{4}{3}$

b $DE^2 + 4.8^2 = 6^2$ *First find DE.*

$DE = 3.6$

i $\sin\alpha = \frac{DE}{DF} = \frac{3.6}{6}$

$\sin\alpha = 0.6$

ii $\cos\alpha = \frac{EF}{DF} = \frac{4.8}{6}$

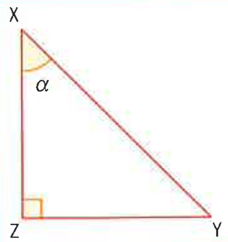
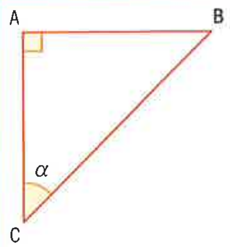
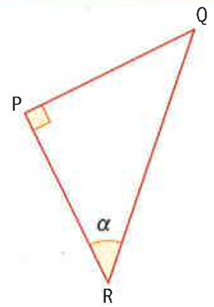
$\cos\alpha = 0.8$

iii $\tan\alpha = \frac{DE}{EF} = \frac{3.6}{4.8}$

$\tan\alpha = 0.75$

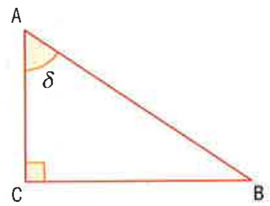
Exercise 3H

1 Copy and complete this table.

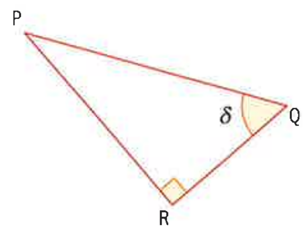
Triangle	Hypotenuse	Side opposite α	Side adjacent to α
			
			
			

2 Write down the three trigonometric ratios for the angle δ in terms of the sides of the triangle.

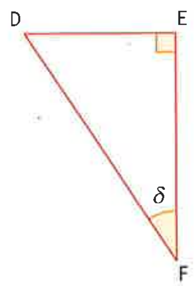
a



b

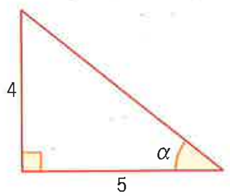


c

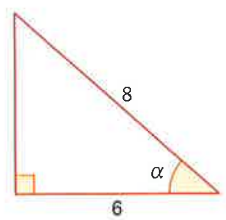


3 In each of these right-angled triangles all lengths are in cm.

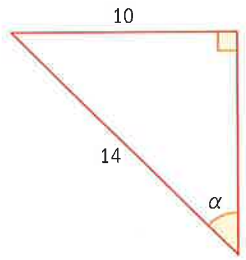
a



b



c

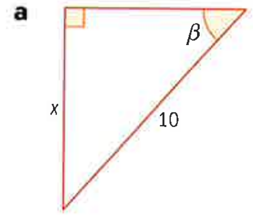


Find the exact value of

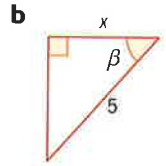
i $\sin\alpha$ **ii** $\cos\alpha$ **iii** $\tan\alpha$.

4 For each triangle write down a trigonometric equation to link angle β and the side marked x.

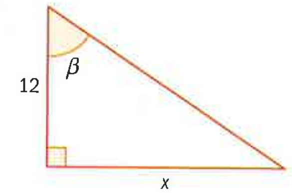
a



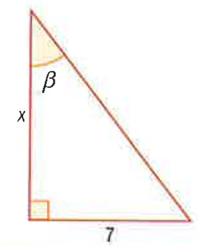
b



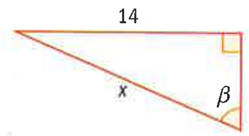
c



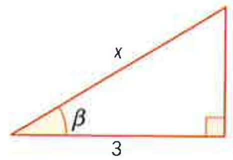
d



e



f



Finding the sides of a right-angled triangle

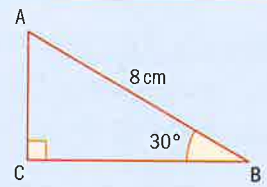
If you know the size of one of the acute angles and the length of one side in a right-angled triangle you can find

- the lengths of the other sides using trigonometric ratios
- the third angle using the sum of the interior angles of a triangle.

Label the sides opposite, adjacent and hypotenuse so you can identify which ones you know.

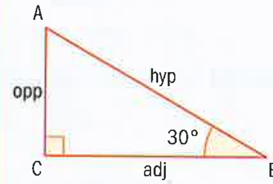
Example 11

Find the length of the unknown sides in triangle ABC.
Give your answer to 3sf.



Answer

To find BC:



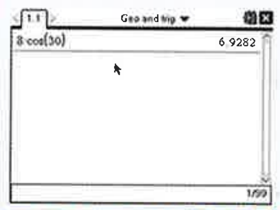
$\cos 30° = \dfrac{BC}{8}$

Cosine links the unknown side BC (adjacent to the 30° angle) and the known side AB (the hypotenuse).

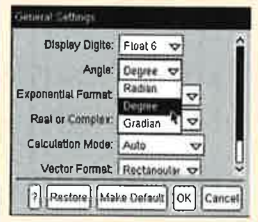
BC = 8cos 30°

Use the GDC to solve for BC.

BC = 6.93 cm (to 3 sf)



Remember to set your GDC in **degrees**. To change to **degree mode** press On and choose 5:Settings & Status | 2:Settings | 1:General



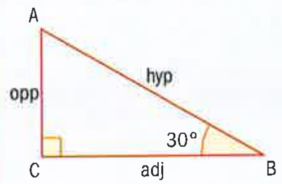
Use the tab key to move to Angle and select Degree. Press enter and then select 4:Current to return to the document.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

▶ Continued on next page

To find AC:

Method 1



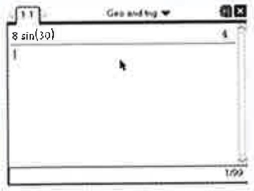
$\sin 30° = \frac{AC}{8}$

Sine links the known and the unknown sides.

$AC = 8\sin 30°$

$= 4\text{cm}$

Solve for AC. Use the GDC:



Method 2

$AC^2 + BC^2 = AB^2$

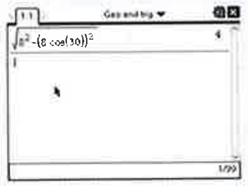
$AC^2 + (8\cos 30°)^2 = 8^2$

$AC = \sqrt{8^2 - (8\cos 30°)^2}$

$= 4\text{cm}$

Use Pythagoras as you already know two sides of the triangle.

Solve for AC. Use the GDC:

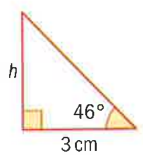


You could also use tangent as you know the angle and the adjacent.

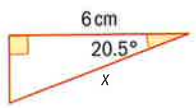
Exercise 3I

Find the lengths of the sides marked with letters. Give your answers correct to two decimal places.

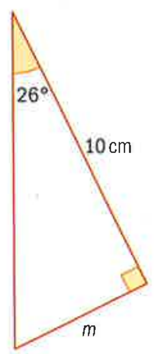
1



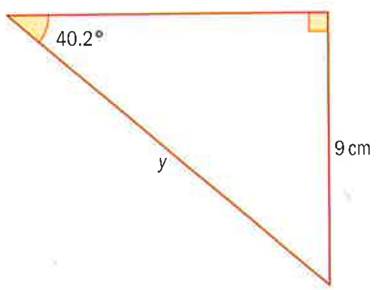
2



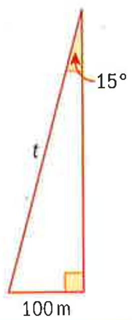
3



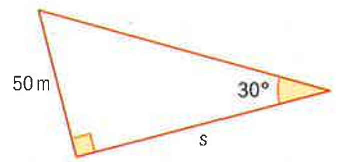
4



5



6



Example 12

In triangle DEF, $\hat{E} = 90°$, $\hat{F} = 50°$ and DE = 7 m

a Represent this information in a clear **labeled** diagram.

b Find the size of $\hat{D}$.

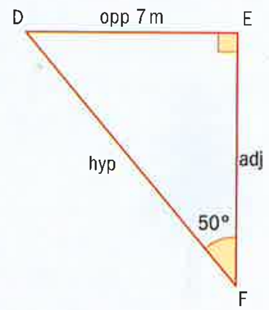
c Find EF.

d Find DF.

Give your answers to 3 sf.

Answers

a



Draw a diagram. Label the triangle in alphabetical order clockwise.

b $\hat{D} + 90° + 50° = 180°$

$\hat{D} = 40°$

The sum of the interior angles of a triangle is 180°.

c $\tan 50° = \frac{7}{EF}$

$EF = \frac{7}{\tan 50°}$

$= 5.87\text{m}$

Tangent links the known and the unknown sides.

Use the GDC to solve for EF.

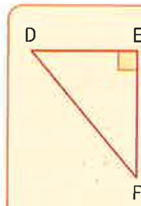
d $\sin 50° = \frac{7}{DF}$

$DF = \frac{7}{\sin 50°} = 9.14\text{m}$

Sine links the known and the unknown sides.

Use the GDC to solve for DF.

The astronomer Aryabhata, born in India is about 476 CE, believed that the Sun, planets and stars circled the Earth in different orbits. He began to invent trigonometry in order to calculate the distances from planets to the Earth.



$\hat{D}$ can also be described as $E\hat{D}F$ or ∠FDE. Make sure you understand all these notations.

Exercise 3J

1 In triangle PQR, $\hat{R} = 90°$, $\hat{P} = 21°$, PR = 15 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{Q}$.

c Find QR.

2 In triangle STU, $\hat{T} = 90°$, $\hat{U} = 55°$, SU = 35 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{S}$.

c Find TU.

3 In triangle ZWV, $\hat{V} = 90°$, $\hat{W} = 15°$, WV = 30 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{Z}$.

c Find VZ.

Label the triangle in alphabetical order clockwise.

EXAM-STYLE QUESTIONS

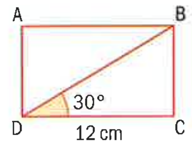
4 In triangle LMN, $\hat{N} = 90°$, $\hat{L} = 33°$, LN = 58 cm.

a Represent this information in a clear and **labeled** diagram.

b Write down the size of $\hat{M}$.

c Find LM.

5 In rectangle ABCD, DC = 12 cm and the diagonal BD makes an angle of 30° with DC.

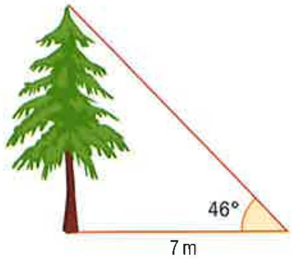


a Find the length of BC.

b Find the perimeter of the rectangle ABCD.

c Find the area of the rectangle ABCD.

6 When the sun makes an angle of 46° with the horizon a tree casts a shadow of 7 m.
Find the height of the tree.



7 A ladder 7 metres long leans against a wall, touching a window sill, and makes an angle of 50° with the ground.

a Represent this information in a clear and **labeled** diagram.

b Find the height of the window sill above the ground.

c Find how far the foot of the ladder is from the foot of the wall.

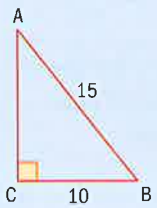
Finding the angles of a right-angled triangle

If you know the lengths of two sides in a right-angled triangle, you can find

- the length of the other side by using Pythagoras
- the size of the two acute angles by using the appropriate trigonometric ratios.

Example 13

Find the sizes of the two acute angles in this triangle.



Answer

Angle $\hat{B}$

$\cos\hat{B} = \frac{10}{15}$

$\hat{B} = \cos^{-1}\left(\frac{10}{15}\right)$

Cosine links adjacent and hypotenuse.

'$\cos^{-1}\left(\frac{10}{15}\right)$' means 'the angle with a cosine of $\frac{10}{15}$'.

$\cos^{-1}\left(\frac{10}{15}\right)$ is read 'inverse cosine of $\frac{10}{15}$'.

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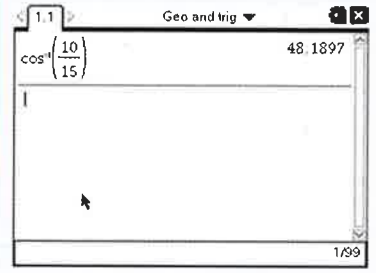
Therefore

$\hat{B} = 48.2°$

Use the GDC.



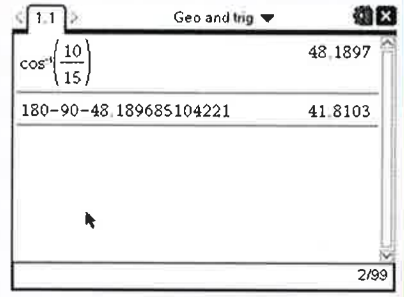
Angle $\hat{A}$

$90° + \hat{B} + \hat{A} = 180°$

$90° + 48.18... + \hat{A} = 180°$

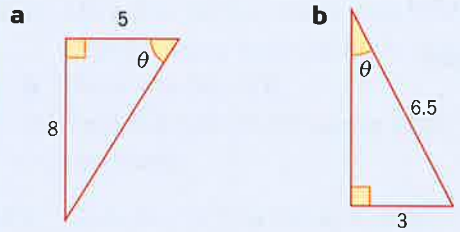
$\hat{A} = 41.8°$

Use the angle sum of a triangle.
Using the GDC:



Example 14

Find the angle marked θ in each triangle.
Give your answers correct to the nearest degree.



Answers

a $\tan\theta = \frac{8}{5}$

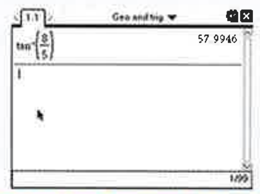
$\theta = \tan^{-1}\left(\frac{8}{5}\right)$

$\theta = 58°$

Use tangent; it links the adjacent and the opposite.

'$\theta = \tan^{-1}\left(\frac{8}{5}\right)$' means 'the angle with a tangent of $\frac{8}{5}$'.

Use the GDC:



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b $\sin\theta = \frac{3}{6.5}$

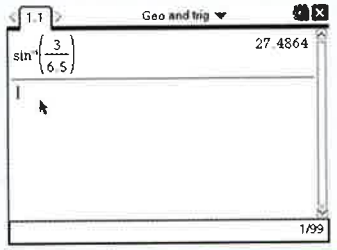
$\theta = \sin^{-1}\left(\frac{3}{6.5}\right)$

$\theta = 27°$

Use sine; it links the opposite and the hypotenuse.

'$\theta = \sin^{-1}\left(\frac{3}{6.5}\right)$' means 'the angle with a sine of $\frac{3}{6.5}$'.

Use the GDC:



Exercise 3K

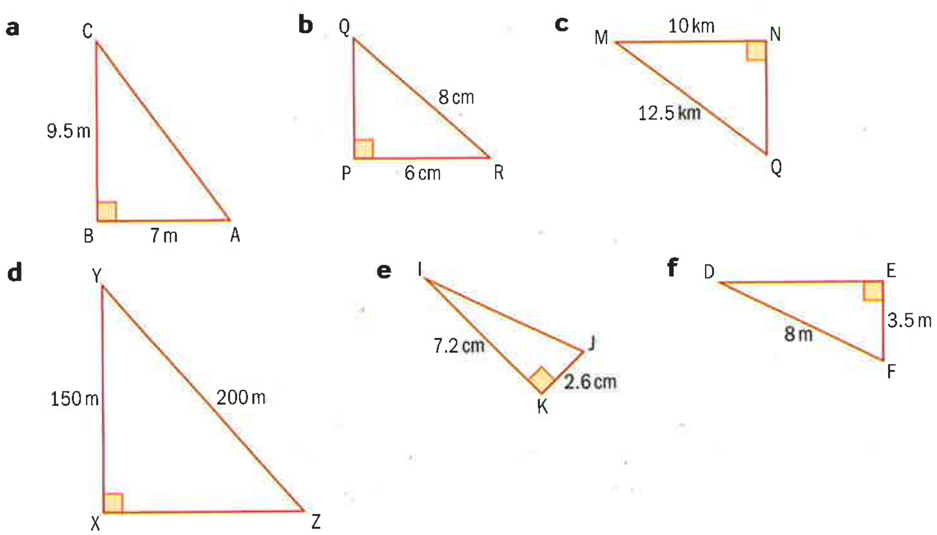
Give your answers correct to 3 sf.

1 Explain the meaning of
a $\sin^{-1}(0.6)$ **b** $\tan^{-1}\left(\frac{1}{2}\right)$ **c** $\cos^{-1}\left(\frac{2}{3}\right)$.

2 Calculate
a $\sin^{-1}(0.6)$ **b** $\tan^{-1}\left(\frac{1}{2}\right)$ **c** $\cos^{-1}\left(\frac{2}{3}\right)$.

3 Find the **acute** angle α if
a $\sin\alpha = 0.2$ **b** $\cos\alpha = \frac{2}{3}$ **c** $\tan\alpha = 1$.

4 Find the sizes of the two acute angles in these triangles.

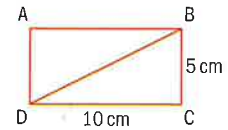


5 In triangle BCD, $\hat{D} = 90°$, BD = 54 cm, DC = 42 cm.
a Represent this information in a clear and **labeled** diagram.
b Find the size of $\hat{C}$.

6 In triangle EFG, $\hat{G} = 90°$, FG = 56 m, EF = 82 m.
a Represent this information in a clear and **labeled** diagram.
b Find the size of $\hat{F}$.

7 In triangle HIJ, $\hat{J} = 90°$, IJ = 18 m, HI = 25 m.
a Represent this information in a clear and **labeled** diagram.
b Find the size of $\hat{H}$.

8 In rectangle ABCD, BC = 5 cm and DC = 10 cm.



Find the size of the angle that the diagonal BD makes with the side DC.

9 The length and width of a rectangle are 20 cm and 13 cm respectively.
Find the angle between a diagonal and the shorter side of the rectangle.

10 A ladder 8 m long leans against a vertical wall.
The base of the ladder is 3 m away from the wall.
Calculate the angle between the wall and the ladder.

EXAM-STYLE QUESTIONS

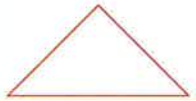
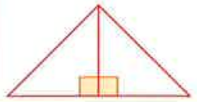
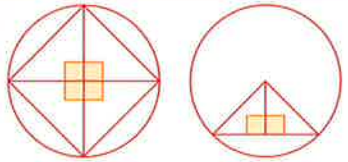
11 a On a pair of Cartesian axes plot the points A(3, 0) and B(0, 4). Use the same scale on both axes.
b Draw the line AB.
c Find the size of the **acute** angle that the line AB makes with the x-axis.

12 a On a pair of Cartesian axes plot the points A(–1, 0) and B(1, 4). Use the same scale on both axes.
b Draw the line AB.
c Find the size of the **acute** angle that the line AB makes with the x-axis.

Finding right-angled triangles in other shapes

So far you have found unknown sides and angles in right-angled triangles. Next you will learn how to find unknown sides and angles in triangles that are not right-angled and in shapes such as rectangles, rhombuses and trapeziums.

The technique is to break down the shapes into smaller ones that contain right-angled triangles.

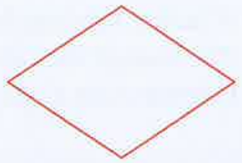
Name of shape	Shape	Where are the right-angled triangles?
Isosceles or equilateral triangles		
Rectangles or squares		
Circle		

Investigation – 2-D shapes

How can you break these shapes into smaller shapes so that at least one of them is a right-angled triangle?

To do this you need to know the properties of 2-D shapes.

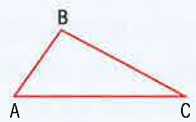
1 *Rhombus*
What is the property of the diagonals of a rhombus?
Make an accurate drawing of a rhombus on squared paper.
Draw the diagonals. How many right-angled triangles do you obtain? Are they congruent? Why?
Comment on your findings.



2 *Kite*
What is the property of the diagonals of a kite?
Make an accurate drawing of a kite on squared paper.
Draw the diagonals. How many right-angled triangles do you obtain?
Are they congruent? Why? Comment on your findings.

3 *Parallelogram*
Draw a parallelogram like this one on squared paper.
There is a rectangle that has the same base and height as this parallelogram. Draw dotted lines where you would cut the parallelogram and rearrange it to make a rectangle.
How many shapes do you obtain? How many of them are right-angled triangles? Comment on your findings.

4 *Triangle*
Draw a triangle like this one.
Every triangle has three heights, one for each base (or side).
Draw the height relative to AC (this is the line segment drawn from B to AC and perpendicular to AC). You will get two right-angled triangles that make up the triangle ABC. Under what conditions would these triangles be congruent? Comment on your findings.



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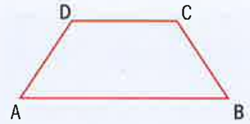
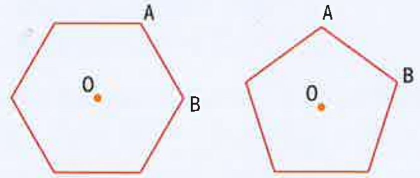
5 *Trapezium*
Draw a trapezium like this one.



Draw a line from D perpendicular to AB and a line from C perpendicular to AB. You will get two right-angled triangles.
What is the condition for these triangles to be congruent?

6 *Regular polygon*
Here are a regular hexagon and a regular pentagon.



O is the center of each polygon.
For **each** polygon:
What type of triangle is ABO? Why? Draw a line from O perpendicular to the side AB to form two right-angled triangles. These two triangles are congruent.
Explain why.

> A **regular polygon** has all sides equal lengths and all angles equal.

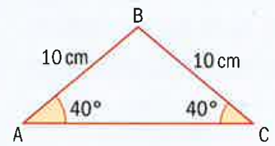
Example 15

Triangle ABC is isosceles. The two equal sides AB and BC are 10 cm long and each makes an angle of 40° with AC.

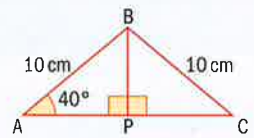
a Represent this information in a clear and labeled diagram.
b Find the length of AC.
c Find the perimeter of triangle ABC.

Answers

a



b



$\cos 40° = \frac{AP}{10}$

In an isosceles triangle the perpendicular from the apex to the base ***bisects*** *the base, making two right-angled triangles.*
$\cos = \frac{adj}{hyp}$

$AP = 10\cos 40°$
$AC = 2 \times 10\cos 40°$
$AC = 15.3\text{ cm}$

Make AP the subject of the equation.
Use the fact that AC = 2 × AP.

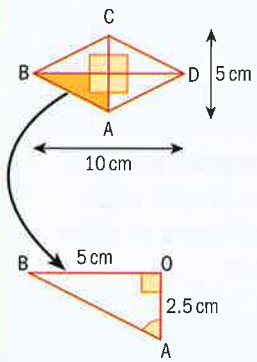
c Perimeter = AB + BC + CA
= 15.32 … + 2 × 10
= 35.3 cm (to 3 sf)

> Bisect means 'cut in half'.

Example 16

The diagonals of a rhombus are 10 cm and 5 cm. Find the size of the **larger** angle of the rhombus.

Answer



Draw a diagram, showing the diagonals.

Let O be the point where the diagonals meet.

$\tan \text{angle OAB} = \frac{5}{2.5}$

In triangle ABO, angle OAB is greater than angle OBA (it is opposite the larger side). So find angle OAB.

Angle OAB $= \tan^{-1}\left(\frac{5}{2.5}\right)$

$\tan = \frac{opp}{adj}$

Angle BAD $= 2 \times$ OAB

$= 2 \times \tan^{-1}\left(\frac{5}{2.5}\right)$

Angle BAD (or BCD) is the larger angle of the rhombus.

Angle BAD = 127° (to 3 sf)

The diagonals of the rhombus bisect each other at right angles.

'Angle OAB' and OÂB are alternative notation for Â.

Investigation – rhombus

1 Use a ruler and a pair of compasses to construct a rhombus with a side length of 6 cm.
2 Construct another rhombus with a side length of 6 cm that is not congruent to the one you drew in **1**.
3 How many different rhombuses with a side length of 6 cm could you construct? In what ways do they differ?

Exercise 3L

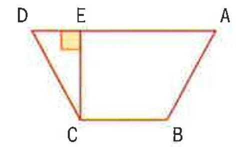
1 Triangle ABC is isosceles. The two equal sides AC and BC are 7 cm long and they each make an angle of 65° with AB.
  a Represent this information in a clear and labeled diagram.
  b Find the length of AB.
  c Find the perimeter of triangle ABC (give your answer correct to the nearest centimetre).

2 The diagonals of a rhombus are 12 cm and 7 cm. Find the size of the **smaller** angle of the rhombus.

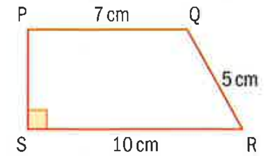
3 The size of the larger angle of a rhombus is 120° and the longer diagonal is 7 cm.
  a Represent this information in a clear and labeled diagram.
  b Find the length of the shorter diagonal.

EXAM-STYLE QUESTIONS

4 In the diagram ABCD is a trapezium where AD || BC, CD = BA = 6 m, BC = 12 m and DA = 16 m
  a Show that DE = 2 m.
  b Find the size of $\hat{D}$.



5 In the diagram PQRS is a trapezium, PQ || SR, PQ = 7 cm, RS = 10 cm, QR = 5 cm and $\hat{S}$ = 90°
  a Find the height, PS, of the trapezium.
  b Find the area of the trapezium.
  c Find the size of angle SRQ.

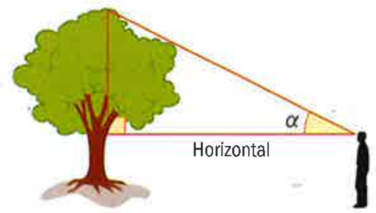


6 The length of the shorter side of a rectangular park is 400 m. The park has a straight path 600 m long joining two opposite corners.
  a Represent this information in a clear and labeled diagram.
  b Find the size of the angle that the path makes with the longer side of the park.

7 a On a pair of Cartesian axes, plot the points A(3, 2), C(−1, −4), and D(−1, 2). Use the same scale on both axes.
  B is a point such that ABCD is a rectangle.
  b i Plot B on your diagram.
    ii Write down the coordinates of the point B.
  c Write down the length of
    i AB **ii** BC.
  d Hence find the size of the angle that a diagonal of the rectangle makes with one of the shorter sides.

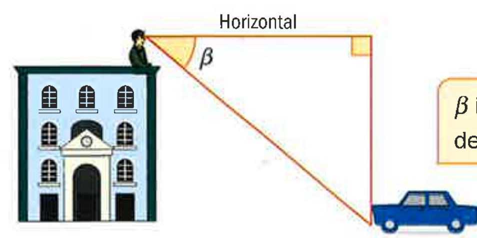
Angles of elevation and depression

→ The **angle of elevation** is the angle you lift your eyes through to look at something above.



α is the angle of elevation.

→ The **angle of depression** is the angle you lower your eyes through to look at something below.



β is the angle of depression.

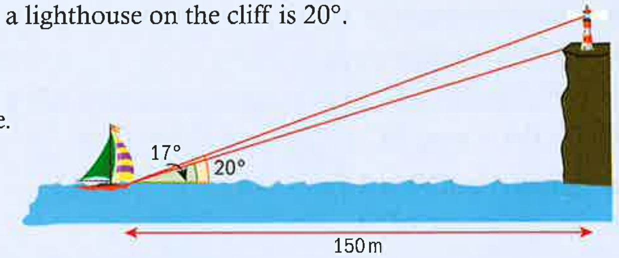
Notice that both the angle of elevation and the angle of depression are measured from the **horizontal**.

Example 17

From a yacht, 150 metres out at sea, the angle of elevation of the top of a cliff is 17°. The angle of elevation to the top of a lighthouse on the cliff is 20°. This information is shown in the diagram.

a Find the height of the cliff.

b Hence find the height of the lighthouse.

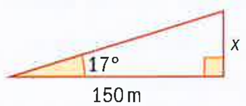


Answers

a Let x be the height of the cliff

$\tan 17° = \frac{x}{150}$

$x = 45.9$ m (to 3 sf)



b Let y be the distance from the top of the lighthouse to the sea.

$\tan 20° = \frac{y}{150}$

$y = 54.5955...$m

height of the lighthouse $= y - x$

$= 8.74$ m (to 3 sf)

Use the unrounded value of x to find y – x.

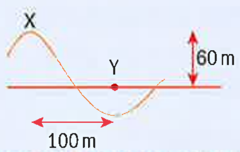
Example 18

A boy standing on a hill at X can see a boat on a lake at Y as shown in the diagram. The vertical distance from X to Y is 60 m and the horizontal distance is 100 m.

Find:

a the shortest distance between the boy and the boat

b the angle of depression of the boat from the boy.



Answers

a $XY^2 = 100^2 + 60^2$

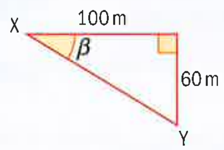
$XY = 117$ m (to 3 sf)

Use Pythagoras.

b $\tan \beta = \frac{60}{100}$

The angle of depression $= 31.0°$ (to 3 sf)

Use $\tan = \frac{opp}{adj}$



The **shortest distance** is the length *XY*.

Exercise 3M

Draw a diagram for each question.

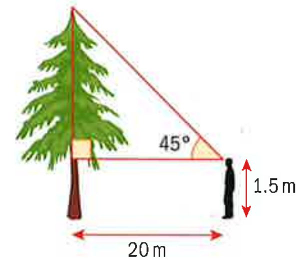
1 Find the angle of elevation of the top of a tree 13 m high from a point 25 m away on level ground.

2 A church spire 81 metres high casts a shadow 63 metres long. Find the angle of elevation of the sun.

3 The angle of depression from the top of a cliff to a ship at sea is 14°. The ship is 500 metres from shore. Find the height of the cliff.

4 Find the angle of depression from the top of a cliff 145 metres high to a ship at sea 1.2 kilometres from the shore.

5 A man whose eye is 1.5 metres above ground level stands 20 metres from the base of a tree. The angle of elevation to the top of the tree is 45°. Calculate the height of the tree.



6 The height of a tree is 61.7 metres and the angle of elevation to the top of the tree from ground level is 62.4°. Calculate the distance from the tree to the point at which the angle was measured.

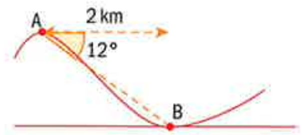
EXAM-STYLE QUESTION

7 The angle of depression of town B from town A is 12°.

a Find the angle of elevation of town A from town B.

The horizontal distance between the towns is 2 km.

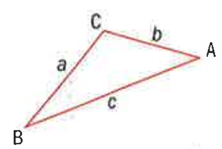
b Find the vertical distance between the towns. Give your answer correct to the nearest metre.



3.4 The sine and cosine rules

The sine and cosine rules are formulae that will help you to find unknown sides and angles in a triangle. They enable you to use trigonomentry in triangles that are **not** right-angled.

The formula and notation are simpler if you label triangles like this.



- The side opposite Â is *a*.
- The side opposite B̂ is *b*.
- The side opposite Ĉ is *c*.

Also notice that

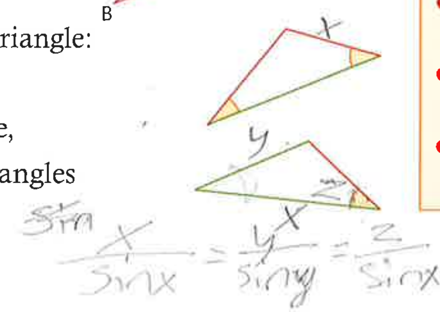
- Â is between sides *b* and *c*.
- B̂ is between sides *a* and *c*.
- Ĉ is between sides *a* and *b*.

The sine rule

If you have this information about a triangle:

- two angles and one side, or
- two sides and a non-included angle,

then you can find the other sides and angles of the triangle.

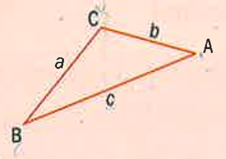


Sine rule

In any triangle ABC with angles $\hat{A}$, $\hat{B}$ and $\hat{C}$, and opposite sides a, b and c respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The sine rule is in the Formula booklet.

Example 19

In triangle ABC, $b = 16$ cm, $c = 10$ cm and $\hat{B} = 135°$.

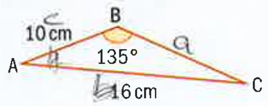
a Represent the given information in a labeled diagram.

b Find the size of angle C.

c Hence find the size of angle A.

Answers

a



b $\frac{16}{\sin 135°} = \frac{10}{\sin \hat{C}}$ — *Substitute in the sine rule.*

$16 \sin \hat{C} = 10 \sin 135°$ — *Cross multiply.*

$\sin \hat{C} = \frac{10 \sin 135°}{16}$ — *Make sin $\hat{C}$ the subject of the formula.*

$\hat{C} = 26.2°$ (to 3 sf) — *Use your GDC.*

c $\hat{A} + \hat{B} + \hat{C} = 180°$

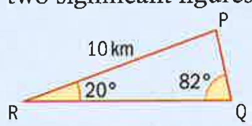
$\hat{A} + 135° + 26.227... = 180°$ — *Use your GDC.*

$\hat{A} = 18.8°$ (to 3 sf)

Cross multiply.
$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

Example 20

In triangle PQR, find the length of RQ. Give your answer correct to two significant figures.



Answer

$\hat{P} = 78°$ — *RQ is opposite angle P so first find the size of angle P.*

$\frac{RQ}{\sin 78°} = \frac{10}{\sin 82°}$ — *Substitute in the sine rule.*

$RQ = \frac{10 \sin 78°}{\sin 82°}$ — *Make RQ the subject of the formula.*

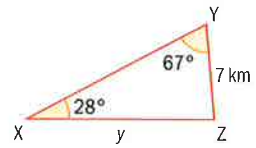
$= 9.9$ km (to 2 sf) — *Use your GDC.*

Ptolemy (c. 90–168 CE), in his 13-volume work *Almagest*, wrote sine values for angles form 0° to 90°. He also included theorems similar to the sine rule.

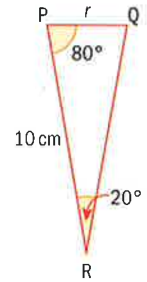
Exercise 3N

1 Find the sides marked with letters.

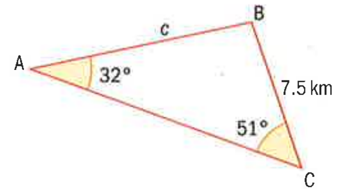
a



b



c



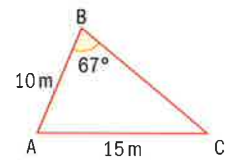
2 In triangle ABC, AC = 12 cm, $\hat{A} = 30°$ and $\hat{B} = 46°$. Find the length of BC.

3 In triangle ABC, $\hat{A} = 15°$, $\hat{B} = 63°$ and AB = 10 cm. Find the length of BC.

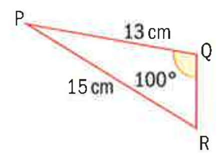
4 In triangle PQR, PR = 15 km, $\hat{P} = 25°$ and $\hat{Q} = 60°$. Find the length of QR.

5 In each triangle, find the angle indicated.

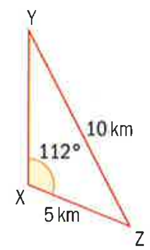
a $\hat{C}$



b $\hat{R}$



c $\hat{Y}$



6 In triangle ABC, BC = 98 m, AB = 67 m and $\hat{A} = 85°$. Find the size of $\hat{C}$.

7 In triangle PQR, PQ = 5 cm, QR = 6.5 cm and $\hat{P} = 70°$. Find the size of $\hat{R}$.

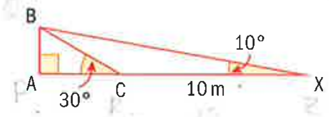
EXAM-STYLE QUESTION

8 In the diagram, $\hat{A} = 90°$, CX = 10 m, $A\hat{C}B = 30°$ and $\hat{X} = 10°$

a Write down the size of angle BCX.

b Find the length of BC.

c Find the length of AB.

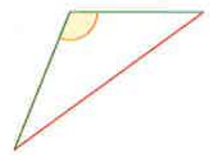


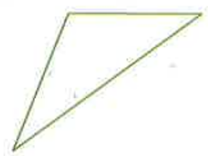
The cosine rule

If you have this information about a triangle:

- two sides and the included angle, or
- the three sides,

then you can find the other side and angles of the triangle.





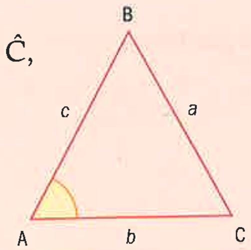
→ Cosine rule

In any triangle ABC with angles Â, B̂ and Ĉ, and opposite sides a, b and c respectively:

$$a^2 = b^2 + c^2 - 2bc\cos \hat{A}$$

This formula can be rearranged to

$$\cos \hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$



These formulae are in the Formula booklet. The first version of the formula is useful when you need to find a side. The second version of the formula is useful when you need to find an angle.

Example 21

In triangle ABC, AC = 8.6 m, AB = 6.3 m and Â = 50°.
Find the length of BC.

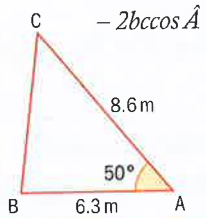
Answer

$BC^2 = 8.6^2 + 6.3^2 - 2 \times 8.6 \times 6.3 \times \cos 50°$

$BC^2 = 43.9975\ldots$

BC = 6.63 m (to 3 sf)

Sketch the triangle.
Use $a^2 = b^2 + c^2 - 2bc\cos \hat{A}$



The cosine rule applies to **any** triangle. For a right-angled triangle A = 90°. What does the formula look like? Do you recognize it? Is the cosine rule a generalization of Pythagoras' theorem?

Example 22

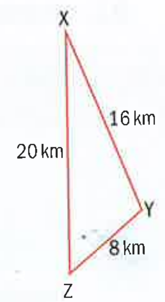
X, Y and Z are three towns. X is 20 km due north of Z.
Y is to the east of line XZ. The distance from Y to X is 16 km and the distance from Z to Y is 8 km.

a Represent this information in a clear and labeled diagram.

b Find the size of angle X.

Answers

a

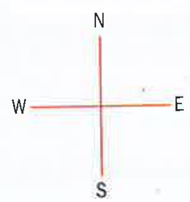


b $\cos \hat{X} = \frac{20^2 + 16^2 - 8^2}{2 \times 20 \times 16}$

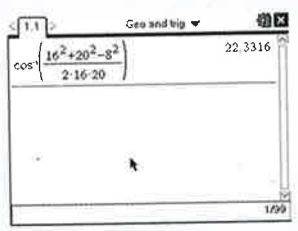
$\hat{X} = \cos^{-1}\left(\frac{20^2 + 16^2 - 8^2}{2 \times 20 \times 16}\right)$

$= 22.3$ (to 3 sf)

Remember:



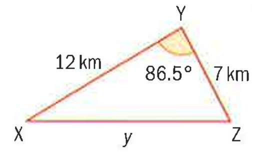
Use $\cos \hat{X} = \frac{y^2 + z^2 - x^2}{2yz}$



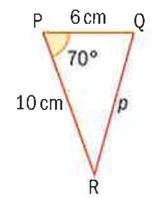
Exercise 30

1 Find the sides marked with letters.

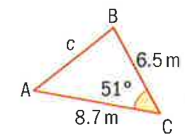
a



b

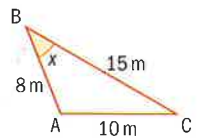


c



2 Find the angles marked with letters.

a



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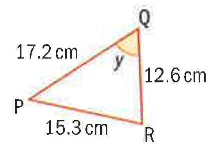
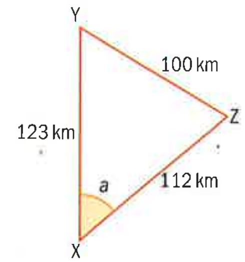


c



3 In triangle ABC, CB = 120 m, AB = 115 m and B̂ = 110°.
Find the length of side AC.

4 In triangle PQR, RQ = 6.9 cm, PR = 8.7 cm and R̂ = 53°.
Find the length of side PQ.

5 In triangle XYZ, XZ = 12 m, XY = 8 m, YZ = 10 m.
Find the size of angle X.

EXAM-STYLE QUESTIONS

6 X, Y and Z are three towns. X is 30 km due south from Y.
Z is to the east of the line joining XY.
The distance from Y to Z is 25 km and the distance from X to Z is 18 km.

a Represent this information in a clear and labeled diagram.

b Find the size of angle Z.

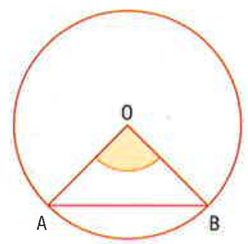
7 Alison, Jane and Stephen are together at point A. Jane walks 12 m due south from A and reaches point J. Stephen looks at Jane, turns through 110°, walks 8 m from A and reaches point S.

a Represent this information in a clear and labeled diagram.

b Find how far Stephen is from Jane.

c Find how far **north** Stephen is from Alison.

8 The diagram shows a circle of radius 3 cm and center O.
A and B are two points on the circumference.
The length AB is 5 cm.
A triangle AOB is drawn inside the circle.
Calculate the size of angle AOB.



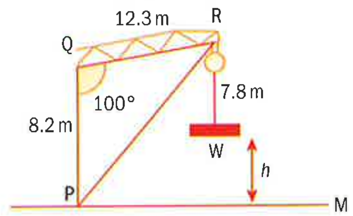
EXAM-STYLE QUESTION

9 The diagram shows a crane PQR that carries a flat box W. PQ is vertical, and the floor PM is horizontal. Given that PQ = 8.2 m, QR = 12.3 m, $P\hat{Q}R = 100°$ and RW = 7.8 m, calculate

a PR

b the size of angle PRQ

c the height, h, of W above the floor, PM.



Extension material on CD: *Worksheet 3 - Cosine and sine rule proofs*

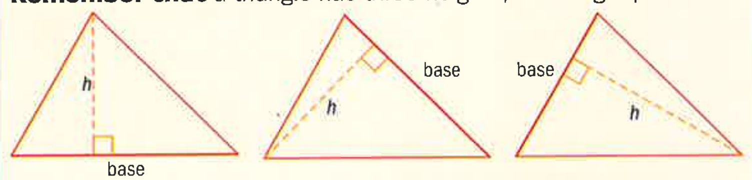
Area of a triangle

If you know one side of a triangle, the base b and the corresponding height h, you can calculate the area of the triangle using the formula

$$A = \frac{1}{2}(b \times h)$$

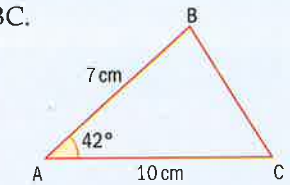
If you do not know the height, you can still calculate the area of the triangle as in the next example.

Remember that a triangle has three heights, one height per side.

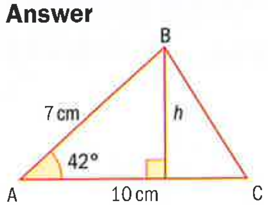


Example 23

Calculate the area of triangle ABC.



Answer



$\sin 42° = \frac{h}{7} \Rightarrow h = 7\sin 42°$

$Area = \frac{1}{2} \times b \times h$

$= \frac{1}{2}(10 \times 7\sin 42°)$

$= 23.4\text{ cm}^2$ (to 3 sf)

Use the formula

$A = \frac{1}{2}(b \times h)$ with AC as the base,

$b = 10$

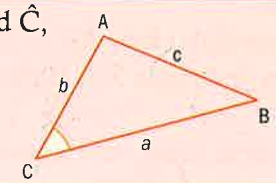
Draw the height, h, the perpendicular to AC from B.

Substitute in the formula for the area of a triangle.

You can use the same method for any triangle.

→ In any triangle ABC with angles $\hat{A}$, $\hat{B}$ and $\hat{C}$, and opposite sides a, b and c respectively, this rule applies:

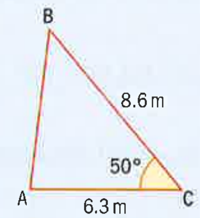
Area of triangle $= \frac{1}{2}ab\sin\hat{C}$



This formula is in the Formula booklet.

Example 24

Calculate the area of the triangle ABC.



Answer

Area of triangle ABC =

$\frac{1}{2} \times 8.6 \times 6.3 \times \sin 50°$

$= 20.8\text{ m}^2$ (to 3sf)

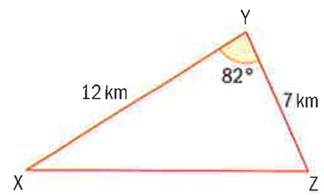
Substitute in the formula

$Area = \frac{1}{2}ab\sin\hat{C}$

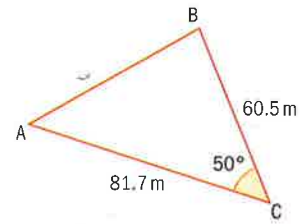
Exercise 3P

1 Calculate the area of each triangle.

a



b

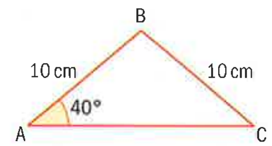


In the first century CE, Hero (or Heron) of Alexandria developed a different method for finding the area of a triangle using the lengths of the triangle's sides.

2 Here is triangle ABC.

a Find the size of angle B.

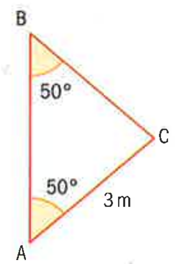
b Calculate the area of triangle ABC.



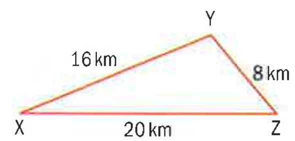
3 Here is triangle ABC.

a Write down the size of angle C.

b Find the area of triangle ABC.



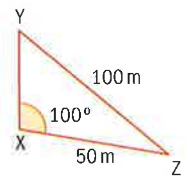
4 Calculate the area of triangle XYZ.



First find the size of one of the angles.

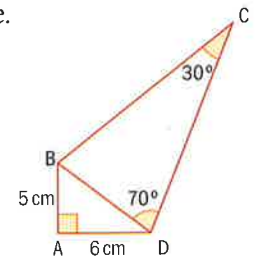
EXAM-STYLE QUESTIONS

5 The diagram shows a triangular field XYZ.
XZ is 50 m, YZ is 100 m and angle X is 100°.
a Find angle Z.
b Find the area of the field. Give your answer correct to the nearest 10 m^2.



6 The area of an isosceles triangle ABC is 4 cm^2. Angle B is 30° and AB = BC = x cm.
a Write down, in terms of x, an expression for the area of the triangle.
b Find the value of x.

7 In the diagram, AB = 5 cm, AD = 6 cm, $B\hat{A}D = 90°$, $B\hat{C}D = 30°$, $B\hat{D}C = 70°$.
a Find the length of DB.
b Find the length of DC.
c Find the area of triangle BCD.
d Find the area of the quadrilateral ABCD.



Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

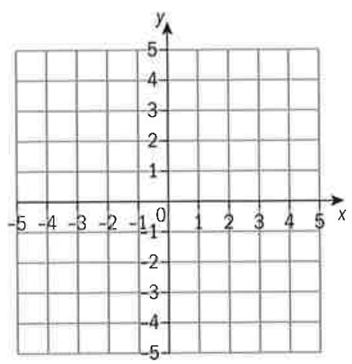
Give answers correct to 3 sf.

1 Line L_1 passes through the points A(1, 3) and B(5, 1).
a Find the gradient of the line AB.

Line L_2 is parallel to line L_1 and passes through the point (0, 4).
b Find the equation of the line L_2.

2 Line L_1 passes through the points A(0, 6) and B(6, 0).
a Find the gradient of the line L_1.
b Write down the gradient of all lines perpendicular to L_1.
c Find the equation of a line L_2 perpendicular to L_1 and passing through O(0, 0).

3 Consider the line L with equation $y = 2x + 3$.
a Write down the coordinates of the point where
i L meets the x-axis **ii** L meets the y-axis.
b Draw L on a grid like this one.
c Find the size of the acute angle that L makes with the x-axis.



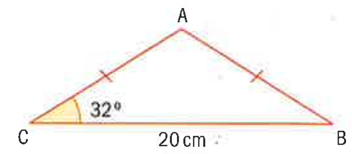
4 Consider the line L_1 with equation $y = -2x + 6$.
a The point $(a, 4)$ lies on L_1. Find the value of a.
b The point $(12.5, b)$ lies on L_1. Find the value of b.

Line L_2 has equation $3x - y + 1 = 0$.
c Find the point of intersection between L_1 and L_2.

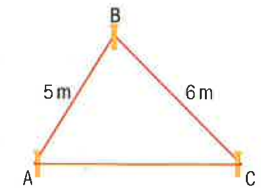
5 The height of a vertical cliff is 450 m. The angle of elevation from a ship to the top of the cliff is 31°. The ship is x metres from the bottom of the cliff.
a Draw a diagram to show this information.
b Calculate the value of x.

EXAM-STYLE QUESTIONS

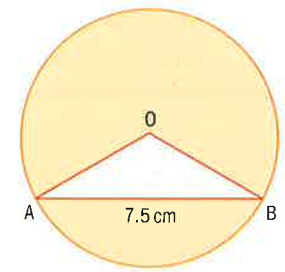
6 In the diagram, triangle ABC is isosceles.
AB = AC, CB = 20 cm and angle ACB is 32°.
Find **a** the size of angle CAB
b the length of AB
c the area of triangle ABC.



7 A gardener pegs out a rope, 20 metres long, to form a triangular flower bed as shown in this diagram.
a Write down the length of AC.
b Find the size of the angle BAC.
c Find the area of the flower bed.



8 The diagram shows a circle with diameter 10 cm and center O. Points A and B lie on the circumference and the length of AB is 7.5 cm. A triangle AOB is drawn inside the circle.
a Find the size of angle AOB.
b Find the area of triangle AOB.
c Find the shaded area.

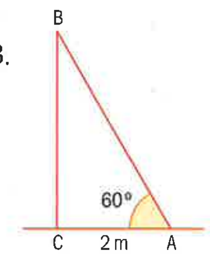


Paper 2 style questions

EXAM-STYLE QUESTIONS

1 **a** On a pair of axes plot the points A(–2, 5), B(2, 2) and C(8, 10).
Use the same scale on both axes.
The quadrilateral ABCD is a rectangle.
b **i** Plot D on the pair of axes used in **a**.
ii Write down the coordinates of D.
c Find the gradient of line BC.
d Hence write down the gradient of line DC.
e Find the equation of line DC in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.
f Find the length of **i** DC **ii** BC.
g Find the size of the angle DBC.

2 The diagram shows a ladder AB. The ladder rests on the horizontal ground AC. The ladder is touching the top of a vertical telephone pole CB. The angle of elevation of the top of the pole from the foot of the ladder is 60°. The distance from the foot of the ladder to the foot of the pole is 2 m.
a Calculate the length of the ladder.
b Calculate the height of the pole.



The ladder is moved in the same vertical plane so that its foot remains on the ground and its top touches the pole at a point P which is 1.5 m below the top of the pole.
c Write down the length of CP.
d Find the new distance from the foot of the ladder to the foot of the pole.
e Find the size of the new angle of elevation of the top of the pole from the foot of the ladder.

EXAM-STYLE QUESTION

3 The diagram shows a cross-country running course. Runners start and finish at point A.

a Find the length of BD.

b Find the size of angle BDC, giving your answer correct to two decimal places.

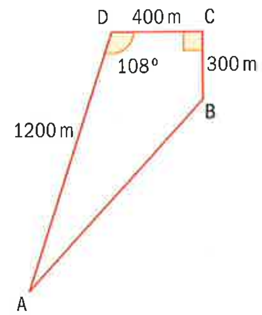
c Write down the size of angle ADB.

d Find the length of AB.

e i Find the perimeter of the course.

ii Rafael runs at a constant speed of $3.8\,\text{m s}^{-1}$. Find the time it takes Rafael to complete the course. Give your answer correct to the nearest minute.

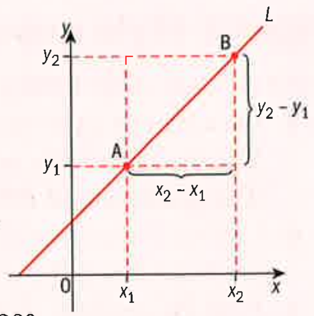
f Find the area of the quadrilateral ABCD enclosed by the course. Give your answer in km^2.



CHAPTER 3 SUMMARY

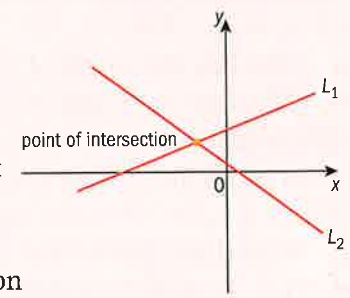
Gradient of a line

- If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points that lie on line L, the gradient of L is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- **Parallel lines** have the **same gradient**. This means that
  - if two lines are parallel then they have the same gradient
  - if two lines have the same gradient then they are parallel.
- Two lines are **perpendicular** if, and only if, they make an angle of 90°. This means that
  - if two lines are perpendicular then they make an angle of 90°
  - if two lines make an angle of 90° then they are perpendicular.
- Two lines are **perpendicular** if the product of their gradients is −1.



Equations of lines

- The equation of a straight line can be written in the form
  - **i** $y = mx + c$, where m is the **gradient** and c is the y-**intercept** (the y-coordinate of the point where the line crosses the y-axis).
  - **ii** $ax + by + d = 0$ where a, b and $d \in \mathbb{Z}$.
- The equation of any vertical line is of the form $x = k$ where k is a constant.
- The equation of any horizontal line is of the form $y = k$ where k is a constant.
- If two lines are parallel then they have the same gradient and do not intersect.
- If two lines L_1 and L_2 are not parallel then they intersect at just one point. To find the point of intersection write $m_1x_1 + c_1 = m_2x_2 + c_2$ and solve for x.



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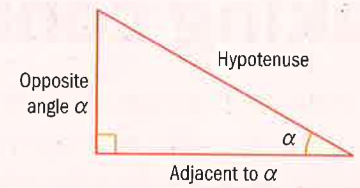
The sine, cosine, and tangent ratios

- Three trigonometric ratios in a right-angled triangle are defined as

$$\sin\alpha = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\cos\alpha = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

$$\tan\alpha = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

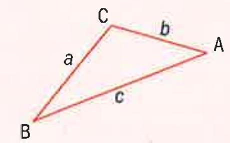


- The **angle of elevation** is the angle you lift your eyes through to look at something above.
- The **angle of depression** is the angle you lower your eyes through to look at something below.

The sine and cosine rules

- In any triangle ABC with angles A, B and C, and opposite sides a, b and c respectively:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

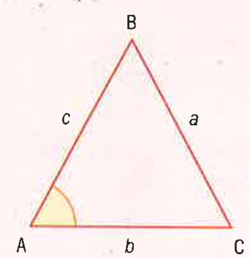


- In any triangle ABC with angles A, B and C, and opposite sides a, b and c respectively:

$$a^2 = b^2 + c^2 - 2bc\cos\hat{A}$$

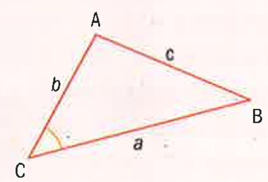
This formula can be rearranged to

$$\cos\hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$



- In any triangle ABC with angles A, B and C, and opposite sides a, b and c respectively, this rule applies:

Area of triangle $= \dfrac{1}{2}ab\sin\hat{C}$



Making connections

Mathematics is often separated into different topics, or fields of knowledge.

- List the different fields of mathematics you can think of.
- Why do humans feel the need to categorize and compartmentalize knowledge?
- Does this help or hinder the search for more knowledge?

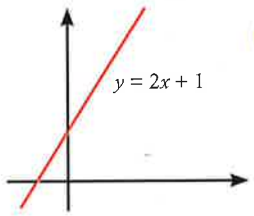
Algebra and geometry

Algebra and geometry are both mathematical disciplines with a very long history.

algebra – generalizes arithmetical operations and relationships by using letters to represent unknown numbers. Possibly originated in solving equations, which goes back (at least) to Babylonian mathematics.

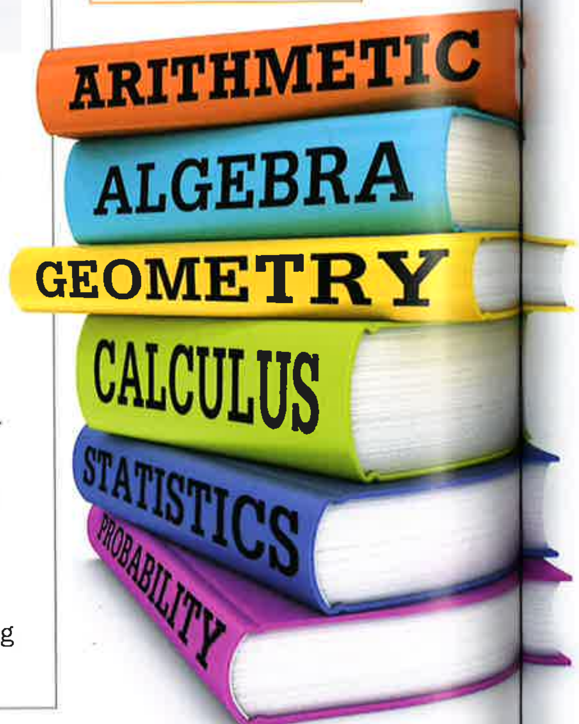
geometry – studies the properties, measurement, and relationships of points, lines, planes, surfaces, angles, and solids. Origins in the very beginning of mathematics.

There was no common ground between algebra and geometry until René Descartes, the French philosopher and mathematician (1596–1650) showed that equations could be represented by lines on a graph, giving people an insight into what these equations mean, and where their solutions are found. Cartesian Geometry – representing equations for given values of x and y on a system of orthogonal (perpendicular) axes, is named after him.



It is said (although the story is probably a myth) that Descartes came up with the idea for his coordinate system while lying in bed and watching a fly crawl on the ceiling of his room.

Algebra and geometry are central to mathematics and school mathematics curricula around the world. Some schools run entirely separate courses on geometry and algebra, while others alternate mathematical topics throughout a course.



Algebra and geometry are both useful in their own right but historically it is the interaction of these two areas that has led to many major mathematical developments and insights in the natural sciences, economics and of course other areas of mathematics.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited, but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

Joseph Louis Lagrange, 1736–1813, French mathematician

Fermat's Last Theorem

Fermat's Last Theorem states that no three positive integers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2. This theorem was first conjectured by Pierre de Fermat in 1637, in a note in a copy of *Arithmetica*, where he claimed he had a proof that was too large to fit in the margin. His proof, if it existed, was never found. It was not solved until 1995, when Andrew Wiles published a proof that he had been working on in secret for seven years.

▼ Andrew Wiles (1953–), British mathematician.

Wiles's complex proof uses the link between what were thought to be two separate areas of mathematics – modular forms and elliptic curves. Don't worry, these are not on the Mathematical Studies syllabus!

Many of the most famous proofs have needed input from different areas of mathematics.

4 Mathematical models

CHAPTER OBJECTIVES:

6.1 Concept of a function, domain, range and graph; function notation; concept of a function as a mathematical model

6.2 Drawing accurate graphs and sketch graphs; transferring a graph from GDC to paper; reading, interpreting and making predictions using graphs

6.3 Linear models: linear functions and their graphs

6.4 Quadratic models: quadratic functions and their graphs (parabolas); properties of a parabola; symmetry, vertex; intercepts; equation of the axis of symmetry

6.5 Exponential models: exponential functions and their graphs; concept and equation of a horizontal asymptote

6.6 Use of a GDC to solve equations involving combinations of the functions above

Before you start

You should know how to:

1 Substitute values into a formula, e.g. Given that $x = -1$, find the value of $y = 3x^2 + 2x$.
$y = 3(-1)^2 + 2(-1) \Rightarrow y = 1$

2 Use your GDC to solve quadratic equations and simultaneous equations in two unknowns, e.g. Solve

a $3x^2 + 9x - 30 = 0 \Rightarrow x = 2, x = -5$

b $\begin{cases} x + y = 4 \\ -2x + y = 1 \end{cases} \Rightarrow x = 1, y = 3$

3 Find the gradient, m, of a line joining two points, e.g. A(3, 5) and B(1, 4).
$y - y_1 = m(x - x_1)$

$m = \dfrac{5-4}{3-1}$

$m = \dfrac{1}{2}$

Skills check

1 a Find the value of $y = 2.5x^2 + x - 1$ when $x = -3$.

b Find the value of $h = 3 \times 2^t - 1$ when $t = 0$.

c Find the value of $d = 2t^3 - 5t^{-1} + 2$ when $t = \dfrac{1}{2}$.

2 Use your GDC to solve:

a $x^2 + x - 3 = 0$

b $2t^2 - t = 2$

c $\begin{cases} x - 2y = 3 \\ 3x - 5y = -2 \end{cases}$

For help, see Chapter 12, Sections 1.1 and 1.2.

3 Find the gradient, m, of a line joining the two points:

a A(7, −2) and B(−1, 4)

b A(−3, −2) and B(1, 8)

The above photo shows the positions of a diver at various moments until he reaches the sea. Initially, the diver is at 40 m above sea level and it takes him 4.5 seconds to reach the sea. We can use mathematics to find a numerical relationship between the time in seconds, t, and the diver's height, h, in metres above sea level. The relationship linking the time, t, and the height, h, is a mathematical model. It can be described using a formula, a graph or a table of values.

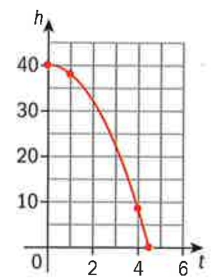
To construct a mathematical model we usually begin by making some assumptions. Here, we assume that the diver is initially at 40 m above sea level and it takes him 4.5 seconds to reach the sea. The formula linking the variables t and h is

$$h = -1.97\,(t^2 - 20.25) \quad \text{where } t \geq 0.$$

You can use this model to calculate the diver's height, h, above sea level at different times, t. Substitute the value of t into the formula to get the corresponding value for h. The table shows three pairs of values for t and h.

t (seconds)	h (metres)
0	40.0
1	38.0
4	8.37

The graph of $h = -1.97(t^2 - 20.25)$, $t \geq 0$, is shown.



The three pairs of values from the table are indicated with a ●.

You can use the formula and/or the graph to answer questions such as:

At what height is the diver after 2 seconds?

How long does it take the diver to reach a height of 20 m above sea level?

4 Mathematical models

CHAPTER OBJECTIVES:

6.1 Concept of a function, domain, range and graph; function notation; concept of a function as a mathematical model

6.2 Drawing accurate graphs and sketch graphs; transferring a graph from GDC to paper; reading, interpreting and making predictions using graphs

6.3 Linear models: linear functions and their graphs

6.4 Quadratic models: quadratic functions and their graphs (parabolas); properties of a parabola; symmetry, vertex; intercepts; equation of the axis of symmetry

6.5 Exponential models: exponential functions and their graphs; concept and equation of a horizontal asymptote

6.6 Use of a GDC to solve equations involving combinations of the functions above

Before you start

You should know how to:

1 Substitute values into a formula, e.g. Given that $x = -1$, find the value of $y = 3x^2 + 2x$.

$y = 3(-1)^2 + 2(-1) \Rightarrow y = 1$

2 Use your GDC to solve quadratic equations and simultaneous equations in two unknowns, e.g. Solve

a $3x^2 + 9x - 30 = 0 \Rightarrow x = 2, x = -5$

b $\begin{cases} x + y = 4 \\ -2x + y = 1 \end{cases} \Rightarrow x = 1, y = 3$

3 Find the gradient, m, of a line joining two points, e.g. A(3, 5) and B(1, 4).

$y - y_1 = m(x - x_1)$

$m = \dfrac{5-4}{3-1}$

$m = \dfrac{1}{2}$

Skills check

1 **a** Find the value of $y = 2.5x^2 + x - 1$ when $x = -3$.

b Find the value of $h = 3 \times 2^t - 1$ when $t = 0$.

c Find the value of $d = 2t^3 - 5t^{-1} + 2$ when $t = \dfrac{1}{2}$.

2 Use your GDC to solve:

a $x^2 + x - 3 = 0$

b $2t^2 - t = 2$

c $\begin{cases} x - 2y = 3 \\ 3x - 5y = -2 \end{cases}$

For help, see Chapter 12, Sections 1.1 and 1.2.

3 Find the gradient, m, of a line joining the two points:

a A(7, –2) and B(–1, 4)

b A(–3, –2) and B(1, 8)

The above photo shows the positions of a diver at various moments until he reaches the sea. Initially, the diver is at 40 m above sea level and it takes him 4.5 seconds to reach the sea. We can use mathematics to find a numerical relationship between the time in seconds, t, and the diver's height, h, in metres above sea level. The relationship linking the time, t, and the height, h, is a mathematical model. It can be described using a formula, a graph or a table of values.

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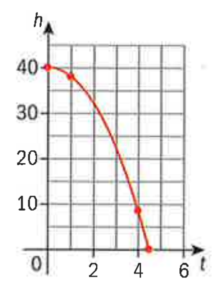


The three pairs of values from the table are indicated with a ●.

You can use the formula and/or the graph to answer questions such as:

At what height is the diver after 2 seconds?

How long does it take the diver to reach a height of 20 m above sea level?

In this chapter you will work with different types of mathematical models called **functions** to represent a range of practical situations. These functions help us to understand and predict the behavior of variables.

4.1 Functions

Mathematical models that link two variables are called functions.

→ A **function** is a relationship between two sets: a **first** set and a **second** set. Each element 'x' of the first set is related to **one and only one** element 'y' of the second set.

Example 1

Antonio and Lola are students at Green Village High School (GVHS). Miu is a student at Japan High School (JHS).
The set of students A = {Antonio, Lola, Miu}.
The set of schools B = {GVHS, JHS}.
Decide whether these relationships are functions. Justify your decisions.

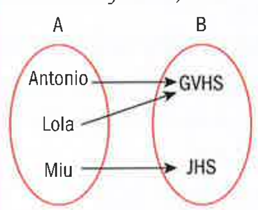
a The relationship between the first set A and the second set B: 'x is a student at school y'.

b The relationship between the first set B and the second set A: 'x is the school where y is a student'.

Answers

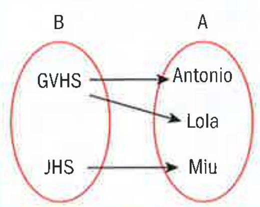
a This relationship is a function because each element of the first set A is related to one and only one element of the second set B; that is, each student studies at only one school.

Draw a mapping diagram to show how the elements of set A, Antonio, Lola and Miu, are related to the elements of set B, GVHS, JHS.



b This relationship is not a function because one element of the first set B, GVHS, is related to more than one element in the second set A, Antonio and Lola.

Draw a mapping diagram to show how the elements of set B are related to the elements of set A.



In Chapter 1 we saw sets where the elements were numbers. However, the elements of a set may be any kind of object.

A mapping diagram is a drawing used to show how the elements in the first set are matched to the elements in the second set.

Example 2

Let A = {1, −1, 0, 2, 4}, B = {1, 0, 4} and C = {1, 0, 4, 16}.
Decide whether these relationships are functions. Justify your decisions.

a The relationship between the first set A and the second set B: 'the square of x is y' or '$y = x^2$'.

b The relationship between the first set A and the second set C: 'the square of x is y' or '$y = x^2$'.

c The relationship between the first set C and the second set A: 'the square root of x is y' or '$y = \sqrt{x}$'.

Answers

a It is not a function because one element of the first set A, 4, is not related to any element of the second set B.

Draw a table of values. The elements of set A are the values of x. Use these to work out the corresponding values of y given $y = x^2$. Check that the values of y match the elements of set B.

A	B
x	$y = x^2$
1	1
−1	1
0	0
2	4
4	

$4^2 = 16$; 16 is not an element of set B.

b It is a function because each element of the first set A is related to one and only one element of the second set C.

A	C
x	$y = x^2$
1	1
−1	1
0	0
2	4
4	16

c It is a function because each element of the first set C is related to one and only one element of the second set A.

C	A
x	$y = \sqrt{x}$
1	1
0	0
4	2
16	4

In Example 2 as the elements of the sets are numbers, the relationships are **numerical**. In Mathematical Studies we work with numerical relationships that can be described using **equations**.

Think of everyday situations where you can define functions between two sets. For example, the relationship between a group of people and their names; the relationship between a tree and its branches; the relationship between the days and the mean temperature of each of these days, etc.

Exercise 4A

1 Mrs. Urquiza and Mr. Genzer both teach mathematics. Mick and Lucy are in Mrs. Urquiza's class. Lidia and Diana are in Mr. Genzer's class.
Let the set of students A = {Mick, Lucy, Lidia, Diana}
and the set of teachers B = {Mrs. Urquiza, Mr. Genzer}.
Decide whether these relationships are functions. Justify your decisions.
  a The relationship between the first set A and the second set B: 'x is in y's mathematics class'.
  b The relationship between the first set B and the second set A: 'x is y's mathematics teacher'.

2 Let A = {3, 7, 50}, B = {12, 16, 49, 100} and C = {49, 100}.
Decide whether these relationships are functions. Justify your decisions.
  a The first set A, the second set B and the relationship 'x is a factor of y'.
  b The first set B, the second set A and the relationship 'x is a multiple of y'.
  c The first set C, the second set A and the relationship 'x is a multiple of y'.

3 Let A = {1, 2, 3, 4}, B = {2, 4, 6} and C = {1, 2, 4, 6}.
  a Decide whether these relationships are functions. Justify your decisions.
    i First set A, second set B and the relationship 'x is half of y'.
    ii First set A, second set C and the relationship 'x is half of y'.
    iii First set C, second set A and the relationship 'x is the double of y'.
    iv First set B, second set C and the relationship 'x is equal to y'.
    v First set C, second set A and the relationship 'x is equal to y'.
  b Draw a diagram to represent the relationships from part **a** that are functions.

4 Describe the following relationships between x and y using equations.
  a y is double x.
  b Half of x is y.
  c The cubic root of x is y.
  d Half of the cube of x is y.

The equation $y = x^2$ describes the relationship 'y is the square of x'.

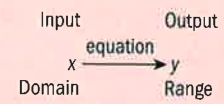
5 Decide whether these relationships are functions. Explain your decision in the cases in which they are not functions.
  a The first set is $\mathbb{R}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = 3x + 1$.
  b The first set is $\mathbb{R}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = x^2$.
  c The first set is $\mathbb{R}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = \sqrt{x}$.
  d The first set is A = $\{x \geq 0, x \in \mathbb{R}\}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = \sqrt{x}$.

$\mathbb{R}$ is the set of real numbers.

Domain and range of a function

A function is a relationship between two sets: a first set and a second set.

- The first set is called the **domain** of the function. The elements of the domain, often thought of as the 'x-values', are the **independent variables**.
- For each value of 'x' (input) there is one and only one output. This value is called the **image** of 'x'. The set of all the images (all the outputs) is called the **range** of the function. The elements of the range, often thought of as the 'y-values', are the **dependent variables**.



In Mathematical Studies the domain will always be the set of real numbers unless otherwise stated.

We write domain and range values as sets inside curly brackets:
Domain = {inputs}
Range = {images or outputs}

Example 3

Consider the function $y = x^2$.
a Find the image of **i** $x = 1$ **ii** $x = -2$.
b Write down the domain.
c Write down the range.

Answers

a **i** $y = 1$

i Substitute $x = 1$ into $y = x^2$
$y = (-1)^2 \Rightarrow y = 1$

ii $y = 4$

ii Substitute $x = -2$ into $y = x^2$
$y = (-2)^2 \Rightarrow y = 4$

b The domain is the set of real numbers, $\mathbb{R}$.

Squaring any real number produces another real number. So the domain is the set of all real numbers.

c The range is $y \geq 0$.

The square of any positive or negative number will be positive and the square of zero is zero. So the range is the set of all real numbers greater than or equal to zero.

The domain is assumed to be $\mathbb{R}$ unless there are any values which x cannot take.

Example 4

Consider the function $y = \frac{1}{x}, x \neq 0$.

a Find the image of: **i** $x = 2$ **ii** $x = -\frac{1}{2}$

b Write down the domain.

c **i** Decide whether $y = 0$ is an element of the range. Justify your decision.

ii Decide whether $y = -5$ is an element of the range. Justify your decision.

Answers

a **i** $y = \frac{1}{2}$

Substitute
i $x = 2$ *and*
ii $x = -\frac{1}{2}$ *into* $y = \frac{1}{x}$.

ii $y = \frac{1}{-\frac{1}{2}} = -2$

b The domain is the set of all real numbers except 0.

Since division by zero is not defined, the domain is the set of all real numbers except zero ($x \neq 0$).

c **i** $0 = \frac{1}{x}$

This equation has no solution. Therefore, $y = 0$ is not an element of the range.

Substitute i $y = 0$ *and*
ii $y = -5$ *into* $y = \frac{1}{x}$.

Is there an input value (x) that gives an output (y) of 0?

ii $-5 = \frac{1}{x}$

$x = -\frac{1}{5}$

So, $y = -5$ is an element of the range as it is the image of $x = -\frac{1}{5}$.

Is there an input value (x) that gives an output (y) of −5?

Gottfried Leibniz first used the mathematical term 'function' in 1673.

Exercise 4B

1 For each of the functions in **a–d**:

i Copy and complete the table. Put a × in any cells that cannot be completed.

ii Write down the domain.

iii Decide if $y = 0$ is in the range of the function. Justify your decision.

a $y = 2x$

x	$-\frac{1}{2}$	0	1	3.5	
$y = 2x$					12

b $y = x^2 + 1$

x	−3	0	2	$\frac{1}{4}$		
$y = x^2 + 1$					5	5

c $y = \frac{1}{x+1}, x \neq -1$

x	−2	−1	0	$\frac{1}{2}$	3	
$y = \frac{1}{x+1}$						$\frac{1}{6}$

d $y = \sqrt{x}, x \geq 0$

x	−3	0	$\frac{1}{4}$		9	
$y = \sqrt{x}$				1		10

2 Decide whether each statement is true or false. Justify each of your decisions.

a $y = 0$ is an element of the range of the function $y = \frac{2}{x}$.

b The equation $y = x^2$ cannot take the value -1.

c The equation $y = x^2 + 3$ cannot take the value 2.

d For the function $y = x^2 - 1$ there are two values of x when $y = 3$.

e For the function $y = \frac{x}{3} - 1$ the image of $x = -3$ is -2.

f For the function $y = 2(-x + 1)$ the image of $x = -1$ is $y = 0$.

Graph of a function

A graph can represent a function.

→ The graph of a function f is the set of points (x, y) on the Cartesian plane where y is the image of x through the function f.

We use different letters to name functions: f, g, h, etc.

Drawing graphs

- Draw a table of values to find some points on the graph.
- On 2 mm graph paper, draw and label the axes with suitable scales.
- Plot the points.
- Join the points with a straight line or a smooth curve.

Cartesian coordinates and the Cartesian plane are named after the Frenchman René Descartes (1596–1650).

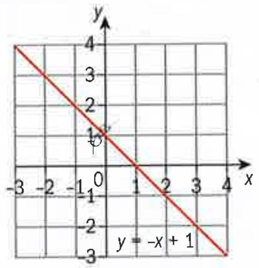
Example 5

a Draw the graph of the function $y = -x + 1$.
b Write down the coordinates of the point where the graph of this function intercepts the **i** x-axis **ii** y-axis.
c Decide whether the point A(200, −199) lies on the graph of this function.
d The point B(6, y) lies on the graph of this function. Find the value of y.

'Draw' means draw accurately on graph paper.

Answers

a



Draw a table of values. Use negative and positive values of x. Use the values of x to work out the corresponding values of y.
When $x = -3$, $y = -(-3) + 1 = 4$.

x	−3	−1	0	1	3
y	4	2	1	0	−2

Use 2 mm graph paper.
Let 1 cm = 1 unit. Label the x- and y-axes.
Plot the coordinate points (−3, 4), (−1, 2), (0, 1), (1, 0) and (3, −2).
Join the points with a straight line.

b **i** x-intercept is (1, 0).
ii y-intercept is (0, 1).

i *To find the x-intercept, read off the point where the graph intersects the x-axis.*
ii *To find the y-intercept, read off the point where the graph intersects the y-axis.*

c $-199 = -200 + 1$
So A(200, 199) lies on the graph.

A(200, −199)
Substitute the x- and y-values into the equation of the line to see if they 'fit'.

A point P lies on the graph of a function if and only if the point satisfies the equation of the function.

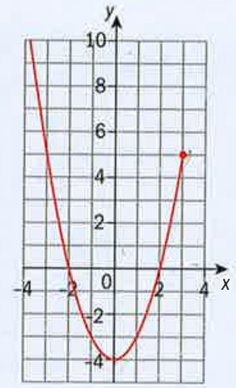
d B(6, y) lies on the graph, so $y = -6 + 1 = -5 \Rightarrow y = -5$

B(6, y)
Substitute x = 6 into the equation of the line to find the value of y at that point.

In the solution to the next example, the set notation $\{x \mid x \leq 3\}$ is used. It is read as: the set of all x such that x is a real number less than or equal to 3.

Example 6

Here is the graph of a function f.
Use the graph to find
a the domain of f
b the range of f
c the points where the graph of f intersects the **i** x-axis and **ii** y-axis.

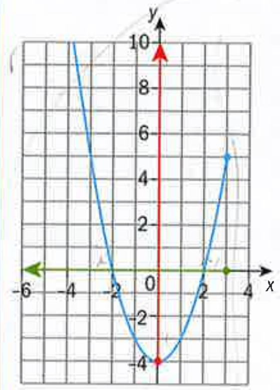


• indicates that an endpoint lies on the graph of a function. In Example 6 the point (3, 5) lies on the graph. To indicate that a point does not lie on the graph of a function use an empty dot, ○.

Answers

a Domain of $f = \{x \mid x \leq 3\}$

To find the domain from a graph of a function, 'squash' or project the graph against the x-axis.



On the graph above, the domain is shown by the green line.

b Range of $f = \{y \mid y \geq -4\}$

To find the range from the graph of a function, 'squash' the graph against the y-axis.
On the graph in part a, the range is shown by the red line.

c **i** x-intercepts: (−2, 0) and (2, 0)
ii y-intercept: (0, −4)

i *On the x-axis the y-coordinate is zero.*
ii *On the y-axis the x-coordinate is zero.*

Is it possible for the graph of a function to cut the y-axis more than once?

Sketching a linear graph

- Draw and label the axes.
- Label the points where the graph crosses the x- or y-axis.

Example 7

Sketch a graph of the function $y = 3x - 1$.

Answer

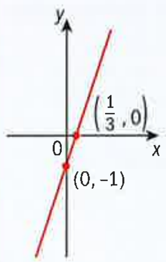
The x-axis intercept is at $\left(\frac{1}{3}, 0\right)$

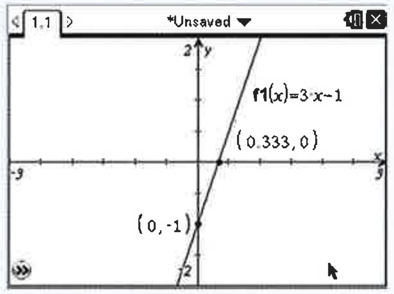
When $y = 0$, $x = \frac{1}{3}$

The y-axis intercept is at $(0, -1)$

When $x = 0$, $y = -1$

Draw the graph on your GDC.





Now sketch the graph:

1. *Draw and label the axes.*
2. *Copy the graph shown on the GDC.*
3. *Label the points where the graph crosses the axes.*

'Sketch' means give a general shape of the graph.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4C

EXAM-STYLE QUESTION

1 **a** Draw the graph of the function $y = 2x - 4$.

b Write down the coordinates of the point where the graph of this function meets

i the x-axis **ii** the y-axis.

c Decide whether the point A(250, 490) lies on the graph of this function. Justify your decision.

d The point B(-3, y) lies on the graph of this function. Find the value of y.

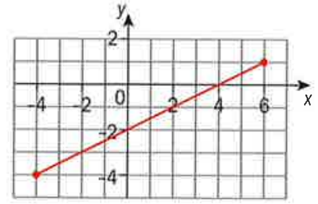
2 For each of the graphs of functions in **a**–**d** write down

i the domain **ii** the range

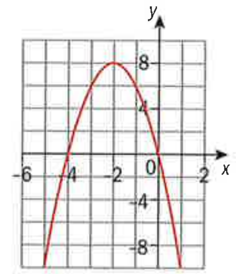
iii the points where the graph meets the x-axis (where possible)

iv the point where the graph meets the y-axis (where possible).

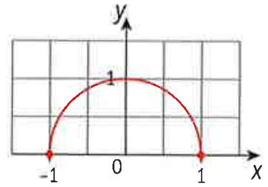
a



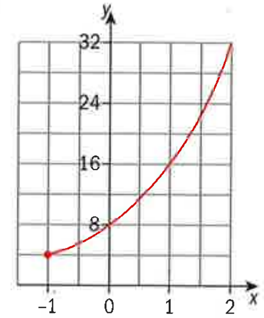
b



c



d



3 Decide whether these statements about the functions drawn in question 2 are true or false.

Function **a**

i Point $(1, -1)$ lies on this graph.

ii The image of $x = -2$ is 0. **iii** When $x = 6$, $y = 1$.

Function **b**

i There are two values of x for which $y = 8$.

ii There are two values of x for which $y = 4$.

iii There is a value of x for which $y = 9$.

Function **c**

i The line $x = 0.5$ intersects the graph of this function twice.

ii The line $y = 0.5$ intersects the graph of this function twice.

iii The image of $x = 0.2$ is the same as the image of $x = -0.8$.

Function **d**

i The line $y = 1$ intersects the graph of this function once.

ii When $x = 16$, $y = 1$.

iii As the values of x increase, so do their corresponding values of y.

4 Sketch a graph for each of these functions.

a $y = 2x + 3$ **b** $y = -x + 2$ **c** $y = 3x - 4$

Function notation

→ $y = f(x)$ means that the image of x through the function f is y. x is the independent variable and y is the dependent variable.

So, for example, if $f(x) = 2x - 5$

- $f(3)$ represents the image of $x = 3$.
  To find the value of $f(3)$ substitute $x = 3$: $f(3) = 2 \times 3 - 5 = 1$
- $f(-1)$ represents the image of $x = -1$.
  To find the value of $f(-1)$ substitute $x = -1$: $f(-1) = 2 \times -1 - 5 = -7$

We use different variables and different letters for functions; for example, $d = v(t)$, $m = C(n)$, etc.

$f(3) = 1$ can be read as: 'f at 3 equals 1' or 'f of 3 equals 1'.

One of the first mathematicians to study the concept of function was French Philosopher Nicole Oresme (1323–82). He worked with independent and dependent variable quantities.

Example 8

Consider the function $f(x) = -x^2 + 3x$.

a Find the image of $x = -2$. **b** Find $f(1)$.

c Show that the point $(4, -4)$ lies on the graph of f.

Answers

a $f(-2) = -(-2)^2 + 3 \times -2 = -10$	*Substitute $x = -2$ into $f(x) = -x^2 + 3x$.*
b $f(1) = -1^2 + 3 \times 1 = 2$	*Substitute $x = 1$ into $f(x)$.*
c $f(4) = -4^2 + 3 \times 4 = -4$ So $(4, -4)$ lies on the graph.	*If $(4, -4)$ lies on the graph of f then $f(4) = -4$. Substitute $x = 4$.*

Exercise 4D

1 Consider the function $f(x) = x(x - 1)(x + 3)$.

a Calculate $f(2)$. **b** Find the image of $x = \frac{1}{2}$.

c Show that $f(-3) = 0$.

d Decide whether the point $(-1, -4)$ lies on the graph of f. Justify your decision.

2 Consider the function $d(t) = 5t - t^2$.

a Write down the independent variable of this function.

b Calculate $d(2.5)$.

c Calculate the image of $t = 1$.

d Show that $d(1)$ and $d(4)$ take the same value.

3 Consider the function $C(n) = 100 - 10n$.

a Calculate $C(2)$.

b The point $(3, b)$ lies on the graph of the function C. Find the value of b.

c The point $(a, 0)$ lies on the graph of the function C. Find the value of a.

EXAM-STYLE QUESTION

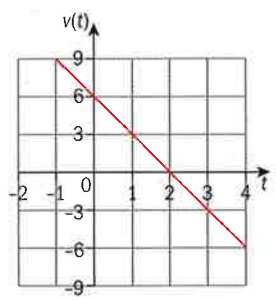
4 Here is the graph of the function $v(t) = -3t + 6$.

a Write down the value of **i** $v(1)$ **ii** $v(3)$.

b The point $(m, 9)$ lies on the graph. Find the value of m.

c Find the value of t for which $v(t) = 0$.

d Find the set of values of t for which $v(t) < 0$.



EXAM-STYLE QUESTION

5 Consider the function $f(x) = 0.5(3 - x)$.

a Draw the graph of f.

b Find the point A where the graph of f meets the x-axis.

c Find the point B where the graph of f meets the y-axis.

d Solve the equation $f(x) = 2$.

6 Consider the function $h(x) = 3 \times 2^{-x}$.

a Calculate **i** $h(0)$ **ii** $h(-1)$. **b** Find x if $h(x) = 24$.

$h(x) = 3 \times 2^{-x}$ is an exponential function. You will learn more about these in section 4.4.

Functions as mathematical models

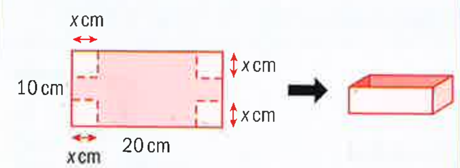
We can use functions to describe real-life situations.

Translate the situation into mathematical language and symbols. → Find the solution using mathematics. → Interpret the solution in the context of the problem.

Example 9

A rectangular piece of card measures 20 cm by 10 cm. Squares of length x cm are cut from each corner. The remaining card is then folded to make an open box. Write a function to model the volume of the box.

Answer



First draw a diagram to represent the information given in the question. Carefully label the dimensions of the open box:
length $(20 - 2x)$ cm
width $(10 - 2x)$ cm
height x cm

$V(x) = (20 - 2x)(10 - 2x)x$

The volume of the box, V, will depend on the value of x.
Volume of cuboid = length × width × height.

Look at Example 9.

1 What is the domain of the function $V(x)$? Can x take any value? Why? Try different values and draw a conclusion.

2 How could the function help you to find the maximum possible volume?

Exercise 4E

EXAM-STYLE QUESTION

1 A rectangular piece of card measures 30 cm by 15 cm. Squares of length x cm are cut from each corner. The remaining card is then folded to make an open box of length l cm and width w cm.

a Write expressions, in terms of x, for

i the length, l **ii** the width, w.

b Find an expression for the volume of the box, V, in terms of x.

i Explain in words the meaning of $V(3)$.

ii Find the value of $V(3)$. **iii** Find the value of $V(3.4)$.

iv Is $x = 8$ in the domain of the function $V(x)$? Justify your decision.

EXAM-STYLE QUESTIONS

2 The perimeter of a rectangle is 24 cm and its length is x cm.

a Find the width of the rectangle in terms of the length, x.

b Find an expression for the area of the rectangle, A, in terms of x.

c **i** Explain the meaning of $A(2)$.

ii Calculate $A(2)$.

d Is $x = 12$ in the domain of the function $A(x)$? Justify your decision.

You can use mathematical functions to represent things from your own life. For example, suppose the number of pizzas your family eats depends on the number of football games you watch. If you eat 3 pizzas during every football game, the function would be 'number of pizzas' (p) = 3 times 'number of football games' (g) or $p = 3g$. Can you think of another real-life function? It could perhaps be about the amount of money you spend or the number of minutes you spend talking on the phone.

3 The Simpsons rent a holiday house costing 300 USD for the security deposit plus 150 USD per day.
Let n be the number of days they stay in the house and C the cost of renting the house.

a Write a formula for C in terms of n.

b How much does it cost to rent the house for 30 days?

The Simpsons have 2300 USD to spend on the rent of the house.

c **i** Write down an inequality using your answer to part **a** to express this condition.

ii Hence, decide whether they have enough money to rent the house for two weeks.

iii Write down the maximum number of days that they can rent the house.

4 An Australian company produces and sells books.
The monthly **cost**, in AUD, for producing x books is modeled by $C(x) = 0.4x^2 + 1500$.
The monthly **income**, in AUD, for selling x books is modeled by $I(x) = -0.6x^2 + 160x$.

a Show that the company's monthly profit can be calculated using the function
$P(x) = -x^2 + 160x - 1500$.

b What profit does the company make on producing and selling six books? Comment on your answer.

c **i** What profit does the company make on producing and selling 40 books?

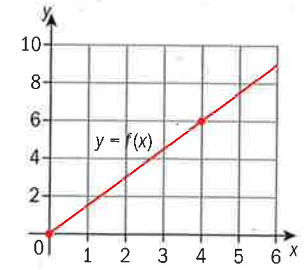
ii Find the selling price of one book when 40 books are produced and sold.
(Assume that all the books have the same price.)

d Use your GDC to find the number of books for which $P(x) = 0$.

Profit = Income – Cost

4.2 Linear models

Linear models of the form $f(x) = mx$



The straight line shown here has a positive gradient and the function $y = f(x)$ is increasing.

$f(0) = 0$ and the line passes through the origin (0, 0).

The gradient of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Using two points on the line, (4, 6) and (0, 0), the gradient is

$$m = \frac{6-0}{4-0} = \frac{3}{2} = 1.5.$$

So $f(x) = 1.5x$.

This type of linear model is used in **conversion graphs**. The two variables which have a fixed relationship between them are in direct proportion, so their graphs are straight lines with a positive gradient passing through the origin.

Conversion graphs can be used to convert one currency to another, or one set of units to another: for example, kilometres to miles, or kilograms to pounds.

Example 10

1 mile is equivalent to 1.6 km.

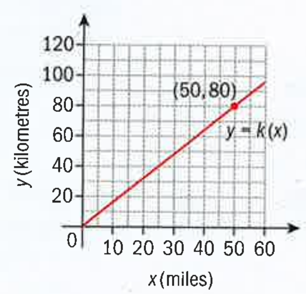
a Draw a conversion graph of miles to km.

b Find the gradient of the line.

c Hence, write down a model for $k(x)$, where $k(x)$ is the distance in km and x is the distance in miles.

Answers

a



Use 2 mm graph paper.
Put miles on the x-axis.
Put kilometres on the y-axis.
Find two points to draw a straight line: 0 miles = 0 km so (0, 0) is on the line.
50 miles is equivalent to 1.6 × 50 = 80 km so (50, 80) is on the line.
Plot the two points and join with a straight line.

b Gradient, $m = \frac{80-0}{50-0} = 1.6$

Use the two points from part **a** *to find the gradient,* $m = \frac{y_2 - y_1}{x_2 - x_1}$.

c The equation of the line is $y = 1.6x$.
Hence, $k(x) = 1.6x$, where $k(x)$ is the distance in km, and x is the distance in miles.

For a general linear function through the origin, $f(x) = mx$. Here, the function is $k(x) = mx$.

The equation $y = 1.6x$ can be rearranged to $x = \frac{y}{1.6}$ or $x = \frac{1}{1.6}y = 0.625y$. Use this to convert km to miles.

Exercise 4F

1 1 kg is equivalent to 2.2 pounds.

a Convert 50 kg into pounds.

b Draw a conversion graph of pounds to kilograms. Use x-values from 0 kg to 100 kg and y-values from 0 pounds to 250 pounds.

> Plot the point you found in part **a**.

c Find the gradient of the line. Hence, write down the model for $p(x)$, where $p(x)$ is the weight in pounds and x is the weight in kg.

d Find $p(75)$ and $p(125)$.

e Find the model for $k(x)$, where $k(x)$ is the weight in kg and x is the weight in pounds.

> Write the formula as $y = \dots$ and rearrange to make x the subject.

f Calculate $k(75)$ and $k(100)$.

2 The exchange rate for pounds sterling (GBP) to Singapore dollars (SGD) is £1 = S\$2.05.

a Find the number of Singapore dollars equivalent to 50 GBP.

b Draw a conversion graph of GBP to SGD. Use x-values from £0 to £100 and y-values from S\$0 to S\$250.

c Find the gradient of the line. Hence, write down the model for $s(x)$, where $s(x)$ is the amount of money in SGD and x is the amount of money in GBP.

d Find $s(80)$ and $s(140)$.

e Find the model for $p(x)$, where $p(x)$ is the amount of money in GBP and x is the amount of money in SGD.

f Calculate $p(180)$.

3 The exchange rate for pounds sterling (GBP) to US dollars (USD) is £1 = \$1.55.

a Find the number of US dollars equivalent to 60 GBP.

b Draw a conversion graph of GBP to USD. Use the x-axis for GBP with $0 \le x \le 80$, and the y-axis for USD with $0 \le y \le 140$.

c Find the gradient of the line. Hence, write down the model for $u(x)$, where $u(x)$ is the amount of money in USD and x is the amount of money in GBP.

d Find $u(300)$ and $u(184)$.

e Find the model for $p(x)$, where $p(x)$ is the amount of money in GBP and x is the amount of money in USD.

f Calculate $p(250)$ and $p(7750)$.

Linear models of the form $f(x) = mx + c$

When two variables are not in direct proportion, their graphs are straight lines that do not pass through the origin, that is, **linear functions**.

> → A **linear function** has the general form
> $$f(x) = mx + c$$
> where m (the gradient) and c are constants.

> You have seen the equation of a line in Chapter 3, Section 3.2.

Example 11

In a chemistry experiment, a liquid is heated and the temperature at different times recorded.
The table of results for one student is shown.

Time (x minutes)	2	4	6	9
Temperature (y °C)	30	40	50	65

a Draw a graph for this data.

b Find a model for $T(x)$, the temperature with respect to time, for these data.

c Use the model to predict:

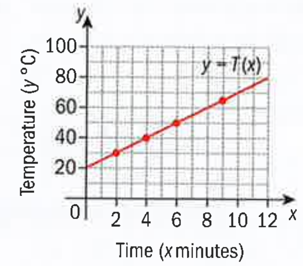
i the temperature of the liquid after 8 minutes

ii the time taken for the liquid to reach 57 °C.

> You can plot the graph on your GDC and find the model for $T(x)$. For help, see Chapter 12, Section 5.4.

Answers

a



Use 2 mm graph paper.
Put time on the x-axis.
Put temperature on the y-axis.
Plot the points from the table e.g. (2, 30) and join them with a straight line.

b Gradient, $m = \dfrac{65-40}{9-4}$

$= \dfrac{25}{5} = 5$

$T(x) = mx + c$

$T(x) = 5x + c$

$T(2) = 5 \times 2 + c = 30$

$10 + c = 30$

$c = 20$

The model will be in the form $T(x) = mx + c$. You need to find the constants m and c.
Use any two points from the table, e.g. (4, 40) and (9, 65), to find the gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Therefore, the model for the temperature is $T(x) = 5x + 20$.

To find the value of c use any point from the table, e.g. (2, 30), which means $T(2) = 30$.

c i At 8 minutes:
$T(8) = 5 \times 8 + 20 = 60$
So, the temperature of the liquid after 8 minutes is 60 °C.

*i A time of 8 minutes means $x = 8$. Substitute $x = 8$ into the function in part **b**.*

ii When $T(x) = 57$ °C:
$57 = 5x + 20$
$5x = 37$
$x = \dfrac{37}{5} = 7.4$
So, it takes 7.4 minutes for the liquid to reach 57 °C.

ii A temperature of 57 °C means $T(x) = 57$. Substitute $T(x) = 57$ and solve for x.

> For Example 11, the equation of the model was $T(x) = 5x + 20$.
> Compare the equation of the model with
> **a** the initial temperature
> **b** the average rise in temperature every minute.
> What conclusions can be drawn?

Exercise 4G

1 In a chemistry experiment, a liquid is heated and the temperature at different times is recorded. Here is a table of results.

Time (x minutes)	3	5	7	9
Temperature (y °C)	130	210	290	410

a Draw a graph for these data.
b What was the initial temperature of the liquid?
c Find the linear model, $T(x)$, for the temperature of the liquid with respect to time.

Use a scale up to 420 on the y-axis.

2 In a physics experiment, a spring is stretched by loading it with different weights, in grams. The results are given in the table.

Weight (x g)	40	50	75	90
Length of spring (y mm)	38	43	55.5	63

a Draw a graph for these data.
b Find the natural length of the spring.
c By how many mm does the spring stretch when the weight increases from 50 g to 90 g?
d Use the answer to part **c** to find the average extension of the spring in mm for each extra gram loaded.
e Find the equation of the linear model, $L(x)$, for the length of the spring with respect to load.

The natural length is the length of the spring with no loading.

3 The temperature of the water in a hot water tank is recorded at 15 minute intervals after the heater is switched on.

Time (x minutes)	15	30	45	60	75	90
Temperature (y °C)	20	30	40	50	60	70

a Plot a graph of these data on your GDC.
b Find the linear model, $T(x)$, for the temperature of the liquid with respect to time.
c Find the temperature of the water after 85 minutes.

Read off the value from your graph.

4 Different weights are suspended from a spring. The length of the spring with each weight attached is recorded in the table.

Weight (x g)	125	250	375	500
Length of spring (y cm)	30	40	50	60

a Plot a graph of these data on your GDC.
b Find the natural length of the spring.
c By how many cm does the spring stretch when the weight is increased from 125 g to 375 g?
d Find the weight that will stretch the spring to a length of 48 cm.
e Find the equation of the linear model, $L(x)$, for the length of the spring with respect to load.

Linear models involving simultaneous equations

Sometimes you cannot find the model from the given data. You may need to write equations to represent the situation and solve them simultaneously.

For a reminder on solving simultaneous equations, see Chapter 13, Section 2.4.

Example 12

A carpenter makes wooden tables and chairs.
He takes 10 hours to make a table and 4 hours to make a chair.
The wood costs \$120 for a table and \$40 for a chair.
Find a model for
a the time required to make the tables and chairs
b the cost of making the tables and chairs.

The given values for a model are called **constraints**.

Answers

a Let t be the time required to make the tables and chairs. The model for the time required is $t = 10x + 4y$.

Let x be the number of tables and y be the number of chairs.
Total number of hours for the tables: 10 hours per table, x tables $\Rightarrow 10 \times x$
Total number of hours for the chairs: 4 hours per chair, y chairs $\Rightarrow 4 \times y$

b Let c be the cost of making the tables and chairs. The model for the cost is $c = 120x + 40y$.

Total cost (\$) for the tables: \$120 per table, x tables $\Rightarrow 120 \times x$
Total cost (\$) for the chairs: \$40 per chair, y chairs $\Rightarrow 40 \times y$

The simultaneous equations arise when you are given values that the model must satisfy.

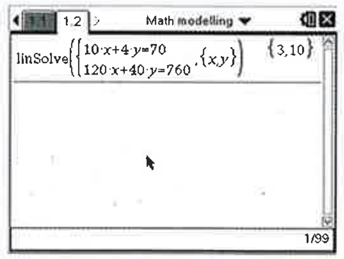
Example 13

The carpenter in Example 12 works 70 hours one week and spends \$760 on wood.
How many tables and chairs can he make?

Answer

From Example 12:
$t = 10x + 4y$
$c = 120x + 40y$
$10x + 4y = 70$
$120x + 40y = 760$
Using a GDC: $x = 3$ and $y = 10$
The carpenter can make 3 tables and 7 chairs.

The model must work for the values: (time) $t = 70$ and (cost) $c = 760$.
Write a set of simultaneous equations.
Solve these either analytically or using a GDC.



For help with using a GDC to solve simultaneous equations see Chapter 12, Sections 1.1 and 3.4.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4H

1 To make a sponge cake you need 80 g of flour and 50 g of fat.
To make a fruit cake you need 60 g of flour and 90 g of fat.
Find a model for
a the amount flour needed to make both cakes
b the amount of fat needed to make both cakes.

Peter has 820 g of flour and 880 g of fat.
c How many of each type of cake can he make?

2 It takes 8 hours to make a table and 3 hours to make a chair.
For a table the wood costs \$100. For a chair the wood costs \$30.
A carpenter has 51 hours and \$570.
How many tables and chairs can she make?

3 A van carries up to 3 people and 7 cases.
A car carries up to 5 people and 3 cases.
How many vans and cars do you need for 59 people and 70 cases?

4 A passenger plane carries 80 passengers and 10 tonnes of supplies.
A transport plane carries 50 passengers and 25 tonnes of supplies.
How many planes of each type do you need to carry 620 people and 190 tonnes of supplies?

5 A school mathematics department has 1440 euros to buy textbooks.
Maths for All volume 1 costs 70 euros. *Maths for All* volume 2 costs 40 euros.
The department wants twice as many copies of volume 1 as volume 2.
How many of each volume can they buy?

Extension material on CD: *Worksheet 4 - Equations*

4.3 Quadratic models

Quadratic functions and their graphs

→ A **quadratic function** has the form
$f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Why $a \neq 0$? What kind of function would you get if $a = 0$?

The domain of a quadratic function can be the entire set of real numbers ($\mathbb{R}$) or any subset of this.

Here are examples of some quadratic functions:

$f(x) = x^2 + 3x + 2$ $(a = 1, b = 3, c = 2)$

$f(x) = x - 3x^2$ $(a = -3, b = 1, c = 0)$

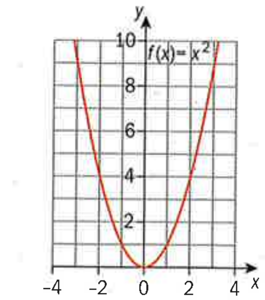
$f(x) = 3x^2 + 12$ $(a = 3, b = 0, c = 12)$

The simplest quadratic function is $f(x) = x^2$.

Here is a table of values for $f(x) = x^2$.

x	−3	−2	−1	0	1	2	3
f(x)	9	4	1	0	1	4	9

Plotting these values gives the graph shown here.



1 The graph is called a **parabola**.
2 The parabola has an **axis of symmetry** (the y-axis).
3 The parabola has a **minimum point** at (0, 0).
The minimum point is called the **vertex** (or turning point) of the parabola.
4 The range of $f(x) = x^2$ is $y \geq 0$.

'Squash' the graph of $f(x) = x^2$ against the y-axis to confirm that the range is $y \geq 0$.

→ The graph of any quadratic function is a **parabola** – a ∪-shaped (or ∩-shaped) curve. It has an **axis of symmetry** and either a **minimum** or **maximum** point, called the **vertex** of the parabola.

The name 'parabola' was introduced by the Greek, Apollonius of Perga (c. 262–190 BCE) in his work on conic sections.

Investigation – the curve $y = ax^2$

1 Draw these curves on your GDC: $y = x^2$ and $y = -x^2$
How are these two curves related?

2 Now draw: $y = 2x^2$ $y = 3x^2$ $y = 0.5x^2$
$y = -2x^2$ $y = -3x^2$ $y = -0.5x^2$
Compare each of these six graphs to $y = x^2$.
Consider:
a Is the curve still a parabola? Is the curve ∪-shaped or ∩-shaped?
b Does it have a vertical line of symmetry?
c What is its vertex? Is the vertex a minimum or maximum point?

3 What is the effect of changing the value of a?
Draw a few more graphs to test your conjecture. (Remember to use positive and negative values of a and also use fractions.)

For help with drawing graphs on your GDC see Chapter 12, Section 4.1.

Without drawing the graph, how do you know that it will be ∩-shaped?

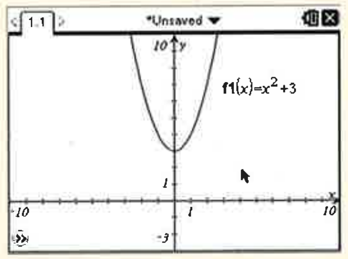
Investigation – the curve $y = x^2 + c$

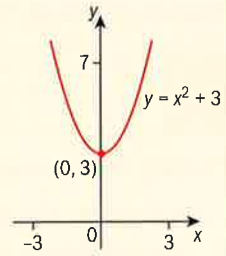
Draw these curves on your GDC: $y = x^2$ $y = x^2 + 2$ $y = x^2 - 4$
$y = x^2 + 3$ $y = x^2 - 2$

Compare each graph to the parabola $y = x^2$. (Use the list of considerations in the preceding investigation as a guide.)
What is the effect of changing the value of c?

Sketching a quadratic graph (1)

- Draw and label the axes.
- Mark any points where the graph intersects the axes (the *x*- and *y*-intercepts). Label these with their coordinates.
- Mark and label the coordinates of any maximum or minimum points.
- Show one or two values on each axis to give an idea of the scale.





Exercise 4I

Use your results from the two previous investigations to help you sketch these graphs.

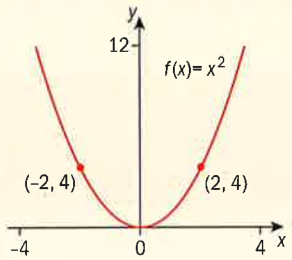
1 $y = 2x^2 + 1$

2 $y = -x^2 + 3$

3 $y = 3x^2 - 2$

4 $y = -2x^2 + 7$

Use this sketch graph of $f(x) = x^2$ to help you.



Investigation – the curves $y = (x + p)^2$ and $y = (x + p)^2 + q$

1 Use your GDC to draw these graphs:
$y = x^2$, $y = (x + 2)^2$, $y = (x + 3)^2$, $y = (x - 1)^2$, $y = (x - 0.5)^2$
Compare each graph to the graph of $y = x^2$.
What is the effect of changing the value of *p*?

2 Use your GDC to draw these graphs:
$y = (x + 2)^2 - 3$, $y = (x - 4)^2 + 2$, $y = (x - 1)^2 - 5$
What is the axis of symmetry of $y = (x + p)^2 + q$?
What are the coordinates of the vertex of $y = (x + p)^2 + q$?

Exercise 4J

For each graph, write down the coordinates of the vertex and the equation of the axis of symmetry.

1 $y = (x + 3)^2 - 2$

2 $y = (x + 5)^2 + 4$

3 $y = (x - 4)^2 - 1$

4 $y = (x - 5)^2 + 7$

5 $y = -(x + 3)^2 + 4$

The equation of the axis of symmetry must be given as '$x = \ldots$'.

Investigation – the curves $y = kx - x^2$ and $y = x^2 - kx$

Part A

1 Use your GDC to draw the graph of $y = 4x - x^2$.
What is the equation of its axis of symmetry?
What are the coordinates of the vertex?
What are the coordinates of the points at which the curve intersects the *x*-axis?

2 Draw these curves: $y = 2x - x^2$, $y = 6x - x^2$, $y = x - x^2$, $y = 5x - x^2$

3 What is the effect of varying the value of *k*?
What is the equation of the axis of symmetry of the curve $y = kx - x^2$?
What are the coordinates of the points at which the curve $y = kx - x^2$ intersects the *x*-axis?

Part B

Draw these curves: $y = x^2 - 2x$, $y = x^2 - 4x$, $y = x^2 - 6x$
Answer the same questions for these as you did for the curves in part A.

Investigation – curves of the form $y = (x - p)(x - q)$

1 Use your GDC to draw the graph of $y = (x - 1)(x - 3)$.
Where does it intersect the *x*-axis?
What is the equation of its axis of symmetry?
What are the coordinates of the vertex?

2 Answer the previous questions for the general curve $y = (x - p)(x - q)$.
(You may wish to draw more graphs of functions of this form.)

Exercise 4K

For each function, write down:

a the equation of the axis of symmetry

b the coordinates of the points at which the curve intersects the *x*-axis

c the coordinates of the vertex.

Do not draw the graphs.

1 $y = x(x - 4)$	2 $y = x(x + 6)$
3 $y = 8x - x^2$	4 $y = 3x - x^2$
5 $y = x^2 - 2x$	6 $y = x^2 - x$
7 $y = x^2 + 4x$	8 $y = x^2 + x$
9 $y = (x + 1)(x - 3)$	10 $y = (x - 5)(x + 3)$
11 $y = (x - 2)(x - 6)$	12 $y = (x + 2)(x - 4)$

Factorize and then use the same method as in questions 1 and 2.

Investigation – the general quadratic curve $y = ax^2 + bx + c$

Part A: *a* = 1

1 Use your GDC to draw the graph of $y = x^2 - 4x + 3$.
Where does it intersect the *x*-axis?
What is the equation of its axis of symmetry?
What are the coordinates of the vertex?

2 Answer the previous questions for the general curve $y = ax^2 + bx + c$.
(You may wish to draw more graphs of functions of this form.)

Part B: varying *a*

Use your GDC to draw the graph of $y = 2x^2 - 4x + 3$ as a starting point.
Consider graphs of this form and answer the questions from Part A.

Exercise 4L

For each function, write down:

a the equation of the axis of symmetry
b the coordinates of the points at which the curve intersects the *x*-axis
c the coordinates of the vertex.

1 $y = x^2 - 2x + 3$ **2** $y = x^2 + 4x - 5$
3 $y = x^2 + 6x + 4$ **4** $y = 3x^2 - 6x + 2$
5 $y = 2x^2 - 8x - 1$ **6** $y = 2x^2 + 6x - 7$
7 $y = 0.5x^2 - x + 2$ **8** $y = 0.5x^2 + 3x - 4$

The general form of a quadratic function is $f(x) = ax^2 + bx + c$.

→
- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- The curve intersects the *y*-axis at $(0, c)$.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$, $a \neq 0$.
- The *x*-coordinate of the vertex is $x = -\frac{b}{2a}$.

A ∪-shaped graph is 'concave up'.
A ∩-shaped graph is 'concave down'.

This formula is in the Formula booklet. You should have found it in the investigation above.

In a parabola, the axis of symmetry passes through the vertex.

→ The factorized form of a quadratic function is

$f(x) = a(x - k)(x - l)$.

- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- The curve intersects the *x*-axis at $(k, 0)$ and $(l, 0)$.
- The equation of the axis of symmetry is $x = \frac{k+l}{2}$.
- The *x*-coordinate of the vertex is also $x = \frac{k+l}{2}$.

→ **Finding the *x*-intercepts**
The function $f(x) = ax^2 + bx + c$ intersects the *x*-axis where $f(x) = 0$. The *x*-values of the points of intersection are the two solutions (or **roots**) of the equation $ax^2 + bx + c = 0$.
(The *y*-values at these points of intersection are zero.)

Example 14

Consider the function $f(x) = x^2 + 6x + 8$.

a Find
 i the point where the graph intersects the y-axis
 ii the equation of the axis of symmetry
 iii the coordinates of the vertex
 iv the coordinates of the point(s) of intersection with the *x*-axis.

b Use the information from part **a** to sketch this parabola.

Answers

a i The graph intersects the *y*-axis at (0, 8).

General form: $f(x) = ax^2 + bx + c$.
Here: $f(x) = x^2 + 6x + 8$
So: $a = 1$, $b = 6$, $c = 8$
The curve intersects the y-axis at (0, c).

ii The equation of the axis of symmetry is $x = -\frac{6}{2(1)} = -3$.

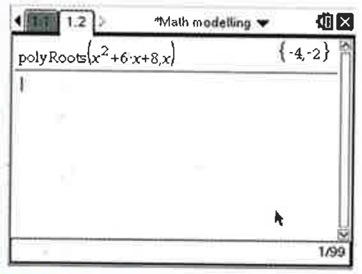
Use $x = -\frac{b}{2a}$, with $a = 1$ and $b = 6$.

iii The *x*-coordinate of the vertex is $x = -3$.
The *y*-coordinate of the vertex is:
$f(-3) = (-3)^2 + 6(-3) + 8$
$= -1$
So, the coordinates of the vertex are $(-3, -1)$.

*The x-coordinate of the vertex is $x = -\frac{b}{2a}$, which we found in part **b** so $x = -3$.*
To find the y-coordinate substitute $x = -3$ into the equation of the function.

iv $x^2 + 6x + 8 = 0$
$f(x) = 0$ when $x = -2$ or -4
The graph intersects the *x*-axis at $(-2, 0)$ and $(-4, 0)$.

The curve intersects the x-axis where $f(x) = 0$, so put $x^2 + 6x + 8 = 0$ and solve using a GDC.



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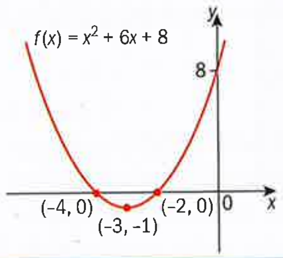
The Indian mathematician Sridhara is believed to have lived in the 9th and 10th centuries. He was one of the first mathematicians to propose a rule to solve a quadratic equation. Research why there is controversy about when he lived.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

b



$a > 0$ ($a = 1$) so the curve is ∪-shaped.
The vertex is $(-3, -1)$ (*part* **a iii**)
The curve intersects the y-axis at $(0, 8)$ (*part* **a i**)
$x = -3$ *is a line of symmetry* (*part* **a ii**)

For $f(x) = x^2 + 6x + 8$
- the vertex is $(-3, -1)$.
- the range of f is $y \geq -1$.

Exercise 4M

For each function $f(x)$ in 1–8:

a Find
- **i** the coordinates of the point of intersection with the y-axis
- **ii** the equation of the axis of symmetry
- **iii** the coordinates of the vertex
- **iv** the coordinates of the point(s) of intersection with the x-axis
- **v** the range of f.

b Sketch the graph of the function.

c Use your GDC to draw the graph to check your results.

1 $f(x) = x^2 + 2x - 3$ **2** $f(x) = x^2 + 8x + 7$
3 $f(x) = x^2 - 6x - 7$ **4** $f(x) = x^2 - 3x - 4$
5 $f(x) = x^2 - 3x - 10$ **6** $f(x) = 2x^2 + x - 3$
7 $f(x) = 2x^2 + 5x - 3$ **8** $f(x) = 3x^2 - x - 4$

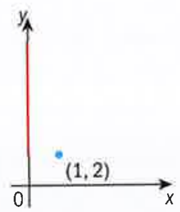
Sketching quadratic graphs

Example 15

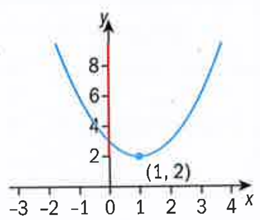
a Sketch the graph of a parabola with vertex $(1, 2)$ and range $y \geq 2$.
b Sketch the graph of a parabola with x-intercepts at $x = -2$ and $x = 3$ and y-intercept at $y = -1$.

Answers

a



Draw and label the axes.
Use a vertical line to show the range ($y \geq 2$) *of the function on the y-axis (shown in red here).*
Plot and label the vertex $(1, 2)$.



Draw a smooth curve through the point $(1, 2)$. *The curve is symmetrical about the vertical line through the vertex, that is,* $x = 1$.

Is the parabola shown in Example 15 part **a** the only one that satisfies the information given? If not, how many are there?

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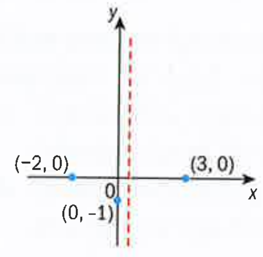
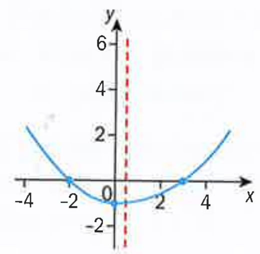
b





Draw and label the axes.
Mark the x-intercepts at $(-2, 0)$ *and* $(3, 0)$. *The axis of symmetry is halfway between the two x-intercepts, at* $x = \frac{1}{2}$. *Draw it in (as shown by the dashed red line here).*
Mark the y-intercept at $(0, -1)$.
Draw a smooth curve through the marked points.
The curve is symmetrical about $x = \frac{1}{2}$, *and the axis of symmetry passes through the vertex.*

The x-intercepts are the points where the graph crosses the x-axis. The y-values at these points are zero. The values at the x-intercepts are called the 'zeros' of the function.
The y-intercepts are the points where the graph crosses the y-axis. The x-values at these points are zero.

Sketching a quadratic graph (2)
- If you are given the function, use your GDC to draw the graph and then copy the information on to a sketch.
- If you are not given the function, use the information you are given and what you know about quadratic curves, that is:
  - They are ∪-shaped or ∩-shaped.
  - They have an axis of symmetry that passes through the vertex.

If a quadratic function only takes negative values between $x = m$ and $x = n$, what can you tell about $x = m$ and $x = n$? What happens at the points where x takes those negative values? Is the parabola ∪-shaped or ∩-shaped?
What can you tell about a quadratic function that only takes positive values between $x = m$ and $x = n$?

Exercise 4N

Sketch the graph of:

1 A parabola with vertex $(1, -3)$ and x-intercepts at -1 and 3.

2 A parabola with vertex $(-1, 2)$ and range $y \leq 2$.

3 A parabola with an axis of symmetry at $x = 0$ and range $y \leq -1$.

4 A parabola with x-intercepts at $x = 3$ and $x = 0$ and range $y \leq 1$.

5 A parabola passing through the points $(0, -2)$ and $(4, -2)$ with a maximum value at $y = 2$.

6 A quadratic function f that takes negative values between $x = 2$ and $x = 5$, and $f(0) = 4$.

Intersection of two functions

→ Two functions $f(x)$ and $g(x)$ intersect at the point(s) where $f(x) = g(x)$.

To find the coordinates of the points of intersection
- use a GDC, or
- equate the two functions algebraically, rearrange to equal zero, and then solve on the GDC.

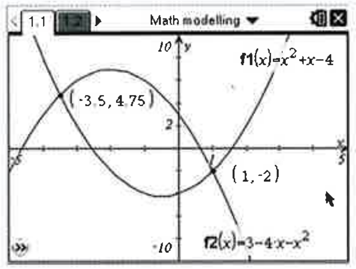
Example 16

Find the points of intersection of $f(x) = x^2 + x - 4$ and $g(x) = 3 - 4x - x^2$.

Answers

Method 1: Graphical

The points of intersection are $(-3.5, 4.75)$ and $(1, -2)$.



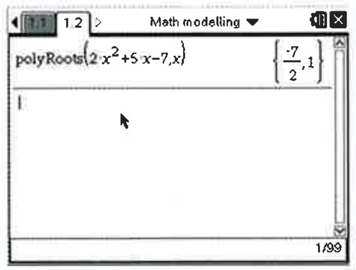
Method 2: Algebraic

$f(x) = g(x)$ — *Equate f (x) and g (x).*

$x^2 + x - 4 = 3 - 4x - x^2$

$2x^2 + 5x - 7 = 0$ — *Rearrange to equal zero. Solve using a GDC.*

$x = 1,\ x = -\frac{7}{2}$



$f(1) = (1)^2 + (1) - 4 = -2$

$f\left(-\frac{7}{2}\right) = \left(-\frac{7}{2}\right)^2 + \left(-\frac{7}{2}\right) - 4 = \frac{19}{4}$ — *Substitute the values of x into the function f (x) to find the y-coordinate of each point.*

So, the points of intersection are $(1, -2)$ and $\left(-\frac{7}{2}, \frac{19}{4}\right)$. — *Write as coordinate pairs.*

For help with using a GDC to find the points of intersection of two curves see Chapter 12, Section 4.5.

For help with using a GDC to solve a quadratic equation see Chapter 12, Section 1.2.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4O

1 Here are two functions $f(x) = x^2 + 3x - 5$ and $g(x) = x - 2$ for the domain $-5 \le x \le 2,\ x \in \mathbb{R}$.

a Use a GDC to draw the graphs of these functions and find the coordinates of their points of intersection.

b Write down $f(x) = g(x)$ and solve it for x. Do you get the same answers as you did for part **a**?

The function $h(x) = 2x - 3$ has the same domain.

c Find the points of intersection of $f(x)$ and $h(x)$

i algebraically **ii** graphically.

> Find the points 'graphically' means draw the graphs on a GDC and read off the coordinates of the points of intersection.

2 Find the coordinates of the points of intersection of the graph of $f(x) = x^2 + 3x - 5$ for the domain $-5 \le x \le 2,\ x \in \mathbb{R}$, and the line $x + y + 5 = 0$.

> First rearrange the linear equation to make y the subject.

3 Find the points of intersection of the graphs of:

a $f(x) = 5 + 3x - x^2$ and $g(x) = 1$

b $f(x) = 5 + 3x - x^2$ and $h(x) = 2x + 3$

4 **a** Use a GDC to draw the graphs of the functions $f(x) = 2x^2 - x - 3$ and $g(x) = x + 1$ for the domain $-3 \le x \le 3,\ x \in \mathbb{R}$.

b State the ranges of f and g on this domain.

c Find the x-coordinates of the points of intersection of the two functions.

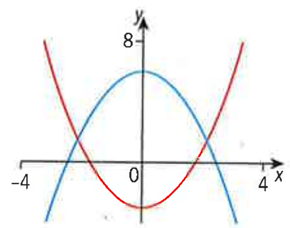
d On the same axes, and for the same domain, draw the graph of the function $h(x) = 2x + 2$.

e Solve the equation $f(x) = h(x)$ both graphically and algebraically.

f Find the coordinates of the points of intersection of the graph of $y = f(x)$ and the line $x + y = 5$.

EXAM-STYLE QUESTION

5 The diagram shows the graphs of the functions $f(x) = x^2 - 3$ and $g(x) = 6 - x^2$ for values of x between -4 and 4.



a Find the coordinates of the points of intersection.

b Write down the set of values of x for which $f(x) < g(x)$.

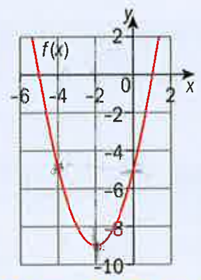
Finding the equation of a quadratic function from its graph

To find the equation of a graph of a quadratic function, use these facts:
For the graph of $f(x) = ax^2 + bx + c$
- the point of intersection of the function with the y-axis is $(0, c)$
- the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

You can use your GDC to find the equation of a quadratic function from its graph. For help see Chapter 12, Section 4.6.

Example 17

Find the equation of the quadratic function shown in the graph.



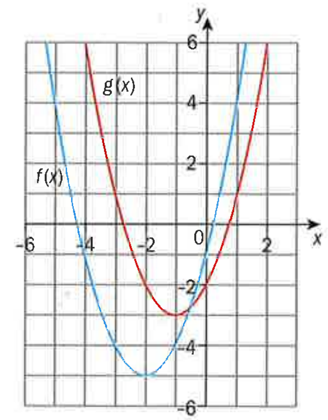
Answer

The general form of a quadratic function is given by $f(x) = ax^2 + bx + c$.	
The function intersects the y-axis at the point $(0, -5)$. So $c = -5$. $\Rightarrow f(x) = ax^2 + bx - 5$	*The function intersects the y-axis at the point (0, c). Read off the value of c from the graph.*
The equation of the axis of symmetry is $x = -2$. So: $-2 = -\frac{b}{2a}$; $-b = -4a$; $b = 4a$	*The equation of the axis of symmetry is given by* $x = -\frac{b}{2a}$. *Substitute the value of x.*
At the vertex, $x = -2$, $y = -9$. So: $f(-2) = a(-2)^2 + b(-2) - 5 = -9$; $4a - 2b - 5 = -9$; $4a - 2b = -4$	*Read off the coordinates of the vertex from the graph: (−2, −9). Substitute the x- and y-values into* $f(x) = ax^2 + bx - 5$.
$\left.\begin{array}{r} b = 4a \\ 4a - 2b = -4 \end{array}\right\} \Rightarrow$ $4a - 2(4a) = -4$; $4a - 8a = -4$; $-4a = -4 \Rightarrow a = 1$; $b = 4a \Rightarrow b = 4$	*Solve the simultaneous equations.*
Therefore, the equation of the quadratic function is: $f(x) = x^2 + 4x - 5$	*Substitute the values of a = 1, b = 4 and c = −5 into* $f(x) = ax^2 + bx + c$.

Exercise 4P

Find the equations of these quadratic functions.

1



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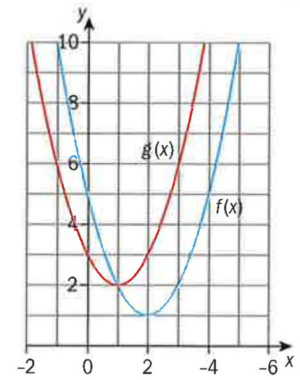

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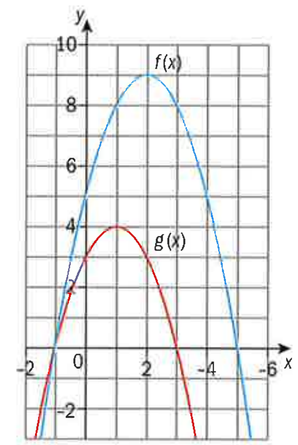

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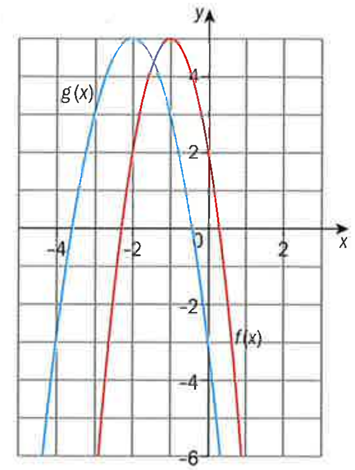

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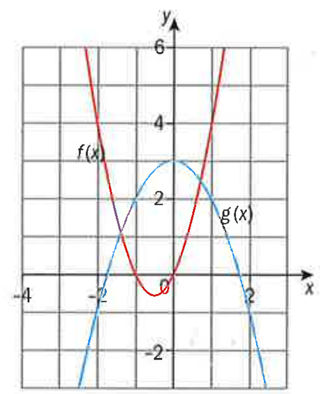
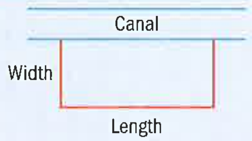


Quadratic models

Many real-life situations can be modeled using a quadratic function.

Example 18

A farmer wishes to fence off the maximum area possible to make a rectangular field.
She has 150 metres of fencing.
One side of the land borders a canal.
Find the maximum area of the field.



Answers

There are three variables:

- the length of the rectangle, l
- the width of the rectangle, w
- the area of the rectangle, A

Start by naming the variables in the problem.

The area of the rectangle $A = lw$.

Area = length × width.

As the total length of fencing is 150 m,

$l + 2w = 150$

$l = 150 - 2w$

Write down an equation for the perimeter of the field. Make l the subject.

Substitute the expression for l into the area formula.

So,

$A = lw$

$A = (150 - 2w)w$

$A = 150w - 2w^2$

Method 1: Using a GDC

The width, w, is 37.5 m.

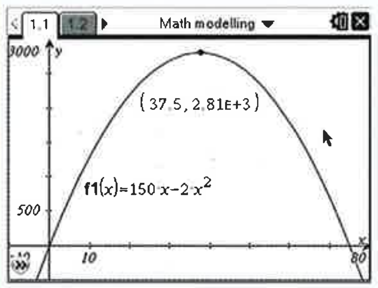
$l = 150 - 2w = 150 - 75 = 75$ m

Maximum area,

$A = lw = 75 \times 37.5$

$= 2812.5\ \text{m}^2$

Graph $A(x) = 150x - 2x^2$ on your GDC and read off the x-coordinate of the vertex: 37.5. This is the value of width, w, that gives the maximum value of A.



Method 2: Algebraic

$w = -\dfrac{150}{2(-2)} = 37.5$

$A = 150 \times 37.5 - 2 \times 37.5^2$

$= 2812.5\ \text{m}^2$

For the quadratic function $f(x) = ax^2 + bx + c$ the x-coordinate of the vertex is given by $x = -\dfrac{b}{2a}$.

The x-coordinate gives us the width, w. Here, the function is $150w - 2w^2$ so $a = -2$ and $b = 150$.

Ancient Babylonians and Egyptians studied quadratic equations like these thousands of years ago, to find solutions to problems involving areas of rectangles.

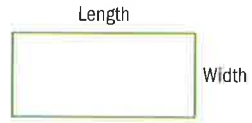
2.81E3 means $2.81 \times 10^3 = 2810$.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can use $A = lw$ or $A = 150w - 2w^2$ to work out the area.

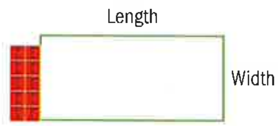
Exercise 4Q

1 a A farmer has 170 metres of fencing to fence off a rectangular area.



Find the length and width that give the maximum area.

b A farmer has 110 metres of fencing to fence off a rectangular area.
Part of one side is a wall of length 15 m.



Find the dimensions of the field that give the maximum area.

1. Identify and name the variables.
2. Use the constraint to find a model for the 'length' (this model will be linear).
3. Find a model for the area (this model will be quadratic).

2 A company's weekly profit, in riyals, is modeled by the function
$P(u) = -0.032u^2 + 46u - 3000$
where u is the number of units sold each week.
Find

a the maximum weekly profit

b the loss for a week's holiday period, where no units are sold

c the number of units sold each week at break-even point for the company.

At break-even point there is no profit and no loss, so $P(u) = 0$.

EXAM-STYLE QUESTION

3 A rocket follows a parabolic trajectory.
After t seconds, the vertical height of the rocket above the ground, in metres, is given by
$H(t) = 37t - t^2$.

a Find the height of the rocket above the ground after 10 seconds.

b Find the maximum height of the rocket above the ground.

c Find the length of time the rocket is in the air.

The trajectory is the path followed by an object.

4.4 Exponential models

Exponential functions and their graphs

→ In an **exponential function**, the independent variable is the **exponent** (or **power**).

Here are some examples of **exponential functions:**

$f(x) = 2^x, \quad f(x) = 5(3)^x + 2, \quad g(x) = 5^{-x} - 3, \quad h(x) = \left(\frac{1}{3}\right)^x + 1$

Investigation – exponential graphs

1 The number of water lilies in a pond doubles every week. In week one there were 4 water lilies in the pond.
Draw a table and write down the number of water lilies in the pond each week up to week 12.
Plot the points from the table on a graph of number of lilies against time.
Draw a smooth curve through all the points.

Time is the dependent variable, so it goes on the horizontal axis.

The graph is an example of an increasing exponential graph.

2 A radioactive substance has a half-life of two hours. This means that every two hours its radioactivity halves.
A Geiger counter reading of the radioactive substance is taken at time $t = 0$. The reading is 6000 counts per second.
Two hours later ($t = 2$) the reading is 3000 counts per second.
What will the readings be at $t = 4$, $t = 6$, $t = 8$ and $t = 10$?
Plot the points on a graph of counts per second against time and join them with a smooth curve.

Could the number of water lilies in a pond keep doubling forever? Will the radioactivity of the substance ever reach zero?

This graph is an example of a decreasing exponential graph.

Does the shape of a ski slope form an exponential function? Investigate ski slopes on the internet to find out what the function is.

Graphs of exponential functions $f(x) = a^x$ where $a \in \mathbb{Q}^+$, $a \neq 1$

$\mathbb{Q}^+$ is the set of positive rational numbers.

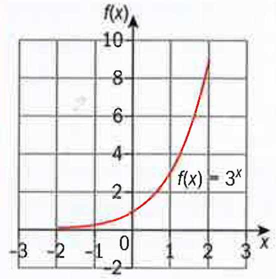
Why can $a \neq 1$? What kind of function would you get if $a = 1$?

Example 19

Draw a graph of the function $f(x) = 3^x$ for $-2 \leq x \leq 2$

Answers

Method 1: By hand

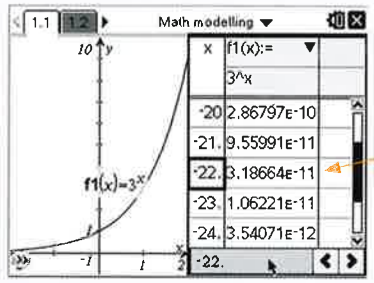


Draw a table of values.

x	−2	−1	0	1	2
f(x)	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Plot the points.
Draw a smooth curve through all the points.
This is an increasing exponential function.

Method 2: Using a GDC



For help with graphing exponential functions on your GDC see Chapter 12, Section 4.3.

3.1866E−11
= 0.000 000 000 031 866

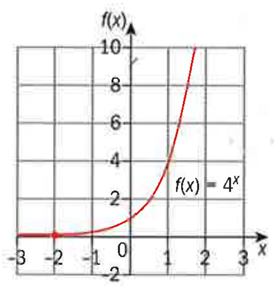
GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can check what happens when the values of x get very small or very large using the table of values on your GDC.

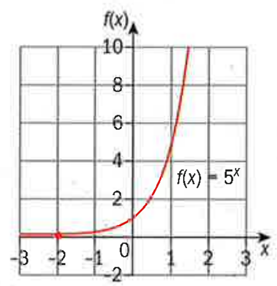
Look at the graph in Example 19. As the values of x get smaller, the curve gets closer and closer to the x-axis. The x-axis ($y = 0$) is a horizontal **asymptote** to the graph. At $x = 0$, $f(x) = 1$. As the values of x get very large, $f(x)$ gets larger even more quickly. We say that $f(x)$ tends to infinity. The function is an **increasing** exponential function.

An asymptote is a line that the curve approaches but never touches.

Here are some more graphs of **increasing** exponential functions.



▲ $f(x) = 4^x$



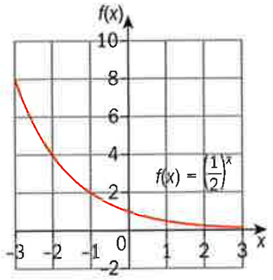
▲ $f(x) = 5^x$

All these graphs pass through the point (0, 1) and have $y = 0$ (the x-axis) as a horizontal asymptote.

Graphs of exponential functions $f(x) = a^x$ where $0 < a < 1$

What happens if a is a positive proper fraction?

Here is the graph of $y = \left(\frac{1}{2}\right)^x$.



A **proper fraction** is a fraction where the numerator is smaller than the denominator.

- For an increasing exponential function, the y-values increase as the x-values increase from left to right.
- For a decreasing exponential function, the y-values decrease as the x-values increase from left to right.

This graph also passes through the point (0, 1) and has $y = 0$ (the x-axis) as a horizontal asymptote. However, this is an example of a **decreasing** exponential function.

Exercise 4R

Draw the graphs of these functions using your GDC.
For each, write down the coordinates of the point where the curve intersects the y-axis and the equation of the horizontal asymptote.

1 $f(x) = 2^x$ **2** $f(x) = 6^x$ **3** $f(x) = 8^x$

4 $f(x) = \left(\frac{1}{3}\right)^x$ **5** $f(x) = \left(\frac{1}{5}\right)^x$

Investigation – graphs of $f(x) = ka^x$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

Use your GDC to draw the graphs of

1 $f(x) = 2(3)^x$ **2** $f(x) = 3\left(\frac{1}{2}\right)^x$ **3** $f(x) = -3(2)^x$

For each graph, write down

a the value of k in the equation $f(x) = ka^x$
b the point where the graph crosses the y-axis
c the equation of the horizontal asymptote.

What do you notice?

Investigation – graphs of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

Use your GDC to draw the graphs of

1 $f(x) = 2^x + 3$ **2** $f(x) = 3\left(\frac{1}{2}\right)^x - 4$ **3** $f(x) = -2(3)^x + 5$

for $-3 \le x \le 3$.

For each graph, write down

a the values of k and c in the equation $f(x) = ka^x + c$
b the point where the graph crosses the y-axis
c the equation of the horizontal asymptote.

Work out $k + c$ for each graph. What do you notice?

→ In general, for the graph of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

- the line $y = c$ is a **horizontal asymptote**
- the curve passes through the point $(0, k + c)$.

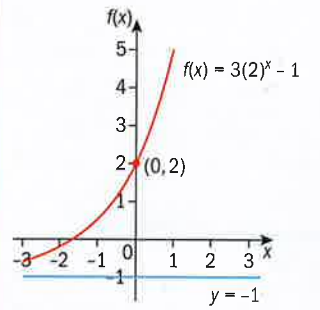
Sketching an exponential graph

- Draw and label the axes.
- Label the point where the graph crosses the y-axis.
- Draw in the asymptotes.

Example 20

Sketch the graph of the function $f(x) = 3(2)^x - 1$

Answer



Comparing $f(x) = 3(2)^x - 1$ to $f(x) = ka^x + c$:
$k = 3$
$a = 2$
$c = -1$
$y = c$ is a horizontal asymptote ⇒ $y = -1$
The curve passes through the point $(0, k + c) \Rightarrow (0, 3 - 1)$ or $(0, 2)$.

Exercise 4S

For each function, write down

a the coordinates of the point where the curve cuts the y-axis
b the equation of the horizontal asymptote.

Hence, sketch the graph of the function.

1 $f(x) = 2^x$ **2** $f(x) = 6^x$

3 $f(x) = \left(\frac{1}{3}\right)^x$ **4** $f(x) = \left(\frac{1}{5}\right)^x$

5 $f(x) = 3(2)^x + 4$

6 $f(x) = -2(4)^x - 1$

7 $f(x) = -1(2)^x + 3$

8 $f(x) = 4(3)^x - 2$

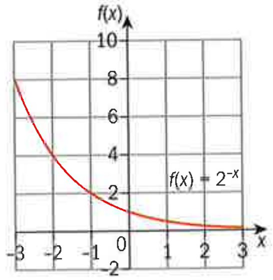
9 $f(x) = 0.5(2)^x + 3$

10 $f(x) = 2(0.5)^x + 1$

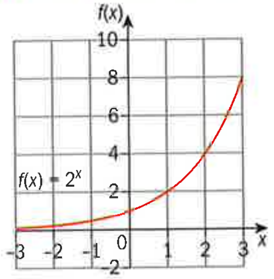
11 $f(x) = 0.4^x + 1$

12 $f(x) = 2(0.1)^x - 1$

Graphs of $f(x) = a^{-x} + c$ where $a \in \mathbb{Q}^+$ and $a \neq 1$

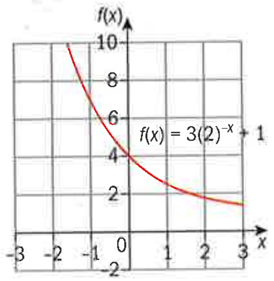


▲ Graph of $f(x) = 2^{-x}$.

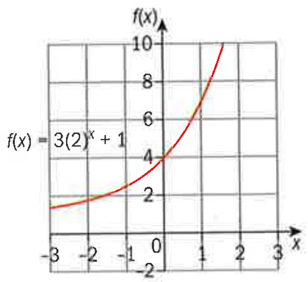


▲ Graph of $f(x) = 2^x$.

The graph of $f(x) = 2^{-x}$ is a reflection in the y-axis of the graph of $f(x) = 2^x$.



▲ Graph of $f(x) = 3(2)^{-x} + 1$.



▲ Graph of $f(x) = 3(2)^x + 1$.

$k = 3$ and $c = 1$.
Notice that $3 + 1 = 4$.

The curves pass through the point (0, 4) and the horizontal asymptote is $y = 1$.

→ In general, for the graph of $f(x) = ka^{-x} + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

- the line $y = c$ is a horizontal asymptote
- the curve passes through the point $(0, k + c)$
- the graph is a reflection in the y-axis of $g(x) = ka^x + c$.

Exercise 4T

For each function, write down

a the coordinates of the point where the curve cuts the y-axis

b the equation of the horizontal asymptote.

Hence, sketch the graph of the function.

1 $f(x) = 4(2)^{-x} + 2$

2 $f(x) = -4^{-x} + 1$

3 $f(x) = -2(2)^{-x} + 3$

4 $f(x) = 3(2)^{-x} - 2$

5 $f(x) = 0.5(3)^{-x} + 2$

6 $f(x) = 0.5^{-x} + 1$

7 $f(x) = 2(0.1)^{-x} - 1$

8 $f(x) = 0.4^{-x} + 2$

9 $f(x) = 3(0.2)^{-x} + 4$

10 $f(x) = 5(3)^{-x} - 2$

Applications of exponential functions

Many real-life situations involving growth and decay can be modeled by exponential functions.

Example 21

The length, l cm, of a pumpkin plant increases according to the equation

$$l = 4(1.2)^t$$

where t is the time in days.

a Copy and complete the table. Give your answers correct to 3 sf.

t	0	2	4	6	8	10	12	14	16
l									

b Draw a graph of l against t for $0 \leq t \leq 20$ and $0 \leq l \leq 100$.

c How long is the pumpkin plant when $t = 0$?

d How long will the pumpkin plant be after 3 weeks?

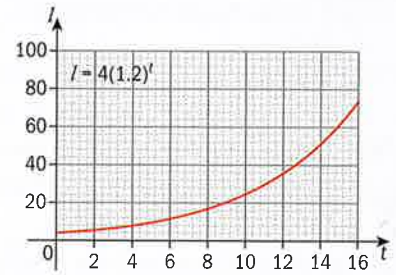
Answers

a

t	0	2	4	6	8	10	12	14	16
l	4	5.8	8.3	11.9	17.2	24.8	35.7	51.4	74

Substitute each value of t into the equation to find the corresponding value of l.

b



Draw and label the axes.
Put t on the horizontal axis.
Put l on the vertical axis.
Plot the points from the table and join with a smooth curve.

c When $t = 0$, $l = 4$ cm.

Read the value of l that corresponds to t = 0 from the table.

d 3 weeks = 21 days
So, $l = 4(1.2)^{21} = 184$ cm (to 3 sf).

For the equation, time is given in days, so convert from weeks.
Substitute t = 21 into the equation.

Example 22

Hubert invests 3000 euros in a bank at a rate of 5% per annum compounded yearly.
Let y be the amount he has in the bank after x years.

a Draw a graph to represent how much Hubert has in the bank after x years. Use a scale of 0–10 years on the x-axis and 2500–5000 euros on the y-axis.

b How much does he have after 4 years?

c How many years is it before Hubert has 4000 euros in the bank?

Answers

a The compound interest formula is:

$$y = 3000\left(1+\frac{5}{100}\right)^x$$

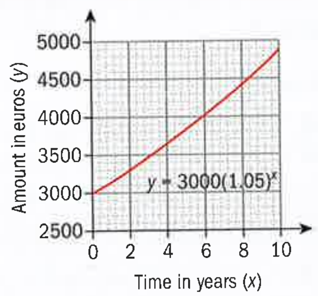
$$y = 3000(1.05)^x$$

where x = number of years.

This problem can be represented by a compound interest function.

Draw a table of values.

Time (x years)	Amount (y euros)
0	3000
2	3307.50
4	3646.52
6	4020.29
8	4432.37
10	4886.68



Draw and label the axes.

Plot the points and join them with a smooth curve.

b After 4 years Hubert has $3000(1.05)^4 = 3646.52$ euros.

Substitute $x = 4$ into the formula.

c Hubert has 4000 euros in the bank after 6 years.

You need to find the value of x for $y = 4000$ euros.
*From the table of values in part **a** you can see that after 6 years the amount is 4020.29.*
Check the amount after 5 years:
$y = 3000(1.05)^5 = 3828.84$
This is less than 4000 euros.

The **compound interest** formula is an exponential (growth) function.

You will learn more about compound interest in Chapter 7.

Exercise 4U

EXAM-STYLE QUESTIONS

1 Sketch the graphs of $f(x) = 2^x + 0.5$ and $g(x) = 2^{-x} + 0.5$ for $-3 \leq x \leq 3$.

a Write down the coordinates of the point of intersection of the two curves.

b Write down the equation of the horizontal asymptote to both graphs.

2 The value of a car decreases every year according to the function

$$V(t) = 26\,000x^t$$

where V is the value of the car in euros, t is the number of years after it was first bought and x is a constant.

a Write down the value of the car when it was first bought.

b After one year the value of the car is 22 100 euros. Find the value of x.

c Calculate the number of years that it will take for the car's value to fall to less than 6000 euros.

3 The equation $M(t) = 150(0.9)^t$ gives the amount, in grams, of a radioactive material kept in a laboratory for t years.

a Sketch the graph of the function $M(t)$ for $0 \leq t \leq 100$.

b Write down the equation of the horizontal asymptote to the graph of $M(t)$.

c Find the mass of the radioactive material after 20 years.

d Calculate the number of years that it will take for the radioactive material to have a mass of 75 grams.

4 The area, A m², covered by a certain weed is measured at 06:00 each day.
On the 1st June the area was 50 m².
Each day the area of the weeds grew by the formula

$$A(t) = 50(1.06)^t$$

where t is the number of days after 1st June.

a Sketch the graph of $A(t)$ for $-4 \leq t \leq 20$.

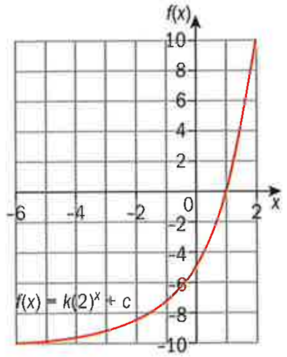
b Explain what the negative values of t represent.

c Calculate the area covered by the weeds at 06:00 on 15th June.

d Find the value of t when the area is 80 m².

EXAM-STYLE QUESTIONS

5 The graph shows the function $f(x) = k(2)^x + c$.
Find the values of c and k.



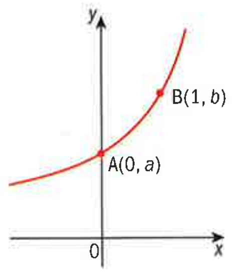
6 The temperature, T, of a cup of coffee is given by the function
$T(t) = 18 + 60(2)^{-t}$
where T is measured in °C and t is in minutes.

a Sketch the graph of $T(t)$ for $0 \le t \le 10$.
b Write down the temperature of the coffee when it is first served.
c Find the temperature of the coffee 5 minutes after serving.
d Calculate the number of minutes that it takes the coffee to reach a temperature of 40 °C.
e Write down the temperature of the room where the coffee is served. Give a reason for your answer.

7 The value, in USD, of a piece of farm machinery depreciates according to the formula
$D(t) = 18\,000(0.9)^t$ where t is the time in years.
a Write down the initial cost of the machine.
b Find the value of the machine after 5 years.
c Calculate the number of years that it takes for the value of the machine to fall below 9000 USD.

8 The graph of the function $f(x) = \frac{2^x}{a}$ passes through the points $(0, b)$ and $(2, 0.8)$. Calculate the values of a and b.

9 The diagram shows the graph of $y = 2^x + 3$. The curve passes through the points A(0, a) and B(1, b).
a Find the value of a and the value of b.
b Write down the equation of the asymptote to the curve.



10 A function is represented by the equation $f(x) = 2(3)^x + 1$.
Here is a table of values of $f(x)$ for $-2 \le x \le 2$.

x	−2	−1	0	1	2
f(x)	1.222	a	3	7	b

a Calculate the value a and the value of b.
b Draw the graph of $f(x)$ for $-2 \le x \le 2$.
c The domain of $f(x)$ is the real numbers. What is the range?

4.5 Graphs of functions of the form $f(x) = ax^m + bx^n + \ldots,\ m, n \in \mathbb{Z}$

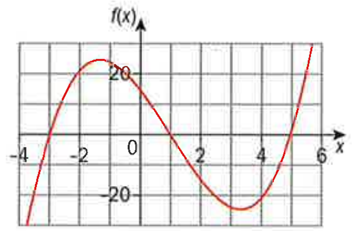
In Sections 4.2 and 4.3, you have seen examples of linear and quadratic functions. What happens when the power of x is an integer larger than 2 or smaller than 0?

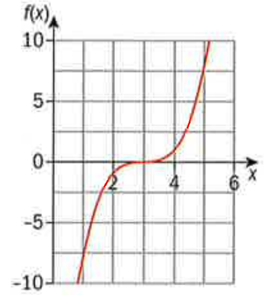
Cubic functions

When the largest power of x is 3 the function is called a **cubic** function.

→ A cubic function has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \ne 0$. The domain is $\mathbb{R}$, unless otherwise stated.

Here are two examples of graphs of cubic functions.



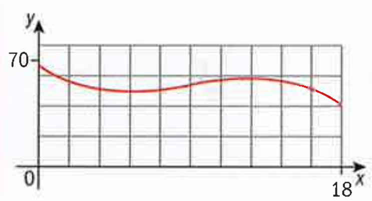


Example 23

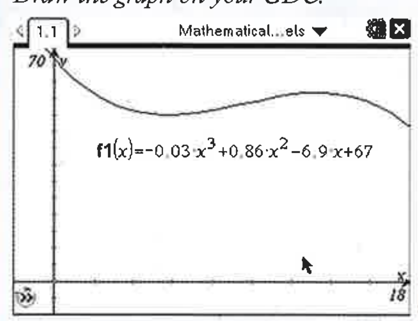
The number of fish, F, in a pond from the period 1995 to 2010 is modeled using the formula
$F(x) = -0.030x^3 + 0.86x^2 - 6.9x + 67$
where x is the number of years after 1995.
a Use your GDC to sketch the function for $0 \le x \le 18$.
b Find the number of fish in the pond after 6 years.
c Find the number of fish in the pond after 13 years.

Answers

a



Draw the graph on your GDC.



Copy the details on to a sketch graph.

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▶ Continued on next page

b $F(6) = -0.030(6)^3 + 0.86(6)^2 - 6.9(6) + 67$
$= -6.48 + 30.96 - 41.4 + 67$
$= 50.08$
So, after 6 years, there are 50 fish in the pond.

Substitute $x = 6$ into the equation. Or, you can use your GDC table of values or the Trace function.

c $F(13)$
$= -0.030(13)^3 + 0.86(13)^2 - 6.9(13) + 67$
$= -65.91 + 145.34 - 89.7 + 67$
$= 56.73$
So, after 13 years, there are 56 fish in the pond.

Substitute $x = 13$ into the equation. Or, you can use your GDC table of values.

Example 24

A pandemic is modeled using the equation

$y = (x - 20)^3 + 5000$

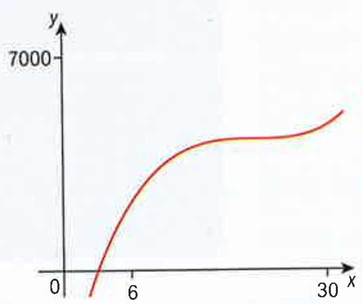
where x is the number of weeks after the outbreak started and y is the number of cases reported.

a Use your GDC to sketch the function for $0 \le x \le 30$.
b Find the number of cases after 10 weeks.
c Find the number of cases after 20 weeks.
d Is this a good model to represent the number of cases of a pandemic?

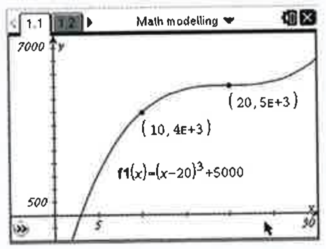
A pandemic is an epidemic of an infectious disease that spreads over several continents.

Answers

a



Draw the graph on your GDC.



Copy the details on to a sketch graph.

b $y = (10 - 20)^3 + 5000 = 4000$
So, after 10 weeks, there are 4000 cases.

Substitute $x = 10$ into the equation.

c $y = (20 - 20)^3 + 5000 = 5000$
So, after 20 weeks, there are 5000 cases.

Substitute $x = 20$ into the equation.

d No, because the number of cases starts to rise again after 20 weeks and will keep on rising.

Consider: Does the graph keep increasing? Would you expect the pandemic to increase forever?

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Can mathematical models accurately model the real world?

Investigation – quartic functions

When the largest power of x is 4 then the function is called a **quartic** function.

A quartic function has the form
$f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$. The domain is $\mathbb{R}$, unless otherwise stated.

Substitute different values of a, b, c, d and e into the equation
$f(x) = ax^4 + bx^3 + cx^2 + dx + e$.
Use your GDC to draw the functions.
What can you say about the shape of a quartic graph?

Exercise 4V

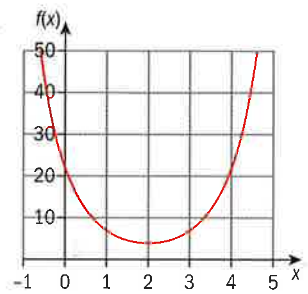
1 The times of high and low tides one day are modeled by the function

$f(x) = -0.0015x^4 + 0.056x^3 - 0.60x^2 + 1.65x + 4$

where x is the number of hours after midnight.

a Use your GDC to sketch the function for $0 \le x \le 20$.
b Find the time of the low tides.
c Find the times of the high tides.

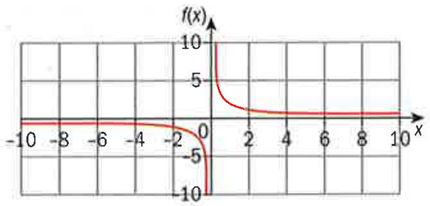
2 Here is the graph of the function $f(x) = (x - 2)^4 + 6$

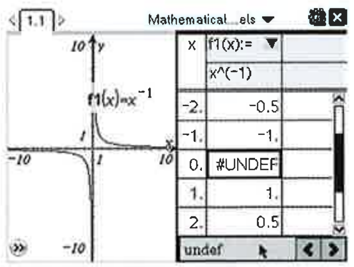


a Find the value of $f(x)$ when $x = 2$.
b Find the values of x when $y = 6$.
c Write down the range of this function.

Graphs of functions when the power of x is a negative integer

Here is the graph of $y = x^{-1}$, $x \neq 0$, for $-10 \leq x \leq 10$





GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

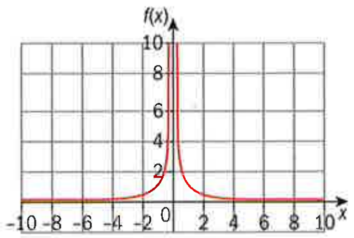
The graph has two branches that do not overlap or cross the y-axis.

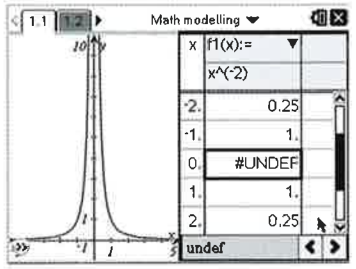
There is no value for y when $x = 0$. We call $x = 0$ a **vertical asymptote**.

When you look at the table of values on the GDC, you usually see UNDEF in the column for y whenever you have a vertical asymptote.

A vertical asymptote occurs when the value of y tends to infinity as x tends to zero. This means that when x approaches 0 from either the negative side or the positive side then y approaches either a very big negative number or a very big positive number.

Here is the graph of $y = x^{-2}$, $x \neq 0$, for $-10 \leq x \leq 10$





There is no value for y when $x = 0$, so $x = 0$ is a vertical asymptote. However, in this graph y tends to a very large positive number when x approaches 0 from either the negative side or the positive side.

Investigation – graphs of $y = ax^{-n}$

1 Use your GDC to draw the graphs of:

- $y = x^{-3}$ for $-10 \leq x \leq 10$
- $y = x^{-4}$ for $-10 \leq x \leq 10$

Compare them to the graphs of $y = x^{-1}$ and $y = x^{-2}$.
What do you notice?

2 Draw the graphs of:

- $y = 2x^{-3}$ for $-10 \leq x \leq 10$
- $y = 3x^{-4}$ for $-10 \leq x \leq 10$

Compare these graphs to the others.
What do you notice?

Example 25

A rectangle has an area of $1.5\,\text{m}^2$.
Let the length of the rectangle be y and the width be x.

a Show that $y = \dfrac{1.5}{x}$.

b Use your GDC to draw the graph of $y = \dfrac{1.5}{x}$ for $0 < x \leq 10$.

c What happens when x gets closer to 0?

d What happens when x gets closer to 10?

e Write down the equations of the vertical and horizontal asymptotes.

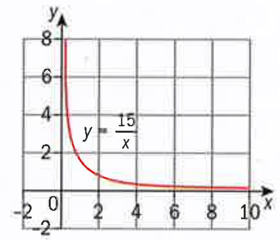
How many different rectangles could you draw with an area of $1.5\,\text{m}^2$?

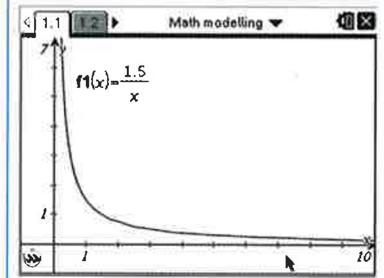
Answers

a $x \times y = 1.5 \Rightarrow y = \dfrac{1.5}{x}$

Area = length × width.
Rearrange the formula to make y the subject.

b





GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

c When x gets closer to 0 then y becomes a very large positive number.

d When x gets closer to 10 then y becomes a very small positive number.

e The vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

Which lines does the curve approach but never meet?

Exercise 4W

1 The temperature of water as it cools to room temperature is modeled by the function

$$f(x) = 21 + \frac{79}{x},\ x \neq 0,$$

where x is the time in minutes and $f(x)$ represents the temperature in °C.

a Use your GDC to sketch the graph of the function for $0 < x \leq 15$.

b Calculate the temperature of the water after 10 minutes.

c How many minutes does it take for the temperature to cool down to 50 °C?

d Write down the equation of the vertical asymptote.

e Write down the equation of the horizontal asymptote.

f Write down the room temperature.

2 Oil is heated on a stove. The temperature is modeled by the function

$$f(x) = 100 - \frac{100}{x},\ x \neq 0,$$

where x is the time in minutes from when the oil began to heat and $f(x)$ represents the temperature in °C.

a Use your GDC to sketch the graph of the function for $0 < x \leq 50$.

b Find the temperature of the oil after 10 minutes.

c Find the number of minutes that it takes the temperature to reach 30 °C.

d Write down the maximum temperature that the oil can reach.

3 **a** Use your GDC to sketch the graph of $f(x) = \frac{5}{x^2},\ x \neq 0$.

b Write down the values of x when $y = 8$.

c Write down the equations of the vertical and horizontal asymptotes to the graph.

d Given that the domain of f is the real numbers, $x \neq 0$, write down the range of f.

4 **a** Use your GDC to sketch the graph of $f(x) = 3 + \frac{6}{x}, x \neq 0$, for $-10 \leq x \leq 10$.

b Find the value of $f(x)$ when $x = 8$.

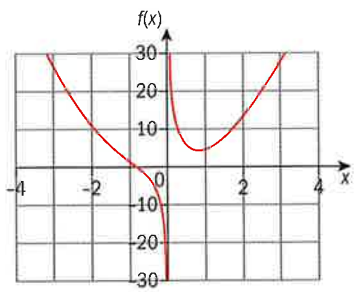
c Find the value of x when $y = 5$.

d Write down the equations of the vertical and horizontal asymptotes to the graph.

e Given that the domain of f is the real numbers, $x \neq 0$, write down the range of f.

Graphs of more complex functions

Here is the graph of $f(x) = 3x^2 + \frac{2}{x}$, $x \neq 0$, for $-4 \leq x \leq 4$.



The graph has two separate branches.
$x = 0$ is a vertical asymptote.
The domain is $-\infty \leq x < 0,\ 0 < x \leq +\infty$.

English mathematician John Wallis (1616–1703) introduced the symbol ∞ for infinity.

Example 26

A taxi company's fares depend on the distance, in kilometres, traveled.
The fares are calculated using the formula

$$f(x) = 2x + \frac{50}{x^2}$$

where x is the number of kilometres traveled ($x \neq 0$) and $f(x)$ is the fare in euros.

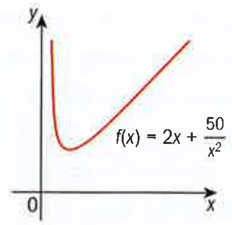
a Sketch the graph of the function for $0 < x \leq 20$.

b Find the cost for a journey of 10 kilometres.

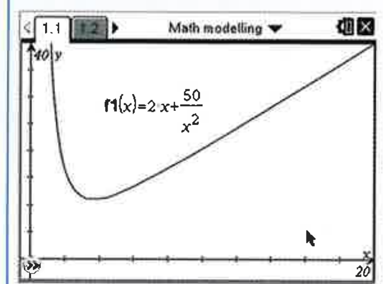
c Find the number of kilometres traveled that gives the cheapest fare.

Answers

a



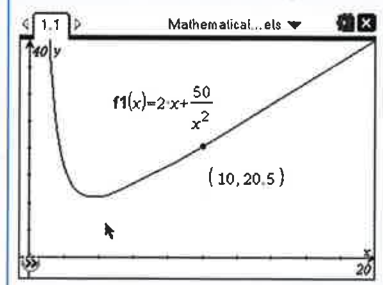
Draw the graph on your GDC.



Copy the details onto a sketch graph.

b The cost for a journey of 10 kilometres is 20.50 euros.

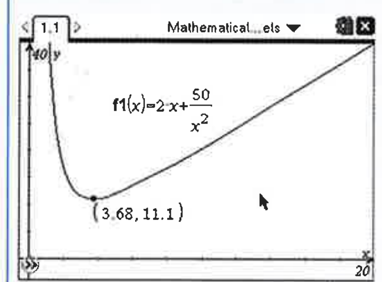
Use a GDC:



Use Trace (or the table) to find the value of f(x) when x = 10.

c The cheapest fare is achieved by a journey of 3.68 kilometres.

Use a GDC:

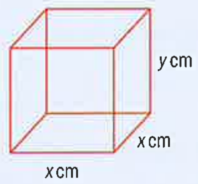


GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

For help with finding the minimum value using a GDC see Chapter 12, Section 4.2, Example 20.

Example 27

A closed cuboid of height y cm has a square base of length x cm.
The volume of the cuboid is $500\,\text{cm}^3$.



a Write down an expression for the volume of the cuboid.

b Hence, find an expression for the surface area, A, of the cuboid in terms of x. Simplify your answer as much as possible.

c Use your GDC to draw a graph of the area function for $0 < x \le 30$.

d Use your GDC to find the dimensions that give a minimum surface area.

Answers

a Volume $= x^2y$

Volume = length × width × height

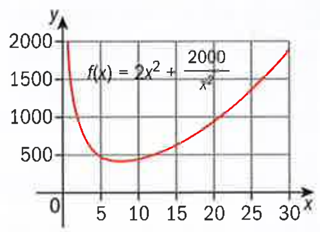
b $A = 2x^2 + 4xy$

$= 2x^2 + 4x \times \dfrac{500}{x^2}$

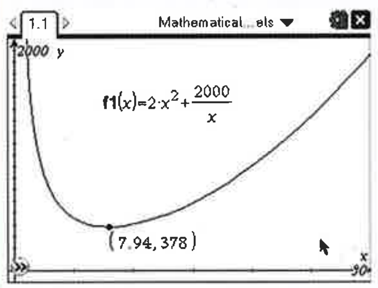
$= 2x^2 + \dfrac{2000}{x}$

2 square faces each with area $x^2 \Rightarrow 2x^2$
4 rectangular faces each with area xy $\Rightarrow 4xy$
*From part **a**:*
Volume $= x^2y$
$500 = x^2y \Rightarrow y = \dfrac{500}{x^2}$
Substitute the expression for y into the formula for A.

c



The area function is:
$f(x) = 2x^2 + \dfrac{2000}{x}$



d The minimum surface area is obtained when $x = 7.937$ and $y = \dfrac{500}{7.937^2} = 7.937$.

Using a GDC the minimum value of base length $x = 7.937$.
Substitute the value of x into the rearranged volume formula to find y.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

For help with finding a minimum value using a GDC see Chapter 12, Section 4.2, Example 20.

Exercise 4X

1 One section of a rollercoaster ride can be modeled by the equation

$$f(x) = \frac{20}{x} + 2x^2,\ x \neq 0,$$

where x is the time in seconds from the start of the ride and $f(x)$ is the speed in m s^{-1}.

a Use your GDC to sketch the graph of the function for $0 < x \le 10$.

b Find the minimum value on the graph.

c Find the speed when $x = 6$.

d Find the times when the speed is $50\,\text{m s}^{-1}$.

2 An open box has the following dimensions:
length $= x$ cm, breadth $= 2x$ cm and height $= y$ cm.
The volume of the box is $300\,\text{cm}^3$.

a Write down an expression for the volume of the box.

b Find an expression for the surface area of the open box in terms of x only.

c Use your GDC to sketch the graph of the area function for $0 < x \le 20$.

d Find the dimensions that make the surface area a minimum.

3 A pyramid has a square base of side x metres. The perpendicular height of the pyramid is h metres. The volume of the pyramid is $1500\,\text{m}^3$.

a Find an expression for the volume of the pyramid using the information given.

b Show that the height of each of the triangular faces is $\sqrt{\left(h^2 + \left(\dfrac{x}{2}\right)^2\right)}$.

c Hence, find an equation for the total surface area of the pyramid.

d Write the equation in part **c** in terms of x only.

e Use your GDC to sketch the graph of this equation for $0 < x \le 30$.

f Find the dimensions that make this surface area a minimum.

4 A fish tank in the shape of a cuboid has length 320 cm. Its length is twice that of its width.
To enhance viewing, the area of the four vertical faces should be maximized.
Find the optimum 'viewing area' of a fish tank that is fixed to the wall so that the area of three faces only should be considered.

Example 28

Consider the function $f(x) = \frac{3x-12}{x}$, $x \neq 0$.

a Write down the domain of $f(x)$.

b Copy and complete the table of values for $f(x)$. Give your answers correct to two significant figures.

x	−24	−12	−4	−1	0	1	2	4	8	12	24
f(x)											

c Draw the graph of $f(x)$ for $-24 \le x \le 24$. Use a scale of 1 cm to represent 4 units on the horizontal axis and 1 cm to represent 2 units on the vertical axis.

d Write down the equation of the vertical asymptote to the graph of $f(x)$.

Answers

a The domain of f is the real numbers, $x \neq 0$.

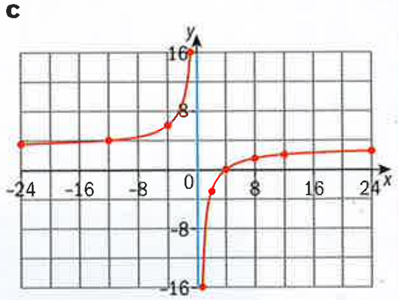
The only value excluded is $x = 0$ (as division by zero is not defined).

b

x	f(x)
−24	3.5
−12	4
−4	6
−1	15
0	
1	−9
2	−3
4	0
8	1.5
12	2
24	2.5

Substitute each value of x into f(x) to find the corresponding value of f(x). x = 0 has no image.

c



*Draw and label the axes. Plot the points from the table in part **b**. The graph has **two** branches. Points to the right of x = 0 are joined up with a smooth curve. Points to the left of x = 0 are joined up with another smooth curve.*

d $x = 0$

*Which vertical line does the curve approach but never meet? (Shown in blue on graph in part **c**.)*

As x gets very large in absolute value the graph of f(x) gets closer and closer to a horizontal line. What is the equation of this line?

Large in absolute value means very large positive numbers (1000, 10 000, etc.) or very large negative numbers (−1000, −10 000, etc.).

For more on absolute value, see Chapter 13, section 2.8.

Exercise 4Y

1 Consider the function $f(x) = 1 + \frac{2}{x}$, $x \neq 0$.

a Write down the domain of $f(x)$.

b Copy and complete the following table.

x	−10	−5	−4	−2	−1	−0.5	−0.2	0	0.2	0.5	1	2	4	5	10
f(x)															

c Draw the graph for $-10 \le x \le 10$. Use a scale of 1 unit to represent 1 cm on each of the axes.

d i Draw the vertical asymptote.

ii Write down the equation of the vertical asymptote.

e i Draw the horizontal asymptote.

ii Write down the equation of the horizontal asymptote.

2 Consider the function $f(x) = 8x^{-1} + 3$, $x \neq 0$.

a Write down the domain of $f(x)$.

b Copy and complete the following table.

x	−10	−8	−5	−4	−2	−1	0	1	2	4	5	8	10
f(x)													

c Draw the graph of $f(x)$ for $-10 \le x \le 10$. Use a scale of 1 cm to represent 2 units on both axes.

d i Draw the vertical asymptote.

ii Write down the equation of the vertical asymptote.

e i Draw the horizontal asymptote.

ii Write down the equation of the horizontal asymptote.

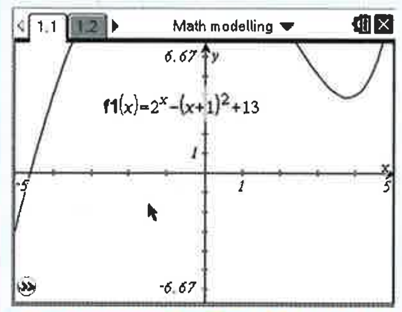
Sketching more complex graphs

Example 29

Sketch the graph of the function $f(x) = 2x - (x+1)^2 + 13$ for $-5 \le x \le 5$.

Answer

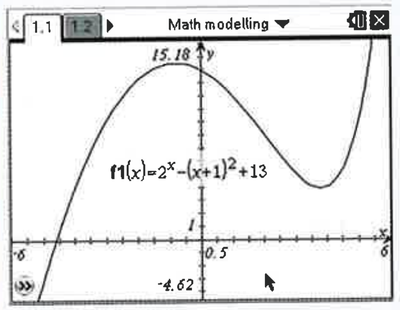
Using a GDC:
Enter the function and adjust the window settings for x.



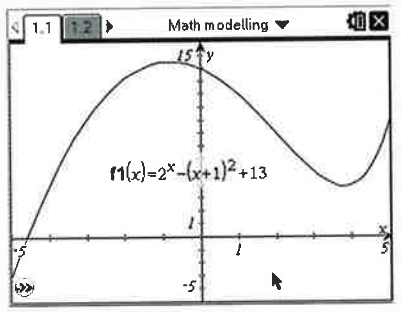
GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

▶ Continued on next page

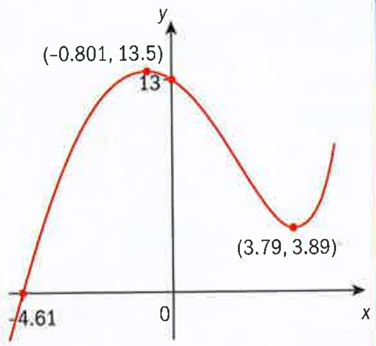
Use Zoom–Fit to adjust the y-axis to include points on the graph.



Choose some integer values to define the window.
For x
Minimum: −5, Maximum: 5
For y
Minimum: −5, Maximum: 15



Copy the details onto a sketch graph.



The range of the function in Example 29 is ℝ.
You can use a table on a GDC to give you an idea of the range of the function.

x	f1(x):= 2^x−(x+1)
−5.	−2.96875
−4.	4.0625
−3.	9.125
−2.	12.25
−1.	13.5

x	f1(x):= 2^x−(x+1)
−1.	13.5
0.	13.
1.	11.
2.	8.
3.	5.

Exercise 4Z

Use your GDC to help you sketch the graph of these functions.
Give the range for each function.

1 $f(x) = -0.5x + 1 + 3^x$

2 $f(x) = 2^x - x^2$

3 $f(x) = x(x - 1)(x + 3)$

4 $f(x) = x^4 - 3x^2 + 1$

5 $f(x) = 0.5^x - x^{-1}, x \neq 0$

4.6 Using a GDC to solve equations

Example 30

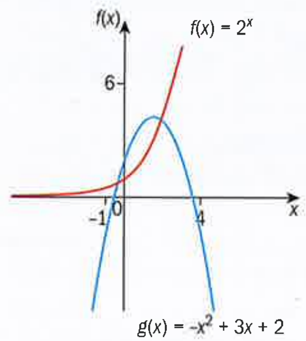
a Use your GDC to sketch the graphs of $f(x) = 2^x$ and $g(x) = -x^2 + 3x + 2$.

b Hence, solve the equation $2^x + x^2 - 3x - 2 = 0$.

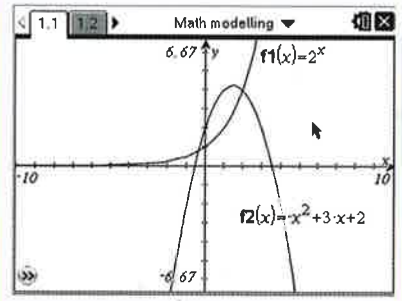
'Hence' means that you should try to use the previous part to answer this part of the question.

Answers

a



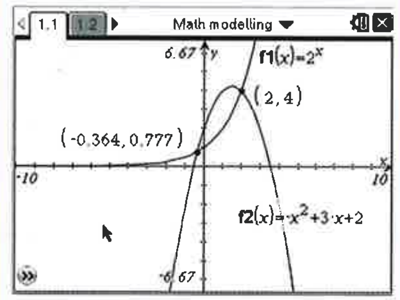
Put $Y_1 = 2^x$ and $Y_2 = -x^2 + 3x + 2$.



A standard window has been used here.

b The solutions are $x = -0.364$ or $x = 2$.

The equation $2^x + x^2 - 3x - 2 = 0$ is the same as $2^x = -x^2 + 3x + 2$. There are 2 points of intersection and we need to find them both.



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4AA

1 **a** On the same graph, sketch the curves $y = x^2$ and $y = 4 - \frac{1}{x}$ for values of x from -8 to 8 and values of y from -2 to 8. Show scales on your axes.

b Find the coordinates of the points of intersection of these two curves.

EXAM-STYLE QUESTIONS

2 The functions f and g are defined by

$$f(x) = 1 + \frac{4}{x},\ x \in \mathbb{R},\ x \neq 0$$

$$g(x) = 3x,\ x \in \mathbb{R}$$

a Sketch the graph of f for $-8 \leq x \leq 8$.

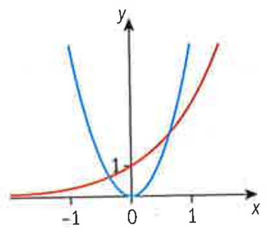
b Write down the equations of the horizontal and vertical asymptotes of the function f.

c Sketch the graph of g on the same axes.

d Hence, or otherwise, find the solutions of $1 + \frac{4}{x} - 3x = 0$.

e Write down the range of function f.

3 The diagram shows the graphs of the functions $y = 5x^2$ and $y = 3^x$ for values of x between -2 and 2.



a Find the coordinates of the points of intersection of the two curves.

b Write down the equation of the horizontal asymptote of the exponential function.

4 Two functions $f(x)$ and $g(x)$ are given by $f(x) = \frac{3}{x},\ x \in \mathbb{R},\ x \neq 0$ and $g(x) = x^3,\ x \in \mathbb{R}$.

a On the same diagram sketch the graphs of $f(x)$ and $g(x)$ using values of x between -4 and 4, and values of y between -4 and 4. You must label each curve.

b State how many solutions exist for the equation $\frac{3}{x} - x^3 = 0$.

c Find a solution of the equation given in part **b**.

5 Sketch the graphs of $y = 3x - 4$ and $y = x^3 - 3x^2 + 2x$. Find all the points of intersection of the graphs.

6 Sketch the graphs of $y = 2^x$ and $y = x^3 + x^2 - 6x$. Find the coordinates of all the points of intersection.

7 Sketch the graphs of $y = x + 2$ and $y = \frac{5}{x},\ x \neq 0$.

a Find the solutions of the equation $\frac{5}{x} = x + 2$.

b Write down the equation of the horizontal asymptote to $y = \frac{5}{x}$.

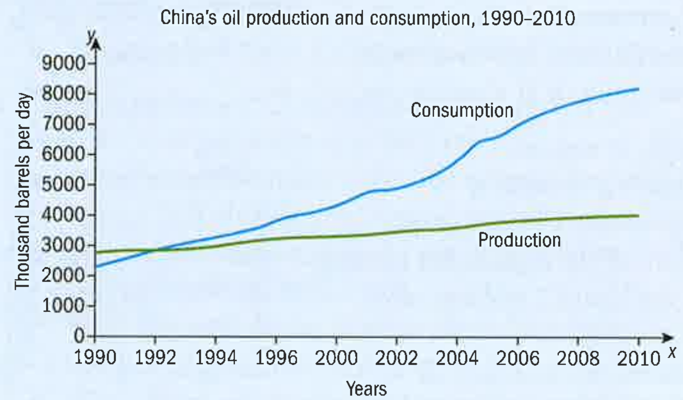
c Write down the equation of the vertical asymptote to $y = \frac{5}{x}$.

4.7 Graphs of real-life situations

Linear and non-linear graphs can be used to represent a range of real-life situations.

Example 31

The graph below shows China's oil production and consumption from 1990 to 2010.



Source: US Energy Information Administration, *International Energy Annual 2006*, 'Short term energy outlook (July 2009)'

a What are the two variables represented by this graph?

b What does the blue curve represent?

c What does the green curve represent?

d Explain the meaning of the point where both curves meet. What is the year at that point?

e Explain what happens before and after 1992.

f What is the tendency of the oil consumption in China?

Answers

a The variables are year and number of thousands of barrels per day.

b The blue curve represents oil consumption per day in China from 1990 to 2010.

c The green curve represents oil production per day in China from 1990 to 2010.

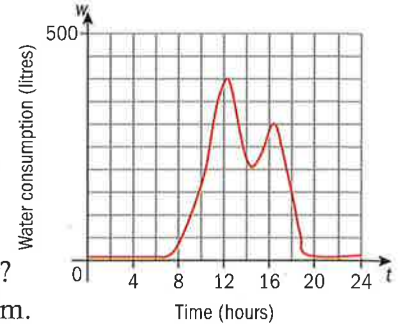
d At the point that the curves meet, oil production and consumption in China were equal. This occurred in 1992.

e Before 1992, oil consumption was less than oil production. After 1992, oil consumption was greater than oil production.

f Oil consumption in China tends to keep increasing.

Can you deduce any further information from this graph?

Exercise 4AB



1 The water consumption in Thirsty High School is represented in the graph.

a Write down are the two variables represented by this graph.

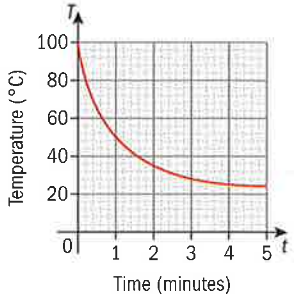
b During what period of time is Thirsty High School open?

c During what intervals of time is consumption increasing?

d During what intervals of time is consumption decreasing?

e Find the time at which the consumption is at a maximum.

f Find the time at which the consumption is at a minimum.



2 The graph represents the temperature, in degrees Celsius, of some coffee after Manuela has heated it.

a Write down the two variables represented by this graph.

b Write down the initial temperature of the liquid after heating.

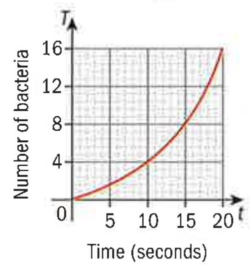
c Write down the temperature of the liquid 2 minutes after heating.

d Find the time it takes for the temperature to reach 68 °C.

e Decide whether the liquid reaches 22 °C during the 5-minute period shown on the graph.

f Write down the room temperature.

EXAM-STYLE QUESTION



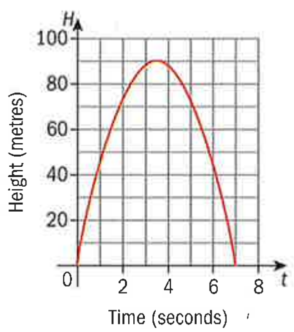
3 Under certain conditions the number of bacteria in a particular culture doubles every 5 seconds as shown by the graph.

a Copy and complete the table below.

Time (t seconds)	0	5	10	15	20
Number of bacteria (N)	1				

b Write down the time it takes for the culture to have 6 bacteria.

c Calculate the number of bacteria in the culture after 1 minute if the conditions remain the same.



4 In a physics experiment a ball is projected vertically into the air from ground level.
The diagram represents the height of the ball at different times.

a Write down the height of the ball after one second.

b Find out how many seconds after being thrown the ball is at 60 metres.

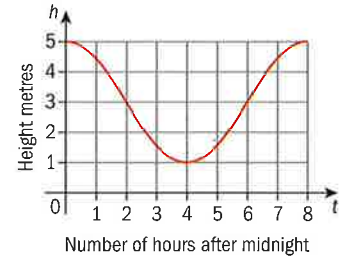
c Write down the interval of time in which the ball is going up.

d Write down the interval of time in which the ball is coming down.

e Write down the maximum height reached by the ball and the time it takes the ball to reach that height.

f Explain what happens at $t = 7$.

EXAM-STYLE QUESTIONS



5 The graph shows the tide heights, h metres, at time t hours after midnight for Blue Coast Harbor.

a Use the graph to find

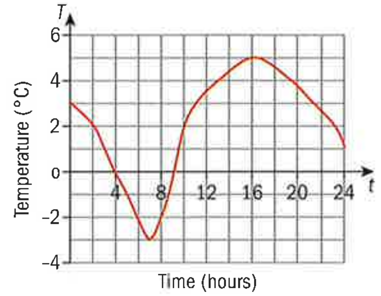
i the height of the tide at 01:30

ii the height of the tide at 05:30

iii the times when the height of the tide is 3 metres.

The best time to catch fish in Blue Coast Harbor is when the tide is below 3 metres.

b Find this best time, giving your answer as an inequality in t.



6 The temperature (°C) during a 24-hour period in a certain city is represented in the graph.

a Determine how many times the temperature is exactly 0 °C during this 24-hour period.

b Write down the interval of time in which the temperature falls below 0 °C.

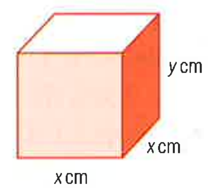
c Write down the time at which the temperature reaches its maximum value.

d Write down the maximum temperature registered during this 24-hour period.

e Write down the interval of time in which the temperature increases from 3 °C to 5 °C.

f Write down the times at which the temperature is 4 °C.

g Can you deduce from this graph whether the behavior of the temperature in the following day will be exactly the same as this day? Why?



7 The diagram represents a box with volume $16\,cm^3$.
The base of the box is a square with sides x cm. The height of the box is y cm.

a Write an expression for the height, y, in terms of x.

b Copy and complete the table below for the function $y = f(x)$ from part **a**. Give your answers correct to two significant figures.

x	0.5	1	2	4	8	10
$y = f(x)$						

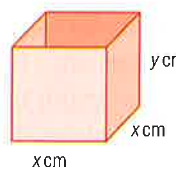
For part **a** use the formula:
volume = length × width × height.

c Draw the graph of f for $0 < x \le 10$. Use a scale of 1 cm to represent 1 unit on the horizontal axis and 1 cm to represent 10 units on the vertical axis.

d What happens to the height of the box as the values of x tend to infinity?

EXAM-STYLE QUESTION

8 The diagram represents an open container with a capacity of 3 litres.
The base of the container is a square with sides x cm. The height of the container is y cm.



a Write down the volume of the box in cm³.
b Find an expression for the height, y, in terms of x.
c Find an expression for the surface area of the container, A, in terms of x.
d Copy and complete the table below. Give your answers correct to 2 significant figures.

x(cm)	5	10	15	20	25	30	35
$A(x)$(cm²)							

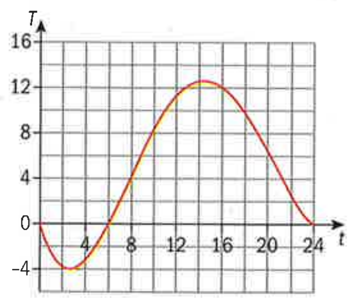
e Draw the graph of A for $0 < x \le 35$. Use a scale of 2 cm to represent 5 units on the horizontal axis and 1 cm to represent 400 units on the vertical axis.
f Use your graph to decide if there is a value of x that makes the surface area of the container a minimum. If there is, write down this value of x.

Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

1 The graph represents the temperature in °C in a certain city last Tuesday.



a Write down the interval of time in which the temperature was below 0 °C.
b Write down the interval of time in which the temperature was above 11 °C.
c Write down the maximum temperature last Tuesday. Give your answer correct to the nearest unit.

2 The cost c, in Singapore dollars (SGD), of renting an apartment for n months is a linear model

$c = nr + s$

where s is the security deposit and r is the amount of rent per month.

Wan Ning rented the apartment for 6 months and paid a total of 35 000 SGD.
Tanushree rented the same apartment for 2 years and paid a total of 116 000 SGD.
Find the value of
a r, the rent per month **b** s, the security deposit.

EXAM-STYLE QUESTIONS

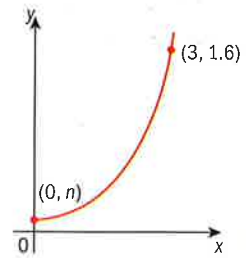
3 Given that $f(x) = x^2 + 5x$
a factorize $x^2 + 5x$
b sketch the graph of $y = f(x)$. Show on your sketch
i the coordinates of the points of intersection with the axes
ii the equation of the axis of symmetry
iii the coordinates of the vertex of the parabola.

4 A signal rocket is fired vertically from ground level by a gun. The height, in metres, of the rocket above the ground is a function of the time t, in seconds, and is defined by:

$h(t) = 30t - 5t^2, 0 \le t \le 6.$

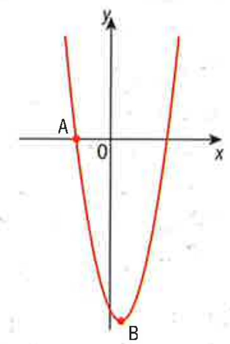
a Find the height of the rocket above the ground after 4 seconds.
b Find the maximum height of the rocket above the ground.
c Use your GDC to find the length of time, in seconds, for which the rocket is at a height of 25 m or more above the ground.

5 The graph of the function $f(x) = \frac{2^x}{m}$ passes through the points (3, 1.6) and (0, n).



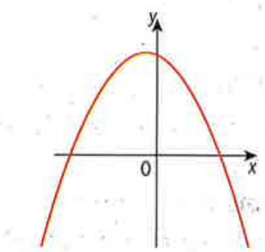
a Calculate the value of m.
b Calculate the value of n.
Find $f(2)$.

6 The diagram shows the graph of $y = x^2 - 2x - 15$.
The graph crosses the x-axis at the point A, and has a vertex at B.



a Factorize $x^2 - 2x - 15$.
b Find the coordinates of the point
i A **ii** B.

7 Consider the graphs of the following functions.
i $y = 8x + x^2$
ii $y = (x - 3)(x + 4)$
iii $y = x^2 - 2x + 5$
iv $y = 5 - 4x - 3x^2$



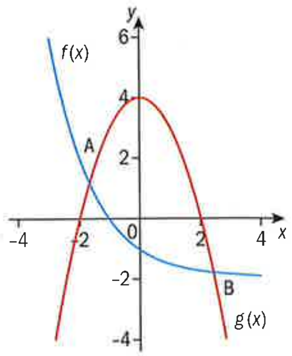
Which of these graphs
a has a y-intercept below the x-axis
b passes through the origin
c does not cross the x-axis
d could be represented by this diagram?

EXAM-STYLE QUESTIONS

8 The figure shows the graphs of the functions

$f(x) = (0.5)^x - 2$ and $g(x) = -x^2 + 4$

for values of x between -3 and 3. The two graphs meet at the points A and B.



a Find the coordinates of

i A **ii** B.

b Write down the set of values of x for which $f(x) < g(x)$.

c Write down the equation of the horizontal asymptote to the graph of $f(x)$.

9 Gabriel is designing a rectangular window with a perimeter of 4.40 m. The length of the window is x m.

a Find an expression for the width of the window in terms of x.

b Find an expression for the area of the window, A, in terms of x.

Gabriel wants to make the amount of light passing through this window a maximum.

c Find the value of x that meets this condition.

10 a On the same graph sketch the curves $y = 3x^2$ and $y = \frac{1}{x}$ for values of x from -4 to 4 and values of y from -4 to 4.

b Write down the equations of the vertical and horizontal asymptotes of $y = \frac{1}{x}$.

c Solve the equation $3x^2 - \frac{1}{x} = 0$.

Paper 2 style questions

EXAM-STYLE QUESTIONS

1 The number (n) of bacteria after t hours is given by the formula $n = 1500(1.32)^t$.

a Copy and complete the table below for values of n and t.

Time (t hours)	0	1	2	3	4
Number of bacteria (n)	1500		2613	3450	

b On graph paper, draw the graph of $n = 1500(1.32)^t$. Use a scale of 2 cm to represent 1 hour on the horizontal axis and 2 cm to represent 1000 bacteria on the vertical axis. Label the graph clearly.

c Find

i the number of bacteria after 2 hours 30 minutes. Give your answer to the nearest ten bacteria.

ii the time it will take to form approximately 5000 bacteria. Give your answer to the nearest 10 minutes.

2 The functions f and g are defined by

$f(x) = \frac{4}{x}$, $x \in \mathbb{R}$, $x \neq 0$

$g(x) = 2x$, $x \in \mathbb{R}$

a Sketch the graph of $f(x)$ for $-8 \le x \le 8$.

b Write down the equations of the horizontal and vertical asymptotes of the function f.

c Sketch the graph of g on the same axes.

d Find the solutions of $\frac{4}{x} = 2x$.

e Write down the range of function f.

3 A function is represented by the equation $f(x) = 2(1.5)^x + 3$. The table shows the values of $f(x)$ for $-3 \le x \le 2$.

x	−3	−2	−1	0	1	2
f(x)	3.59	3.89	a	5	6	b

a Calculate the values for a and b.

b On graph paper, draw the graph of $f(x)$ for $-3 \le x \le 2$, taking 1 cm to represent 1 unit on both axes.

The domain of the function $f(x)$ is the real numbers, $\mathbb{R}$.

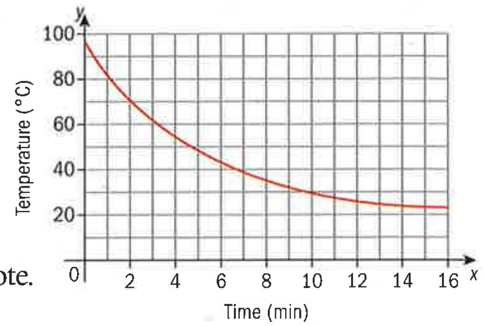
c Write down the range of $f(x)$.

d Find the approximate value for x when $f(x) = 10$.

e Write down the equation of the horizontal asymptote of $f(x) = 2(1.5)^x + 3$.

EXAM-STYLE QUESTIONS

4 The graph shows the temperature, in degrees Celsius, of Leonie's cup of hot chocolate t minutes after pouring it. The equation of the graph is $f(t) = 21 + 77(0.8)^t$ where $f(t)$ is the temperature and t is the time in minutes after pouring the hot chocolate out.



a Find the initial temperature of the hot chocolate.

b Write down the equation of the horizontal asymptote.

c Write down the room temperature.

d Find the temperature of the hot chocolate after 8 minutes.

5 Consider the functions

$f(x) = x^2 - x - 6$ and $g(x) = -2x + 1$

a On the same diagram draw the graphs of $f(x)$ and $g(x)$ for $-10 \le x \le 10$.

b Find the coordinates of the local minimum of the graph of $f(x)$.

c Write down the gradient of the line $g(x)$.

d Write down the coordinates of the point where the graph of $g(x)$ cuts the y-axis.

e Find the coordinates of the points of intersection of the graphs of $f(x)$ and $g(x)$.

f Hence, or otherwise, solve the equation $x^2 + x - 7 = 0$.

6 **a** Sketch the graph of $f(x) = x^2 - \frac{3}{x}$, for $-4 \le x \le 4$.

b Write down the equation of the vertical asymptote of $f(x)$.

c On the same diagram draw the graph of $g(x) = -3(2)^x + 9$, for $-4 \le x \le 4$.

d Write down the equation of the horizontal asymptote of $g(x)$.

e Find the coordinates of the points of intersection of $f(x)$ and $g(x)$.

EXAM-STYLE QUESTIONS

7 The profit (P) in euros made by selling homemade lemonade is modeled by the function

$$P = -\frac{x^2}{10} + 10x - 60$$

where x is the number of glasses of lemonade sold.

a Copy and complete the table.

x	0	10	20	30	40	50	60	70	80	90
P		30			180			150	100	

b On graph paper draw axes for x and $P(x)$, placing x on the horizontal axis and $P(x)$ on the vertical axis. Draw the graph of $P(x)$ against x by plotting the points.

c **Use your graph** to find

i the maximum possible profit

ii the number of glasses that need to be sold to make the maximum profit

iii the number of glasses that need to be sold to make a profit of 160 euros

iv the amount of money initially invested.

8 **a** Sketch the graph of the function $f(x) = x^2 - 7$, $x \in \mathbb{R}$, $-4 \le x \le 4$. Write down the coordinates of the points where the graph of $y = f(x)$ intersects the axes.

b On the same diagram sketch the graph of the function $g(x) = 7 - x^2$, $x \in \mathbb{R}$, $-4 \le x \le 4$.

c Solve the equation $f(x) = g(x)$ in the given domain.

d The graph of the function $h(x) = x + c$, $x \in \mathbb{R}$, $-4 \le x \le 4$, where c is a positive integer, intersects twice with both $f(x)$ and $g(x)$ in the given domain.
Find the possible values for c.

9 The functions f and g are defined by $f(x) = \frac{x^2}{2}$ and $g(x) = -\frac{x^2}{2} + 2x$, $x \in \mathbb{R}$.

a Calculate the coordinates of the points of intersection of the graphs $f(x)$ and $g(x)$.

b Find the equation of the axis of symmetry of the graph of $y = g(x)$.

c The straight line with equation $y = k$, $k \in \mathbb{R}$, is a tangent to the graph of g. Find the value of k.

d Sketch the graph of $f(x)$ and the graph of $g(x)$, using a rectangular Cartesian coordinate system with 1 cm as a unit. Show the coordinates of any points of intersection with the axes.

e Find the values of x for which $f(x) < g(x)$.

CHAPTER 4 SUMMARY

Functions

- A **function** is a relationship between two sets: a **first** set and a **second** set. Each element 'x' of the first set is related to **one and only one** element 'y' of the second set.
- The first set is called the **domain** of the function. The elements of the domain, the 'x-values', are the **independent variables**.
- For each value of 'x' (input) there is one and only one output. This value is called the **image** of 'x'. The set of all the images (all the outputs) is called the **range** of the function. The elements of the range, the 'y-values', are the **dependent variables**.
- The graph of a function f is the set of points (x, y) on the Cartesian plane where y is the image of x through the function f.
- $y = f(x)$ means that the image of x through the function f is y. x is the independent variable and y is the dependent variable.

Linear models

- A **linear function** has the general form $f(x) = mx + c$, where m (the gradient) and c are constants.
- When $f(x) = mx$ the graph passes through the origin, $(0, 0)$.

Quadratic models

- A **quadratic function** has the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.
- The graph of any quadratic function is a **parabola** – a ∪-shaped (or ∩-shaped) curve. It has an **axis of symmetry** and either a **minimum** or **maximum** point, called the **vertex** of the parabola.
- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- The curve intersects the y-axis at $(0, c)$.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$, $a \neq 0$.
- The x-coordinate of the vertex is $x = -\frac{b}{2a}$.
- The factorized form of a quadratic function is $f(x) = a(x - k)(x - l)$.
- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- A ∪-shaped graph is 'concave up'. A ∩-shaped graph is 'concave down'.
- The curve intersects the x-axis at $(k, 0)$ and $(l, 0)$.
- The equation of the axis of symmetry is $x = \frac{k+l}{2}$.
- The x-coordinate of the vertex is also $x = \frac{k+l}{2}$.
- The function $f(x) = ax^2 + bx + c$ intersects the x-axis where $f(x) = 0$. The x-values of the points of intersection are the two solutions (or **roots**) of the equation $ax^2 + bx + c$. (The y-values at these points of intersection are zero.)
- Two functions $f(x)$ and $g(x)$ intersect at the point(s) where $f(x) = g(x)$.

Continued on next page

Exponential models

- In an **exponential function,** the independent variable is the **exponent** (or **power**).
- In general, for the graph of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$
  - the line $y = c$ is a **horizontal asymptote**
  - the curve passes through the point $(0, k + c)$.
- In general, for the graph of $f(x) = ka^{-x} + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$
  - the line $y = c$ is a horizontal asymptote
  - the curve passes through the point $(0, k + c)$
  - the graph is a reflection in the y-axis of $g(x) = ka^x + c$.

Cubic functions

- A cubic function has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. The domain is $\mathbb{R}$, unless otherwise stated.

The language of mathematics

Mathematics is described as a language. It has vocabulary (mathematical symbols with precise meaning) and grammar (an order in which we combine these symbols together to make them meaningful).

- Mathematics is often considered a 'universal language'. Can a language ever be truly universal?

Precise and concise

Mathematical language is precise and explicit, with no ambiguity. It uses its own set of rules for manipulating its statements, so it is completely abstract.

$3 \le x < 7$

$k = t^3 - 6t^2 + 12t + 2$

$D = \{(x, y) \mid x + y = 5\}$

$2 + 2 = 4$

$6 \times 9 = 54$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x + 18$

Mathematics can describe and represent ideas that are not easily expressed by conventional written or spoken words.

These two statements are equivalent:

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

(Euclid, Elements, II.4, c.300 BCE)

<=>

$(a + b)^2 = a^2 + b^2 + 2ab$

Mathematics says it far more simply!

- Draw and label a diagram to show that these statements are equivalent.

Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say.

Bertrand Russell, *The Scientific Outlook*, 1931

'Mathematics is the abstract key which turns the lock of the physical universe.'

John Polkinghorne, *One World: The Interaction of Science and Theology*, 2007

Abstract language

- What does '1' mean?

You can probably answer that with confidence. '1' is part of our language, we use it every day. Its meaning is clear to us. We can easily picture '1' banana.

But the language of mathematics has continued to expand to encompass more abstract concepts. Mathematicians call the square root of -1, '*i*'.

- What does this mean? Can you use *i* in everyday life?

What about pi (π)? Lots of people know this number.

It is the ratio $\frac{\text{circumference of a circle}}{\text{diameter of a circle}}$.

- What does this 'mean'? Can you picture 'π' bananas?
- Do π and *i* exist?

1 is an abstract concept of mathematics that has also become part of our everyday, English language too. *i* or π are also abstract concepts of mathematics, but have not become part of everyday language. Mathematicians need and use these numbers. They are not any more abstract than the number 1. They appear in a mathematical context and allow us to think mathematically and communicate these ideas, to perform manipulations, to express results and model real-life occurrences in a simple way.

Simple and beautiful equations that model the world

Here are some famous equations

Einstein's equation: $E = mc^2$

Newton's second law: $F = ma$

Boyle's law: $V = \frac{k}{p}$

Schrödinger's equation: $\hat{H}\psi = E\psi$

Newton's law of universal gravitation: $F = G\frac{m_1 m_2}{r^2}$

These are simple equations (although they were not simple to derive!). Isn't it startling that so much of what happens in the universe can be described using equations like these?

These equations have helped to put a man on the moon and bring him back, develop wireless internet and understand the workings of the human body.

- Do you think that mathematics and science will one day discover the ultimate 'theory of everything'? A theory that fully explains and links together all known physical phenomena? A theory that can predict the outcome of any experiment that could be carried out?
- What will mathematicians and scientists do then?

5 Statistical applications

CHAPTER OBJECTIVES:

4.1 The normal distribution; random variables; the parameters μ and σ; diagrammatic representation; normal probability calculations; expected value; inverse normal calculations

4.2 Bivariate data: correlation; scatter diagrams; line of best fit; Pearson's product-moment correlation coefficient, r

4.3 The regression line for y on x

4.4 The χ^2 test for independence: null and alternative hypotheses; significance levels; contingency tables; expected frequencies; degrees of freedom; p-values

Before you start

You should know how to:

1 Find the mean and standard deviation of a set of data and comment on the relationship between them, e.g. for the data set

4, 5, 6, 8, 12, 13, 2, 5, 6, 9, 10, 9, 8, 3, 5:

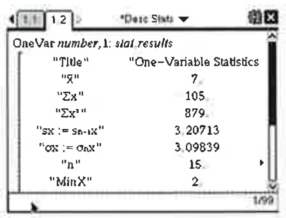
Mean =

$$\frac{(4+5+6+8+12+13+2+5+6+9+10+9+8+3+5)}{15}$$

$$= \frac{105}{15} = 7$$

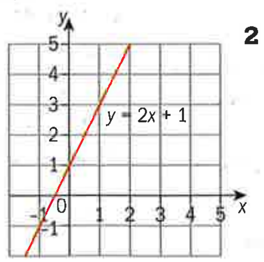
On a GDC, the mean is indicated by $\bar{x}$.

Using a GDC, standard deviation $(\sigma_x) = 3.10$ (to 3 sf).



The small standard deviation implies that the data are close to the mean.

2 Sketch the graph of the equation of a straight line, e.g. the straight line $y = 2x + 1$ passes through the point (0, 1) and has gradient 2.



Skills check

1 Find the mean and standard deviation of these sets of data. Comment on your answers.

a 2, 4, 3, 6, 3, 2, 5, 3, 2, 5, 4, 4, 3, 5, 2, 3, 4, 5

b

x	Frequency
12	1
13	2
14	23
15	2
16	1

For help, see Chapter 2, Sections 2.4 and 2.7.

2 Sketch the graphs of:

a $y = -3x + 4$

b $y = 2x - 6$

The people in this photograph are a sample of a population and a source of valuable data. Like a lot of data on natural phenomena, people's heights and weights fit a 'normal distribution', which you will study in this chapter. Medical statisticians use this information to plot height and weight charts, and establish guidelines on healthy weight.

The information can also be used to chart changes in a population over time. For example, the data can be analyzed to determine whether people, on the whole, are getting taller or heavier. These results may affect or even determine government health policy. Moreover, manufacturing and other industries may use the information to decide whether to, for example, make door frames taller or aircraft seats wider.

You may think that some data might be related, for example, people's height and shoe size, or perhaps a child's height and their later adult height. This chapter shows you how to investigate correlation and the strength of relationships between data sets.

Investigation – related data?

Do you think that height and shoe size are related?
Collect the height and shoe size of at least 60 students in your school.
Plot these data points on a graph. Use the x-axis for 'Height' and the y-axis for 'Shoe size'. Do not join up the points.
Does the data support your original hypothesis on height and shoe size?

The graph you will draw in this investigation is called a **scatter diagram**. You will find out more about scatter diagrams and the correlation between data sets in Section 5.2 of this chapter.

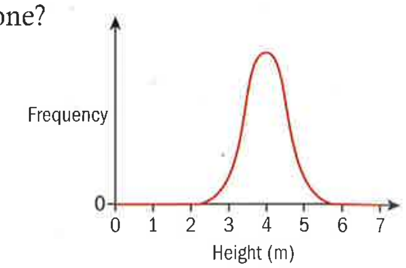
5.1 The normal distribution

For his Mathematical Studies Project, Pedro measures the heights of all the apple trees in his father's orchard. There are 150 trees.

If Pedro drew a diagram to represent the frequency of the heights of all 150 trees, what do you think it would look like?

Pedro then measures the heights of the apple trees in his uncle's orchard. If he drew a diagram of the frequencies of these heights, do you think that this diagram would look different to the previous one?

In both orchards there would probably be a few very small trees and a few very large trees – but those would be the exception. Most of the trees would fall within a certain range of heights. They would roughly fit a bell-shaped curve that is symmetrical about the mean. We call this a **normal distribution**.



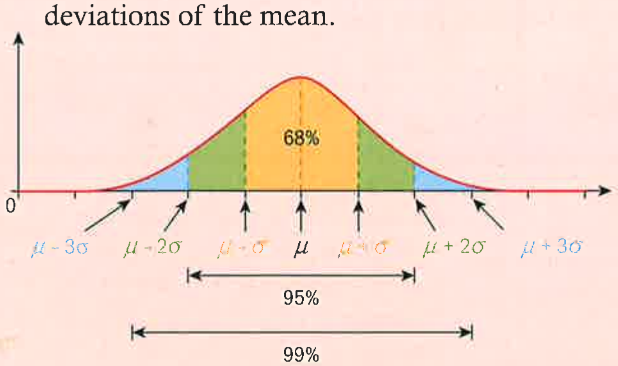
▲ Normal distribution diagram for the tree heights measured by Pedro

Many events fit this type of distribution: for example, the heights of 21-year-old males, the results of a national mathematics examination, the weights of newborn babies, etc.

The properties of a normal distribution

→ The **normal distribution** is the most important continuous distribution in statistics. It has these properties:

- It is a bell-shaped curve.
- It is symmetrical about the mean, μ. (The mean, the mode and the median all have the same value.)
- The x-axis is an asymptote to the curve.
- The total area under the curve is 1 (or 100%).
- 50% of the area is to the left of the mean and 50% to the right.
- Approximately 68% of the area is within 1 standard deviation, σ, of the mean.
- Approximately 95% of the area is within 2 standard deviations of the mean.
- Approximately 99% of the area is within 3 standard deviations of the mean.

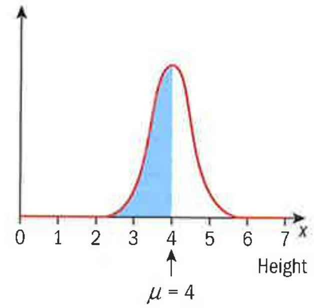


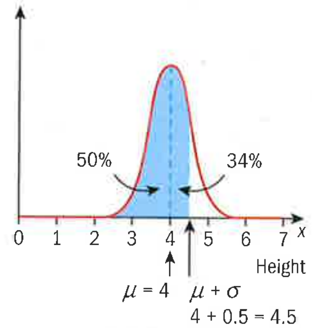
The normal curve is sometimes called the 'Gaussian curve' after the German mathematician Carl Friedrich Gauss (1777–1855). Gauss used the normal curve to analyze astronomical data in 1809. A portrait of Gauss and the normal curve appeard on the old German 10 Deutschmark note.

You can calculate the probabilities of events that follow a normal distribution.

Returning to Pedro and the apple trees, imagine that the mean height of the trees is 4 m and the standard deviation is 0.5 m.

Let the height of an apple tree be x.





From the properties of the normal distribution: Area to left of μ = 50%. Area between μ and $\mu + \sigma$ = 34% (68% ÷ 2).

The probability that an apple tree is less than 4 m is $P(x < 4) = 50\%$ or 0.5. And $P(x < 4.5) = 50\% + 34\% = 84\%$ or 0.84.

→ The **expected value** is found by multiplying the number in the sample by the probability.

For example, if we chose 100 apple trees at random, the expected number of trees that would be less than 4 m = 100 × 0.5 = 50.

Example 1

The waiting times for an elevator are normally distributed with a mean of 1.5 minutes and a standard deviation of 20 seconds.

a Sketch a normal distribution diagram to illustrate this information, indicating clearly the mean and the times within one, two and three standard deviations of the mean.

b Find the probability that a person waits longer than 2 minutes 10 seconds for the elevator.

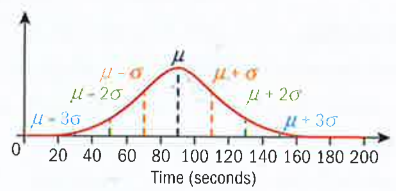
c Find the probability that a person waits less than 1 minute 10 seconds for the elevator.

200 people are observed and the length of time they wait for an elevator is noted.

d Calculate the number of people expected to wait less than 50 seconds for the elevator.

Answers

a



1.5 minutes = 90 seconds

μ = mean = 90 seconds

σ = standard deviation = 20 seconds

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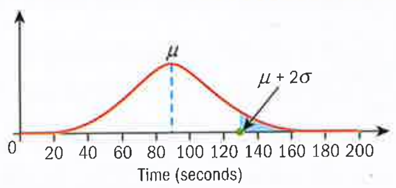
b



P(waiting longer than 2 minutes 10 seconds) = 2.5%, or 0.025.

2 minutes 10 seconds = 130 seconds

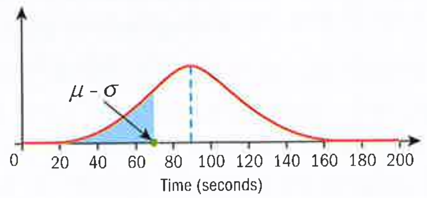
Using symmetry about μ:

Area to right of μ = 50%

Area between μ and $\mu + 2\sigma$ = 47.5% (95% ÷ 2)

Area to right of $\mu + 2\sigma$ = 50% − 47.5% = 2.5%

c



P(waiting less than 1 minute 10 seconds) = 16%, or 0.16.

1 minute 10 seconds = 70 seconds

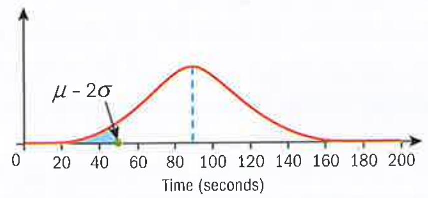
Using symmetry about μ:

Area to left of μ = 50%

Area between μ and $\mu - \sigma$ = 34% (68% ÷ 2)

Area to left of $\mu - \sigma$ = 50% − 34% = 16%

d



P(waiting less than 50 seconds) = 2.5%, or 0.025

So, the expected number of people = 200 × 0.025 = 5.

First find the probability of waiting less than 50 seconds.

Using symmetry about μ:

Area to left of μ = 50%

Area between μ and $\mu - 2\sigma$ = 47.5% (95% ÷ 2)

Area to left of $\mu - 2\sigma$ = 50% − 47.5% = 2.5%

There are 200 people in the sample.

Example 2

The heights of 250 twenty-year-old women are normally distributed with a mean of 1.68 m and standard deviation of 0.06 m.

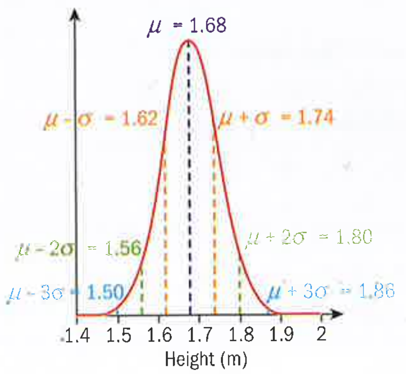
a Sketch a normal distribution diagram to illustrate this information, indicating clearly the mean and the heights within one, two and three standard deviations of the mean.

b Find the probability that a woman has a height between 1.56 m and 1.74 m.

c Find the expected number of women with a height greater than 1.8 m.

Answers

a



Let

μ = mean = 1.68 m

σ = standard deviation = 0.06 m

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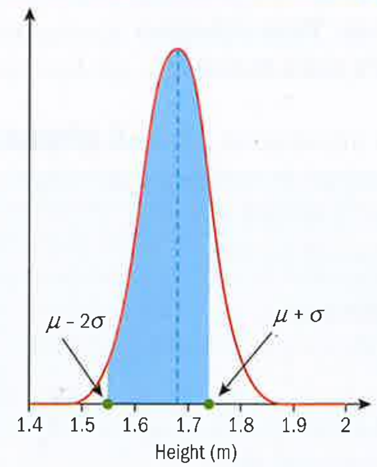
b



P(height between 1.56 m and 1.74 m) = 81.5%, or 0.815.

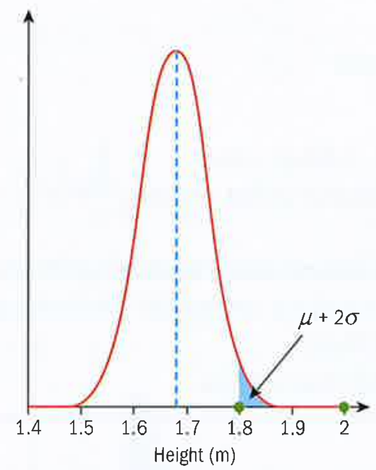
Using symmetry about μ:

Area between μ and $\mu + \sigma$ = 34% (68% ÷ 2)

Area between μ and $\mu - 2\sigma$ = 47.5% (95% ÷ 2)

Area between 1.56 m and 1.74 m = 34% + 47.5% = 81.5%

c



P(height greater than 1.8 m) = 2.5%, or 0.025.

So, the expected number of women = 250 × 0.025 = 6.25, or 6 women.

First find the probability of a woman being taller than 1.8 m.

Using symmetry about μ:

Area to right of μ = 50%

Area between μ and $\mu + 2\sigma$ = 47.5% (95% ÷ 2)

Area to right of $\mu + 2\sigma$ = 50% − 47.5% = 2.5%

There are 250 women in the sample.

Exercise 5A

EXAM-STYLE QUESTION

1 The heights of 200 lilies are normally distributed with a mean of 40 cm and a standard deviation of 3 cm.

a Sketch a normal distribution diagram to illustrate this information. Indicate clearly the mean and the heights within one, two and three standard deviations of the mean.

b Find the probability that a lily has a height less than 37 cm.

c Find the probability that a lily has a height between 37 cm and 46 cm.

d Find the expected number of lilies with a height greater than 43 cm.

EXAM-STYLE QUESTIONS

2 100 people were asked to estimate the length of one minute. Their estimates were normally distributed with a mean time of 60 seconds and a standard deviation of 4 seconds.

- **a** Sketch a normal distribution diagram to illustrate this information, indicating clearly the mean and the times within one, two and three standard deviations of the mean.
- **b** Find the percentage of people who estimated between 52 and 64 seconds.
- **c** Find the expected number of people estimating less than 60 seconds.

3 60 students were asked how long it took them to travel to school. Their travel times are normally distributed with a mean of 20 minutes and a standard deviation of 5 minutes.

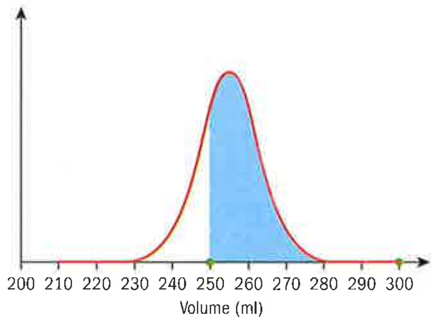
- **a** Sketch a normal distribution diagram to illustrate this information, indicating clearly the mean and the times within one, two and three standard deviations of the mean.
- **b** Find the percentage of students who took longer than 25 minutes to travel to school.
- **c** Find the expected number of students who took between 15 and 25 minutes to travel to school.

4 Packets of coconut milk are advertised to contain 250 ml. Akshat tests 75 packets. He finds that the contents are normally distributed with a mean volume of 255 ml and a standard deviation of 8 ml.

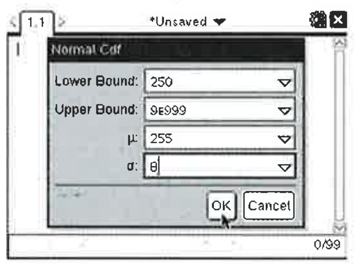
- **a** Sketch a normal distribution diagram to illustrate this information, indicating clearly the mean and the volumes within one, two and three standard deviations of the mean.
- **b** Find the probability that a packet contains less than 239 ml.
- **c** Find the expected number of packets that contain more than 247 ml.

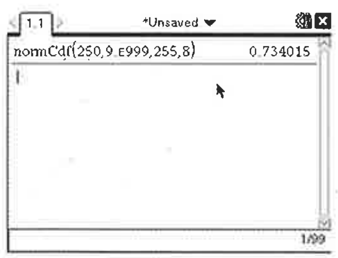
You can use your GDC to calculate values that are not whole multiples of the standard deviation.

For example, in question 4 of Exercise 5A, suppose we wanted to find the probability that a packet contains more than 250 ml.
First sketch a normal distribution diagram.



In a Calculator page press MENU 5:Probability | 5:Distributions | 2:Normal Cdf and enter the lower bound (250), the upper bound (9×10^{999} – a *very* large number), the mean (255) and the standard deviation (8) in the wizard.





To enter 9×10^{999} you need to type 9E999, but you cannot use the E key. Instead, you must use the EE key.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

So, 73.4% of the packets contain more than 250 ml of coconut milk. Alternatively, enter normCdf, the lowest value, the highest value, the mean and the standard deviation directly into the calculator screen.

For a very small number enter -9×10^{999}

Example 3

The lifetime of a light bulb is normally distributed with a mean of 2800 hours and a standard deviation of 450 hours.

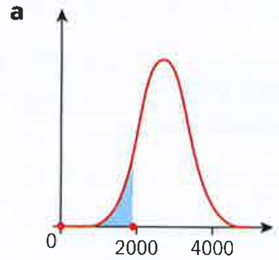
- **a** Find the percentage of light bulbs that have a lifetime of less than 1950 hours.
- **b** Find the percentage of light bulbs that have a lifetime between 2300 and 3500 hours.
- **c** Find the probability that a light bulb has a lifetime of more than 3800 hours.

120 light bulbs are tested.

- **d** Find the expected number of light bulbs with a lifetime of less than 2000 hours.

Answers

a



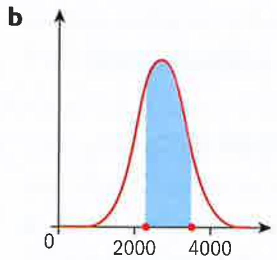
2.95% of the light bulbs have a lifetime of less than 1950 hours.

μ = mean = 2800 hours
σ = standard deviation = 450 hours

Lifetime less than 1950 hours:
lower bound = -9×10^{999}
upper bound = 1950

From GDC:
normCdf(−9E999, 1950, 2800, 450) = 0.02945
= 2.95%

b



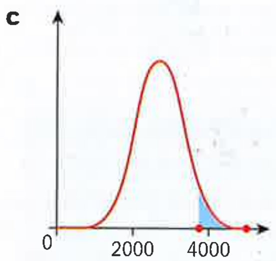
80.7% of the light bulbs have a lifetime between 2300 and 3500 hours.

Lifetime between 2300 and 3500 hours:
lower bound = 2300
upper bound = 3500

Remember not to use −9E999 notations in an exam.

From GDC:
normCdf(2300, 3500, 2800, 450) = 0.8068 = 80.7%

c



Only 1.31% of the light bulbs have a lifetime of more than 3800 hours.

Lifetime more than 3800 hours:
lower bound = 3800
upper bound = 9×10^{999}

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

From GDC:
normCdf(3800, 9E999, 2800, 450) = 0.0131 = 1.31%

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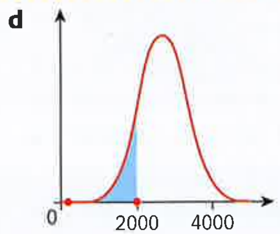
d  P(lifetime less than 2000 hours) = 3.77% Expected number = 120 × 0.0377 = 4.524 So, you would expect 4 or 5 light bulbs to have a lifetime of less than 2000 hours.	*First find P(lifetime less than 2000 hours):* *lower bound* = -9×10^{999} *upper bound* = 2000 *From GDC:* *normCdf(−9E999, 2000, 2800, 450) = 0.0377 = 3.77%* *120 light bulbs are tested.*

Exercise 5B

EXAM-STYLE QUESTION

1 Jordi delivers daily papers to a number of homes in a village. The time taken to deliver the papers follows a normal distribution with mean 80 minutes and standard deviation 7 minutes.

a Sketch a normal distribution diagram to illustrate this information.

b Find the probability that Jordi takes longer than 90 minutes to deliver the papers.

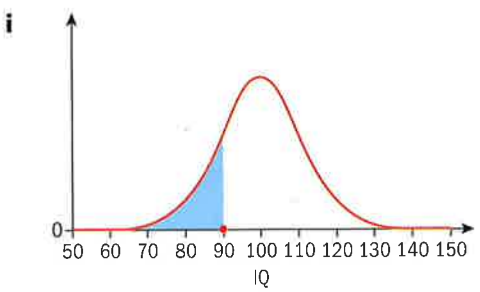
Jordi delivers papers every day of the year (365 days).

c Calculate the expected number of days on which it would take Jordi longer than 90 minutes to deliver the papers.

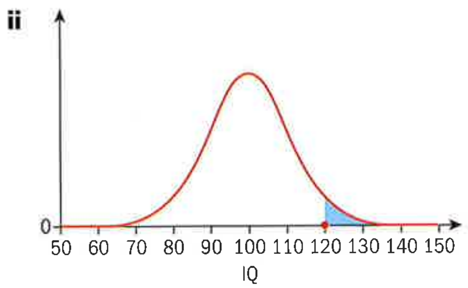
2 A set of 2000 IQ scores is normally distributed with a mean of 100 and a standard deviation of 10.

a Calculate the probability that is represented by each of the following diagrams.

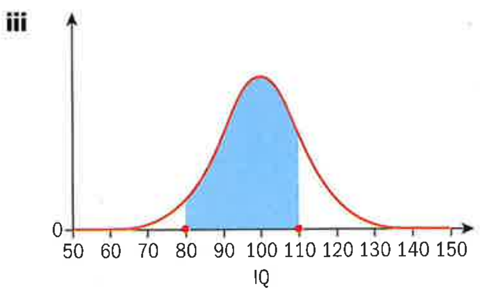
i



ii



iii



Lambert Quételet (1796–1874), a Flemish scientist, was the first to apply the normal distribution to human characteristics. He noticed that measures such as height, weight and IQ were normally distributed.

b Find the expected number of people with an IQ of more than 115.

3 A machine produces washers whose diameters are normally distributed with a mean of 40 mm and a standard deviation of 2 mm.

a Find the probability that a washer has a diameter less than 37 mm.

b Find the probability that a washer has a diameter greater than 45 mm.

Every week 300 washers are tested.

c Calculate the expected number of washers that have a diameter between 35 mm and 43 mm.

EXAM-STYLE QUESTIONS

4 In a certain school, the monthly incomes of members of staff are normally distributed with a mean of 2500 euros and a standard deviation of 400 euros.

a Sketch a normal distribution diagram to illustrate this information.

b Find the probability that a member of staff earns less than 1800 euros per month.

The school has 80 members of staff.

c Calculate the expected number of staff who earn more than 3400 euros.

5 The lengths of courgettes are normally distributed with a mean of 16 cm and a standard deviation of 0.8 cm.

a Find the percentage of courgettes that have a length between 15 cm and 17 cm.

b Find the probability that a courgette is longer than 18 cm.

The lengths of 100 courgettes are measured.

c Calculate the expected number of courgettes that have a length less than 14.5 cm.

6 At a market, the weights of bags of kiwi fruit are normally distributed with a mean of 500 g and a standard deviation of 8 g.
A man picks up a bag of kiwi fruit at random.
Find the probability that the bag weighs more than 510 g.

EXAM-STYLE QUESTIONS

7 The scores in a Physics test follow a normal distribution with mean 70% and standard deviation 8%.

a Find the percentage of students who scored between 55% and 80%.

30 students took the physics test.

b Calculate the expected number of students who scored more than 85%.

8 A machine produces pipes such that the length of each pipe is normally distributed with a mean of 1.78 m and a standard deviation of 2 cm.
Any pipe whose length is greater than 1.83 m is rejected.

a Find the probability that a pipe will be rejected.

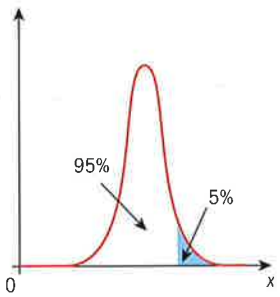
500 pipes are tested.

b Calculate the expected number of pipes that will be rejected.

Inverse normal calculations

Sometimes you are given the percentage area under the curve, i.e. the probability or proportion, and are asked to find the value corresponding to it. This is called an inverse normal calculation.

Always make a sketch to illustrate the information given.
You must always remember to use the area to the **left** when using your GDC. If you are given the area to the **right** of the value, you must subtract this from 1 (or 100%) before using your GDC.
For example, an area of 5% above a certain value means there is an area of 95% below it.

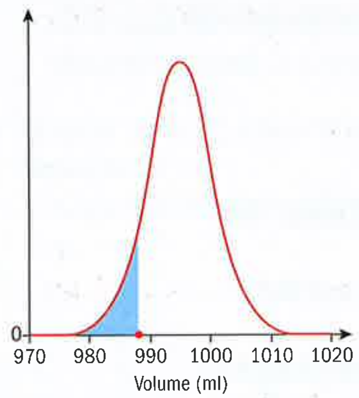


In examinations, inverse normal questions will not involve finding the mean or standard deviation.

Example 4

The volume of cartons of milk is normally distributed with a mean of 995 ml and a standard deviation of 5 ml.
It is known that 10% of the cartons have a volume less than x ml.
Find the value of x.

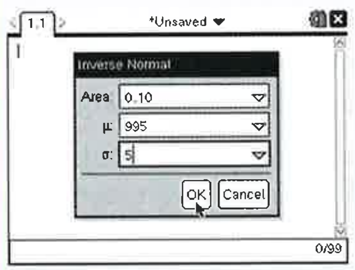
Answer

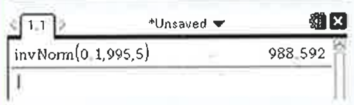


x = 989 ml (to 3 sf)

First sketch a diagram. The shaded area represents 10% of the cartons.
Using the GDC:

In a Calculator page press MENU 5:Probability | 5:Distributions | 3:Inverse Normal...
Enter the percentage given (as a decimal, 0.1), the mean (995) and the standard deviation (5).





x = 989 (3 sf)
x = 989 ml means that 10% of the cartons have a volume less than 989 ml.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Example 5

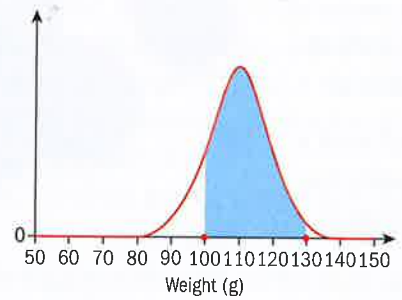
The weights of pears are normally distributed with a mean of 110 g and a standard deviation of 8 g.
a Find the percentage of pears that weigh between 100 g and 130 g.

It is known that 8% of the pears weigh more than m g.
b Find the value of m.

250 pears are weighed.
c Calculate the expected number of pears that weigh less than 105 g.

Answers

a

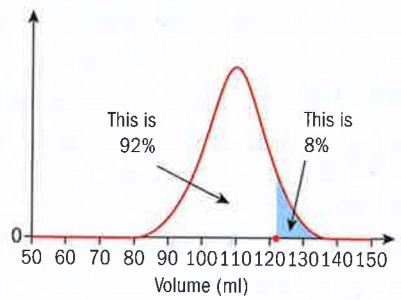


Sketch a diagram.
μ = mean = 110 g
σ = standard deviation = 8 g
Weight between 100 g and 130 g:
lower bound = 100
upper bound = 130

88.8% of the pears weigh between 100 g and 130 g.

From GDC:
normCdf(100, 130, 110, 8) = 0.888 = 88.8%

b

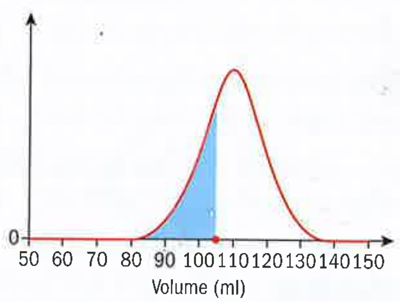


8% weighing more than m g is the same as saying that 92% weigh less than m g.
From GDC:
invNorm(0.92, 110, 8) = 121

m = 121 g

m = 121 g means that 8% of the pears weigh more than 121 g.

c



Weight less than 105 g:
lower bound = -9×10^{999}
upper bound = 105

P(weight less than 105 g) = 0.266
Expected number = 250 × 0.266 = 66.5
So, you would expect 66 or 67 pears to weigh less than 105 g.

From GDC:
normCdf(−9E999, 105, 110, 8) = 0.266
250 pears are weighed.

Exercise 5C

1 The mass of coffee grounds in Super-strength coffee bags is normally distributed with a mean of 5 g and a standard deviation of 0.1 g. It is known that 25% of the coffee bags weigh less than p grams. Find the value of p.

2 The heights of Dutch men are normally distributed with a mean of 181 cm and a standard deviation of 5 cm. It is known that 35% of Dutch men have a height less than h cm. Find the value of h.

3 The weight of kumquats is normally distributed with a mean of 20 g and a standard deviation of 0.8 g. It is known that 15% of the kumquats weigh more than k grams. Find the value of k.

4 The weight of cans of sweetcorn is normally distributed with a mean of 220 g and a standard deviation of 4 g. It is known that 30% of the cans weigh more than w grams. Find the value of w.

EXAM-STYLE QUESTIONS

5 The weights of cats are normally distributed with a mean of 4.23 kg and a standard deviation of 0.76 kg.

a Write down the weights of the cats that are within one standard deviation of the mean.

A vet weighs 180 cats.

b Find the number of these cats that would be expected to be within one standard deviation of the mean.

c Calculate the probability that a cat weighs less than 3.1 kg.

d Calculate the percentage of cats that weigh between 3 kg and 5.35 kg.

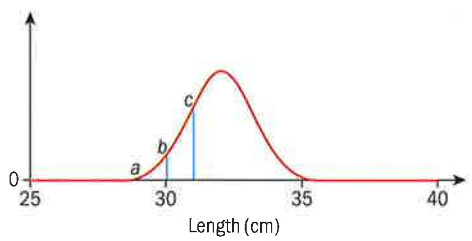
It is known that 5% of the cats weigh more than w kg.

e Find the value of w.

6 A manufacturer makes drumsticks with a mean length of 32 cm. The lengths are normally distributed with a standard deviation of 1 cm.

a Calculate the values of a, b and c shown on the graph.

b Find the probability that a drumstick has a length greater than 30.6 cm.



It is known that 80% of the drumsticks have a length less than d cm.

c Find the value of d.

One week 5000 drumsticks are tested.

d Calculate the expected number of drumsticks that have a length between 30.5 cm and 32.5 cm.

7 The average lifespan of a television set is normally distributed with a mean of 8000 hours and a standard deviation of 1800 hours.

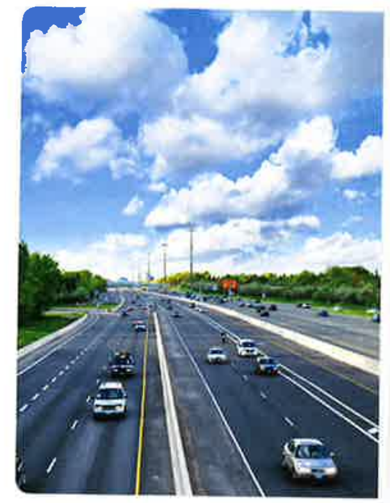
a Find the probability that a television set will break down before 2000 hours.

b Find the probability that a television set lasts between 6000 and 12 000 hours.

c It is known that 12% of the television sets break down before t hours.
Find the value of t.

EXAM-STYLE QUESTIONS

8 The speed of cars on a motorway is normally distributed with a mean of 120 km h^{-1} and a standard deviation of 10 km h^{-1}.



a Draw a normal distribution diagram to illustrate this information.

b Find the percentage of cars that are traveling at speeds of between 105 km h^{-1} and 125 km h^{-1}.

It is known that 8% of the cars are traveling at a speed of less than p km h^{-1}.

c Find the value of p.

One day 800 cars are checked for their speed.

d Calculate the expected number of cars that will be traveling at speeds of between 96 km h^{-1} and 134 km h^{-1}.

The speed limit is 130 km h^{-1}.

e Find the number of cars that are expected to be exceeding the speed limit.

9 The weights of bags of rice are normally distributed with a mean of 1003 g and a standard deviation of 2 g.

a Draw a normal distribution diagram to illustrate this information.

b Find the probability that a bag of rice weighs less than 999 g.

The manufacturer states that the bags of rice weigh 1 kg.

c Find the probability that a bag of rice is underweight.

400 bags of rice are weighed.

d Calculate the expected number of bags of rice that are underweight.

5% of the bags of rice weigh more than p g.

e Find the value of p.

EXAM-STYLE QUESTION

10 The weights of babies are normally distributed with a mean of 3.8 kg and a standard deviation of 0.5 kg.

a Find the percentage of babies who weigh less than 2.5 kg.

In a space of 15 minutes two babies are born. One weighs 2.34 kg and the other weighs 5.5 kg.

b Calculate which event is more likely to happen.

One month 300 babies are weighed.

c Calculate the number of babies expected to weigh more than 4.5 kg.

It was found that 10% of the babies weighed less than w kg.

d Find the value of w.

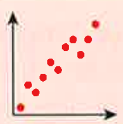
5.2 Correlation

When two sets of data appear to be connected, that is, one set of data is dependent on the other, then there are various methods that can be used to check whether or not there is any **correlation**. One of these methods is the scatter diagram.

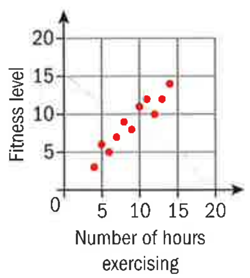
Data can be plotted on a scatter diagram with the **independent variable** on the horizontal axis and the **dependent variable** on the vertical axis. The pattern of dots will give a visual picture of how closely, if at all, the variables are related.

Types of correlation

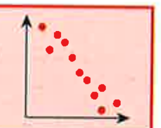
→ In a **positive** correlation the dependent variable increases as the independent variable increases.



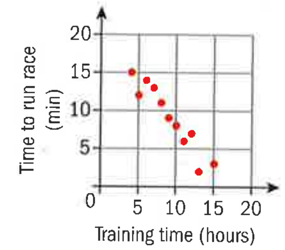
For example, fitness levels (dependent variable) increase as the number of hours spent exercising (independent variable) increase:



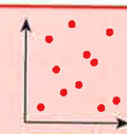
→ In a **negative** correlation the dependent variable decreases as the independent variable increases.



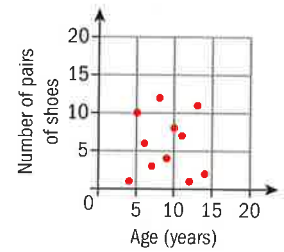
For example, the time taken to run a race (dependent variable) decreases as the training time (independent variable) increases:



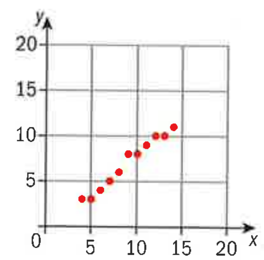
→ When the points are scattered randomly across the diagram there is **no** correlation.



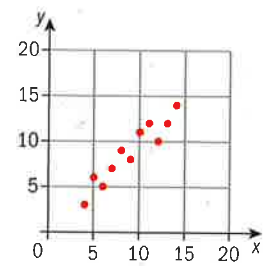
For example, the number of pairs of shoes that a person owns is not related to their age:



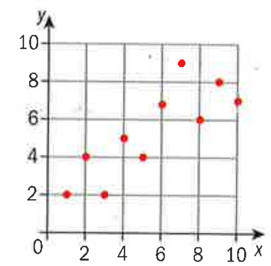
→ Correlations can also be described as strong, moderate or weak.



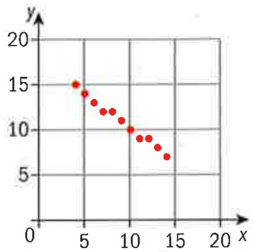
This is an example of a **strong positive** correlation.



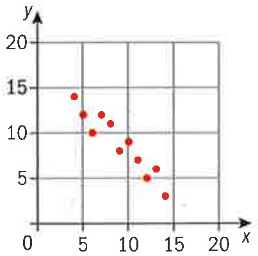
This is an example of a **moderate positive** correlation.



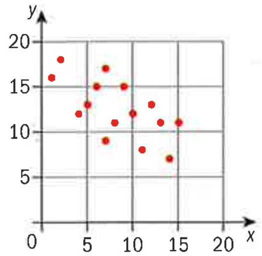
This is an example of a **weak positive** correlation.



This is an example of a **strong negative** correlation.

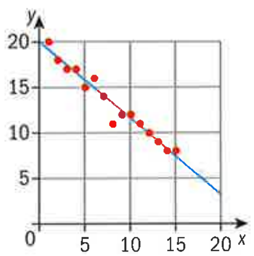


This is an example of a **moderate negative** correlation.

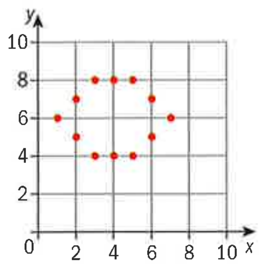


This is an example of a **weak negative** correlation.

Correlations can also be classed as linear or non-linear.



This is an example of a **linear** correlation.



This is an example of a **non-linear** correlation.

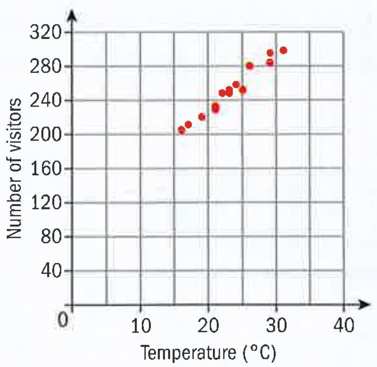
For Mathematical Studies, you will only need to learn about **linear** correlations. However you may use other types of correlation in your project.

Example 6

The manager of a recreation park thought that the number of visitors to the park was dependent on the temperature.
He kept a record of the temperature and the numbers of visitors over a two-week period.
Plot these points on a scatter diagram and comment on the type of correlation.

Temperature (°C)	16	22	31	19	23	26	21	17	24	29	21	25	23	29
Number of visitors	205	248	298	223	252	280	233	211	258	295	229	252	248	284

Answer



Draw the x-axis 'Temperature (°C)' from 0 to 40 and the y-axis 'Number of visitors' from 0 to 320.

Plot the points.

There is a strong positive correlation between temperature and the number of visitors to the park.

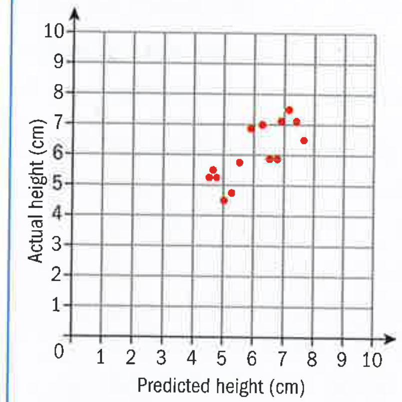
Describe the correlation.

Example 7

A Mathematical Studies student wanted to check if there was a correlation between the predicted heights of daisies and their actual heights.
Draw a scatter diagram to illustrate the data and comment on the correlation.

Predicted height (cm)	5.3	6.2	4.9	5.0	4.8	6.6	7.3	7.5	6.8	5.5	4.7	6.8	5.9	7.1
Actual height (cm)	4.7	7.0	5.3	4.5	5.6	5.9	7.2	6.5	7.2	5.8	5.3	5.9	6.8	7.6

Answer

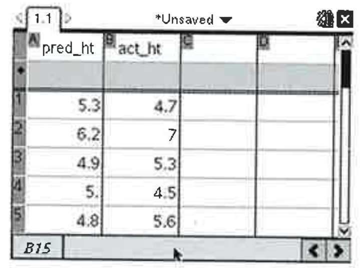


Draw x- and y-axes from 0 to 10.
Plot 'Predicted height' on the horizontal axis and 'Actual height' on the vertical axis.

There is a moderate positive correlation between predicted height and actual height.

Describe the correlation.

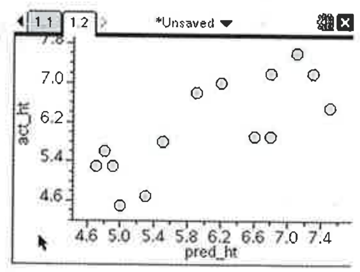
You can also use a GDC to plot a scatter diagram. For Example 7:



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

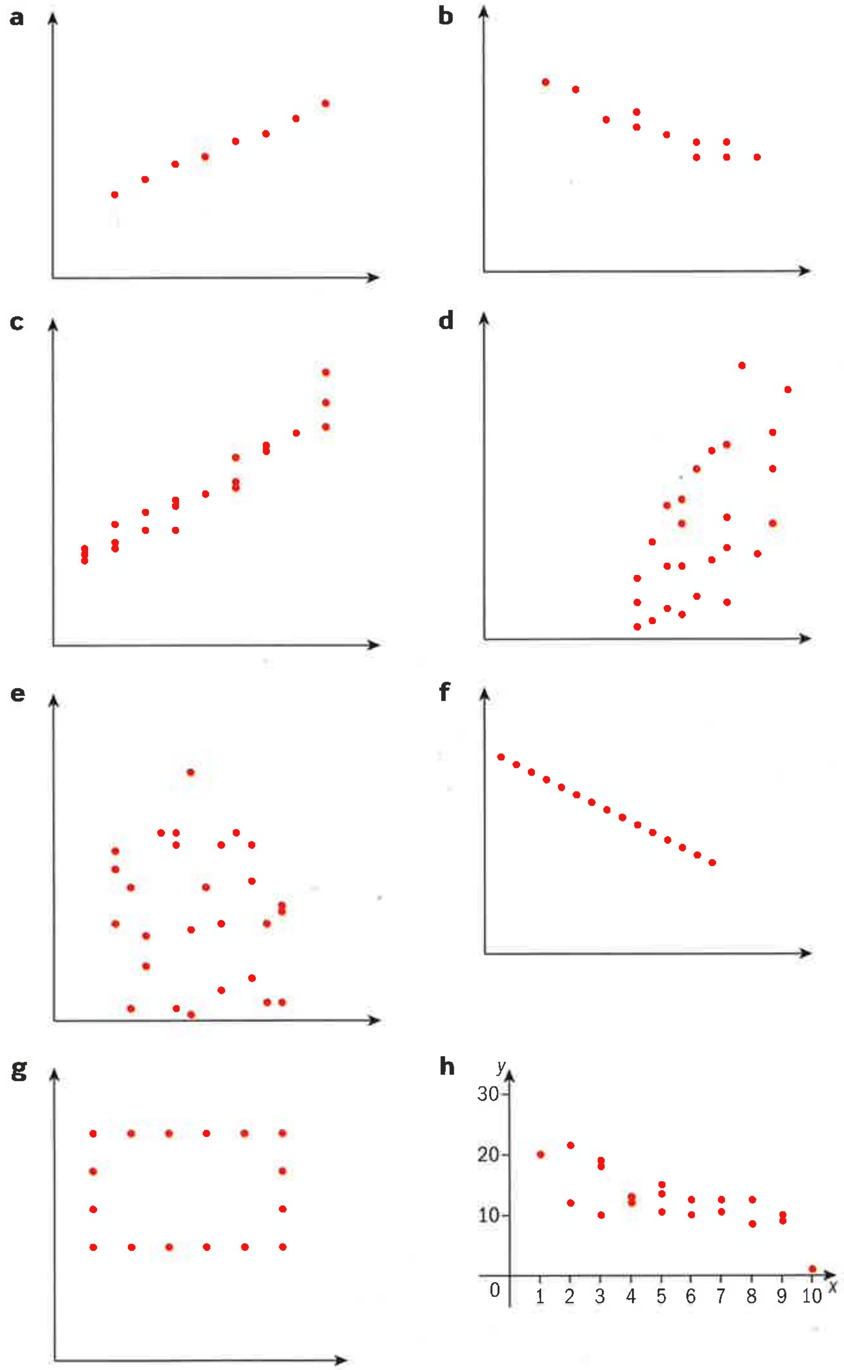
First enter the data in two lists on a List & Spreadsheet page.

Then enter the variables onto the axes on a Data and Statistics page to draw the scatter diagram.



Exercise 5D

1 For each diagram, state the type of correlation (positive/negative and linear/non-linear) and the strength of the relationship (perfect/strong/moderate/weak/none).



A **perfect correlation** is one where all the plotted points lie on a straight line.

2 For each set of data, plot the points on a scatter diagram and describe the type of correlation.

a

x	28	30	25	35	19	38	25	33	41	22	35	44
y	24	36	30	40	15	34	28	34	44	23	37	45

b

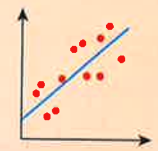
x	3	7	7	11	16	15	17	17	18	20
y	16	11	12	9	6	7	3	9	5	6

Line of best fit

A **line of best fit** is a line that is drawn on a graph of two sets of data, so that approximately as many points lie above the line as below it.

→ To draw the **line of best** fit by eye:

- Find the mean of each set of data and plot this point on your scatter diagram.
- Draw a line that passes through the mean point and is close to all the other points – with approximately an equal number of points above and below the line.



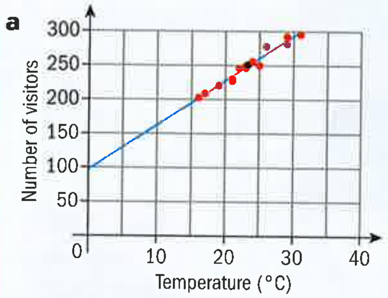
The line of best fit does not need to go through the origin and, in fact, in most cases it will not go through the origin.

Example 8

a For Example 6 draw the line of best fit on the diagram.
b For Example 7 draw the line of best fit on the diagram.

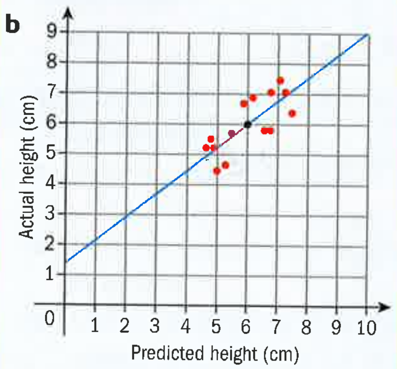
Answers

a



Calculate the means using your GDC. The mean temperature is 23.3, and the mean number of visitors is 251. Plot the mean point (23.3, 251) on the scatter diagram. Draw a line of best fit through the mean point so that there are roughly an equal number of points above and below the line.

b



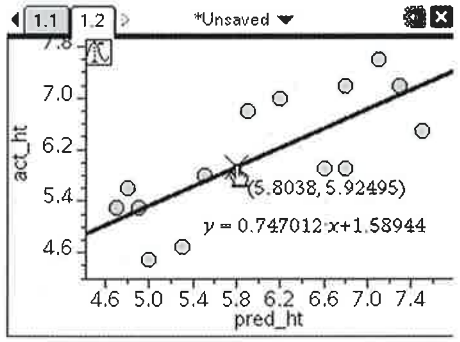
The mean predicted height is 6.03, and the mean actual height is 6.09. Plot the mean point (6.03, 6.09) on the scatter diagram and draw a straight line through it so that there are roughly an equal number of points above and below the line.

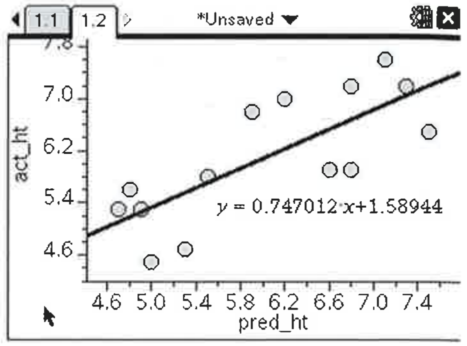
Geosciences use a line of best fit in

- flood frequency curves
- earthquake forecasting
- meteorite impact prediction
- climate change.

You can also use a GDC to draw a line of best fit.
For Example 7:
Select MENU 4:Analyze | 6:Regression | 2:Show Linear $(ax + b)$.

Given a value of predicted height, use trace (MENU 4:Analyze | A:Graph Trace) to find the value of the actual height from the graph.





In the Data & Statistics mode it is not possible to find *exact* values when using trace

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

There is often a lot of confusion between the concepts of *causation* and *correlation*. However, they should be easy enough to distinguish.
One action can *cause* another (such as smoking can cause lung cancer), or it can *correlate* with another (for example, blue eyes are correlated with blonde hair).

If one action *causes* another, then they are also *correlated*. But if two things are *correlated* it does *not* mean that one *causes* the other. For example, there could be a strong *correlation* between the predicted grades that teachers give and the actual grades that the students achieve. However, the achieved grades are not *caused* by the predicted grades.

Can you think of other examples?
Can you find articles in newspapers, magazines or online where *cause* is used incorrectly?

Exercise 5E

1 For each set of data:

- **i** Plot the points on a scatter diagram and describe the type of correlation.
- **ii** Find the mean of x and the mean of y.
- **iii** Plot the mean point on your diagram and draw a line of best fit by eye.

a

x	2	4	6	8	10	12	14	16	18	20	22	24
y	14	15	18	21	24	25	27	29	30	32	35	39

b

x	12	13	14	15	16	17	18	19	20	21
y	32	29	30	25	22	22	15	10	10	7

EXAM-STYLE QUESTIONS

2 The following table gives the heights and weights of 12 giraffes.

Height (xm)	4.8	4.1	4.2	4.7	5.0	5.0	4.8	5.2	5.3	4.3	5.5	4.5
Weight (ykg)	900	600	650	750	1100	950	850	1150	1100	650	1250	800

- **a** Plot the points on a scatter diagram and describe the correlation.
- **b** Find the mean height and the mean weight.
- **c** Plot the mean point on your diagram and draw a line of best fit by eye.
- **d** Use your diagram to estimate the weight of a giraffe of height 4.6 m.

3 Fourteen students took a test in Chemistry and ITGS (Information Technology in a Global Society).
The results are shown in the following table.

Chemistry (%)	45	67	72	34	88	91	56	39	77	59	66	82	96	42
ITGS (%)	42	76	59	44	76	88	55	45	69	62	58	94	85	58

- **a** Plot the points on a scatter diagram and describe the correlation.
- **b** Find the mean score for each test.
- **c** Plot the mean point on your diagram and draw a line of best fit by eye.
- **d** Use your diagram to estimate the result for an ITGS test when the chemistry score was 50%.

4 Twelve mothers were asked how many hours per day, on average, they held their babies and how many hours per day, on average, the baby cried. The results are given in the following table.

Baby held (hours)	1	2	3	3	4	4	5	6	6	7	8	9
Baby cried (hours)	6	6	5	5.5	4	3	3.5	2	2.5	2	1.5	1

- **a** Plot the points on a scatter diagram and describe the correlation.
- **b** Find the mean number of hours held and the mean number of hours spent crying.
- **c** Plot the mean point on your diagram and draw a line of best fit by eye.
- **d** Use your diagram to estimate the number of hours a baby cries if it is held for 3.5 hours.

EXAM-STYLE QUESTION

5 The table shows the size of a television screen and the cost of the television.

Size (inches)	32	37	40	46	50	55	59
Cost ($)	450	550	700	1000	1200	1800	2000

a Plot the points on a scatter diagram and describe the correlation.

b Find the mean screen size and the mean cost.

c Plot the mean point on your diagram and draw a line of best fit by eye.

d Use your diagram to estimate the cost of a 52-inch TV.

Pearson's product-moment correlation coefficient

Karl Pearson (1857–1936) was an English lawyer, mathematician and statistician.

His contributions to the field of statistics include the product-moment correlation coefficient and the chi-squared test.

Pearson's career was spent largely on applying statistics to the field of biology.

He founded the world's first University statistics department at University College London in 1911.

▶ Karl Pearson

It is useful to know the **strength** of the relationship between any two sets of data that are thought to be related.

Pearson's product-moment correlation coefficient, r, is one way of finding a numerical value that can be used to determine the strength of a linear correlation between the two sets of data.

In examinations you will only be expected to use your GDC to find the value of r.

→ **Pearson's product-moment correlation coefficient**, r, can take all values between −1 and +1 inclusive.

- When $r = -1$, there is a **perfect negative** correlation between the data sets.
- When $r = 0$, there is **no** correlation.
- When $r = +1$, there is a **perfect positive** correlation between the data sets.
- A **perfect correlation** is one where **all** the plotted points lie on a straight line.

When r is between

- 0 and 0.25, the correlation is very weak
- 0.25 and 0.5, the correlation is weak
- 0.5 and 0.75, there is a moderate correlation
- 0.75 and 1, the correlation is strong.

The formula for Pearson's product–moment correlation coefficient for two sets of data, x and y, is: $r = \frac{s_{xy}}{s_x s_y}$

where s_{xy} is the covariance (beyond the scope of this course) and s_x and s_y are the standard deviations of x and y respectively.

You will be expected to use this formula to enhance your project.

Other formulae that you will need are:

$$s_{xy} = \sum \frac{(x-\bar{x})(y-\bar{y})}{n} \text{ or } \frac{\sum xy}{n} - \frac{\sum x}{n}\frac{\sum y}{n}$$

$$s_x = \sqrt{\frac{\sum (x-\bar{x})^2}{n}} \text{ or } \sqrt{\left(\frac{\sum x^2}{n} - \bar{x}^2\right)} \qquad s_y = \sqrt{\frac{\sum (y-\bar{y})^2}{n}} \text{ or } \sqrt{\left(\frac{\sum y^2}{n} - \bar{y}^2\right)}$$

Example 9

The data given below for a first-division football league show the position of the team and the number of goals scored.

Find the correlation coefficient, r, and comment on this value.

Position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Goals	75	68	60	49	59	50	55	46	57	49	48	39	44	56	54	37	42	37	40	27

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

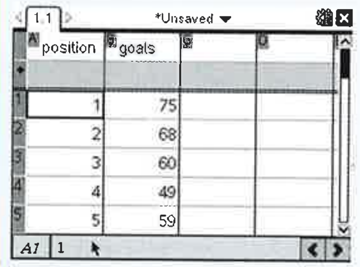
Answer

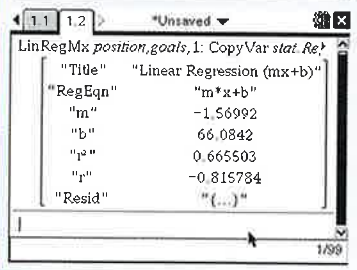
$r = -0.816$ (to 3 sf)

So, there is a **strong negative** correlation between the position of the team and the number of goals scored.

Using a GDC:

First enter 'Position' numbers and 'Goals' into two lists (X and Y respectively).





Your GDC also gives r^2, **the coefficient of determination**. This is an indication of how much of the variation in one set of data, y, can be explained by the variation in the other set of data, x. For example, if $r^2 = 0.821$, this means that 82.1% of the variation in set y is caused by the variation in set x. Here, either $r = 0.906$ which is a strong positive linear correlation, or $r = -0.906$ which is a strong negative linear correlation.

Example 10

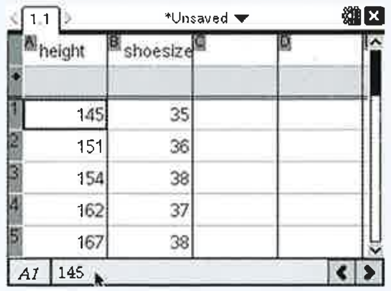
The heights and shoe sizes of the students at Learnwell Academy are given in the table below. Find the correlation coefficient, r, and comment on your result.

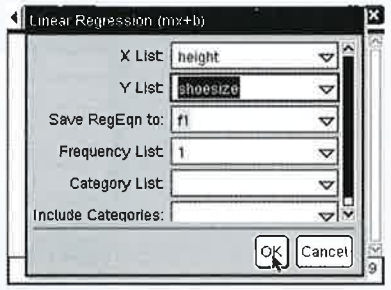
Height (x cm)	145	151	154	162	167	173	178	181	183	189	193	198
Shoe size	35	36	38	37	38	39	41	43	42	45	44	46

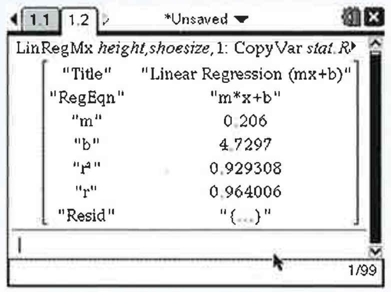
Answer

$r = 0.964$ (to 3 sf)

This means that there is a **strong positive** correlation between height and shoe size.







GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 5F

1 The table gives the temperature (°C) at midday and the number of ice creams sold over a period of 21 days.

Temperature (°C)	22	23	22	19	20	25	23	20	17	18	23	24	22	26	19	19	20	22	23	22	20
Number of ice creams sold	59	61	55	40	51	72	55	45	39	35	59	72	63	77	37	41	44	50	59	48	38

Find the correlation coefficient, r, and comment on this value.

2 A chicken farmer selected a sample of 12 hens. During a two-week period, he recorded the number of eggs each hen produced and the amount of feed each hen ate. The results are given in the table.

a Find the correlation coefficient, r.

b Comment on the value of the correlation coefficient.

Number of eggs	Units of feed eaten
11	6.2
10	4.9
13	7.1
10	6.2
11	5.0
15	7.9
9	4.8
12	6.9
11	5.3
12	5.9
13	6.5
9	4.5

3 The table gives the average temperature for each week in December, January and February and the corresponding number of hours that an average family used their central heating.

Average temperature (°C)	4	1	3	−2	−9	−12	−8	−9	−2	1	3	5
Hours of heating	43	45	51	52	58	64	57	60	55	43	40	30

Find the correlation coefficient, r, and comment on this value.

4 Eight students complete examination papers in Economics and Biology. The results are shown in the table.

Student	A	B	C	D	E	F	G	H
Economics	64	55	43	84	67	49	92	31
Biology	53	42	44	79	75	52	84	29

Find the correlation coefficient, r, and comment on your result.

5 The table shows the age of a baby, measured in days, and the weight, in kilograms, at 08:00 on the corresponding day.

Age (days)	0	7	14	21	28	35	42
Weight (kg)	3.50	3.75	3.89	4.15	4.42	4.55	5.02

Find the correlation coefficient, r, and comment on your result.

6 The heights and weights of 10 students selected at random are shown in the table.

Height (x cm)	155	161	173	150	182	165	170	185	175	145
Weight (y kg)	50	75	80	46	81	79	64	92	74	108

Find the correlation coefficient, r, and comment on your answer.

7 The table shows the mock examination results and the actual results of 15 students at Top High College.

Mock	32	35	28	24	19	39	44	41	23	29	28	35	38	43	21
Actual	33	34	30	25	18	36	43	42	24	27	29	36	39	44	22

Find the correlation coefficient, r, and comment on your result.

8 The ages of 14 people and the times it took them to run 1 km are shown in the table.

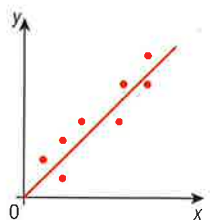
Age (years)	9	12	13	15	16	19	21	29	32	43	48	55	61	66
Time (minutes)	7.5	6.8	7.2	5.3	5.1	4.9	5.2	4.6	4.9	6.8	6.2	7.5	8.9	9.2

Find the correlation coefficient, r, and comment on your result.

5.3 The regression line

→ The **regression line for y on x** is a more accurate version of a line of best fit, compared to best fit by eye.

The regression line for y on x, where y is the dependent variable, is also known as the least squares regression line. It is the line drawn through a set of points such that the sum of the squares of the distance of each point from the line is a minimum.



→ If there is a strong or moderate correlation, you can use the regression line for y on x to predict values of y for values of x within the range of the data.

You should only calculate the equation of the regression line if there is a moderate or strong correlation coefficient.

In your project you can work out the equation of the regression line for y on x using the formula:

$$(y-\bar{y})=\frac{s_{xy}}{(s_x)^2}(x-\bar{x})$$

where $\bar{x}$ and $\bar{y}$ are the means of the x and y data values respectively, s_x is the standard deviation for the x data values, and s_{xy} is the covariance.

In examinations you will only be expected to use your GDC to find the equation of the regression line.

Example 11

Ten students train for a charity walk.
The table shows the average number of hours per week that each member trains and the time taken to complete the walk.

Training time (hours)	9	8	12	3	25	6	10	5	6	21
Time to complete walk (minutes)	15.9	14.8	15.3	18.4	13.8	16.2	14.1	16.1	16	14.2

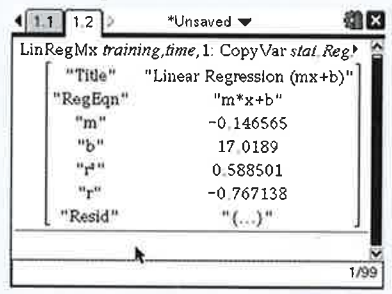
a Find the correlation coefficient, r.
b Find the equation of the regression line.
c Using your equation, estimate how many minutes it will take a student who trains 18 hours per week to complete the walk.

British scientist and mathematician Francis Galton (1822–1911) coined the term 'regression'.

Answers

a $r = -0.767$ (to 3 sf)

First enter the data into two lists and then compute the results.



GDC help on CD: Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.

b The equation of the regression line is:
$y = -0.147x + 17.0$

The general form of the equation is:
$y = mx + c$
From the GDC:
$m = -0.147$ (to 3 sf)
$c = 17.0$ (to 3 sf)

c $y = -0.147(18) + 17.0 = 14.4$ (to 3 sf)
Therefore, the time taken is approximately 14.4 minutes.

*Substitute 18 (hours) for x in the equation from part **b**.*

In this book we use $y = mx + c$ as the general form of a linear equation. The GDC uses $y = mx + b$ as the general form. Some people use $y = ax + b$.

Example 12

The table shows the number of mice for sale in a pet shop at the end of certain weeks.

Time (x weeks)	3	5	6	9	11	13
Number of mice (y)	41	57	61	73	80	91

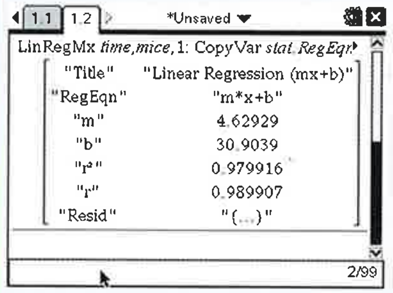
a Find the correlation coefficient, r.
b Find the equation of the regression line for y on x.
c Use your regression line to predict the number of mice for sale after 10 weeks.
d Can you accurately predict the number of mice after 20 weeks?

How do we know what we know? How sure can we be of our predictions? What predictions are made about population, or the climate?

Answers

a $r = 0.990$ (to 3 sf)

First enter the data into two lists.



b The equation of the regression line is:
$y = 4.63x + 30.9$

The general form of the equation is:
$y = mx + c$
From the GDC:
$m = 4.63$ *(to 3 sf)*
$c = 30.9$ *(to 3 sf)*

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

c $y = 4.63(10) + 30.9 = 77.2$
$= 77$
After 10 weeks, the number of mice is 77.

Substitute 10 (weeks) for x in the equation from part **b**.

d No, because it is too far away from the data in the table.

Remember that you cannot use the regression line to predict values beyond the region of the given data.

Exercise 5G

EXAM-STYLE QUESTION

1 The table shows the distance travelled by train between various places in India and the cost of the journey.

Distance (km)	204	1407	1461	793	1542	343	663	780
Cost (rupees)	390	2200	2270	1390	2280	490	1200	1272

a Find the correlation coefficient, r, and comment on your result.
b Find the equation of the regression line.
c Use your equation to estimate the cost of a 1000 km train journey.

EXAM-STYLE QUESTIONS

2 Different weights were attached to a vertical spring and the length of the spring measured. The results are shown in the table.

Load (x kg)	0	2	3	5	6	7	9	11
Length (y cm)	15	16.5	17.5	18.5	18.8	19.2	20	20.4

a Find the correlation coefficient, r.
b Find the equation of the regression line.
c Use your equation to estimate the length of the spring when a weight of 8 kg is added.

3 Lijn is a keen swimmer. For his Mathematical Studies Project he wants to investigate whether or not there is a correlation between the length of the arm of a swimmer and the time it takes them to swim 200 m.
He selects 15 members of a swimming club to swim 200 m.
Their times (y seconds) and arm lengths (x cm) are shown in the table below.

Length of arm (x cm)	78	72	74	67	79	58	62	67	71	69	75	65	73	59	60
Time (y seconds)	130	135	132	143	133	148	140	139	135	145	129	140	130	145	142

a Calculate the mean and standard deviation of x and y.
b Calculate the correlation coefficient, r.
c Comment on your value for r.
d Calculate the equation of the regression line for y on x.
e Using your equation, estimate how many seconds it will take a swimmer with an arm length of 70 cm to swim 200 m.

4 Saif asked his classmates how many minutes it took them to travel to school and their stress level, out of 10, for this journey. The results are shown in the table.

Travel time (x minutes)	14	28	19	22	24	8	16	5	18	20	25	10
Stress level (y)	3	7	5	6	6	2	3	2	4	5	6	6

a Find the correlation coefficient, r.
b Find the equation of the regression line.
c Use your equation to estimate the stress level of a student who takes 15 minutes to travel to school.

EXAM-STYLE QUESTION

5 The table shows the weight (g) and the cost (Australian dollars) of various candy bars.

Weight (x g)	62	84	79	65	96	58	99	48	73	66
Cost (y AUD)	1.45	1.83	1.78	1.65	1.87	1.42	1.82	1.15	1.64	1.55

a Calculate the equation of the regression line for y on x.

b Use your equation to estimate the cost of a candy bar weighing 70 g.

6 Ten students in Mr Craven's PE class did pushups and situps. Their results are shown in the following table.

Number of pushups (x)	23	19	31	53	34	46	45	22	39	27
Number of situps (y)	31	26	35	51	36	48	45	28	41	30

a Find the equation of the regression line.

A student can do 50 pushups.

b Use your equation to estimate the number of situps the student can do.

7 Fifteen students were asked for their average grade at the end of their last year of high school and their average grade at the end of their first year at university. The results are shown in the table below.

High school grade (x)	44	49	53	47	52	58	67	73	75	79	82	86	88	91	97
University grade (y)	33	52	55	48	51	60	71	72	69	83	84	89	96	92	89

a Find the equation of the regression line.

A student scores 60 at the end of their last year of high school.

b Use your equation to estimate the average university grade for the student.

8 A secretarial agency has a new computer software package. The agency records the number of hours it takes people of different ages to master the package. The results are shown in the table.

Age (x)	32	40	21	45	24	19	17	21	27	54	33	37	23	45
Time (y hours)	10	12	8	15	7	8	6	9	11	16	12	13	9	17

a Find the equation of the regression line.

b Using your equation, estimate the time it would take a 40-year-old person to master the package.

5.4 The chi-squared test

You may be interested in finding out whether or not certain sets of data are independent. Suppose you collect data on the favorite color of T-shirt for men and women. You may want to find out whether color and gender are independent or not. One way to do this is to perform a **chi-squared test (χ^2)** for independence.

To perform a chi-squared test (χ^2) there are four main steps.

Step 1: Write the **null (H_0)** and **alternative (H_1) hypotheses**.

H_0 states that the data sets are independent.
H_1 states that the data sets are not independent.

For example, the hypotheses for color of T-shirt and gender could be:
H_0: Color of T-shirt is independent of gender.
H_1: Color of T-shirt is not independent of gender.

Step 2: Calculate the chi-squared test statistic.

Firstly, you may need to put the data into a **contingency table**, which shows the frequencies of two variables. The elements in the table are the **observed** data. The elements should be frequencies (not percentages).

For the example above, the contingency table could be:

	Black	**White**	**Red**	**Blue**	**Totals**
Male	48	12	33	57	150
Female	35	46	42	27	150
Totals	83	58	75	84	300

If you are given the contingency table, you may need to extend it to include an extra row and column for the 'Totals'.

From the observed data, you can calculate the **expected frequencies**. Since you are testing for independence, you can use the formula for the probability of independent events to calculate the expected values. So:

The expected number of men who like black T-shirts is

$$\frac{150}{300} \times \frac{83}{300} \times 300 = 41.5.$$

The expected number of men who like white T-shirts is

$$\frac{150}{300} \times \frac{58}{300} \times 300 = 29 \text{ and so on.}$$

The expected table of values would then look like this:

	Black	**White**	**Red**	**Blue**	**Totals**
Male	41.5	29	37.5	42	150
Female	41.5	29	37.5	42	150
Totals	83	58	75	84	300

When two variables are independent, one does not affect the other. Here, you are finding out whether a person's gender influences their colour choice. You will learn more about mathematical independence in Chapter 8.

The main entries in this table form a 2×4 **matrix** (array of numbers) - do not include the row and column for the totals.

In examinations, the largest contingency table will be a 4×4.

Note:
- The expected values can **never** be less than 1.
- The expected values must be 5 or higher.
- If there are entries between 1 and 5, you can combine table rows or columns.

For calculations by hand, you need the expected frequencies to find the χ^2 value.

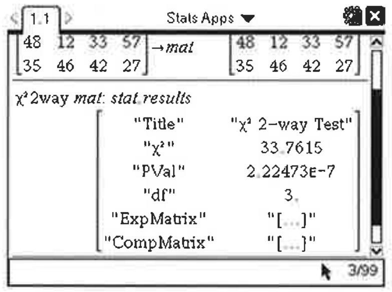
→ To calculate the χ^2 value use the formula $\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e}$, where f_o are the observed frequencies and f_e are the expected frequencies.

For our example,

$$\chi^2_{calc} = \frac{(48-41.5)^2}{41.5} + \frac{(12-29)^2}{29} + \frac{(33-37.5)^2}{37.5} + \frac{(57-42)^2}{42} + \frac{(35-41.5)^2}{41.5} + \frac{(46-29)^2}{29} + \frac{(42-37.5)^2}{37.5} + \frac{(27-42)^2}{42}$$

$= 33.8$

Using your GDC to find the χ^2 value, enter the contingency table as a matrix (array) and then use the matrix with the χ^2 2-way test.



From the screenshot, you can see that $\chi^2_{calc} = 33.8$ (to 3 sf). This confirms our earlier hand calculation.

Step 3: Calculate the critical value.

First note the **level of significance**. This is given in examination questions but you have to decide which level to use in your project. The most common levels are 1%, 5% and 10%.

Now you need to calculate the number of **degrees of freedom**.

→ To find the degrees of freedom for the chi-squared test for independence, use this formula based on the contingency table:

Degrees of freedom = (number of rows − 1) (number of columns − 1)

So, in our ongoing example, the number of degrees of freedom is $(2 - 1) \times (4 - 1) = 3$

In examinations, you will only be expected to use your GDC to find the χ^2 value.

Your GDC calculates the expected values for you but you must know how to find them by hand in case you are asked to show one or two calculations in an exam question. To see the matrix for the expected values, type 'stat.' and then select 'expmatrix' from the menu that pops up.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

If the number of degrees of freedom is 1, you will be expected to use **Yates' continuity correction** to work out the chi-squared value. (In examinations the degrees of freedom will always be greater than 1.)

The level of significance and degrees of freedom can be used to find the critical value. However, in examinations, the **critical value** will always be given.

For our example, at the 1% level, the critical value is 11.345. At the 5% level, the critical value is 7.815. At the 10% level, the critical value is 6.251.

Step 4: Compare χ^2_{calc} against the critical value.

→ If χ^2_{calc} is **less than** the critical value then **do not reject** the null hypothesis.
If χ^2_{calc} is **more than** the critical value then **reject** the null hypothesis.

In our example, at the 5% level, $33.8 > 7.815$. Therefore, we reject the null hypothesis that T-shirt color is independent of gender.

Using a GDC, you can compare the p-value against the significance level.

→ If the p-value is **less** than the significance level then **reject** the null hypothesis.
If the p-value is **more** than the significance level then **do not reject** the null hypothesis.

Use the significance level as a decimal, so 1% = 0.01, 5% = 0.05 and 10% = 0.1.

So, for our example, p-value = 0.000 000 2 (see the GDC screenshot on page 234).

$0.0000002 < 0.05$, so we reject the null hypothesis.

→ **To perform a χ^2 test:**

1. Write the null (H_0) and alternative (H_1) hypotheses.
2. Calculate χ^2_{calc}:
   - **a** using your GDC (examinations)
   - **b** using the χ^2_{calc} formula (project work)
3. Determine:
   - **a** the p-value by using your GDC
   - **b** the critical value (given in examinations)
4. Compare:
   - **a** the p-value against the significance level
   - **b** χ^2_{calc} against the critical value

The p-value is the probability value. It is the probability of evidence against the null hypothesis.

Investigation – shoe size and gender

Use the information that you collected at the beginning of this chapter to test if shoe size is independent of gender.

Example 13

One hundred people were interviewed outside a chocolate shop to find out which flavor of chocolate cream they preferred. The results are given in the table, classified by gender.

	Strawberry	Coffee	Orange	Vanilla	Totals
Male	23	18	8	8	57
Female	15	6	12	10	43
Totals	38	24	20	18	100

Perform a χ^2 test, at the 5% significance level, to determine whether the flavor of chocolate cream is independent of gender.

a State the null hypothesis and the alternative hypothesis.

b Show that the expected frequency for female and strawberry flavor is approximately 16.3.

c Write down the number of degrees of freedom.

d Write down the χ^2_{calc} value for this data.

The critical value is 7.815.

e Using the critical value or the p-value, comment on your result.

Answers

a H_0: Flavor of chocolate cream is independent of gender.
H_1: Flavor of chocolate cream is not independent of gender.

Write H_0 using 'independent of'.
Write H_1 using 'not independent of'.

b $\frac{43}{100} \times \frac{38}{100} \times 100 = 16.34$

So, the expected frequency for female and strawberry flavor is approximately 16.3.

From the contingency table:
Total for 'female' row = 43
Total for 'strawberry' column = 38
Total surveyed = 100

c Degrees of freedom = $(2 - 1)(4 - 1) = 3$

Degrees of freedom = (number of rows − 1) (number of columns − 1)
Here, there are 2 rows and 4 columns in the observed matrix of the contingency table.

d $\chi^2_{calc} = 6.88$

Using your GDC:
Enter the contingency table as a matrix. Use the matrix with χ^2 2-way test. Read off χ^2 value.
The p-value = 0.0758.

e $6.88 < 7.815$; therefore, we do not reject the null hypothesis. There is enough evidence to conclude that flavor of chocolate cream is independent of gender.

Using the given critical value, check:
χ^2_{calc} < critical value → do not reject, or
χ^2_{calc} > critical value → reject.
Or, using the p-value, check:
p-value < significance level → reject, or
p-value > significance level → do not reject.
Significance level = 5% = 0.05. So, 0.0758 > 0.05 and we do not reject the null hypothesis.

Example 14

Members of a club are required to register for one of three games: billiards, snooker or darts.
The number of club members of each gender choosing each game in a particular year is shown in the table.

	Billiards	Snooker	Darts
Male	39	16	8
Female	21	14	17

Perform a χ^2 test, at the 10% significance level, to determine if the chosen game is independent of gender.

a State the null hypothesis and the alternative hypothesis.

b Show that the expected frequency for female and billiards is approximately 27.1.

c Write down the number of degrees of freedom.

d Write down the χ^2_{calc} value for this data.

The critical value is 4.605.

e Using the critical value or the p-value, comment on your result.

Answers

a H_0: The choice of game is independent of gender.
H_1: The choice of game is not independent of gender.

b $\left(\frac{52}{115}\right)\left(\frac{60}{115}\right)(115) = 27.130$

≈ 27.1

So, the expected frequency for female and billiards is approximately 27.1.

Expected value table from the GDC:

	Billiards	Snooker	Darts
Male	32.9	16.4	13.7
Female	27.1	13.6	11.3

c Degrees of freedom = $(2 - 1)(3 - 1) = 2$

d $\chi^2_{calc} = 7.79$

e $7.79 > 4.605$; therefore, we reject the null hypothesis. There is enough evidence against H_0 to conclude that the choice of game is not independent of gender.

The p-value = 0.0203
Or, using the p-value,
0.0203 < 0.10. Therefore, we reject the null hypothesis.

Exercise 5H

EXAM-STYLE QUESTIONS

1 300 people were interviewed and asked which genre of books they mostly read. The results are given below in a table of observed frequencies, classified by age.

		Genre			
		Fiction	**Non-fiction**	**Science fiction**	**Totals**
Age	**0–25 years**	23	16	41	80
	26–50 years	54	38	38	130
	51+ years	29	43	18	90
	Totals	106	97	97	300

Perform a χ^2 test, at the 5% significance level, to determine whether genre of book is independent of age.

a State the null hypothesis and the alternative hypothesis.

b Show that the expected frequency for science fiction and the 26–50 age group is 42.

c Write down the number of degrees of freedom.

d Write down the χ^2_{calc} value for this data.

The critical value is 9.488.

e Using the critical value or the *p*-value, comment on your result.

2 Tyne was interested in finding out whether natural hair color was related to eye color. He surveyed all the students at his school. His observed data is given in the table below.

		Hair color			
		Black	**Brown**	**Blonde**	**Totals**
Eye color	**Brown/Black**	35	43	12	90
	Blue	8	27	48	83
	Green	9	20	25	54
	Totals	52	90	85	227

Perform a chi-squared test, at the 10% significance level, to determine if hair color and eye color are independent.

a State the null hypothesis and the alternative hypothesis.

b Find the expected frequency of a person having blonde hair and brown eyes.

c Write down the number of degrees of freedom.

d Write down the chi-squared value for this data.

The critical value is 7.779.

e Using the critical value or the *p*-value, comment on your result.

EXAM-STYLE QUESTIONS

3 Three different flavors of dog food were tested on different breeds of dog to find out if there was any connection between favorite flavor and breed. The results are given in the table.

	Beef	**Chicken**	**Fish**	**Totals**
Poodle	13	11	8	32
Boxer	15	10	10	35
Terrier	16	12	9	37
Great Dane	17	11	8	36
Totals	61	44	35	140

A χ^2 test, at the 5% significance level, is performed to investigate the results.

a State the null hypothesis and the alternative hypothesis.

b Show that the expected frequency of a Boxer's favorite food being chicken is 11.

c Show that the number of degrees of freedom is 6.

d Write down the χ^2_{calc} value for this data.

The critical value is 12.59.

e Using the critical value or the *p*-value, comment on your result.

4 Eighty people were asked to identify their favorite film genre. The results are given in the table below, classified by gender.

	Adventure	**Crime**	**Romantic**	**Sci-fi**	**Totals**
Male	15	12	2	12	41
Female	7	9	18	5	39
Totals	22	21	20	17	80

A χ^2 test, at the 1% significance level, is performed to decide whether film genre is independent of gender.

a State the null hypothesis and the alternative hypothesis.

b Show that the expected frequency of a female's favorite film genre being crime is 10.2.

c Write down the number of degrees of freedom.

d Write down the chi-squared value for this data.

The critical value is 11.345.

e Using the critical value or the *p*-value, comment on your result.

EXAM-STYLE QUESTIONS

5 Kyu Jin was interested in finding out whether or not the number of hours spent playing computer games per week had an influence on school grades. He collected the following information.

	Low grades	Average grades	High grades	Totals
0–9 hours	6	33	57	96
10–19 hours	11	35	22	68
> 20 hours	23	22	11	56
Totals	40	90	90	220

Perform a chi-squared test, at the 5% significance level, to decide whether the grade is independent of the number of hours spent playing computer games.

a State the null hypothesis and the alternative hypothesis.
b Show that the expected frequency of a high grade and 0–9 hours of playing computer games is 39.3.
c Show that the number of degrees of freedom is 4.
d Write down the χ^2_{calc} value for this data.

The critical value is 9.488.

e Using the critical value or the p-value, comment on your result.

6 The local authority conducted a survey in schools in Rotterdam to determine whether the employment grade in the school was independent of gender. The results of the survey are given in the table.

	Directors	Management	Teachers	Totals
Male	26	148	448	622
Female	6	51	1051	1108
Totals	32	199	1499	1730

Perform a χ^2 test, at the 10% significance level, to determine whether the employment grade is independent of gender.

a State the null hypothesis and the alternative hypothesis.
b Write down the table of expected frequencies.
c Write down the number of degrees of freedom.
d Write down the chi-squared value for this data.

The critical value is 4.605.

e Using the critical value or the p-value, comment on your result.

EXAM-STYLE QUESTIONS

7 Ayako had a part-time job working at a sushi restaurant. She calculated the average amount of sushi sold per week to be 2000. She decided to find out if there was a relationship between the day of the week and the amount of sushi sold. Her observations are given in the table.

	< 1700	1700–2300	> 2300	Totals
Monday–Wednesday	38	55	52	145
Thursday–Friday	39	65	55	159
Saturday–Sunday	43	60	63	166
Totals	120	180	170	470

Perform a χ^2 test, at the 5% significance level, to determine whether the amount of sushi sold is independent of the day of the week.

a State the null hypothesis and the alternative hypothesis.
b Show that the expected frequency of selling over 2300 sushi on Monday–Wednesday is 52.4.
c Write down the number of degrees of freedom.
d Write down the χ^2_{calc} value for this data.

The critical value is 9.488.

e Using the critical value or the p-value, comment on your result.

8 Haruna wanted to investigate the connection between the weight of dogs and the weight of their puppies. Her observed results are given in the table.

		Puppy			
		Heavy	**Medium**	**Light**	**Totals**
Dog	**Heavy**	23	16	11	50
	Medium	10	20	16	46
	Light	8	15	22	45
	Totals	41	51	49	141

Perform a χ^2 test, at the 1% significance level, to determine whether a puppy's weight is independent of its parent's weight.

a State the null hypothesis and the alternative hypothesis.
b Show that the expected frequency of a medium dog having a heavy puppy is 13.4.
c Write down the number of degrees of freedom.
d Write down the χ^2_{calc} value for this data.

The critical value is 13.277.

e Using the critical value or the p-value, comment on your result.

Extension material on CD: *Worksheet 5 - Useful statistical techniques for the project*

Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

1 It is stated that the content of a can of drink is 350 ml. The content of thousands of cans is tested and found to be normally distributed with a mean of 354 ml and a standard deviation of 2.5 ml.

a Sketch a normal distribution diagram to illustrate this information.

b Find the probability that a can contains less than 350 ml.

100 cans are chosen at random.

c Find the expected number of cans that contain less than 350 ml.

2 6000 people were asked how far they lived from their work. The distances were normally distributed with a mean of 4.5 km and a standard deviation of 1.5 km.

a Find the percentage of people who live between 2 km and 4 km from their work.

b Find the expected number of people who live less than 1 km from their work.

3 The weights of bags of tomatoes are normally distributed with a mean of 1.03 kg and a standard deviation of 0.02 kg.

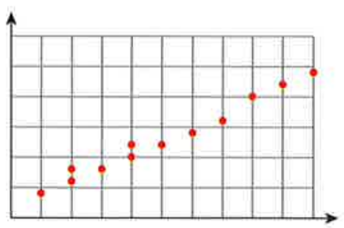
a Find the percentage of bags that weigh more than 1 kg.

It is known that 15% of the bags weigh less than p kg.

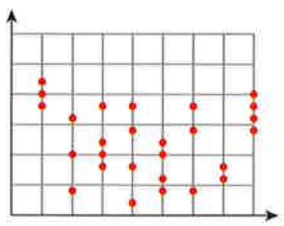
b Find the value of p.

4 For each diagram, state the type of correlation.

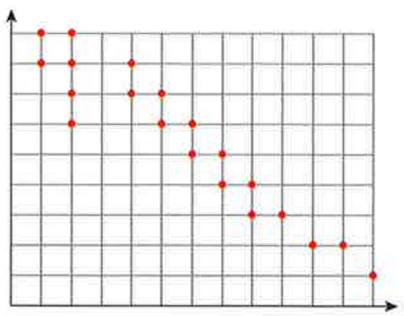
a



b



c



5 Plot these points on a diagram.

x	6	8	10	12	14	16
y	20	21	24	27	28	30

a State the nature of the correlation.

b Find the mean of the x-values and the mean of the y-values.

Plot this mean point on your diagram.

c Draw the line of best fit by eye.

d Find the expected value for y when $x = 9$.

6 The heights and arm lengths of 10 people are shown in the table.

Height (cm)	145	152	155	158	160	166	172	179	183	185
Arm length (cm)	38	42	45	53	50	59	61	64	70	69

a Find the correlation coefficient, r, and comment on your result.

b Write down the equation of the regression line.

c Use your equation to estimate the arm length of a person of height 170 cm.

7 The time taken to eat three doughnuts and the person's age is recorded in the table.

Age (years)	8	12	15	18	21	30	33	35	44	52	63	78
Time (seconds)	23	21	17	14	15	18	20	21	23	25	27	35

a Find the correlation coefficient, r, and comment on your result.

b Write down the equation of the regression line.

c Use your equation to estimate the time taken by a 40-year-old to eat three doughnuts.

8 100 people are asked to identify their favorite flavor of ice cream. The results are given in the contingency table, classified by age (x).

	$x < 25$	$25 \le x < 45$	$x \ge 45$	Totals
Vanilla	14	13	10	37
Strawberry	11	9	8	28
Chocolate	13	10	12	35
Totals	38	32	30	100

Perform a chi-squared test, at the 5% significance level, to determine whether flavor of ice cream is independent of age. State clearly the null and alternative hypotheses, the expected values and the number of degrees of freedom.

9 60 students go ten-pin bowling. They each have one throw with their right hand and one throw with their left. The number of pins knocked down each time is noted. The results are collated in the table.

	0–3	4–7	8–10	Totals
Right hand	8	28	24	60
Left hand	12	30	18	60
Totals	20	58	42	120

A χ^2 test is performed at the 10% significance level.

a State the null hypothesis.

b Write down the number of degrees of freedom.

c Show that the expected number of students who knock down 0–3 pins with their right hand is 10.

The p-value is 0.422.

d Write down the conclusion reached at the 10% significance level.
Give a clear reason for your answer.

10 Erland performs a chi-squared test to see if there is any association between the preparation time for a test (short time, medium time, long time) and the outcome (pass, does not pass). Erland performs this test at the 5% significance level.

a Write down the null hypothesis.

b Write down the number of degrees of freedom.

The p-value for this test is 0.069.

c What conclusion can Erland make?
Justify your answer.

Paper 2 style questions

EXAM-STYLE QUESTIONS

1 The heights of Dutch men are normally distributed with a mean of 181 cm and a standard deviation of 9 cm.

a Sketch a normal distribution diagram to illustrate this information.

b Find the probability that a man chosen at random has a height less than 175 cm.

c Find the probability that a man chosen at random has a height between 172 cm and 192 cm.

Sixty men are measured.

d Find the expected number of men with a height greater than 195 cm.

It is known that 5% of the men have a height less than k cm.

e Find the value of k.

2 The weights of bags of sweets are normally distributed with a mean of 253 g and a standard deviation of 3 g.

a Sketch a diagram to illustrate this information clearly.

b Find the percentage of bags expected to weigh less than 250 g.

Three hunderd bags are weighed.

c Find the expected number of bags weighing more than 255 g.

3 The heights and weights of 10 students selected at random are shown in the table.

Height (x cm)	158	167	178	160	152	160	173	181	185	155
Weight (y kg)	50	75	80	46	61	69	64	86	74	68

a Plot this information on a scatter graph. Use a scale of 1 cm to represent 25 cm on the x-axis and 1 cm to represent 10 kg on the y-axis.

b Calculate the mean height.

c Calculate the mean weight.

d **i** Find the equation of the regression line.
ii Draw the regression line on your graph.

e Use your line to estimate the weight of a student of height 170 cm.

4 An employment agency has a new computer software package. The agency investigates the number of hours it takes people of different ages to reach a satisfactory level using this package. Fifteen people are tested and the results are given in the table.

Age (x)	33	41	22	46	25	18	16	23	26	55	37	34	25	48	17
Time (y hours)	8	10	7	16	8	9	7	10	12	15	11	14	10	16	7

a Find the product-moment correlation coefficient, r, for these data.

b What does the value of the correlation coefficient suggest about the relationship between the two variables?

c Write down the equation of the regression line for y on x in the form $y = mx + c$.

d Use your equation for the regression line to predict the time that it would take a 35-year-old person to reach a satisfactory level. Give your answer correct to the nearest hour.

EXAM-STYLE QUESTIONS

5 Ten students were asked for their average grade at the end of their last year of high school and their average grade at the end of their first year at university. The results were put into a table as follows.

Student	High school grade, x	University grade, y
1	92	3.8
2	76	2.9
3	83	3.4
4	71	1.8
5	93	3.9
6	84	3.2
7	96	3.5
8	77	2.9
9	91	3.7
10	86	3.8

a Find the correlation coefficient, r, giving your answer to one decimal place.

b Describe the correlation between the high school grades and the university grades.

c Find the equation of the regression line for y on x in the form $y = mx + c$.

6 Several bars of chocolate were purchased and the following table shows the weight and the cost of each bar.

	Yum	Choc	Marl	Twil	Chuns	Lyte	BigM	Bit
Weight (x grams)	58	75	70	68	85	52	94	43
Cost (y euros)	1.18	1.45	1.32	1.05	1.70	0.90	1.53	0.95

a Find the correlation coefficient, r, giving your answer correct to two decimal places.

b Describe the correlation between the weight of a chocolate bar and its cost.

c Calculate the equation of the regression line for y on x.

d Use your equation to estimate the cost of a chocolate bar weighing 80 g.

7 The heights and dress sizes of 10 female students selected at random are shown in the table.

Height (x cm)	175	160	180	155	178	159	166	185	189	173
Dress size (y)	12	14	14	8	12	10	14	16	16	14

a Write down the equation of the regression line for dress size (y)
on height (x), giving your answer in the form $y = ax + b$.

b Use your equation to estimate the dress size of a student of height 170 cm.

c Write down the correlation coefficient.

d Describe the correlation between height and dress size.

EXAM-STYLE QUESTIONS

8 Members of a certain club are required to register for one of three games: badminton, table tennis or darts.
The number of club members of each gender choosing each game in a particular year is shown in the table.

	Badminton	Table tennis	Darts
Male	37	16	28
Female	32	10	19

Use a chi-squared test, at the 5% significance level, to test whether choice of game is independent of gender. State clearly the null and alternative hypotheses, the expected values and the number of degrees of freedom.

9 For his Mathematical Studies Project a student gave his classmates a questionnaire to find out which extra-curricular activity was the most popular. The results are given in the table below, classified by gender.

	Reading	Surfing	Skating	
Female	22	16	22	(60)
Male	14	18	8	(40)
	(36)	(34)	(30)	

The table below shows the expected values.

	Reading	Surfing	Skating
Female	p	20.4	18
Male	q	r	12

a Calculate the values of p, q and r.

The chi-squared test, at the 10% level of significance, is used to determine whether the extra-curricular activity is independent of gender.

b **i** State a suitable null hypothesis.

ii Show that the number of degrees of freedom is 2.

The critical value is 4.605.

c Write down the chi-squared statistic.

d Do you accept the null hypothesis? Explain your answer.

EXAM-STYLE QUESTIONS

10 A company conducted a survey to determine whether position in upper management was independent of gender. The results of this survey are tabulated below.

	Managers	Junior executives	Senior executives	Totals
Male	135	90	75	300
Female	45	130	25	200
Totals	180	220	100	500

The table below shows the expected number of males and females at each level, if they were represented proportionally to the total number of males and females employed.

	Managers	Junior executives	Senior executives	Totals
Male	a	c	60	300
Female	b	d	40	200
Totals	180	220	100	500

a **i** Show that the expected number of male managers (a) is 108.

ii Hence, write down the values of b, c and d.

b Write suitable null and alternative hypotheses for these data.

c **i** Find the chi-squared value.

ii Write down the number of degrees of freedom.

iii Given that the critical value is 5.991, what conclusions can be drawn regarding gender and position in upper management?

11 In the small town of Schiedam, population 8000, an election was held. The results were as follows.

	Urban voters	Rural voters
Candidate A	1950	1730
Candidate B	1830	1360
Candidate C	500	630

In **a**–**d** below, use a chi-squared test, at the 1% significance level, to decide whether the choice of candidate depends on where the voter lives.

H_0: The choice of candidate is independent of where the voter lives.

a Write down the alternative hypothesis.

b Show that the expected number of rural voters for candidate A is 1711.

c **i** Calculate the chi-squared value.

ii Write down the number of degrees of freedom.

The critical value is 9.21.

d **i** State your conclusion.

ii Explain why you reached your conclusion.

EXAM-STYLE QUESTION

12 This table of observed results gives the number of candidates taking a Mathematics examination classified by gender and grade obtained.

	6 or 7	4 or 5	1, 2 or 3	Totals
Males	34	50	6	90
Females	40	60	10	110
Totals	74	110	16	200

The question posed is whether gender and grade obtained are independent.

a Show that the expected number of males achieving a grade of 4 or 5 is 49.5.

A chi-squared test is set up at the 5% significance level.

b **i** State the null hypothesis.

ii State the number of degrees of freedom.

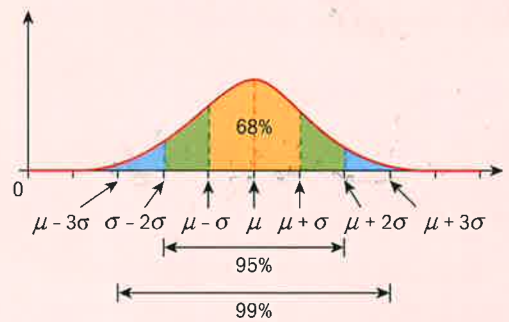
iii Write down the chi-squared value.

The critical value is 5.991.

c What can you say about gender and grade obtained?

CHAPTER 5 SUMMARY
The normal distribution

- The **normal distribution** is the most important continuous distribution in statistics. It has these properties:
  - It is a bell-shaped curve.
  - It is symmetrical about the mean, μ. (The mean, the mode and the median all have the same value.)
  - The x-axis is an asymptote to the curve.
  - The total area under the curve is 1 (or 100%).
  - 50% of the area is to the left of the mean and 50% to the right.
  - Approximately 68% of the area is within 1 standard deviation, σ, of the mean.
  - Approximately 95% of the area is within 2 standard deviations of the mean.
  - Approximately 99% of the area is within 3 standard deviations of the mean.



- The **expected value** is found by multiplying the number in the sample by the probability.

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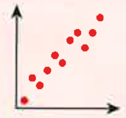
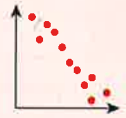
Correlation

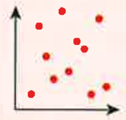
- In a **positive** correlation the dependent variable increases as the independent variable increases.



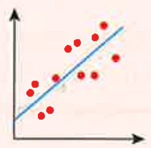
- In a **negative** correlation the dependent variable decreases as the independent variable increases.



- When the points are scattered randomly across the diagram there is **no** correlation.



- Correlations can also be described as strong, moderate or weak.
- To draw the **line of best** fit by eye:
  - Find the mean of each set of data and plot this point on your scatter diagram.
  - Draw a line that passes through the mean point and is close to all the other points – with approximately an equal number of points above and below the line.



- **Pearson's product-moment correlation coefficient**, r, can take all values between −1 and +1 inclusive.
  - When $r = -1$, there is a **perfect negative** correlation between the data sets.
  - When $r = 0$, there is **no** correlation.
  - When $r = +1$, there is a **perfect positive** correlation between the data sets.
  - A **perfect correlation** is one where **all** the plotted points lie on a straight line.

The regression line

- The **regression line for *y* on *x*** is a more accurate version of a line of best fit, compared to best fit by eye.
- If there is a strong or moderate correlation, you can use the regression line for y on x to predict values of y for values of x within the range of the data.

The chi-squared test

- To calculate the χ^2 value use the formula $\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e}$, where f_o are the observed frequencies and f_e are the expected frequencies.
- To find the degrees of freedom for the chi-squared test for independence, use this formula based on the contingency table:

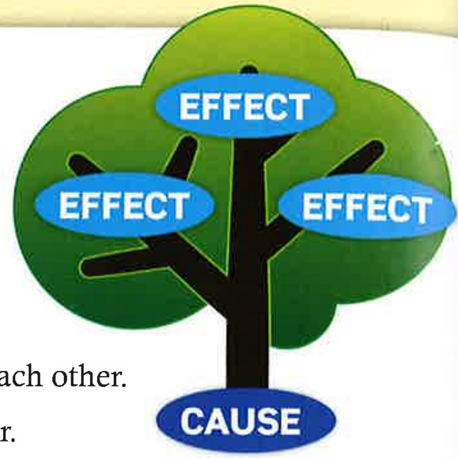
  Degrees of freedom = (number of rows − 1)(number of columns − 1)

Continued on next page

- If χ^2_{calc} is **less than** critical value, **do not reject** the null hypothesis.
  If χ^2_{calc} is **more than** critical value, **reject** the null hypothesis.
- If the p-value is **less than** significance level, **reject** the null hypothesis.
  If the p-value is **more than** significance level, **do not reject** the null hypothesis.
- To perform a χ^2 test:
  1. Write the null (H_0) and alternative (H_1) hypotheses.
  2. Calculate χ^2_{calc}: **a** using your GDC (examinations), or **b** using the χ^2_{calc} formula (project work).
  3. Determine: **a** the p-value using your GDC, or **b** the critical value (given in examinations).
  4. Compare: **a** the p-value against the significance level, or **b** χ^2_{calc} against the critical value.

Theory of knowledge

Correlation or causation?



Correlation shows how closely two variables vary with each other.

Causation is when two variables directly affect each other.

Shaving less than once a day increases risk of stroke by 70%!

In 2003 British researchers found that there was a correlation between men's shaving habits and their risk of a stroke. This link emerged from a 20-year study of over 2,000 men aged 45–59 in Caerphilly, South Wales.

A strong **correlation** between two variables does not mean that one **causes** the other. There may be a cause and effect relation between the two variables, but you cannot claim this if they are only correlated. This is the **fallacy of correlation** – one of the most common logical fallacies.

Do you think a man could decrease his chance of having a stroke by shaving more? This seems silly, and suggests there might be a hidden intermediary variable at work.

In this case, the researchers think that shaving and stroke risk are linked by another variable – hormone levels. For example, testosterone has already been used to explain the link between baldness and a higher risk of heart disease.

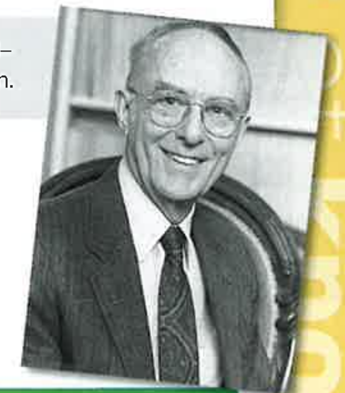
If there is a correlation between two variables, be careful about assuming that there is a relationship between them. There may be no logical or scientific connection at all.

Analyse these examples of assumed correlation or causation.
Which illustrate the fallacy of correlation?

- Joining the military made me a disciplined and strong person
- I wore a hat today on my way to school and I was involved in a car accident; I will not be wearing that red hat again
- People who own washing machines are more likely to die in a car accident.

Anscombe's Quartet

Anscombe's Quartet is a group of four data sets that provide a useful caution against applying individual statistical methods to data without first graphing them. They have identical simple statistical properties (mean, variance, etc.) but look totally different when graphed.



▶ Francis Anscombe (1918–2001), British statistician.

- Find the mean of x, the mean of y, the variance of x and the variance of y and the r-value for each data set.

Set 1		Set 2		Set 3		Set 4	
x	y	x	y	x	y	x	y
4	4.26	4	3.1	4	5.39	8	6.58
5	5.68	5	4.74	5	5.73	8	5.76
6	7.24	6	6.13	6	6.08	8	7.71
7	4.82	7	7.26	7	6.42	8	8.84
8	6.95	8	8.14	8	6.77	8	8.47
9	8.81	9	8.77	9	7.11	8	7.04
10	8.04	10	9.14	10	7.46	8	5.25
11	8.33	11	9.26	11	7.81	8	5.56
12	10.84	12	9.13	12	8.15	8	7.91
13	7.58	13	8.74	13	12.74	8	6.89
14	9.96	14	8.1	14	8.84	19	12.5

1. Write down what you think the graphs and their regression lines will look like.
2. Using your GDC, sketch the graph of each set of points on a different graph.
3. Draw the regression line on each graph.
4. Explain what you notice.

6 Introduci different calculus

CHAPTER OBJECTIVES:

7.1 Concept of the derivative as a rate of change; tangent to a curve
7.2 The principle that $f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$; the derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where all exponents are integers
7.3 Gradients of curves for given values of x; values of x where $f'(x)$ is given; equation of the tangent at a given point; equation of the line perpendicular to the tangent at a given point (normal)
7.4 Values of x where the gradient of a curve is zero; solution of $f'(x) = 0$; stationary points; local maximum and minimum points
7.5 Optimization problems

Before you start

You should know how to:

1 Use function notation, e.g. If $f(x) = 3x + 7$ what is $f(2)$? $f(2) = 3 \times 2 + 7 = 13$

2 Rearrange formulae, e.g. Make x the subject of the formula:
$y = 3x + 7$
$y - 7 = 3x \Rightarrow \frac{y-7}{3} = x$

3 Use index notation, e.g. Write without powers
$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

4 Use the laws of indices, e.g. Simplify:
$5^2 \times 5^4 = 5^{2+4} = 5^6$
$5^4 \div 5^6 = 5^{4-6} = 5^{-2}$

5 Find the equation of a straight line given its gradient and a point, e.g. The line passing through the point (2, 13) with gradient 3
$(y - 13) = 3(x - 2)$
$y - 13 = 3x - 6$
$y = 3x + 7$

Skills check

1 **a** $f(z) = 3 - 2z$, evaluate $f(5)$ and $f(-5)$
b $f(t) = 3t + 5$, evaluate $f(2)$ and $f(-3)$
c $g(y) = y^2$, evaluate $g(5)$ and $g\left(\frac{1}{2}\right)$
d $g(z) = \frac{3}{z}$, evaluate $g(2)$ and $g(15)$
e $f(z) = \frac{z^2}{z+1}$, evaluate $f(4)$ and $f(-3)$

2 Make r the subject of the formula:
a $C = 2\pi r$ **b** $A = \pi r^2$ **c** $A = 4\pi r^2$
d $V = \frac{\pi r^2 h}{3}$ **e** $V = \frac{2\pi r^3}{3}$ **f** $C = \frac{2A}{r}$

3 Write these without powers.
a 4^2 **b** 2^{-3} **c** $\left(\frac{1}{2}\right)^4$

4 Write each expression in the form x^n:
a $\frac{1}{x}$ **b** $\frac{1}{x^4}$ **c** $\frac{x^3}{x}$ **d** $\frac{x^2}{x^5}$ **e** $\frac{(x^2)^3}{x^5}$

5 Find the equation of the line that passes through
a the point (5, −3) with gradient 2
b the point (4, 2) with gradient −3.

The invention of the differential calculus, in the 17th century, was a milestone in the development of mathematics.

At its simplest it is a method of finding the gradient of a **tangent** to a curve. The gradient of the tangent is a measure of how quickly the function is changing as the x-coordinate changes.

All things move, for example, the hands on a clock, the sprinter in a 100 m race, the molecules in a chemical reaction, the share values on the stock market. Mathematics can be used to model all of these situations. Since each situation is dynamic, the models will involve differential calculus.

For more on the history of calculus, see pages 292–3.

In this chapter, you will investigate certain functions to discover for yourself the method of finding the gradient of a tangent to a curve, and check that this method can be applied to all similar curves. You will apply this technique in a variety of situations, to solve problems about graphs and to use mathematical models in 'real-world' problems.

In the photograph, all the cans have the same basic cylindrical shape. However, they are all different sizes. By the end of this chapter you will be able to determine the optimal design of a cylindrical can – one that uses the smallest amount of metal to hold a given capacity.

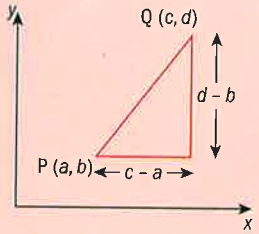
6.1 Introduction to differentiation

You have already met the concept of the gradient of a straight line. **Differentiation** is the branch of mathematics that deals with **gradient functions** of curves.

The gradient measures how fast y is increasing compared to the rate of increase of x.

The gradient of a straight line is constant, which means its direction never changes. The y-values increase at a constant rate.

→ If P is the point (a, b) and Q is (c, d) then the gradient, m, of the straight line PQ is $m = \frac{d-b}{c-a}$.

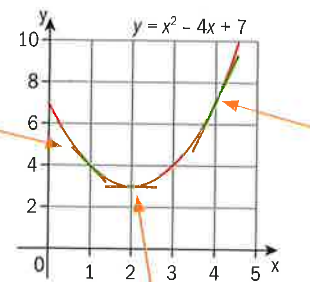


To calculate the gradient of a curve at a particular point you need to draw a tangent at that point. A tangent is a line that just touches the curve.

Here is the curve $y = x^2 - 4x + 7$.
It is a **quadratic** function. Its **vertex** is at the point (2, 3).

The three tangents to the curve are shown in blue.

At the point (1, 4), the curve is decreasing, the gradient of the curve is negative and the tangent to the curve has a negative gradient.



At the point (4, 7), the curve is increasing, the gradient of the curve is positive and the tangent to the curve has a positive gradient.

At the point (2, 3), the gradient of the curve is zero, and the tangent to the curve is horizontal.

The direction of a tangent to the curve changes as the x-coordinate changes. Therefore the gradient of the curve is not constant.

So, for any curve $y = f(x)$ which is not a straight line, its gradient changes for different values of x. The gradient can be expressed as a **gradient function**.

→ Differentiation is a method used to find the equation of the gradient function for a given function, $y = f(x)$.

Extension material on CD: *Worksheet 6 - More about functions*

Investigation – tangents and the gradient function

The tangent to a graph at a given point is the straight line with its gradient equal to that of the curve **at that point**. If you find the gradient of the tangent, then you have also found the gradient of the curve at that point. Repeating this for different points, we can use the data obtained to determine the gradient function for the curve.

GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

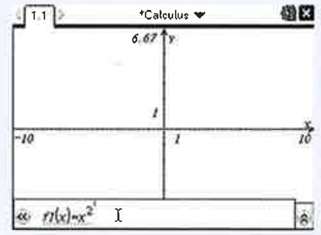
1 Plot the curve $y = x^2$ on your GDC

Open a new document and add a Graphs page.

Save the document as 'Calculus'.

Enter x^2 into the function $f1(x)$.

Press enter del.

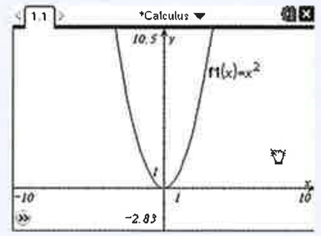


To get a better view of the curve, you should pan the axes in order to see more of it.

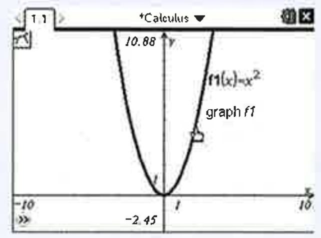
Click down on the touchpad in an area away from the axes, function or any labels.

The ➘ will change to ✋.

Move the grasping hand with the touchpad. The window view will pan with it.



Click the touchpad when the window is in the required position.

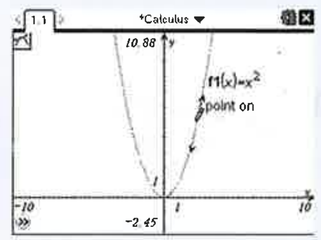


2 Add a tangent to the curve

Press menu 7:Points & Lines | 7:Tangent

Press enter.

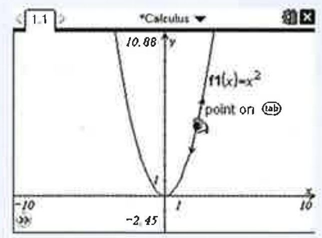
Move the ➘ with the touchpad towards the curve. It will change to a ☝ and the curve will be highlighted.



Click the touchpad.

Choose a point on the curve by clicking the touchpad.

Now you have a tangent drawn at a point on the curve that you can move round to any point on the curve. To get some more information about the tangent, you need the coordinates of the point and the equation of the tangent.



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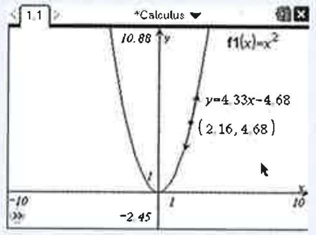
Move the ↖ with the touchpad towards the point.
It will change to a ☜ and you will see 'point on tab'.
Press ctrl menu and select 7: Coordinates and Equations. Press enter.

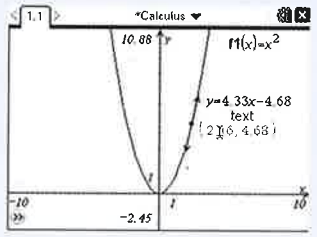
3 Find the equation of the tangent.



Move the ↖ with the touchpad towards the arrow at the end of the tangent. It will change to a ☜ and you will see 'line tab'.

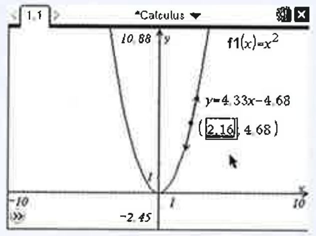
Press ctrl menu and select 7:Coordinates and Equations. Press enter.
You should now have the coordinates of the point and the equation of the tangent labeled.

4 Edit the *x*-coordinate so that the point moves to (1, 1)



Move the ↖ with the touchpad towards the arrow at the *x*-coordinate of the point. It will change to a ☜ and you will see the numbers lighten and the word 'text' appears.

If you move the ↖ very slightly with the touchpad it will change to a I. When it does, click the touchpad.

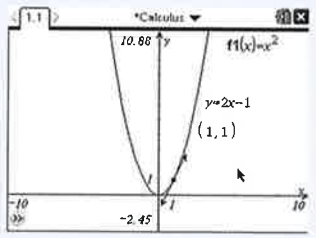


This is quite tricky and may take a bit of practice. If it does not work, press esc and start again.

The *x*-coordinate is now ready for editing.

Use the del key to delete the current value and type 1. Press enter.

You have drawn the tangent to the curve $y = x^2$ at the point (1, 1)



Its equation is $y = 2x - 1$, so gradient of the tangent is 2.

Remember:
In the equation of a straight line $y = mx + c$, m is the gradient

5 Record this information in a table.

$y = x^2$

x-coordinate	−3	−2	−1	0	1	2	3	4	x
Gradient of tangent					2				

Worksheet on CD: *This table is Worksheet 6.1 on the CD.*

6 Complete the table

Go back to the graph and edit the *x*-coordinate again. Change it to 2. Write the gradient of the tangent at the point where the *x*-coordinate is 2 in your table. Repeat this until you have completed the table for all values of *x* between −3 and 4.

Continued on next page

7 Look for a simple formula that gives the gradient of the tangent for any value of *x*

Write this formula in the bottom right cell in your copy of the table.
Is this formula valid for all values of *x*? Try positive, negative and fractional values.

8 Repeat Steps 1–7 for the curve $y = 2x^2$

Draw the curve, then the tangents and complete this table.

$y = 2x^2$

x-coordinate	−3	−2	−1	0	1	2	3	4	x
Gradient of tangent									

Worksheet on CD: *This table is Worksheet 6.1 on the CD.*

Again, look for a simple formula that gives the gradient of the tangent for any value of *x*. Write it down.

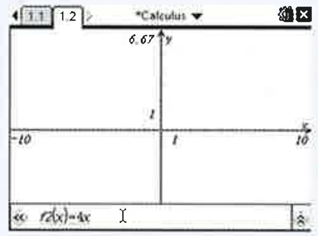
You can repeat this process for other curves, but there is an approach that will save time. The formulae you found in the investigation are called the **gradient functions** of the curves. The gradient function can be written in several ways:

$$\frac{dy}{dx}, \quad \frac{d}{dx}(f(x)), \quad \text{or } f'(x).$$

You can use your GDC to draw a graph of the gradient function for any curve.

Investigation – GDC and the gradient function

1 Use the GDC to draw the gradient function of $y = 4x$



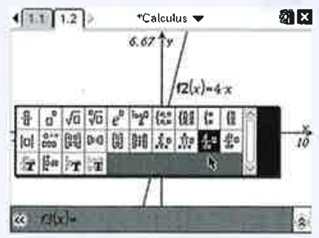
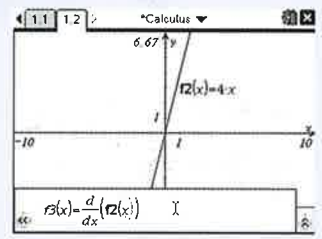
Add a new Graphs page to your document.

Enter 4*x* into the function *f*2(*x*).

Press enter.

GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

2 Enter the gradient function in *f*3(*x*)

Click the » symbol using the touchpad to open the entry line at the bottom of the work area.

Press ⊞ and use the ◄►▲▼ keys to select the $\frac{d}{d\square}$ template.

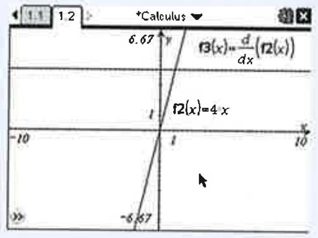
Press enter.

Enter *x* and *f*2(*x*) in the template as shown.
Press enter.

Continued on next page

You should have this diagram, with a horizontal line across the graph.

The graph plotter gives you a picture of the gradient function – you have to find the equation of this function.

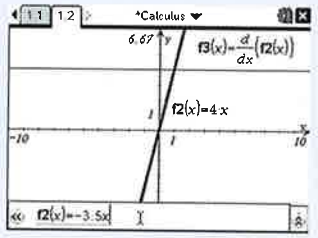


The GDC drew the line $y = 4$.

The gradient of the line $y = 4x$ is '4'.

3 Repeat for other functions

Click the symbol using the touchpad to open the entry line at the bottom of the work area.



Use the ▲ key to select $f2(x)$.

Enter a new function to replace $4x$.

In this way find the gradient functions for these straight lines.

a $y = -3.5x$

b $y = 2x + 4$

c $y = 5$

d $y = 3 - x$

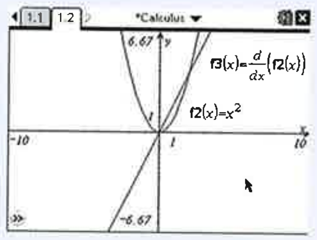
e $y = -3.5$

f $y = 2 - \frac{1}{2}x$

Take care to use the key to enter the – in $-3.5x$ and the key to enter the – in $y = 3 - x$.

4 Change the function to $y = x^2$

A straight line will appear on your screen as in the diagram on the right.



Write down the equation of this new straight line.

The GDC drew the line $y = 2x$.

The gradient function of the curve $y = x^2$ is '$2x$'.

This is the same result that you found by observation in the previous investigation.

Repeat for the curves $y = 2x^2$ and $y = 3x^2$ and write down the gradient functions for these curves.

5 Tabulate your results

You are now building up a set of results that you can use to generalize. To help with this, summarize your findings in a table. You should be able to see patterns in the results.

Continued on next page

Curve	$y = 4x$	$y = -3.5x$	$y = 2x + 4$	$y = 5$	$y = 3 - x$	$y = -3.5$	$y = 2 - \frac{1}{2}x$
Gradient function	4						

Curve	$y = x^2$	$y = 2x^2$	$y = 3x^2$	$y = 4x^2$	$y = -x^2$	$y = -2x^2$	$y = \frac{1}{2}x^2$
Gradient function	$2x$						

6 Extend your results

Complete this table for the curve $y = x^2 + 3x$ using the method from the first Investigation, on page 257.

Worksheet on CD: *This table is Worksheet 6.2 on the CD.*

$y = x^2 + 3x$

x-coordinate	−3	−2	−1	0	1	2	3	4
Gradient of tangent								

What is the algebraic rule that connects the answers for the gradient to the x-coordinates?

Check that your answer is correct by entering $x^2 + 3x$ in $f2(x)$ in the graphs page (Step 2 of this investigation) so that the GDC draws the gradient function.

What is the equation of this straight line?
Is its equation the same as the rule you found?

These should be the same! If they are not, check with your teacher.

Use your GDC to find the gradient functions for the curves below. Look for a pattern developing.

a $y = x^2 + 3x$ **b** $y = x^2 - 5x$ **c** $y = 2x^2 - 3x$ **d** $y = 3x^2 - x$

e $y = 5x - 2x^2$ **f** $y = 2x - x^2$ **g** $y = x^2 + 4$ **h** $y = x^2 - 2$

i $y = 3 - x^2$ **j** $y = x^2 + x - 2$ **k** $y = 2x^2 - x + 3$ **l** $y = 3x - x^2 + 1$

Compare each curve to its gradient function and so determine the formula for the gradient function for the general quadratic curve

$$y = ax^2 + bx + c$$

Write down the gradient functions of the following curves **without using the GDC**.

1 $y = 5x^2 + 7x + 3$

2 $y = 5x + 7x^2 - 4$

3 $y = 3 + 0.5x^2 - 6x$

4 $y = 4 - 1.5x^2 + 8x$

Do not proceed until you have answered these questions correctly.

Investigation – the gradient function of a cubic curve

Now consider the simplest cubic curve $y = x^3$.

Change the function to $y = x^3$ using the GDC.

To enter x^3, press ▸.

(You will need to press the ▸ key to get back to the base line from the exponent.)

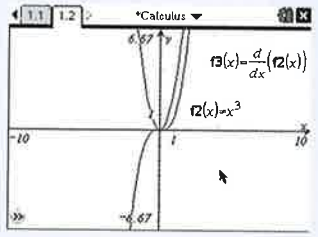
This time a curve appears, instead of a straight line.

Find the equation of the curve.

This is the gradient function of $y = x^3$.

Once you have the equation of the curve, find the gradient function of $y = 2x^3$, $y = 3x^3$, …
Write down your answers in the worksheet copy of the table.

GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*



Have a guess at the equation of the curve. Enter your guess to the gradient function. Adjust your equation until it fits. Then delete it.

Curve	$y = x^3$	$y = 2x^3$	$y = 3x^3$	$y = 4x^3$	$y = -x^3$	$y = -2x^3$	$y = \frac{1}{2}x^3$
Gradient function							

Extend your investigation so that you can find the gradient function of **any** cubic.
Be systematic, so try simple cubic curves first…

Worksheet on CD: *This table is Worksheet 6.3 on the CD.*

Curve	$y = x^3 + 4$	$y = 2x^3 - 3$	$y = x^3 + 5x$	$y = x^3 - 2x$	$y = x^3 + 2x^2$	$y = 2x^3 + \frac{1}{2}x^2$
Gradient function						

Then move on to more complicated cubic curves…

Curve	$y = x^3 + 3x^2 + 2$	$y = x^3 + 4x^2 + 3x$	$y = x^3 + 5x^2 - 4x + 1$	$y = x^3 - x^2 - 5x - 4$
Gradient function				

Generalize your results to determine the formula for the gradient function for the general cubic curve $y = ax^3 + bx^2 + cx + d$

You now have results for the gradient functions of linear functions, quadratic functions and cubic functions. Complete the worksheet copy of the table with these.

Function	Formula	Gradient function
Constant	$y = a$	
Linear	$y = ax + b$	
Quadratic	$y = ax^2 + bx + c$	
Cubic	$y = ax^3 + bx^2 + cx + d$	

Continued on next page

Investigation – the gradient function of any curve

In this investigation you find the gradient function of **any** curve.

Again, take a systematic approach.

1 Find the gradient function of $y = x^4$
2 Find the gradient function of $y = x^5$
3 Generalize these results to find the gradient function of $y = x^n$

Up to this point, all the powers in your curve have been **positive**.

Consider the curves $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, … as well.

To enter $\frac{1}{x}$ on your GDC use the key and select from the template menu.

GDC instructions on CD: *These instructions are for the TI-Nspire GDC. Instructions for the TI-84 Plus and Casio FX-9860GII GDCs, and using a graph plotter, are on the CD.*

Remember that $\frac{1}{x} = x^{-1}$

The final result

Function	Gradient function
$y = ax^n$	

Check this result with your teacher. Do not go on until you have done so.

The process of finding the gradient function of a curve is known as **differentiation.** In these investigations, you have learned for yourself how to differentiate.

Finding this result by investigation is not the same as *proving* it to be true. How, without proof, do we know that a result arrived at by pattern building is **always** true?

6.2 The gradient function

Differentiation is the algebraic process used to find the gradient function of a given function.

Two forms of notation are used for differentiation. The notation that you use will depend on the notation used in the question.

Calculus was discovered at almost the same time by both the British mathematician Isaac Newton (1642–1727), and the German mathematician Gottfried Leibniz (1646–1716). The controversy over the rival claims lasted for decades.

→ To differentiate a function, find the gradient function:

Function	Gradient function
$y = ax^n$	$\frac{dy}{dx} = nax^{n-1}$
$f(x) = ax^n$	$f'(x) = nax^{n-1}$

The process is valid for **all** values of n, both positive and negative.

The $\frac{dy}{dx}$ notation was developed by Leibniz. Newton's notation is now only used in physics. How important is mathematical notation in enhancing your understanding of a subject?

Example 1

Given $y = 4x^7$, find $\frac{dy}{dx}$.

Answer

$\frac{dy}{dx} = 7 \times 4x^{7-1}$

$\frac{dy}{dx} = 28x^6$

$y = ax^n$

$\frac{dy}{dx} = nax^{n-1}$

$a = 4,\ n = 7$

Example 2

Given $f(x) = 3x^5$, find $f'(x)$.

Answer

$f'(x) = 5 \times 3x^{5-1}$

$f'(x) = 15x^4$

$f(x) = ax^n$

$f'(x) = nax^{n-1}$

$a = 3,\ n = 5$

The $f'(x)$ notation is from Euler (1707–83), who was perhaps the greatest mathematician of all.

Example 3

Given $f(x) = 3x - 4x^2 + x^3$, find $f'(x)$.

Answer

$f'(x) = 3x^{1-1} - 2 \times 4x^{2-1} + 3 \times x^{3-1}$

$f'(x) = 3 - 8x + 3x^2$

Differentiate each term separately.

Remember that $x^1 = x$ and that $x^0 = 1$.

Exercise 6A

1 Find $\frac{dy}{dx}$.

a $y = 4x^2$ **b** $y = 6x^3$ **c** $y = 7x^4$ **d** $y = 5x^3$

e $y = x^4$ **f** $y = 5x$ **g** $y = x$ **h** $y = 12x$

i $y = 9x^2$ **j** $y = \frac{1}{2}x^3$ **k** $y = \frac{1}{2}x^2$ **l** $y = \frac{3}{4}x^4$

2 Differentiate

a $y = 7$ **b** $y = -3x^3$ **c** $y = -\frac{1}{4}x^4$ **d** $y = -\frac{2}{3}x^3$

e $y = -x$ **f** $y = -3$ **g** $y = 5x^6$ **h** $y = -7x^9$

i $y = \frac{1}{2}x^8$ **j** $y = \frac{3}{4}x^{12}$ **k** $y = -\frac{2}{3}x^9$ **l** $y = \frac{3}{4}$

3 Find $f'(x)$.

a $f(x) = 3x^2 + 5x^3$ **b** $f(x) = 5x^4 - 4x$

c $f(x) = 9x - 11x^3$ **d** $f(x) = x^4 + 3x + 2$

4 Find y'.

a $y = 8 - 5x + 4x^6$ **b** $y = 9x^2 - 5x + \frac{1}{2}$

c $y = 7x + 4x^5 - 101$ **d** $y = x(2x + 3)$

y' is another way of writing $\frac{dy}{dx}$.

You can use letters other than x and y for the variables. This changes the notation but not the process.

Example 4

Given $v = 3.5t^8$, find $\frac{dv}{dt}$.

Answer

$\frac{dv}{dt} = 8 \times 3.5t^{8-1}$

$\frac{dv}{dt} = 28t^7$

$v = at^n$

$\frac{dv}{dt} = nat^{n-1}$

$a = 3.5,\ n = 8$

Example 5

Given $f(z) = \frac{3z^4}{2}$, find $f'(z)$.

Answer

$f(z) = \frac{3z^4}{2} = \frac{3}{2} \times z^4$

$f'(z) = 4 \times \frac{3}{2}z^{4-1}$

$f'(z) = 6z^3$

$f(z) = az^n$

$f'(z) = naz^{n-1}$

$a = \frac{3}{2},\ n = 4$

Example 6

Given $f(t) = (3t - 1)(t + 4)$, find $f'(t)$.

Answer

$f(t) = 3t^2 + 12t - t - 4$

$f(t) = 3t^2 + 11t - 4$

$f'(t) = 6t + 11$

Multiply out the brackets.

Differentiate each term separately.

Exercise 6B

1 Find $\frac{dA}{dt}$.

a $A = 4t(9 - t^2)$ **b** $A = 6(2t + 5)$

c $A = t^2(t - 5)$ **d** $A = (t + 2)(2t - 3)$

e $A = (5 - t)(3 + 2t)$ **f** $A = (6t + 7)(3t - 5)$

g $A = (t^2 + 3)(t - 1)$ **h** $A = 3(t + 3)(t - 4)$

2 Find $f'(r)$.

a $f(r) = \frac{1}{2}(r + 3)(2r - 6)$ **b** $f(r) = (r + 3)^2$

c $f(r) = (2r - 3)^2$ **d** $f(r) = (5 - 2r)^2$

e $f(r) = 3(r + 5)^2$ **f** $f(r) = 5(7 - r)^2$

You can also differentiate functions which have powers of x in the denominator of a fraction. First you must write these terms using negative indices.

Example 7

Given $y = \dfrac{4}{x^2}$, find $\dfrac{dy}{dx}$.

Answer

$$y = 4 \times \frac{1}{x^2} = 4x^{-2}$$
Write the function in index form: $\frac{1}{x^2} = x^{-2}$.

$$\frac{dy}{dx} = -2 \times 4x^{-2-1}$$
$a = 4$ and $n = -2$

$$\frac{dy}{dx} = -8x^{-3}$$
Remember the rules for multiplying negative numbers.

$$\frac{dy}{dx} = \frac{-8}{x^3}$$
Rewrite in the original form.

Example 8

Given $f(x) = \dfrac{12}{5x^3}$, find $f'(x)$.

Answer

$$f(x) = \frac{12}{5} \times \frac{1}{x^3} = \frac{12}{5}x^{-3}$$
Write the function in index form. *$a = \frac{12}{5}$ and $n = -3$*

$$f'(x) = -3 \times \frac{12}{5} \times x^{-3-1}$$
Be ***very*** *careful with minus signs.*

$$f'(x) = \frac{-36}{5} \times x^{-4}$$
Simplify.

$$f(x) = \frac{-36}{5x^4}$$
Rewrite in the original form.

Exercise 6C

Differentiate the following with respect to x.

Remember to use the same notation as the question.

1 $y = \dfrac{3}{x^2}$ **2** $f(x) = \dfrac{2}{x^4}$ **3** $y = \dfrac{7}{x}$

4 $f(x) = \dfrac{2}{x^8}$ **5** $y = \dfrac{5}{x^7}$ **6** $y = 9 + \dfrac{2}{x}$

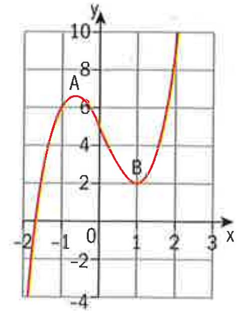
7 $f(x) = 7x^2 + \dfrac{4}{x^5}$ **8** $y = 7 - 4x + \dfrac{5}{2x^2}$ **9** $g(x) = x^3 + \dfrac{3}{x^2}$

10 $y = 4x - \dfrac{3}{x}$ **11** $g(x) = 5x^3 - \dfrac{1}{x^4}$ **12** $y = \dfrac{x^4}{2} - \dfrac{3}{4x^8}$

13 $y = \dfrac{x^4}{8} + 3x^2 + \dfrac{5}{6x^4}$ **14** $g(x) = 2x^3 - x^2 + 2 - \dfrac{3}{2x^2}$ **15** $A(x) = x^2 - \dfrac{5}{2x} + \dfrac{3}{4x^2}$

6.3 Calculating the gradient of a curve at a given point

→ You can use the gradient function to determine the exact value of the gradient at any specific point on the curve.



Here is the curve $y = 2x^3 - x^2 - 4x + 5$ with **domain** $-2 \le x \le 2$. The curve intersects the y-axis at $(0, 5)$.

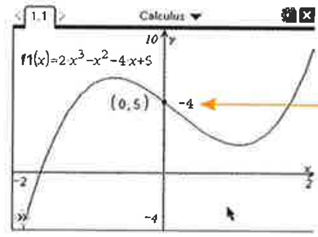
At $x = -2$ the function has a negative value.
It increases to a point A, then decreases to a point B and after $x = 1$ it increases again.

The gradient function of the curve will be negative between points A and B and positive elsewhere.

Will the gradient function be positive or negative at point A and at point B?

Differentiating, the gradient function is $\dfrac{dy}{dx} = 6x^2 - 2x - 4$.

At the y-intercept $(0, 5)$ the x-coordinate is 0. Substituting this value into $\dfrac{dy}{dx}$: at $x = 0$, $\dfrac{dy}{dx} = 6(0)^2 - 2(0) - 4 = -4$



−4 is the gradient at the point (0, 5). Move the point along the curve to find the gradient at other points.

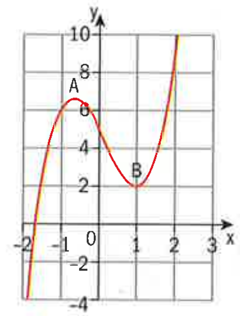
You can check this on your GDC. See Chapter 12, Section 6.1, Example 33.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can use this algebraic method to find the gradient of the curve at other points. For example,

at $x = -1$, $\dfrac{dy}{dx} = 6(-1)^2 - 2(-1) - 4$

$\dfrac{dy}{dx} = 4$



The gradient of the curve at $x = -1$ is 4 and at $x = 0$ it is −4

This result agrees with what can be seen from the graph.

Exercise 6D

These questions can be answered using the algebraic method or using a GDC. Make sure you can do both.

1 If $y = x^2 - 3x$, find $\dfrac{dy}{dx}$ when $x = 4$.

2 If $y = 6x - x^3 + 4$, find $\dfrac{dy}{dx}$ when $x = 0$.

3 If $y = 11 - 2x^4 - 3x^3$, find $\dfrac{dy}{dx}$ when $x = -3$.

4 If $y = 2x(5x + 4)$, find the value of $\frac{dy}{dx}$ when $x = -1$.

5 Find the gradient of the curve $y = x^3 - 5x$ at the point where $x = 6$.

6 Find the gradient of the curve $y = 10 - \frac{1}{2}x^4$ at the point where $x = -2$.

7 Find the gradient of the curve $y = 3x(7 - 4x^2)$ at the point (1, 9).

8 Find the gradient of the curve $y = 3x^2 - 5x + 6$ at the point (–2, 28).

9 $s = 40t - 5t^2$

Find $\frac{ds}{dt}$ when $t = 0$.

10 $s = t(35 + 6t)$

Find $\frac{ds}{dt}$ when $t = 3$.

11 $v = 80t + 7$

Find $\frac{dv}{dt}$ when $t = -4$.

12 $v = 0.7t - 11.9$

Find $\frac{dv}{dt}$ when $t = 0.7$.

13 $A = 14h^3$

Find $\frac{dA}{dh}$ when $h = \frac{2}{3}$.

14 $W = 7.25p^3$

Find $\frac{dW}{dp}$ at $p = -2$.

15 $V = 4r^2 + \frac{18}{r}$.

Find $\frac{dV}{dr}$ at $r = 3$.

16 $A = 5r + \frac{8}{r^2}$

Find $\frac{dA}{dr}$ at $r = 4$.

17 $V = 7r^3 - \frac{8}{r}$

Find $\frac{dV}{dr}$ at $r = 2$.

18 $A = \pi r^2 - \frac{2\pi}{r}$

Find $\frac{dA}{dr}$ at $r = 1$.

19 $V = 6r + \frac{15}{2r}$

Find $\frac{dV}{dr}$ at $r = 5$.

20 $C = 45r + \frac{12}{r^3}$

Find $\frac{dC}{dr}$ at $r = 1$.

By working backwards you can find the coordinates of a specific point on a curve with a particular gradient.

Example 9

Point A lies on the curve $y = 5x - x^2$ and the gradient of the curve at A is 1. Find the coordinates of A.

Answer	
$\frac{dy}{dx} = 5 - 2x$	*First find $\frac{dy}{dx}$*
at A $\frac{dy}{dx} = 1$ so $5 - 2x = 1$ $x = 2$	*Solve the equation to find x.*
$y = 5(2) - (2)^2 = 6$ A is (2, 6)	*Substitute x = 2 into the equation of the curve to find y.*

Exercise 6E

1 Point P lies on the curve $y = x^2 + 3x - 4$. The gradient of the curve at P is equal to 7.

a Find the gradient function of the curve.

b Find the x-coordinate of P.

c Find the y-coordinate of P.

2 Point Q lies on the curve $y = 2x^2 - x + 1$. The gradient of the curve at Q is equal to –9.

a Find the gradient function of the curve.

b Find the x-coordinate of Q.

c Find the y-coordinate of Q.

3 Point R lies on the curve $y = 4 + 3x - x^2$ and the gradient of the curve at R is equal to –3.

a Find the gradient function of the curve.

b The coordinates of R are (a, b), find the value of a and of b.

EXAM-STYLE QUESTIONS

4 Point R lies on the curve $y = x^2 - 6x$ and the gradient of the curve at R is equal to 6.
Find the gradient function of the curve.
The coordinates of R are (a, b)
Find the value of a and of b.

5 Find the coordinates of the point on the curve $y = 3x^2 + x - 5$ at which the gradient of the curve is 4.

6 Find the coordinates of the point on the curve $y = 5x - 2x^2 - 3$ at which the gradient of the curve is 9.

7 There are **two** points on the curve $y = x^3 + 3x + 4$ at which the gradient of the curve is 6.
Find the coordinates of these two points.

8 There are **two** points on the curve $y = x^3 - 6x + 1$ at which the gradient of the curve is –3.
Find the coordinates of these two points.
Find the equation of the straight line that passes through these two points.

EXAM-STYLE QUESTION

9 There are **two** points on the curve $y = x^3 - 12x + 5$ at which the gradient of the curve is zero.
Find the coordinates of these two points.
Find the equation of the straight line that passes through these two points.

EXAM-STYLE QUESTIONS

10 Point P (1, b) lies on the curve $y = x^2 - 4x + 1$.

a Find the value of b.

b Find the gradient function of the curve.

c Show that at P the gradient of the curve is also equal to b.

d Q (c, d) is the point on the curve at which the gradient of the curve is equal to -2. Show that $d = -2$.

11 Point P (5, b) lies on the curve $y = x^2 - 3x - 3$.

a Find the value of b.

b Find the gradient function of the curve.

c Show that at P the gradient of the curve is also equal to b.

d Q (c, d) is the point on the curve at which the gradient of the curve is equal to -3.
Show that d is also equal to -3.

12 Consider the function $f(x) = 4x - x^2 - 1$.

a Write down $f'(x)$.

b Show that at $x = 5$, $f(x) = f'(x)$.

c Find the coordinates of a second point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

13 Consider the function $f(x) = 2x^2 - x + 1$.

a Write down $f'(x)$.

b Show that at $x = 2$, $f(x) = f'(x)$.

c Find the coordinates of a second point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

14 Consider the function $f(x) = 3x - x^2 - 1$.

a Write down $f'(x)$.

b Show that at $x = 1$, $f(x) = f'(x)$.

c Find the coordinates of a second point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

15 Consider the function $f(x) = 2x^2 - x - 1$.

a Write down $f'(x)$.

b Find the coordinates of the points on the curve $y = f(x)$ for which $f(x) = f'(x)$.

16 Consider the function $f(x) = x^2 + 5x - 5$.

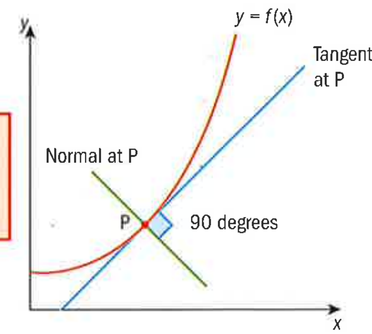
a Write down $f'(x)$.

b Find the coordinates of the points on the curve $y = f(x)$ for which $f(x) = f'(x)$.

17 Consider the function $f(x) = x^2 + 4x + 5$.
Find the coordinates of the point on the curve $y = f(x)$ for which $f(x) = f'(x)$.

6.4 The tangent and the normal to a curve

Here is a curve $y = f(x)$ with a point, P, on the curve.



→ The tangent to the curve at any point P is the straight line which passes through P with gradient equal to the gradient of the curve at P.

The **normal** to the curve at P is the straight line which passes through P that is **perpendicular** to the tangent.

The tangent and the curve are closely related because, at P:

- the x-coordinate of the tangent is equal to the x-coordinate of the curve
- the y-coordinate of the tangent is equal to the y-coordinate of the curve
- the gradient of the tangent is equal to the gradient of the curve.

You can use differentiation to find the equation of the tangent to any curve at a point, P(a, b), provided that you know both the equation of the curve and the x-coordinate, a, of the point P.

→ To find the equation of the tangent to the curve at P(a, b):

1. Calculate b, the y-coordinate of P, using the equation of the curve.
2. Find the gradient function $\frac{dy}{dx}$.
3. Substitute a, the x-coordinate of P, into $\frac{dy}{dx}$ to calculate, m, the value of the gradient at P.
4. Use the equation of a straight line $(y - b) = m(x - a)$.

For more on the equation of a straight line, see Chapter 3.

Example 10

Point P has an x-coordinate 2. Find the equation of the tangent to the curve $y = x^3 - 3$ at P.
Give your answer in the form $y = mx + c$.

Answer

At $x = 2$, $y = (2)^3 - 3 = 5$	*Use $y = x^3 - 3$ to calculate the y-coordinate of P.*
$\frac{dy}{dx} = 3x^2$	*Find the gradient function $\frac{dy}{dx}$.*
At $x = 2$, $\frac{dy}{dx} = 3(2)^2 = 12$ $m = 12$	*Substitute 2, the x-coordinate at P, into $\frac{dy}{dx}$ to calculate m, the value of the gradient at P.*
At P (2, 5) $(y - 5) = 12(x - 2)$ $y - 5 = 12x - 24$ $y = 12x - 19$	*Use the equation $(y - b) = m(x - a)$ with $a = 2$, $b = 5$, $m = 12$. Simplify.*

You can check the equation of the tangent using your GDC.

Exercise 6F

1 Find the equation of the tangent to the given curve at the stated point, P. Give your answers in the form $y = mx + c$.

a $y = x^2$; P(3, 9) **b** $y = 2x^3$; P(1, 2)
c $y = 6x - x^2$; P(2, 8) **d** $y = 3x^2 - 10$; P(1, −7)
e $y = 2x^2 - 5x + 4$; P(3, 7) **f** $y = 10x - x^3 + 5$; P(2, 17)
g $y = 11 - 2x^2$; P(3, −7) **h** $y = 5 - x^2 + 6x$; P(2, 13)
i $y = 4x^2 - x^3$; P(4, 0) **j** $y = 5x - 3x^2$; P(−1, −8)
k $y = 6x^2 - 2x^3$; P(2, 8) **l** $y = 60x - 5x^2 + 7$; P(2, 107)
m $y = \frac{1}{2}x^4 - 7$; P(4, 121) **n** $y = 17 - 3x + 5x^2$; P(0, 17)
o $y = 2x(5 - x)$; P(0, 0) **p** $y = \frac{1}{4}x^3 - 4x$; P(2, −6)
q $y = \frac{3}{4}x^2 + 3$; P(−2, 6) **r** $y = \frac{2}{3}x^3 + \frac{1}{3}$; $P\left(-1, -\frac{1}{3}\right)$
s $y = \frac{1}{4}x^3 - 7x^2 + 5$; P(−2, −25)

2 Find the equation of the tangent to the given curve at the stated point. Give your answers in the form $ax + by + c = 0$

a $y = \frac{12}{x^2}$; (2, 3) **b** $y = 5 + \frac{6}{x^3}$; (1, 11)
c $y = 6x - \frac{8}{x^2}$; (−2, −14) **d** $y = x^3 + \frac{6}{x^2}$; (−1, 5)
e $y = 5x - \frac{8}{x}$; (4, 18)

To find the equation of the normal to a curve at a given point you need to do one extra step.

→ The normal is perpendicular to the tangent so its gradient, m', is found using the formula $m' = \frac{-1}{m}$, where m is the gradient of the tangent.

Example 11

Point P has x-coordinate −4.
Find the equation of the normal to the curve $y = \frac{12}{x}$ at P.
Give your answer in the form
$ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$.

Answer

At $x = -4$, $y = \frac{12}{(-4)} = -3$	*Use $y = \frac{12}{x}$ to calculate the y-coordinate of P.*
$\frac{dy}{dx} = -\frac{12}{x^2}$	*Find the gradient function $\frac{dy}{dx}$. (Remember, $y = 12x^{-1}$.)*

▶ Continued on next page

You learned about gradient of a perpendicular line in Chapter 3.

At $x = -4$, $\frac{dy}{dx} = -\frac{12}{(-4)^2} = -\frac{3}{4}$ The gradient of the **tangent**, $m = -\frac{3}{4}$	*Substitute the value of x into $\frac{dy}{dx}$ to calculate, m, the value of the gradient at P.*
Hence, the gradient of the normal, $m' = \frac{4}{3}$	*The normal is perpendicular to the tangent.*
The equation of the normal to $y = \frac{12}{x}$ at P(−4, −3) is $(y - (-3)) = \frac{4}{3}(x - (-4))$	*Use the equation of a straight line $(y - b) = m(x - a)$ with $a = -4$, $b = -3$, $m = \frac{4}{3}$*
$3(y + 3) = 4(x + 4)$ $3y + 9 = 4x + 16$ $4x - 3y + 7 = 0$	*Simplify.* *Rearrange to the form $ax + bx + c = 0$, where $a, b, c \in \mathbb{Z}$*

The gradient of a line perpendicular to a line whose gradient is m is $-\frac{1}{m}$.

You cannot find the equation of a normal directly from the GDC.

Exercise 6G

Find the equation of the normal to the given curve at the stated point P. Give your answers in the form $ax + by + c = 0$

1 $y = 2x^2$; P(1, 2) **2** $y = 3 + 4x^3$; P(0.5, 3.5)
3 $y = \frac{x}{2} - x^2$; P(2, −3) **4** $y = \frac{3x^2}{2} + x$; P(−2, 4)
5 $y = (x + 2)(5 - x)$; P(0, 10) **6** $y = (x + 2)^2$; P(0, 4)
7 $y = \frac{4}{x}$; P(2, 2) **8** $y = \frac{6}{x^2}$; P(−1, 6)
9 $y = 6x + \frac{8}{x}$; P(1, 14) **10** $y = x^4 - \frac{3}{x^3}$; P(−1, 4)
11 $y = 4 - 2x - \frac{1}{x}$; P(0.5, 1) **12** $y = 5x - \frac{9}{2x}$; P(3, 13.5)

Example 12

The gradient of the tangent to the curve $y = ax^2$ at the point P (3, b) is 30. Find the values of a and b.

Answer

$\frac{dy}{dx} = 2ax$	*As the gradient of the tangent is given, find $\frac{dy}{dx}$.*
$2a(3) = 30$ $\Rightarrow a = 5$	*When $x = 3$, $\frac{dy}{dx} = 30$*
The equation of the curve is $y = 5x^2$. $b = 5(3)^2 \Rightarrow b = 45$	*Substitute $x = 3$ to find b.*

Exercise 6H

1 Find the equation of the tangent to the curve $y = (x - 4)^2$ at the point where $x = 5$.

EXAM-STYLE QUESTIONS

2 Find the equation of the tangent to the curve $y = x(x^2 - 3)$ at the point where $x = -2$.

3 Find the equation of the normal to the curve $y = x + \frac{6}{x}$ at the point where $x = 4$.

4 Find the equation of the normal to the curve $y = x^2 - \frac{1}{x^2}$ at the point where $x = -1$.

5 Find the equations of the tangents to the curve $y = 3x^2 - 2x$ at the points where $y = 8$.

6 Find the equations of the tangents to the curve $y = 2x(3 - x)$ at the points where $y = -20$.

7 Find the equation of the normal to the curve $y = 7 - 5x - 2x^3$ at the point where it intersects the x-axis.

8 Find the equation of the normal to the curve $y = x^3 + 3x - 2$ at the point where $y = -6$.

9 **a** Find the value of x for which the gradient of the tangent to the curve $y = (4x - 3)^2$ is zero.

b Find the equation of the tangent at this point.

EXAM-STYLE QUESTION

10 **a** Find the value of x for which the gradient of the tangent to the curve $y = x^2 + \frac{16}{x}$ is zero.

b Find the equation of the tangent at this point.

11 **a** Find the value of x for which the gradient of the tangent to the curve $y = \frac{x^2}{2} + x - 3$ is 5.

b Find the equation of the tangent at this point.

12 **a** Find the value of x for which the gradient of the tangent to the curve $y = x^4 + 3x - 3$ is 3.

b Find the equation of the tangent at this point.

c Find the equation of the normal at this point.

13 **a** Find the value of x for which the gradient of the tangent to the curve $y = 4x + \frac{3}{x^4}$ is 16.

b Find the equation of the tangent at this point.

c Find the equation of the normal at this point.

14 There are two points on the curve $y = 2x^3 + 9x^2 - 24x + 5$ at which the gradient of the curve is equal to 36. Find the equations of the tangents to the curve at these points.

EXAM-STYLE QUESTION

15 The gradient of the tangent to the curve $y = x^2 + kx$ at the point P $(3, b)$ is 7.
Find the value of k and the value of b.

16 The gradient of the tangent to the curve $y = x^2 + kx$ at the point P $(-2, b)$ is 1.
Find the value of k and that of b.

17 The gradient of the tangent to the curve $y = kx^2 - 2x + 3$ at the point P $(4, b)$ is 2.
Find the value of k and that of b.

18 The gradient of the tangent to the curve $y = 4 + kx - x^3$ at the point P $(-2, b)$ is -5.
Find the value of k and that of b.

19 The gradient of the tangent to the curve $y = px^2 + qx$ at the point P $(2, 5)$ is 7.
Find the value of p and that of q.

20 The gradient of the tangent to the curve $y = px^2 + qx - 5$ at the point P $(-3, 13)$ is 6.
Find the value of p and that of q.

6.5 Rates of change

The gradient function, $f'(x)$, of a function $f(x)$ is a measure of how $f(x)$ changes as x increases. We say that $f'(x)$ measures the **rate of change of *f* with respect to *x*.**

In general, the **rate of change** of one variable with respect to another is the gradient function.

→ For the graph $y = f(x)$, the gradient function $\frac{dy}{dx} = f'(x)$ gives the rate of change of y with respect to x.

Other variables can also be used, for example:

if $A = f(t)$, then $\frac{dA}{dt} = f'(t)$ measures the **rate of change of *A* with respect to *t*.**

If the variable t represents time, then the gradient function measures the rate of change with respect to the *time* that passes.

This is an important concept. If you measure how a variable changes as time is passing then you are applying mathematics to situations that are **dynamic** – to situations that are moving.

For example, if C represents the value of a car (measured on a day-to-day basis) we can say that C is a function of time: $C = f(t)$.

Then, $\frac{dC}{dt} = f'(t)$ represents the rate at which the value of the car is changing – it measures the rate of change of C with respect to t, the rate of inflation or deflation of the price of the car.

Similarly, if s represents the distance measured from a fixed point to a moving object then s is a function of time: $s = g(t)$ and $\frac{ds}{dt} = g'(t)$ measures the rate of change of this distance, s, with respect to t.

$\frac{ds}{dt}$ measures the **velocity** of the object at time t.

If v is the velocity of an object, what does $\frac{dv}{dt}$ represent?

Example 13

The volume of water in a container, V cm^3, is given by the formula $V = 300 + 2t - t^2$, where t is the time measured in seconds.

a What does $\frac{dV}{dt}$ represent?

b What units are used for $\frac{dV}{dt}$?

c Find the value of $\frac{dV}{dt}$ when $t = 3$.

d What does the answer to **c** tell you?

Answers

a $\frac{dV}{dt}$ represents the rate of change of the volume of water in the container.	*The rate at which the water is entering (or leaving) the container.*
b $\frac{dV}{dt}$ is measured in cm^3 per second (cm^{3}s^{-1}).	*The volume is measured in cm^3 and time is measured in seconds.*
c $\frac{dV}{dt} = 2 - 2t$ At $t = 3$, $\frac{dV}{dt} = 2 - 2(3) = -4$	*$\frac{dV}{dt}$ is negative, so*
d Since this value is **negative**, the water is **leaving** the container at 4 cm^3 per second.	*the volume is decreasing.*

How would you decide by considering $\frac{dv}{dt}$ whether the water was **entering** or **leaving** the container?

Example 14

A company mines copper, where the mass of copper, x, is measured in thousands of tonnes. The company's profit, P, measured in millions of dollars, depends on the amount of copper mined. The profit is given by the function $P(x) = 2.3x - 0.05x^2 - 12$

a Find $P(0)$ and $P(6)$ and interpret these results.

b Find $\frac{dP}{dx}$. What does $\frac{dP}{dx}$ represent?

c Find the value of P and $\frac{dP}{dx}$ when $x = 20$ and when $x = 25$.

d Interpret the answers to **c**.

e Find the value of x for which $\frac{dP}{dx} = 0$.

f Determine P for this value of x, and interpret this value.

You can graph any function on the GDC. This could give you further insight into the problem.

Answers

a $P(0) = -12$; a loss of 12 million dollars. $P(6) = 0$; there is no profit and no loss, this is the break-even point.	*Substitute $x = 0$ in to $P(x)$.*
b $\frac{dP}{dx} = -0.1x + 2.3$ $\frac{dP}{dx}$ represents the rate of change of the profit as the amount of copper mined increases.	*$\frac{dP}{dx}$ measures the rate of change of P with respect to x.*
c At $x = 20$, $P = 14$ and $\frac{dP}{dx} = 0.3$ At $x = 25$, $P = 14.25$ and $\frac{dP}{dx} = -0.2$	*Substitute $x = 20$ and $x = 25$ into $P(x)$ and $\frac{dP}{dx}$.*
d At both points the company is profitable. At $x = 20$, $\frac{dP}{dx} > 0$ so a further increase in production will make the company **more profitable**.	*At $x = 20$, $P(x)$ is increasing.*
At $x = 25$, $\frac{dP}{dx} < 0$ so a further increase in production will make the company **less profitable**.	*At $x = 25$, $P(x)$ is decreasing.*
e $\frac{dP}{dx} = -0.1x + 2.3 = 0$ $0.1x = 2.3$ $x = \frac{2.3}{0.1} = 23$	*Set $\frac{dP}{dx}$ equal to 0.* *Solve for x.*
23 000 tonnes of copper needs to be mined to maximize the company's profit.	*x is measured in thousands of tonnes.*
f $P(23) = 14.45$ 14.45 million dollars is the **maximum** profit that the company can make.	*Substitute $x = 23$ into $P(x)$.*

Exercise 6I

EXAM-STYLE QUESTION

1 The volume of water in a container, V cm^3, is given by the formula $V = 100 + 2t + t^3$, where t is the time measured in seconds.

- **a** How much water is there in the container initially?
- **b** How much water is there in the container when $t = 3$?
- **c** What does $\frac{dV}{dt}$ represent?
- **d** Find the value of $\frac{dV}{dt}$ when $t = 3$.
- **e** Use your answers to **b** and **d** to explain what is happening to the volume of water in the tank.

Initially $t = 0$

2 The area, A, of a pool of water forming under a leaking pipe is $A = 4t + t^2$ cm^2 after t seconds.

- **a** What is the area of the pool initially?
- **b** What is the area of the pool when $t = 5$?
- **c** What does $\frac{dA}{dt}$ represent?
- **d** Find the value of $\frac{dA}{dt}$ when $t = 5$.
- **e** Use your answers to **b** and **d** to explain what is happening to the area of the pool.

3 The weight of oil in a storage tank, W, varies according to the formula $W = 5t^2 + \frac{640}{t} + 40$ where W is measured in tonnes and t is the time measured in hours, $1 \le t \le 10$.

- **a** Find the weight of oil in the tank at $t = 1$.
- **b** Find $\frac{dW}{dt}$.
- **c** Find the rate of change of the weight of the oil in the tank when
  - **i** $t = 3$ **ii** $t = 5$.
- **d** What does your answer to **c** tell you?
- **e** Find the value of t for which $\frac{dW}{dt} = 0$.
- **f** Interpret your answer to **e**.

4 The volume of water, V, measured in m^3, in a swimming pool after t minutes, where $t > 0$, is $V = 10 + 6t + t^2$.

- **a** Find the rate at which the volume is increasing when $t = 1$.
- **b** Find the rate at which the volume is increasing when there are 65 m^3 of water in the pool.

5 Water is flowing out of a tank. The depth of the water, y cm, at time t seconds is given by $y = 500 - 4t - t^3$.

- **a** Find the rate at which the depth is decreasing at 2 seconds and at 3 seconds.
- **b** Find the time at which the tank is empty.

6 The area, A cm^2, of a blot of ink is growing so that, after t seconds, $A = \frac{3t^2}{4} + \frac{t}{2}$.

- **a** Find the rate at which the area is increasing after 2 seconds.
- **b** Find the rate at which the area is increasing when the area of the blot is 30 cm^2.

7 The weight of oil in a storage tank, W, varies according to the formula $W = 10t + \frac{135}{t^2} + 4$ where W is measured in tonnes and t is the time measured in hours, $1 \le t \le 10$.

- **a** Find the rate at which the weight is changing after 2 hours.
- **b** Find the value of t for which $\frac{dW}{dt} = 0$.

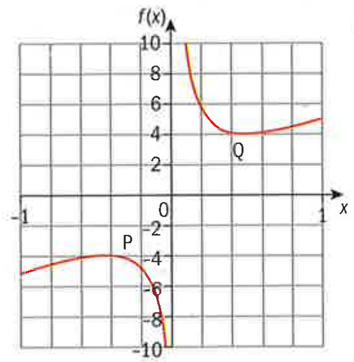
8 The angle turned through by a rotating body, θ degrees, in time t seconds is given by the relation $\theta = 4t^3 - t^2$.

- **a** Find the rate of increase of θ when $t = 2$.
- **b** Find the value of t at which the body changes direction.

9 A small company's profit, P, depends on the amount x of 'product' it makes. This profit can be modeled by the function $P(x) = -10x^3 + 40x^2 + 10x - 15$. P is measured in thousands of dollars and x is measured in tonnes.

- **a** Find $P(0)$ and $P(5)$ and interpret these results.
- **b** Find $\frac{dP}{dx}$.
- **c** Find the value of P and $\frac{dP}{dx}$ when **i** $x = 2$ **ii** $x = 3$.
- **d** Interpret your answers to **c**.
- **e** Find the value of x and of P for which $\frac{dP}{dx} = 0$. What is the importance of this point?

6.6 Local maximum and minimum points (turning points)



Here is the graph of the function

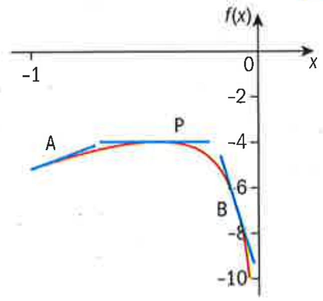
$$f(x) = 4x + \frac{1}{x}, \quad x \neq 0$$

The graph has two branches, because the function is **not defined** at the point $x = 0$.

First, look at the left-hand branch of the graph, for the **domain** $x < 0$.

$\frac{1}{0}$ is undefined; it has no value.

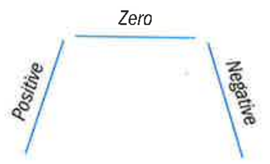
As x increases, the curve increases to the point P. After point P, the curve decreases. P is said to be a local **maximum point**.



You can determine that P is a local maximum point because just before P (for example, at A) the gradient of the curve is positive, and just after P (for example, at B) the gradient of the curve is negative.

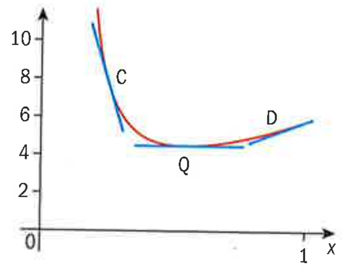
At P itself, most importantly, the gradient of the curve is zero.

→ At a local maximum, the curve stops increasing and changes direction so that it 'turns' and starts decreasing. So, as x increases, the three gradients occur in the order: positive, zero, negative. Where the gradient is zero is the maximum point.



Now look at the right-hand branch of the graph, with the domain $x > 0$.

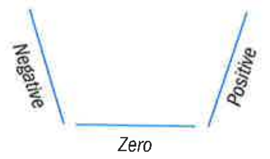
As x increases, the curve decreases to the point Q. After Q, the curve increases. Q is said to be a local **minimum point**.



You can determine that Q is a local minimum point because just before Q (for example, at C) the gradient of the curve is negative and just after Q (for example, at D) the gradient of the curve is positive.

At Q itself, the gradient of the curve is zero.

→ At a local minimum, the curve stops decreasing and changes direction; it 'turns' and starts increasing. So, as x increases, the three gradients occur in the order: negative, zero, positive. Where the gradient is zero is the minimum point.



Local maximum and local minimum points are known as **stationary points** or **turning points**.

→ At any stationary or turning point – either local maximum or local minimum – $f'(x)$ is zero.

At a stationary point, if $y = f(x)$ then $\frac{dy}{dx} = 0$.

To find the coordinates of P (the local maximum) and of Q (the local minimum) for the function $f(x) = 4x + \frac{1}{x}$, use the fact that at each of these points $f'(x)$ is zero.

$f(x) = 4x + \frac{1}{x}$, so $f'(x) = 4 - \frac{1}{x^2}$

Remember that $\frac{1}{x} = x^{-1}$.

Set $f'(x) = 0$ which gives $4 - \frac{1}{x^2} = 0$

Adding $\frac{1}{x^2}$: $\quad 4 = \frac{1}{x^2}$

Multiplying by x^2: $\quad 4x^2 = 1$

Dividing by 4: $\quad x^2 = \frac{1}{4}$

Taking square roots: $x = \frac{1}{2}$ or $x = -\frac{1}{2}$

Substitute each x-value into $f(x)$ to find the y-coordinate of each turning point.

At $x = \frac{1}{2}$, $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) + \frac{1}{\left(\frac{1}{2}\right)} = 4$

At $x = -\frac{1}{2}$, $f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right) + \frac{1}{\left(-\frac{1}{2}\right)} = -4$

You can find local maximum and local minimum points using a GDC, without using differentiation. See Chapter 12, Section 6.3.

So, the coordinates of the turning points are $\left(\frac{1}{2}, 4\right)$ and $\left(-\frac{1}{2}, -4\right)$

To determine which is the local maximum and which is the local minimum, look at the graph of the function: $\left(\frac{1}{2}, 4\right)$ is the local minimum and $\left(-\frac{1}{2}, -4\right)$ the local maximum.

You cannot decide which is the maximum and which is the minimum simply by looking at the coordinates.

→ To find turning points, first set the gradient function equal to zero and solve this equation. This gives the x-coordinate of the turning point.

Exercise 6J

Find the values of x for which $\frac{dy}{dx} = 0$. Verify your answers by using your GDC.

1 $y = x^2 - 6x$

2 $y = 12x - 2x^2$

3 $y = x^2 + 10x$

4 $y = 3x^2 + 15x$

5 $y = x^3 - 27x$

6 $y = 24x - 2x^3$

7 $y = 4x^3 - 3x$

8 $y = 3x - 16x^3$

9 $y = 2x^3 - 9x^2 + 12x - 7$

10 $y = 5 + 9x + 6x^2 + x^3$

11 $y = x^3 - 3x^2 - 45x + 11$

12 $y = 12x^2 + x^3 + 36x - 8$

13 $y = 2x^3 - 6x^2 + 7$

14 $y = 17 + 30x^2 - 5x^3$

15 $f(x) = x + \frac{1}{x}$

16 $y = x + \frac{4}{x}$

17 $y = 4x + \frac{9}{x}$

18 $y = 8x + \frac{1}{2x}$

19 $y = 27x + \frac{4}{x^2}$

20 $y = x + \frac{1}{2x^2}$

Once you have found the x-coordinate of any turning point, you can then calculate the y-coordinate of the point and decide if it is a maximum or minimum.

Example 15

Find the coordinates of the turning points of the curve $y = 3x^4 - 8x^3 - 30x^2 + 72x + 5$. Determine the nature of these points.

'Determine the nature' means decide whether the point is a local maximum or a local minimum.

Answer

$y = 3x^4 - 8x^3 - 30x^2 + 72x + 5$

$\frac{dy}{dx} = 12x^3 - 24x^2 - 60x + 72$ *Differentiate.*

$\frac{dy}{dx} = 12x^3 - 24x^2 - 60x + 72 = 0$ *At each turning point $\frac{dy}{dx} = 0$.*

▶ Continued on next page

$x = -2, x = 1, x = 3$
At $x = -2$,
$y = 3(-2)^4 - 8(-2)^3 - 30(-2)^2 + 72(-2) + 5 = -95$
so $(-2, -95)$ is a turning point.
At $x = 1$, $y = 3(1)^4 - 8(1)^3 - 30(1)^2 + 72(1) + 5 = 42$
so $(1, 42)$ is a turning point.
At $x = 3$, $y = 3(3)^4 - 8(3)^3 - 30(3)^2 + 72(3) + 5 = -22$
so $(3, -22)$ is a turning point.

Solve this equation with your GDC.

Substitute the three values of x to find the y-coordinates.

x-coordinate		−2		1		3	
Gradient		0		0		0	

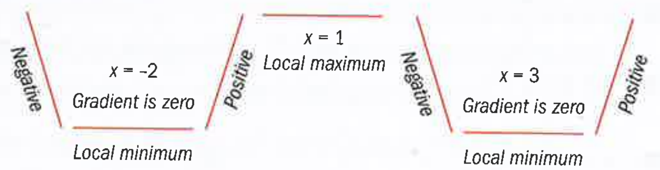
To decide if points are maximum or minimum (without using the GDC) find the gradient at points on each side of the turning points. First, fill in the information on the turning points.

$x = -10$ for $x < -2$ $f'(-10) = -12\,268$
$x = 0$ for $-2 < x < 1$ $f'(0) = 72$
$x = 2$ for $1 < x < 3$ $f'(2) = -48$
$x = 5$ for $x > 3$ $f'(5) = 672$

Now choose x-coordinates of points on each side of the turning points. Calculate the gradient at each point and enter them in the table.

x-coordinate	−10	−2	0	1	2	3	5
Gradient	−12 268	0	72	0	−48	0	672

Choose points close to the stationary point.



Sketch the pattern of the gradients from the table.

$(-2, -95)$ is a local **minimum**.

As the curve moves through (−2, −95), the gradient changes negative → zero → positive.

$(1, 42)$ is a local **maximum**.

As the curve moves through (1, 42), the gradient changes positive → zero → negative.

$(3, -22)$ is also a local **minimum**.

As the curve moves through (3, −22), the gradient changes negative → zero → positive.

Exercise 6K

Determine the coordinates of any turning points on the given curves.
For each, decide if it is a maximum or minimum.
Check your answers by using your GDC.

1 $y = x^3 - 9x^2 + 24x - 20$ **2** $y = x^3 + 6x^2 + 9x + 5$
3 $y = x(9 + 3x - x^2)$ **4** $y = x^3 - 3x^2 + 5$

5 $y = x(27 - x^2)$ **6** $y = x^2(9 - x)$
7 $f(x) = x + \frac{1}{x}$ **8** $f(x) = x + \frac{9}{x}$
9 $f(x) = \frac{x}{2} + \frac{8}{x}$ **10** $f(x) = \frac{9}{x} + \frac{x}{4}$
11 $f: x \to = x^2 - \frac{16}{x}$ **12** $f: x \to = 9x + \frac{1}{6x^2}$

'$f: x \to$' is read as 'f such that x maps to' and means the same as '$f(x) =$'.

You can sometimes determine the nature of a turning point without checking points on either side.

Example 16

Find the coordinates of any turning points of the curve $y = 9x - 3x^2 + 8$ and determine their nature.

Answer
At turning points:
$\frac{dy}{dx} = 9 - 6x = 0$
$x = 1.5$

Solve for x.

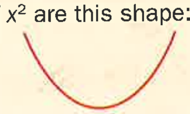
$y = 9(1.5) - 3(1.5)^2 + 8$
$= 14.75$

Substitute x = 1.5 into y = 9x − 3x² + 8.

The turning point is $(1.5, 14.75)$.
The turning point is a local maximum.

Quadratic graphs with a negative coefficient of x² are this shape:

Quadratic graphs with a positive coefficient of x^2 are this shape:



Exercise 6L

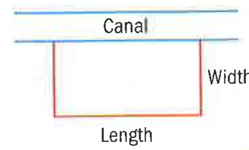
Find the coordinates of the local maximum or local minimum point for each quadratic curve.
State the nature of this point.

1 $y = x^2 - 4x + 10$ **2** $y = 18x - 3x^2 + 2$ **3** $y = x^2 + x - 3$
4 $y = 8 - 5x + x^2$ **5** $y = 3x + 11 - x^2$ **6** $y = 20 - 6x^2 - 15x$
7 $y = (x - 3)(x - 7)$ **8** $y = x(x - 18)$ **9** $y = x(x + 4)$

6.7 Using differentiation in modeling: optimization

An introductory problem

In Chapter 4, you used quadratic functions to model various situations. One of the optimization problems was to maximize the area of a rectangular field that bordered a straight canal and was enclosed on three sides by 120 m of fencing.



A model is a mathematical function that describes the situation. In this case, we need a model for the area of the field (the rectangle) for different widths.

First, identify the **variables** in the problem.
These are:

- the width of the field
- the length of the field
- the area of the field.

Second, identify any **constraints** in the problem. The constraint in this problem is that 120 m of fencing is used for three sides.

It often helps to try a few numerical examples in order to put the problem in context and to indicate the method. For example

1 If the width were 20 m, then the length would be $120 - 2(20) = 80$ m
the area would be $20 \times 80 = 1600$ m^2

2 If the width were 50 m, then the length would be $120 - 2(50) = 20$ m
the area would be $50 \times 20 = 1000$ m^2

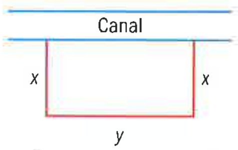
Note that, although the length of the fencing is constant, the size of the enclosed area varies.

Setting up the model

The model is for the area of the field and is a function of *both* its width and its length.

1 Define the variables.
Let A be the area of the field, x be the width of the field and y be the length of the field.
Then $A = xy$

2 Write the constraint algebraically.
$120 = 2x + y$



$x + x + y = 2x + y = 120$

3 Use the formula for the constraint to write the area function using just one variable.
Rearrange the constraint: $y = 120 - 2x$
Substitute in the area function: $A = xy = x(120 - 2x)$

If you define the variables in a different way, you obtain a different function. Here you could have defined the **length** to be x and the width as y. The area $A(x)$ would then have been a different – but correct – function.

So a model for the area of the field is $A(x) = x(120 - 2x)$, where x is the width of the field.

To determine the maximum area (the optimum solution) set the gradient function to zero.

The formula for the area is: $A(x) = x(120 - 2x)$

Expand the brackets: $A(x) = 120x - 2x^2$

Differentiate: $\frac{dA}{dx} = A'(x) = 120 - 4x$

Equate $\frac{dA}{dx}$ to zero: $120 - 4x = 0$

Solve: $4x = 120 \Rightarrow x = 30$

The quadratic function $A(x)$ has a negative coefficient of x^2 so the turning point is a maximum.

The width of the optimum rectangle is 30 m. To find the length substitute $x = 30$ into $y = 120 - 2x$.

$120 - 2(30) = 60$ m

The dimensions of the rectangle are width 30 m and length 60 m.

To find the maximum area substitute $x = 30$ into $A(x) = x(120 - 2x)$.

The maximum area is $A(30) = (30)(120 - 2(30)) = 1800$ m^2

→ In optimization problems, use differentiation to find an optimal value (either the maximum or the minimum) of a function as two variables interact.

You need to find an equation for this function in terms of these two variables and a constraint formula which links the variables. The constraint formula is used to remove one of the variables.

You can only use differentiation in functions with one variable.

Example 17

Optimize the function $A = 3xy$ subject to the constraint $x + y = 20$.

Answer	
$y = 20 - x$	*Rearrange the constraint so y is the subject.*
$A = 3xy = 3x(20 - x)$	*Substitute y into the function.*
$A(x) = 60x - 3x^2$	*Simplify.*
$\frac{dA}{dx} = 60 - 6x$	*Differentiate.*
$60 - 6x = 0 \Rightarrow x = 10$	*Set $\frac{dA}{dx}$ to zero and solve for x.*
$A(10) = 60(10) - 3(10)^2 = 300$ The optimal value of A is 300.	*Substitute the value of x into A(x) to find the optimal value of A.*

$A(x)$ is a quadratic function. Is the value 300 a maximum or a minimum?

Exercise 6M

1 $A = bh$, subject to the constraint $b - h = 7$.
a Use the constraint to express b in terms of h.
b Express A in terms of h.

2 $V = 3xt$ subject to the constraint $x + t = 10$.
a Use the constraint to express x in terms of t.
b Express V in terms of t.

3 $p = x^2y$ subject to the constraint $2x + y = 5$.
a Use the constraint to express y in terms of x.
b Express p in terms of x.

4 $R = \frac{1}{2}nr^2$ subject to the constraint $n - r = 25$.
a Express R in terms of r. **b** Express R in terms of n.

Choosing which variable to eliminate is an important skill. A bad choice will make the function more complicated.

5 $L = 2m(m + x)$ subject to the constraint $\frac{1}{2}(x + 5m) = 50$.

a Express L in terms of m. **b** Express L in terms of x.

6 $V = \pi r^2 h$ and $2r + h = 17$

a Express V in terms of r. **b** Express V in terms of h.

7 $y = 5x^2 + c$ and $12x - 2c = 3$

a Express y in terms of x.

b Use differentiation to find $\frac{dy}{dx}$.

c Hence find the minimum value of y.

d Find the value of c that corresponds to this minimum value.

How do you know, without testing the gradient, that it is a minimum?

8 $N = 2n(5 - x)$ and $12n + 10x = 15$

a Express N in terms of n.

b Use differentiation to find $\frac{dN}{dn}$.

c Hence find the minimum value of N.

d Find the value of x that corresponds to this minimum value.

9 Given $A = \frac{1}{2}LB$ and $3L - 5B = 18$, express A in terms of L. Hence find the minimum value of A and the value of B that corresponds to this minimum value.

10 Given $C = \pi fr$ and $r = 30 - 3f$, express C in terms of either f or r. Hence find the maximum value of C and the values of f and of r that correspond to this maximum value.

11 Given $a - b = 10$ and $X = 2ab$, find the minimum value of X.

12 Given $x + 2t = 12$, find the maximum/minimum value of tx and determine the nature of this optimum value.

Let $A = tx$

13 Given $3y + x = 30$, find the maximum/minimum value of $2xy$ and determine its nature.

14 Given $2M - L = 28$, find the values of L and M which give $3LM$ a maximum/minimum value. Find this optimum value, and determine its nature.

15 Given $c + g = 8$, express $c^2 + g^2$ in terms of g only. Hence find the minimum value of $c^2 + g^2$ subject to the constraint $c + g = 8$.

Let $A = c^2 + g^2$

16 The sum of two numbers is 6. Find the values of these numbers such that the sum of their squares is a minimum.

17 Given that $r + h = 6$, express r^2h in terms of r only. Hence find the maximum value of r^2h subject to the constraint $r + h = 6$.

18 Given that $m + n = 9$, find the maximum/minimum values of m^2n and distinguish between them.

At the beginning of this chapter we defined the optimal design of a can as the one that uses the smallest amount of metal to hold a given capacity. Example 18 calculates the minimum surface area for a can holding 330 cm³.

Example 18

Find the minimum surface area of a cylinder which has a volume of 330 cm³.

Answer

Let A be the total surface area of the cylinder. r be the radius of the base of the cylinder. h be the height of the cylinder.	*Define the variables.*
Then $A = 2\pi r^2 + 2\pi rh$ $\pi r^2 h = 330$	*The constraint is that the volume of the cylinder is 330 cm³.*
$h = \frac{330}{\pi r^2}$	*Rearrange to make h the subject.*
$A = 2\pi r^2 + 2\pi rh$ $= 2\pi r^2 + 2\pi r\left(\frac{330}{\pi r^2}\right)$	*Substitute the expression for h into the area function to reduce it to just one variable.*
$= 2\pi r^2 + \frac{660}{r}$	*Simplify.*
$A = 2\pi r^2 + 660r^{-1}$	*Write using indices.*
$\frac{dA}{dr} = 4\pi r + (-1)660r^{-2}$	*Differentiate.*
$\frac{dA}{dr} = 4\pi r - \frac{660}{r^2}$	*Simplify.*
$4\pi r - \frac{660}{r^2} = 0$	*Equate $\frac{dA}{dx}$ to zero to find the minimum.*
$4\pi r = \frac{660}{r^2}$	*Solve.*
$4\pi r^3 = 660$ $r^3 = \frac{660}{4\pi}$	
$r^3 = \frac{165}{\pi} \Rightarrow r = \sqrt[3]{\frac{165}{\pi}}$	You could solve this using a GDC.
$r = 3.74$ cm to 3 sf.	

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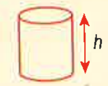
Is a drinks can perfectly cylindrical? What modeling assumptions do you need to make?

The surface area of a cylinder, $A = 2\pi r^2 + 2\pi rh$



The volume of a cylinder, $V = \pi r^2 h$

At $r = 1$, $\frac{dA}{dr} = 4\pi(1) - \frac{660}{(1)^2} < 0$

At $r = 10$, $\frac{dA}{dr} = 4\pi(10) - \frac{660}{(10)^2} > 0$

So, the minimum surface area is

$A = 2\pi(3.74)^2 + \frac{660}{(3.74)} = 264$ cm^2

Check that the answer for r gives a local minimum point by checking the gradient on each side of r = 3.74.

r = 3.74 is a minimum.

You could find the height of the cylinder with this area by substituting $r = 3.74$ into $h = \frac{330}{\pi r^2}$.

Exercise 6N

Draw a diagram first.

1 A gardener wishes to enclose a rectangular plot of land using a roll of wire-netting that is 40 m long. One side of the plot is to be the wall of the garden.
How should he bend the wire-netting to enclose the maximum area?

2 The sum of two numbers is 20. Let the first number be x. Write down an expression for the second number in terms of x.
Find the value of x given that twice the square of the first number added to three times the square of the second number is a minimum.

EXAM-STYLE QUESTIONS

3 An **open** rectangular box has its length double its width. The total surface area of the box is 150 cm^2.
The width of the box is x cm, and its height is h cm. Express the total surface area of the box in terms of x and h.
Use this expression (constraint) to find the volume of the box in terms of x only.
Hence, find the greatest possible volume of the box, and the width, length and height of the box required to give this volume.

4 A piece of wire 24 cm long is to be bent to form a rectangle with just one side duplicated for extra strength. Find the dimensions of the rectangle that give the maximum area.

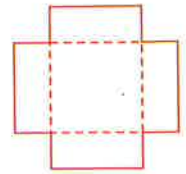
5 A long strip of metal 120 cm wide is bent to form the base and two sides of a chute with a rectangular cross-section.
Find the width of the base that makes the area of the cross-section a maximum.

6 The sum of the height and the radius of the base of a cone is 12 cm. Find the maximum volume of the cone and the values of the height and the radius required to give this volume.

7 A closed box with a square base is to be made out of 600 cm^2 of metal. Find the dimensions of the box so that its volume is a maximum. Find the value of this maximum volume.

8 The total surface area of a closed cylindrical tin is to be 600 cm^2.
Find the dimensions of the tin if the volume is to be a maximum.

9 A square sheet of metal of side 24 cm is to be made into an open tray of depth x cm by cutting out of each corner a square of side x cm and folding up along the dotted lines as shown in the diagram.
Show that the volume of the tray is $4x(144 - 24x + x^2)$ cm^3.
Find the value of x for this volume to be a maximum.



10 A rectangular sheet of metal measures 16 cm by 10 cm. Equal squares of side x cm are cut out of each corner and the remainder is folded up to form a tray of depth x cm. Show that the volume of the tray is $4x(8 - x)(5 - x)$ cm^3, and find the maximum volume.

11 A tin of soup is made in the shape of a cylinder so that the amount of metal used in making the tin is a minimum. The volume of the tin is 350 cm^3.

a If the radius of the base of the tin is 5 cm, find the height of the tin.

b If the radius of the base of the tin is 2 cm, find the height of the tin.

c **i** Use the volume of the tin to write down the constraint between the radius of the tin and its height.

ii Show that the constraint can be written as $h = \frac{350}{\pi r^2}$

iii Find an expression for A, the total surface area of a cylinder, in terms of r only.

iv Find the dimensions of the tin that minimize the total surface area of the tin.

v Find the value of this minimum area.

The metal used in making the tin is the surface area of the cylinder.

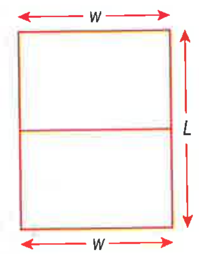
12 The diagram shows a rectangular field with an area of 50 000 m^2. It has to be divided in half and also fenced in.
The most efficient way to enclose the area is to construct the fencing so that the total length of the fence is minimized.

a If the length (L) of the field is 200 m, what is the width?

b Find the total length of the fencing in this case.

c Use the fixed area to write down the problem constraint algebraically.

d Find the dimensions of the field that make the length of fencing a minimum.
Find the perimeter of the field in this case.



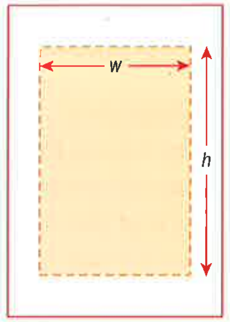
13 A second rectangular field is identical to that in question **12**.
The cost of the fencing around the perimeter is \$3 per metre.
The cost of the dividing fence is \$5 per metre. The most efficient way to enclose the area minimizes the total **cost** of the fence.

a Find the total cost of the fencing when the length (L) is 200 m.

b Use the fixed area to write down the problem constraint algebraically.

c Find the dimensions of the field that make the **cost** of the fencing a minimum. Find the cost in this case.

14 The page of a mathematics book is designed to have a printable area of $144\,cm^2$ plus margins of 2 cm along each side and 3 cm at the top and the bottom. The diagram for this is shown with the printable area shaded.



a If the width of the printable area (w) is 9 cm, find its height (h). Using these values, find the area of the page.

b If the width of the printable area is 14 cm, calculate the area of the page.

c Write down an expression for the printable area in terms of w and h.

d Write down an expression for P, the area **of the page** in terms of w and h.

e Use the results of **c** and **d** to show that $P = 168 + 4h + \frac{864}{h}$.

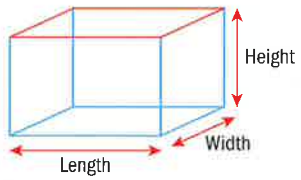
f Find the dimensions of the page that minimize the page area.

15 A fish tank is to be made in the shape of a cuboid with a rectangular base, with a length twice the width. The volume of the tank is fixed at 225 litres. The tank is to be made so that the total length of steel used to make the frame is minimized.

a **i** If the length of the base is 100 cm, what is its width?

ii Show that in this case, the height of the tank is 45 cm.

iii Find the total length of the steel frame.



b If the width of the tank is x, find an expression for the volume of the tank in terms of x and h, the height of the tank.

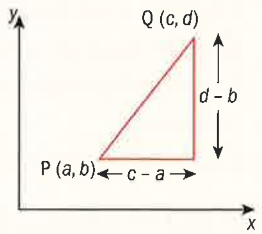
c Show that L, the total length of the steel frame, can be written as $L = 6x + \frac{450\,000}{x^2}$.

d Find the dimensions of the tank that minimize the length of the steel frame. Find also the length of the frame in this case.

CHAPTER 6 SUMMARY

Introduction to differentiation

- If P is the point (a, b) and Q is (c, d) then the gradient, m, of the straight line PQ is $m = \frac{d-b}{c-a}$.



The gradient function

- To differentiate a function, find the gradient function:

Function	Gradient function
$y = ax^n$	$\frac{dy}{dx} = nax^{n-1}$
$f(x) = ax^n$	$f'(x) = nax^{n-1}$

The process is valid for **all** values of n, both positive and negative.

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Calculating the gradient of a curve at a given point

- You can use the gradient function to determine the exact value of the gradient at any specific point on the curve.
- At a local maximum or minimum, $f'(x) = 0$ $\left(\frac{dy}{dx} = 0\right)$

The tangent and the normal to a curve

- The tangent to the curve at any point P is the straight line which passes through P with gradient equal to the gradient of the curve at P.
- To find the equation of the tangent to the curve at P(a, b):
  1. Calculate b, the y-coordinate of P, using the equation of the curve.
  2. Find the gradient function $\frac{dy}{dx}$.
  3. Substitute a, the x-coordinate of P, into $\frac{dy}{dx}$ to calculate, m, the value of the gradient at P.
  4. Use the equation of a straight line $(y - b) = m(x - a)$.
- The normal is perpendicular to the tangent so its gradient, m', is found using the formula $m' = \frac{-1}{m}$, where m is the gradient of the tangent.

Rates of change

- For the graph $y = f(x)$, the gradient function $\frac{dy}{dx} = f'(x)$ gives the rate of change of y with respect to x.

Local maximum and minimum points (turning points)

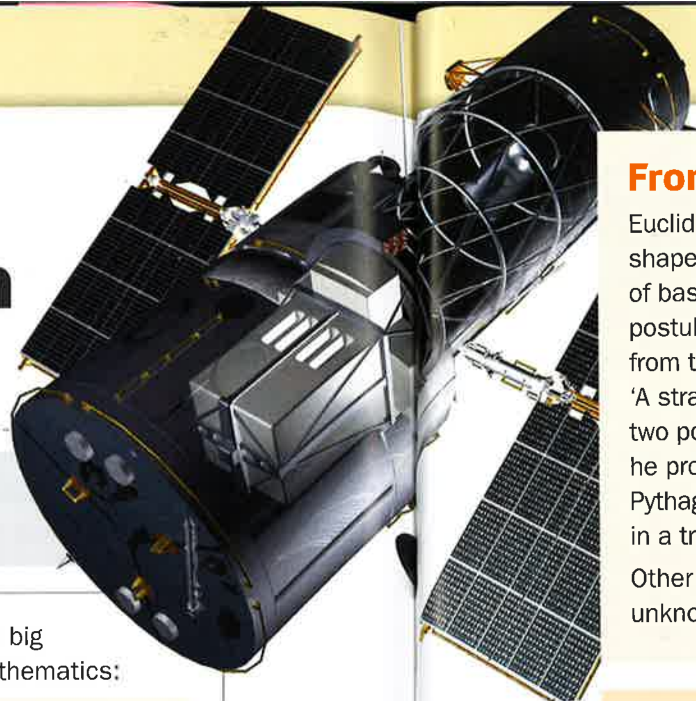
- At a local maximum, the curve stops increasing and changes direction so that it 'turns' and starts decreasing. So, as x increases, the three gradients occur in the order: positive, zero, negative. Where the gradient is zero is the maximum point.
- At a local minimum, the curve stops decreasing and changes direction; it 'turns' and starts increasing. So, as x increases, the three gradients occur in the order: negative, zero, positive. Where the gradient is zero is the minimum point.
- At any stationary or turning point – either local maximum or local minimum – $f'(x)$ is zero.

Using differentiation in modeling: optimization

- In optimization problems, use differentiation to find an optimal value (either the maximum or the minimum) of a function as two variables interact.

Theory of knowledge

Mathematics – invention or discovery?



The invention of the Hubble Telescope, which has been orbiting the Earth since 1990, has allowed astronomers to discover quasars, the existence of dark energy and the age of the universe.

- Write down
  - 3 'things' that have been **invented**
  - 3 'things' that have been **discovered**.

Maybe under 'inventions' you have included such things as the wheel, the electric motor, the mp3 player. In 'discoveries' you could have included friction, electricity, magnetism, the fanged frog, the source of the Nile.

From these lists it appears that inventions are generally objects we can touch and feel, whereas discoveries are generally naturally occurring phenomena. People create inventions with their hands and with machinery. They seek new discoveries (often using new inventions to do so).

The laws of nature are but the mathematical thoughts of God.

Euclid

This brings us to one of the big questions in TOK about mathematics:

- Is **mathematics** invented – just made-up, agreed-upon-conventions – or is it something humans somehow discover about the outside world?

We can use mathematics successfully to model real-world processes.

If mathematics is simply an invention of the human mind how can there be such wonderful applications in the outside world?

- Is this because we create mathematics to mirror the world?
- Or is the world intrinsically mathematical?

In chapter 7 TOK you can see how the chambers of a nautilus shell relate to Fibonacci spirals.

From Euclidean to non-Euclidean geometry

Euclid formalized the rules of geometric shapes on flat planes. He began with a set of basic assumptions – his axioms and postulates – that seemed to come naturally from the observed world. For example, 'A straight line can be drawn between any two points'. Building upon these foundations, he proved properties of shapes, such as Pythagoras' Theorem and that interior angles in a triangle sum to 180°.

Other interesting geometrical properties are unknowable through Euclidean geometry.

For example, the angles of a triangle drawn with straight lines on the flat, 2-D surface of a sphere add up to more than 180°. Thus, non-Euclidean geometry was born, with different systems relying on new axioms.

- Does this suggest that mathematics is an invention?
- Can anyone start with any set of (non-contradictory) axioms that they want and create their very own mathematical system of rules, laws and theorems?

Axiomatic systems

You can create a system of axioms, but if they match the fundamental truths of the natural universe then the rules and laws arising from them are also bound by these fundamental principles. The conclusions (like Pythagoras' Theorem) already exist – whether you discover them or not. And if your system is consistent, no other conclusions are possible.

- Does this suggest that mathematics is a discovery?

Axiomatic systems can be thought of as inventions, but they also reveal new truths about the nature of numbers – and that part is the discovery.

An invention like the internal combustion engine is bound by the law of conservation of energy.

Newton vs. Leibniz

The development of calculus was truly a culmination of centuries of work by mathematicians all over the world.

The 17th century mathematicians Isaac Newton (English) and Gottfried Wilhelm Leibniz (German) are recognized for the actual development of calculus. One of the most famous conflicts in mathematical history is the argument over which one of them invented or discovered calculus first and whether any plagiarism was involved.

Today it is generally believed that Newton and Leibniz did develop calculus independently of one another.

Modern-day calculus emerged in the 19th century, due to the efforts of mathematicians such as Augustin-Louis Cauchy (French), Bernhard Riemann (German), Karl Weierstrass (German), and others.

- What are some consequences when people seek personal acclaim for their work?
- Suppose that Newton and Leibniz did develop calculus independently of one another. Would this offer support to the idea that calculus was discovered?
- Did the work of these mathematicians arise from the need to solve certain real-world problems or purely from intellectual curiosity?