10 Geometry and trigonometry 2

CHAPTER OBJECTIVES:

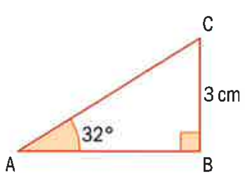
5.4 Geometry of three-dimensional solids; distance between two points; angle between two lines or between a line and a plane

5.5 Volumes and surface areas of three-dimensional solids

Before you start

You should know how to:

1 Use trigonometry in a right-angled triangle, e.g.



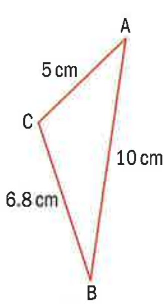
$$\sin 32° = \frac{3}{AC}$$

$$AC = \frac{3}{\sin 32°}$$

$$AC = 5.66\text{ cm (3 sf)}$$

2 Find an angle, a side or the area of any triangle, e.g.

a using the cosine rule
$c^2 = a^2 + b^2 - 2ab\cos C$:



$$\cos \hat{C} = \frac{6.8^2 + 5^2 - 10^2}{2 \times 6.8 \times 5}$$

$$\cos \hat{C} = -0.4229\ldots$$

$$\hat{C} = 115°\text{ (3 sf)}$$

b using the formula

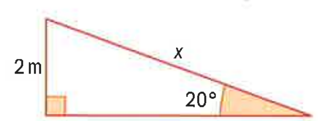
$$A = \frac{1}{2}ab\sin C$$

$$A = \frac{1}{2} \times 6.8 \times 5 \times \sin 115°$$

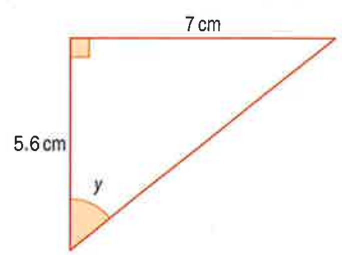
$$= 15.4\text{ cm}^2$$

Skills check

1 **a** Find x in this triangle.



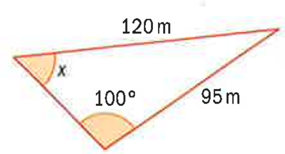
b Find angle y.



2 In this triangle

a find the angle x

b find the area.



Goods are transported all around the world in containers like this. These cuboid-shaped metal boxes come in uniform sizes, so they can be moved from lorry to train to ship using standard equipment.

A company using containers for transport needs to work out how many of their products will fit into one container, and so how many containers they will need. They might need to calculate the maximum possible length of pipe that would fit into a container, on the diagonal.

A company manufacturing containers needs to know how many square metres of metal are needed to make each container.

In Chapter 5 you used geometry and trigonometry to solve problems in two dimensions. In this chapter you will learn how to calculate lengths and angles and solve problems in three dimensions.

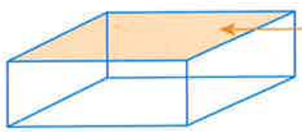
10.1 Geometry of three-dimensional solids

Geometry is the study of points, lines, planes, surfaces and solids.

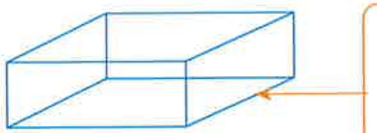
No dimensions	One dimension	Two dimensions	Three dimensions
point	line	plane	solid

Can you draw a point with no dimensions?

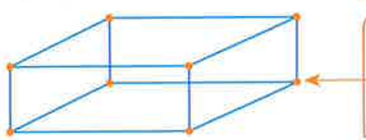
A **plane** is a flat surface.



All the **faces** of a solid together make up the **surface** of the solid. A face of the solid may be **plane** or curved. A cuboid has 6 plane faces.



An **edge** is a line segment where two faces of a solid meet. This cuboid has 12 edges. The edges form the framework of the solid.



A **vertex** is a point where three or more edges meet. This cuboid has 8 vertices.

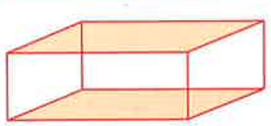
Euclid – the 'father of geometry'
Euclid (c325–c265 BCE) founded a school of mathematics in Alexandria, Egypt, and wrote thirteen volumes of *The Elements of Geometry*. These were the standard mathematics textbook for over 2000 years.

There are two groups of solids:

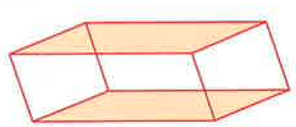
- Solids with all faces **plane**:
  - prisms
  - pyramids
- Solids with at least one **curved** face, e.g. cylinder, cone, sphere

Right prisms

→ In a **right prism** the end faces are the same shape and size and are parallel. All the other faces are rectangles that are **perpendicular** to the end faces.



This is a right prism.



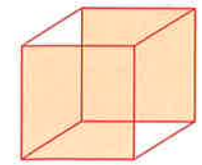
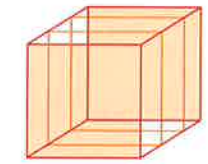
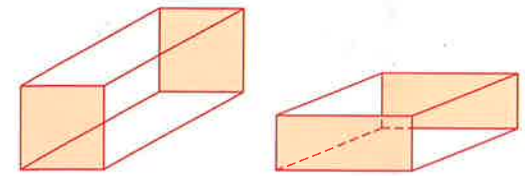
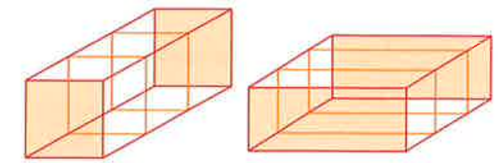
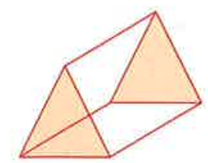
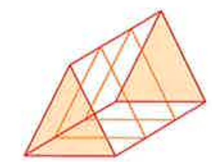
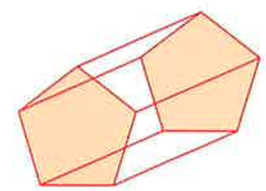
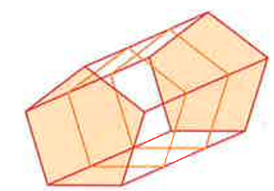
This is **not** a right prism. The end faces are not perpendicular to the other faces.

Remember that two figures with the same shape and size are said to be **congruent**. In a prism the end faces are congruent.

In Mathematical Studies you will only study right prisms.

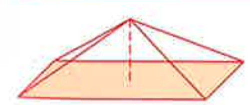
→ If you cut parallel to the end faces of a right prism, the **cross-section** will always be the same shape and size.

Can any cube be a cuboid? Can any cuboid be a cube?

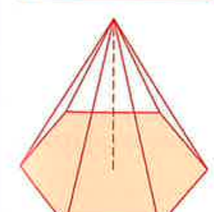
Right prism	**Cross-section**
 Cube	 Square
 Cuboid	 Square or rectangle
 Triangular prism	 Triangle
 Pentagonal prism	 Pentagon

Pyramids

→ The base of a **pyramid** is a polygon. The other faces are triangles that meet at a point called the **apex**. In a **right pyramid** the apex is vertically above the center of the base.

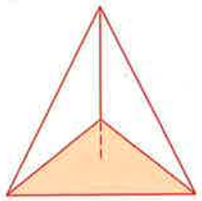


Rectangular-based pyramid
The base is a rectangle.



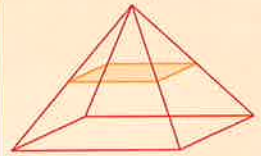
Hexagonal-based pyramid
The base is a hexagon.

The pyramid at Giza in Egypt is the oldest of the seven wonders of the ancient world. It remained the tallest built structure for over 3800 years. What is the tallest structure now? How has mathematics been used in its design?



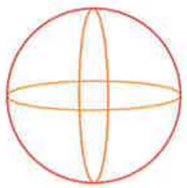
Triangular-based pyramid
The base is a triangle.

The **cross-sections** parallel to the base of a pyramid are the same shape as the base, but different sizes.

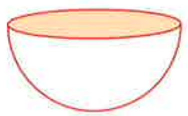


Solids with at least one curved face

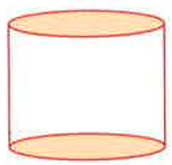
In these solids the plane faces are shaded.



A **sphere** has one curved face.



A **hemisphere** has two faces, one plane and one curved.



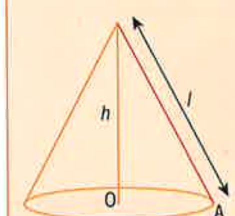
A **cylinder** has three faces, two plane and one curved.

Vertex or apex

A **cone** has two faces, one plane and one curved.

In Mathematical Studies you only study **right cones**. In a right cone the apex is vertically above the center of the base.
In a right cone:

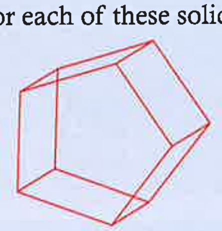
- the **vertical height** h is the distance from the apex to the center of the base
- the **slant height** l is the distance from the apex to any point on the circumference of the base.



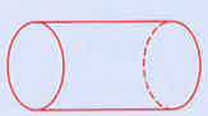
Example 1

For each of these solids

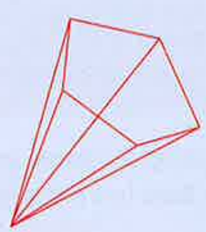
a



b



c



i Write down its name.
ii Write down the number of faces, the number of edges and the number of vertices.
iii Write down the number of plane faces and the number of curved faces.

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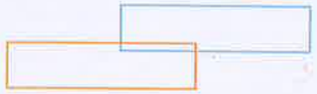
Answers

i	**a Pentagonal prism**	**b Cylinder**	**c Pentagonal-based pyramid**
ii Faces	7	3	6
Edges	15	2	10
Vertices	10	0	6
iii Plane faces	7	2	6
Curved faces	0	1	0

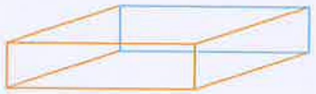
Investigation – drawing a prism

Step 1 Draw one of the end faces.

Step 2 Draw the other end face. Remember that the end faces are congruent.



Step 3 Join up corresponding vertices with parallel lines.

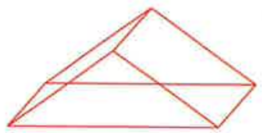


Now use this method to draw a triangular prism.

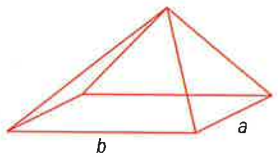
Exercise 10A

1 For each of these solids

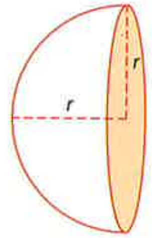
a



b



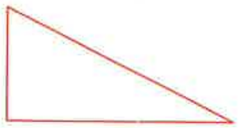
c



i Write down its name.
ii Write down the number of faces, the number of edges and the number of vertices.
iii Write down the number of plane faces and the number of curved faces.

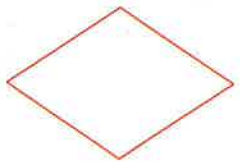
2 Draw prisms with these end faces.

a



b

c

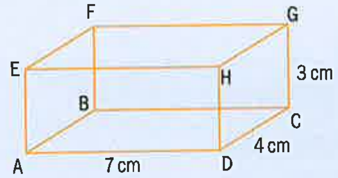


10.2 Distance between points in a solid

You might need to calculate the distance between two vertices in a solid, or the distance between a vertex and the midpoint of an edge, or the distance between the midpoints of two lines. To do this you need first to identify right-angled triangles and then use Pythagoras' theorem.

Example 2

The diagram shows a cuboid ABCDEFGH, where AD = 7 cm, DC = 4 cm, and CG = 3 cm.



a Find the length of

i AH **ii** AC **iii** DG **iv** AG.

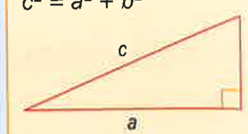
b Find the distance between

i the midpoint of CG and A

ii the midpoint of AD and the midpoint of CG.

Pythagoras' theorem

$c^2 = a^2 + b^2$

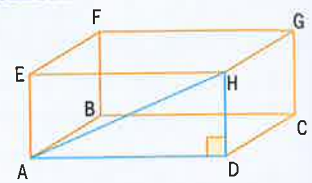


Answers

a i $AH^2 = 7^2 + 3^2$

$AH = \sqrt{58}$ cm = 7.62 cm (3 sf)

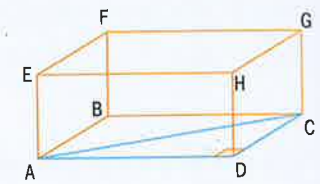
In the right-angled triangle ADH, AH is the hypotenuse.



ii $AC^2 = 7^2 + 4^2$

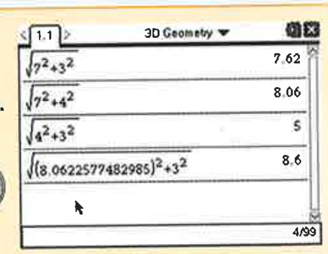
$AC = \sqrt{65}$ cm = 8.06 cm (3 sf)

In the right-angled triangle ABC, AC is the hypotenuse.



Enter the formula directly into your GDC. Use cut and paste for intermediate values.

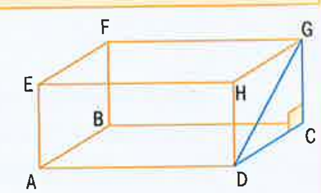
GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*



iii $DG^2 = 4^2 + 3^2$

DG = 5 cm

In the right-angled triangle DCG, DG is the hypotenuse.

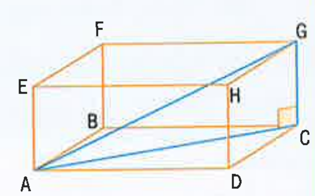


iv $AG^2 = AC^2 + CG^2$

$= \left(\sqrt{65}\right)^2 + 3^2$

$AG = \sqrt{74} = 8.60$ cm (3 sf)

In the right-angled triangle ACG, AG is the hypotenuse.



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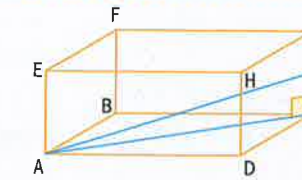
b i $AM^2 = AC^2 + CM^2$

$= \left(\sqrt{65}\right)^2 + 1.5^2$

AM = 8.20 cm (3 sf)

Let M be the midpoint of CG. In the right-angled triangle ACM, AM is the hypotenuse.



ii $MN^2 = MC^2 + CN^2$

$CN^2 = CD^2 + DN^2$

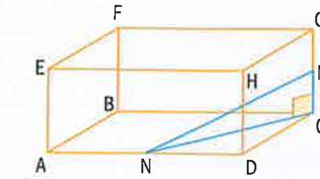
$CN^2 = 4^2 + 3.5^2$

$CN = \sqrt{28.25}$ cm

$MN^2 = 1.5^2 + \left(\sqrt{28.25}\right)^2$

MN = 5.52 cm (3 sf)

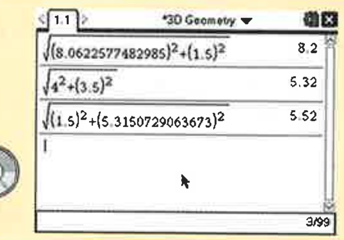
Let N be the midpoint of AD. In the right-angled triangle MCN, MN is the hypotenuse.



Finding CN:

Use cut and paste for the intermediate value CN.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*



Example 3

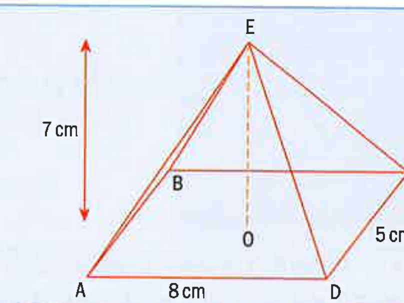
In the diagram, ABCD is the rectangular base of a right pyramid with apex E. The sides of the base are 8 cm and 5 cm, and the height OE of the pyramid is 7 cm.

Find the length of

a AC

b EC

c EM, where M is the midpoint of CD.

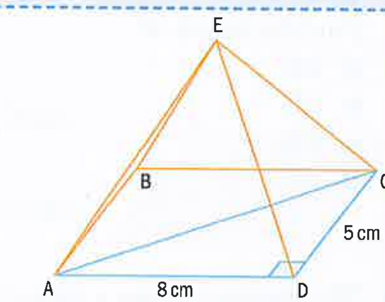


Answers

a $AC^2 = 8^2 + 5^2$

$AC = \sqrt{89}$ cm = 9.43 cm (3 sf)

In the right-angled triangle ADC, AC is the hypotenuse.



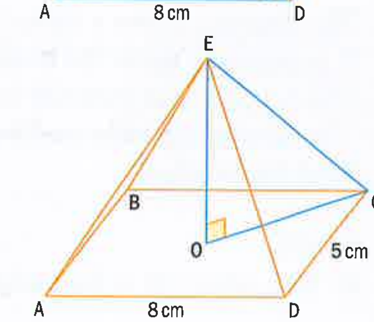
b $OC = \frac{AC}{2} = \frac{\sqrt{89}}{2}$

$EC^2 = OC^2 + OE^2$

$= \left(\frac{\sqrt{89}}{2}\right)^2 + 7^2$

EC = 8.44 cm (3 sf)

O is the center of the base where diagonals of the base meet. OC is half AC. OE is perpendicular to the base, therefore triangle EOC is right-angled. EC is the hypotenuse.



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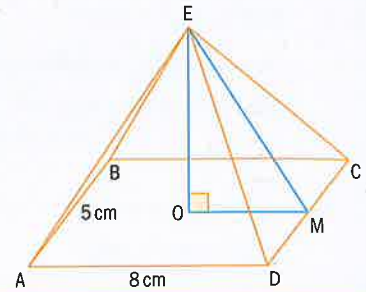
c $OM = \frac{AD}{2} = 4$

$EM^2 = 7^2 + 4^2$

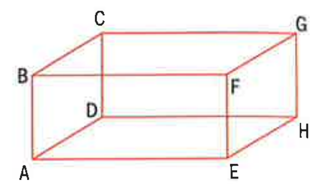
$EM = \sqrt{65}$ cm = 8.06 cm (3 sf)

EOM is a right-angled triangle.
EM is the hypotenuse.

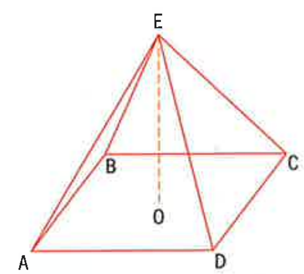


Exercise 10B

1 Copy the cuboid shown in the diagram.
In different sketches mark clearly these right-angled triangles:
a triangle ACD
b triangle AGH
c triangle HBA
d triangle MCD, where M is the midpoint of EH.

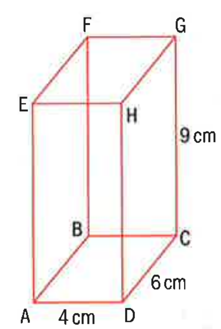


2 Copy the right pyramid shown in the diagram.
In different sketches mark clearly:
a triangle BCD
b triangle EOC
c triangle EOM, where M is the midpoint of CD.

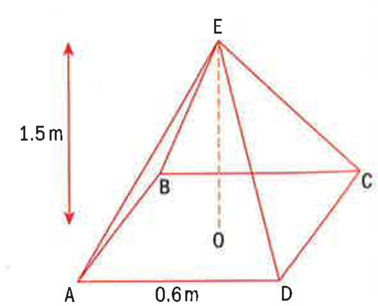


EXAM-STYLE QUESTIONS

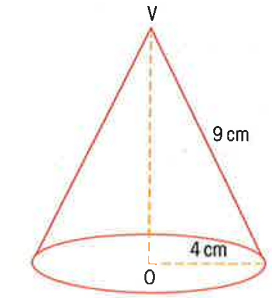
3 The diagram shows a cuboid ABCDEFGH, where AD = 4 cm, CD = 6 cm and CG = 9 cm.
Find the length of
a DB
b ED
c DG
d DF.



4 The diagram shows a square-based pyramid.
E is vertically above the middle of the base, O.
The height of the pyramid is 1.5 m.
The sides of the base are 0.6 m.
Find the length of
a AC
b ED
c EM, where M is the midpoint of CD.

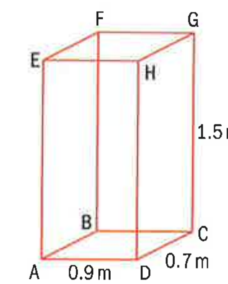


5 The diagram shows a cone with base center O and radius 4 cm. The slant height of the cone is 9 cm.
Find OV, the height of the cone.

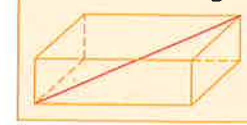


EXAM-STYLE QUESTIONS

6 The diagram represents a cupboard in a gym.
It has the dimensions shown.
a Calculate the length of AC.
b Find the length of the longest fitness bar that can fit in the cupboard.



The longest length in a cuboid is the diagonal.



Sketch the pyramid and label the lengths you know.

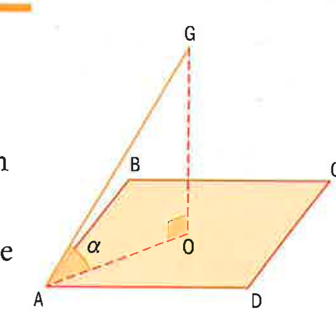
7 The Great pyramid of Giza has a square base. At the present time the length of one side of the base is 230.4 m and the height is 138.8 m.
a Calculate the length of the base diagonal.
b Calculate the distance from the apex to the midpoint of a side of the base.
c Calculate the length of one sloping edge of the pyramid.

10.3 Angles between two lines, or between a line and a plane

To calculate angles start by identifying right-angled triangles. Then use trigonometry.

In the diagram, ABCD is a plane and AG is part of a line. To find the angle α that AG makes with the plane ABCD:

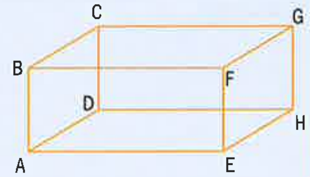
1 Drop a perpendicular from G to the plane.
2 Label the point where the perpendicular meets the plane.
3 Draw the right-angled triangle AOG. Angle α is opposite OG.
4 Use trigonometry to find α.



The angle between the plane ABCD and the line AG is also the angle between the lines OA and AG.

Example 4

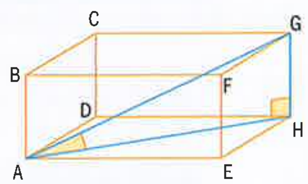
Copy the cuboid shown in the diagram.
Mark the angles described. Use a different diagram for each angle.



a The angle that the plane ADHE makes with the line AG
b The angle that the plane ADHE makes with the line AC
c The angle that the plane ABCD makes with the line CE
d The angle between the lines BH and HA

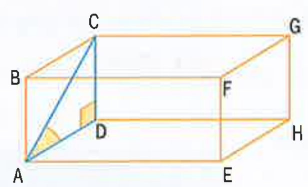
Answers

a



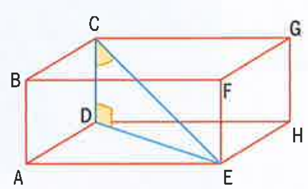
The edge GH is perpendicular to the face ADHE. The angle is opposite GH.

b



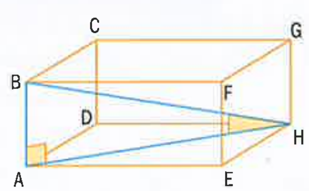
The edge CD is perpendicular to the face ADHE. The angle is opposite CD.

c



DE is perpendicular to the face ABCD. The angle is opposite DE.

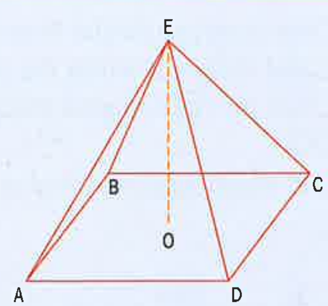
d



Draw both BH and HA to obtain angle AHB.

Example 5

Copy the diagram of a rectangular-based pyramid.
E is vertically above the middle of the base, O. Mark the angles described. Use a different diagram for each angle.

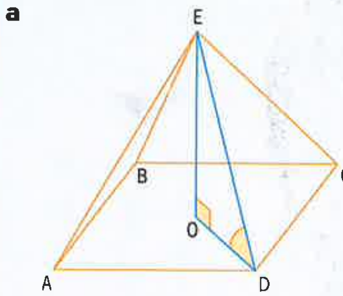


a The angle that the base ABCD makes with the edge DE
b The angle that the base ABCD makes with ME, where M is the midpoint of CD
c The angle between the lines BE and ED
d The angle between the lines DE and EC

▶ Continued on next page

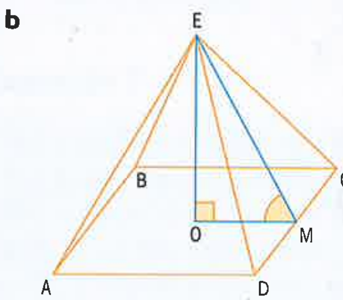
Answers

a



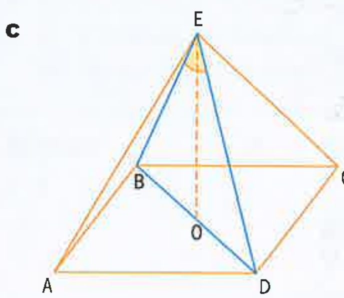
Drop a perpendicular from E to O. The angle is ODE.

b



Drop a perpendicular from E to O. The angle is OME.

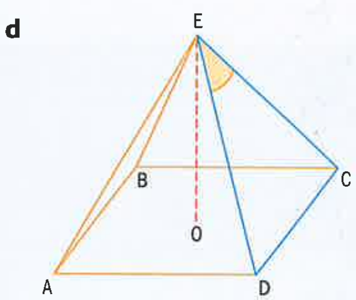
c



The angle is BED.

Notice that BED is an isosceles triangle.

d

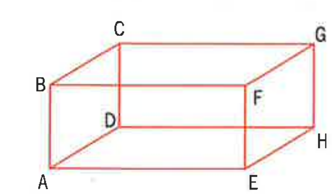


The angle is DEC.

Notice that DEC is an isosceles triangle.

Exercise 10C

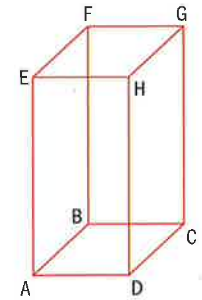
1 Copy the cuboid shown and mark the angles described. Use a different diagram for each angle.



a The angle that the face ADHE makes with the line EG
b The angle that the face ADHE makes with the line EC
c The angle that the face EFGH makes with the line CE
d The angle between the lines CE and CF
e The angle between the lines CE and EA

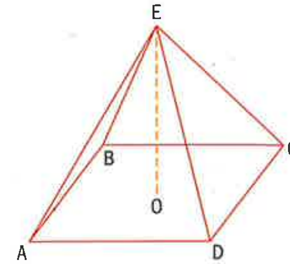
2 Copy the cuboid and mark the angles described.
Use a different diagram for each angle.

- **a** The angle between the face AEHD and DG
- **b** The angle between the face AEHD and DF
- **c** The angle between the lines CF and CA
- **d** The angle between the lines AH and HG



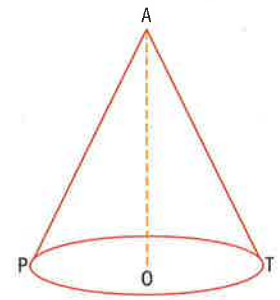
3 Copy the diagram of a square-based right pyramid.
Mark the angles described. Use a different diagram for each angle.

- **a** The angle between the base of the pyramid and the edge EC
- **b** The angle between the edges EC and AE
- **c** The angle between the line ME and the base, where M is the midpoint of CD



4 The diagram shows a right cone with base center O.
A is the apex. T and P are on the circumference of the base and O is the midpoint of PT.
On a copy of the diagram mark these angles. Use a different diagram for each angle.

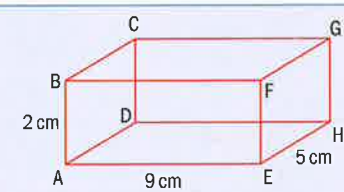
- **a** The angle that the sloping edge AT makes with the base
- **b** The angle that the sloping edge AT makes with PT. What is the relationship between this angle and the angle described in part **a**?
- **c** The angle between the sloping edges AT and AP. What type of triangle is PAT?



Example 6

The diagram shows the cuboid ABCDEFHG. AE is 9 cm, AB is 2 cm and EH is 5 cm.
Calculate the angle

- **a** that the plane ADHE makes with the line AG
- **b** between the lines BH and HE.



Answers

a $\tan G\hat{A}H = \dfrac{GH}{HA}$

AGH is a right-angled triangle with $G\hat{H}A = 90°$ and GH = 2 cm. Calculate $G\hat{A}H$.

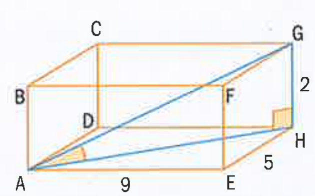
$HA^2 = 9^2 + 5^2$

$HA = \sqrt{106}$ cm

$\tan G\hat{A}H = \dfrac{2}{\sqrt{106}}$

$G\hat{A}H = 11.0°$ (3 sf)

The tangent links the sides GH and HA. Find HA using Pythagoras. Keep the exact value $\sqrt{106}$ for the next calculation to get the final answer as accurate as possible. Substitute for HA in the tangent. Round to 3 sf in the last step.



$\tan = \dfrac{\text{opposite}}{\text{adjacent}}$

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b $\tan B\hat{H}E = \dfrac{BE}{EH}$

As BCHE is a rectangle, triangle BEH is right-angled. So BEH is a right-angled triangle with $B\hat{E}H = 90°$. Calculate angle BHE. The tangent links the sides BE and EH.

$BE^2 = 2^2 + 9^2$

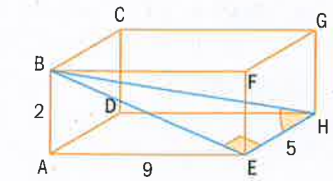
$BE = \sqrt{85}$ cm

Find BE using Pythagoras.

$\tan B\hat{H}E = \dfrac{\sqrt{85}}{5}$

Substitute for BE in the tangent.

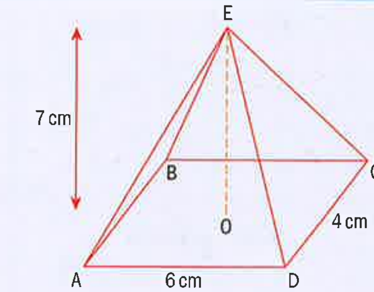
$B\hat{H}E = 61.5°$ (3 sf)



Example 7

The diagram shows the right pyramid ABCDE.
The base is a rectangle with AD = 6 cm and CD = 4 cm.
The height of the pyramid is 7 cm.

- **a** **i** Calculate angle AEO.
  - **ii** Calculate angle AEC.
- **b** Calculate angle EMO, where M is the midpoint of CD.
- **c** **i** Calculate the length of ED.
  - **ii** Hence calculate angle DEC.



Answers

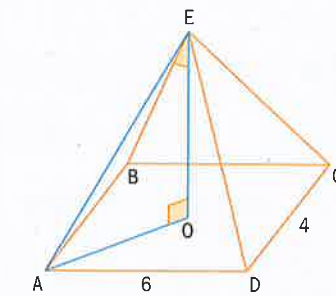
a **i** $\tan A\hat{E}O = \dfrac{AO}{EO}$

AOE is a right-angled triangle with $\hat{O} = 90°$. We are looking for angle AEO. The tangent links AO (half of AC) and EO, the height. Find AC using Pythagoras and then halve it.

$AC^2 = 6^2 + 4^2$

$AC = \sqrt{52}$ cm

$AO = \dfrac{\sqrt{52}}{2}$ cm



$\tan A\hat{E}O = \dfrac{\frac{\sqrt{52}}{2}}{7}$

Substitute for AO and EO in the tangent.

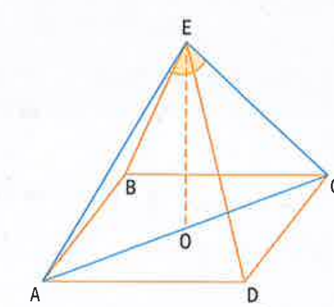
$A\hat{E}O = 27.3°$ (3 sf)

ii $A\hat{E}C = 2 \times A\hat{E}O$

$= 2 \times 27.252\ldots$

$A\hat{E}C = 54.5°$ (3 sf)

Triangle AEC is isosceles, so EO is a line of symmetry. EO bisects angle AEC. So $A\hat{E}C$ is twice $A\hat{E}O$.



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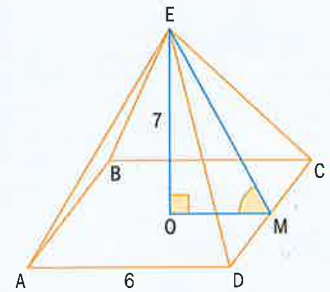
b $\tan E\hat{M}O = \frac{EO}{OM}$

$\tan E\hat{M}O = \frac{7}{3}$

$E\hat{M}O = 66.8°$ (3 sf)

EMO is a right-angled triangle with $\hat{O} = 90°$.
The tangent links EO and OM.
OM is half of AD = $\frac{6}{2} = 3$.



c **i** $ED^2 = OD^2 + OE^2$

$OD = \frac{BD}{2}$

$BD = AC = \sqrt{52}$ cm

So $OD = \frac{\sqrt{52}}{2}$ cm

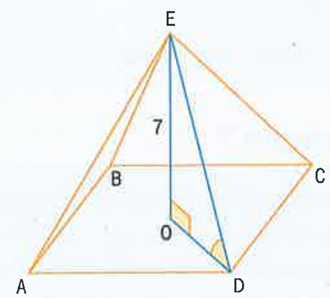
$ED^2 = \left(\frac{\sqrt{52}}{2}\right)^2 + 7^2$

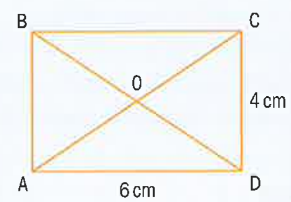
$ED = \sqrt{62}$ cm

$= 7.87$ cm (3 sf)

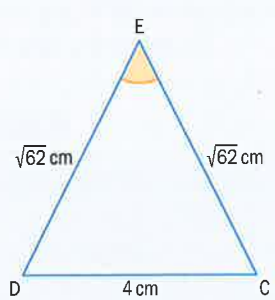
In the right-angled triangle EOD with $\hat{O} = 90°$, ED is the hypotenuse.
Apply Pythagoras in triangle EOD. You need to find OD.
OD is half of BD, which is the same length as AC, which you have already found in ***a i***.

Substitute for ED and OE in Pythagoras' theorem.

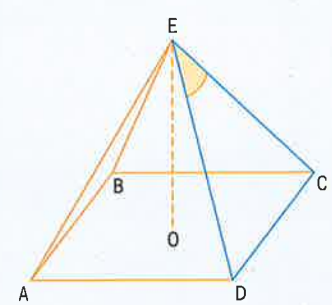




ii



This is angle DEC.
Triangle DEC is isosceles and you know the lengths of the three sides (ED = EC = $\sqrt{62}$ cm from ***c i****).*



$\cos D\hat{E}C = \frac{(\sqrt{62})^2 + (\sqrt{62})^2 - 4^2}{2 \times (\sqrt{62}) \times (\sqrt{62})}$

$D\hat{E}C = 29.4°$ (3 sf)

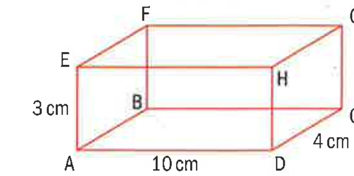
Use the cosine rule in triangle DEC.

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Exercise 10D

EXAM-STYLE QUESTIONS

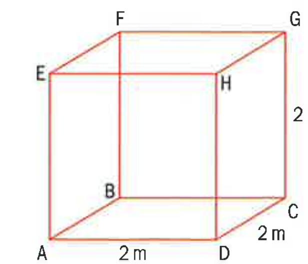
1 In the cuboid ABCDEFGH, AD = 10 cm, CD = 4 cm and AE = 3 cm.



a **i** Calculate the length of AC.
ii Calculate the angle that AG makes with the face ABCD.

b **i** Calculate the length of AF.
ii Find the angle that the face AEFB makes with the line AG.

2 The diagram shows cube ABCDEFGH with side length 2 m.

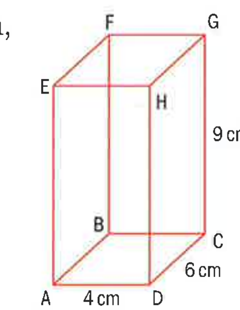


a Calculate the length of BD.
b Find the angle that DF makes with the face ABCD.

Let M be the midpoint of BF.

c Find the angle that MD makes with the face ABCD.

3 The diagram shows a cuboid ABCDEFGH, where AD = 4 cm, CD = 6 cm and CG = 9 cm.

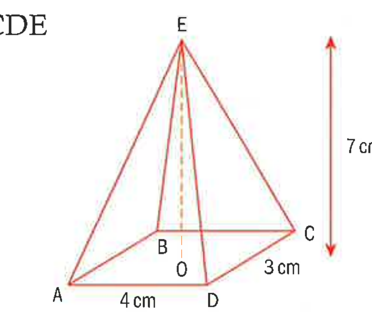


a **i** Calculate the length of BD.
ii Find the angle that AF makes with the face BFGC.

b Find the angle that AF makes with the face ABCD.

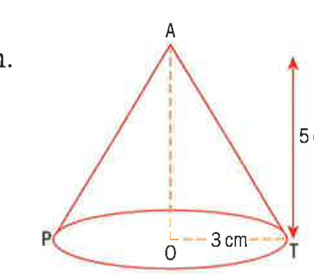
c **i** Calculate the length of AC.
ii Calculate the length of FC.
iii Find the angle between the lines AF and FC.

4 The diagram shows the rectangular-based right pyramid ABCDE with AD = 4 cm, CD = 3 cm and EO = 7 cm.



a Find the length of AC.
b Find the length of AE.
c Find angle AEC.
d Find the angle that AE makes with the base of the pyramid.
e Find the angle that the base of the pyramid makes with EM, where M is the midpoint of CD.

5 The diagram shows a cone with base center O and radius 3 cm. A is 5 cm vertically above O. T and P are on the circumference of the base and O is the midpoint of PT.



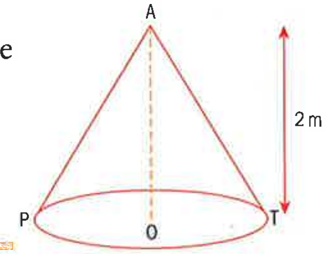
a Find AT, the slant height of the cone.
b Find the angle that AT makes with the base of the cone.
c Find angle PAT.

EXAM-STYLE QUESTION

6 A beach tent has the shape of a right cone. The center of the base is O and the base area is $5\,\text{m}^2$. The tent is $2\,\text{m}$ high. It is attached to the sand at points P and T, and O is the midpoint of PT.

a Find the radius of the base.

b Find angle PAT.



10.4 Surface areas of three-dimensional solids

→ The **surface area** of a solid is the sum of the areas of all its faces. Surface area is measured in square units, e.g. cm^2, m^2.

To calculate surface areas, first sketch the solid.

There are two types of solid:

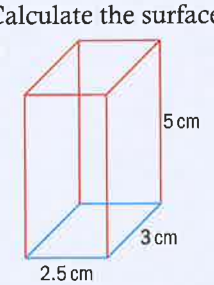
- Solids with all their faces plane
  e.g. prisms (except cylinders), pyramids (except cones), or combinations of these
- Solids with at least one curved face
  e.g. cylinders, spheres, hemispheres, cones, or combinations of these

Surface areas of solids with all faces plane

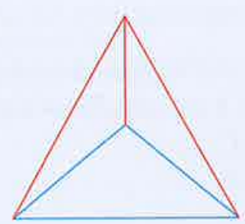
Example 8

Calculate the surface areas of these solids.

a



b



A triangular-based right pyramid with all edges 5 cm

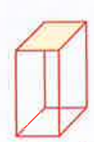
This solid is called a **regular tetrahedron**.

Answers

a Surface area of cuboid
$= 2 \times 2.5 \times 3 + 2 \times 3 \times 5 + 2 \times 2.5 \times 5$
$= 70\,\text{cm}^2$

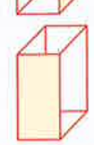
There are 6 rectangular faces:

- *2 faces of 2.5×3*



- *2 faces of 3×5*

- *2 faces of 2.5×5*



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b Surface area of tetrahedron
$= 4 \times$ area of triangle

Area of one triangle $= \frac{1}{2} \times 5 \times 5\sin 60°$
$= 10.825\ldots\,\text{cm}^2$

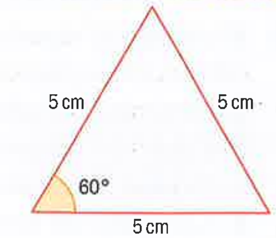
Surface area $= 4 \times 10.825\ldots$
$= 43.3\,\text{cm}^2$ (3 sf)

There are 4 identical faces. Each face is an equilateral triangle:

Use the formula for the area of a triangle

$A = \frac{1}{2}ab\sin C$

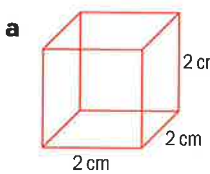
Remember not to round until the end of the calculation.



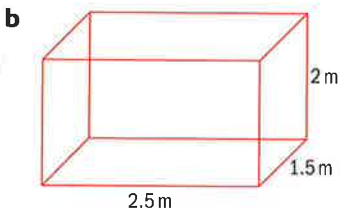
Exercise 10E

1 Calculate the surface areas of these solids.

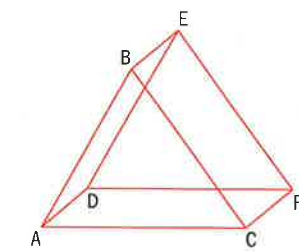
a



b



c ABCDEF is a prism. CF is 5 cm and triangle ABC is equilateral with sides 4 cm.



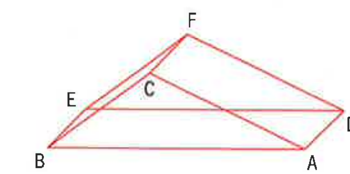
EXAM-STYLE QUESTIONS

2 ABCDEF is a right prism. BE is 4 cm, and triangle ABC is isosceles with AC = CB = 3 cm and angle BCA = 120°.

a Find the area of triangle ABC.

b Find the length of the edge AB.

c Find the surface area of the prism.

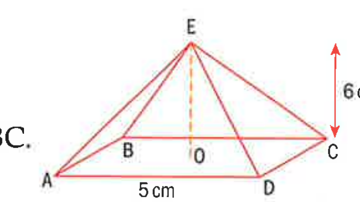


3 ABCDE is a square-based right pyramid and O is the middle of the base. The side length of the base is 5 cm. The height of the pyramid is 6 cm.

a Calculate the length of EM, where M is the midpoint of BC.

b Calculate the area of triangle CDE.

c Calculate the surface area of the pyramid.



4 The surface area of a cube is $600\,\text{m}^2$. Calculate its side length. Give your answer in cm.

EXAM-STYLE QUESTION

5 Each edge of a cube is 5.4 m.

a Calculate the surface area of the cube.

b Give your answer to part **a** in the form $a \times 10^k$ where $1 \le a < 10$, $k \in \mathbb{Z}$.

EXAM-STYLE QUESTION

6 The diagram represents Jamal's room, which is in the shape of a cuboid. He is planning to paint all the surfaces except the floor, the door and the window.

The door is 2 metres high and 1.3 metres wide, and the window is a square with a side length of 1 metre.

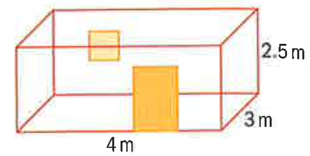
a Calculate the surface area that Jamal intends to paint.

Jamal needs 1.2 litres of paint to cover 1 m^2.

b Calculate the number of litres of paint that Jamal needs. Round **up** your answer to the next whole litre.

One litre of paint costs US$4.60.

c Calculate how much Jamal will spend on paint. Give your answer correct to 2 decimal places.



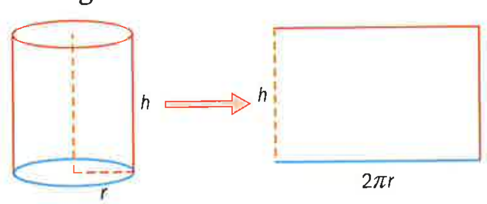
Round up your answer to the next whole number as you buy paint in litres.

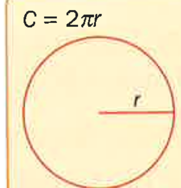
Surface areas of solids with at least one curved face

Cylinder

A cylinder has three faces: one curved and two plane. If you cut the curved face and open it out, you get a rectangle. The length of the curved face and open it out, you get a rectangle. The length of the rectangle is the circumference of the base of the cylinder.

If h is the height and r is the radius of the base





This formula is in the Formula booklet.

→ Area of curved surface of a cylinder = $2\pi rh$

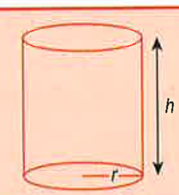
Area of a circle = πr^2

The cylinder has two equal circular faces

Area of two circles = $2\pi r^2$

Therefore

→ Total surface area of a cylinder = $2\pi rh + 2\pi r^2$



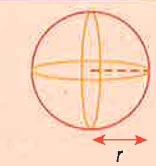
Was π invented or discovered? When was it first used? Why is it denoted with a Greek letter?

Sphere

A sphere has one curved face.

Let r be the radius of the sphere, then

→ Surface area of a sphere = $4\pi r^2$



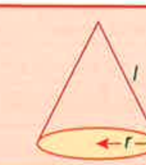
This formula is in the Formula booklet.

Cone

A cone has two faces: one plane and one curved.

Let r be the radius and l the slant height of the cone, then

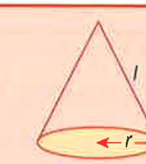
→ Area of curved surface of a cone = πrl



This formula is in the Formula booklet.

The base of a cone is a circle, therefore

→ Total surface area of a cone = $\pi rl + \pi r^2$



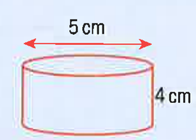
Example 9

For each of these solids, calculate

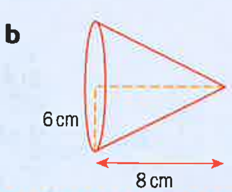
i the area of the curved surface

ii the total surface area.

a



b



Answers

a **i** Area of curved surface
$= 2\pi \times 2.5 \times 4$
$= 20\pi = 62.8$ cm^2 (3 sf)

Area of curved surface of cylinder $= 2\pi rh$

Base radius = 2.5 cm

ii Area of two circular faces
$= 2\pi \times 2.5^2$
$= 39.26...$ cm^2

Total surface area of cylinder
$= 62.8... + 39.26...$
$= 102$ cm^2 (3 sf)

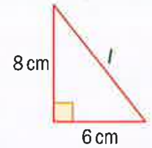
Total surface area $= 2\pi rh + 2\pi r^2$

Continued on next page

b i $l^2 = 6^2 + 8^2$

$l = 10$ cm

Use Pythagoras to find the slant height, l, of the cone.



Curved surface area of cone $= \pi \times 6 \times 10$

$= 60\pi = 188$ cm² (3 sf)

Curved surface area of cone $= \pi rl$

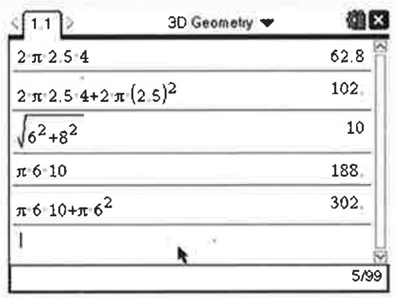
ii Total surface area of cone

$= 60\pi + \pi \times 6^2$

$= 96\pi$ cm²

$= 302$ cm² (3 sf)

Total surface area of cone $= \pi rl + \pi r^2$



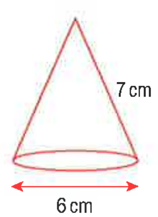
GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 10F

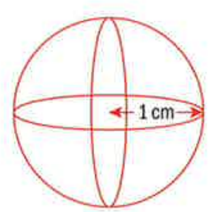
EXAM-STYLE QUESTIONS

1 Calculate the surface area of each solid.

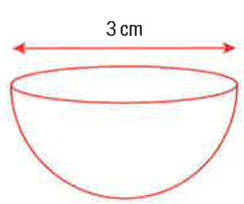
a



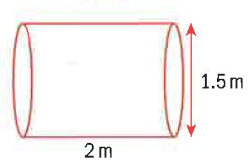
b



c



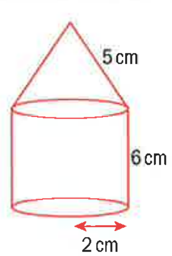
d



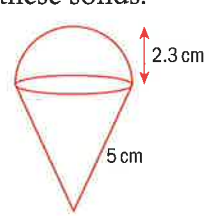
2 Calculate the surface areas of these solids.

Split each solid into two solids.

a



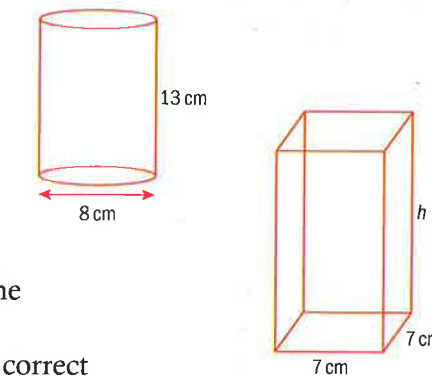
b



3 The surface area of a sphere is 1000 cm². Find its radius.

EXAM-STYLE QUESTION

4 The first diagram shows a cylindrical pencil container made of leather. The base diameter is 8 cm and the height is 13 cm.



a Calculate the area of leather needed to make this pencil container.

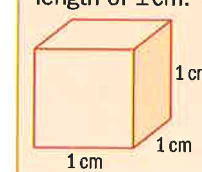
Another container is made in the shape of a cuboid as shown. The square base has sides of 7 cm. This container uses the same area of leather as the cylindrical one.

b Find the height, h, of the cuboid. Give your answer correct to 2 significant figures.

10.5 Volumes of three-dimensional solids

→ The volume of a solid is the amount of space it occupies and is measured in cubic units, e.g. cm³, m³, etc.

Remember that one cubic centimetre is the space occupied by a cube with edge length of 1 cm.

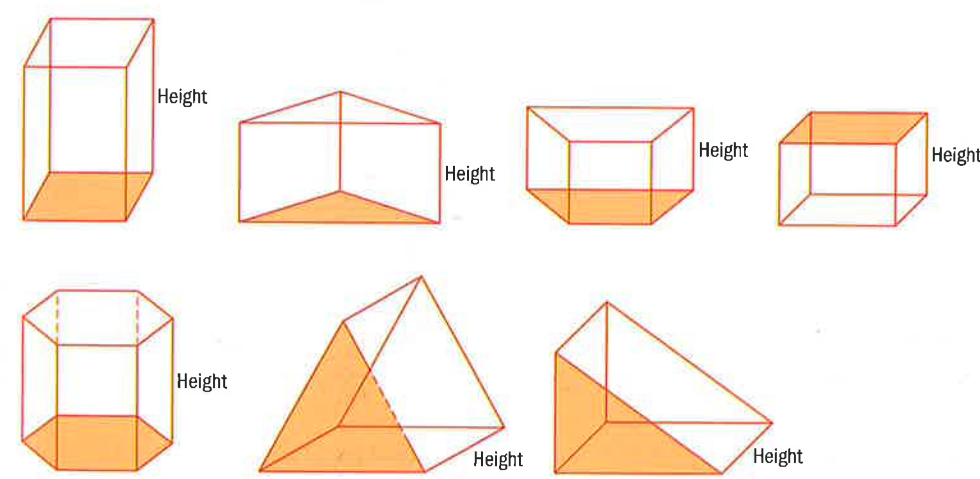


Volume of a prism

To calculate the volume of a prism you need to know

- the area of the **cross-section** of the prism (the end face)
- the **height** (distance between the two end faces).

In each of the prisms shown one end face is shaded and the height is labeled.



→ Volume of a prism is
V = area of cross-section × height

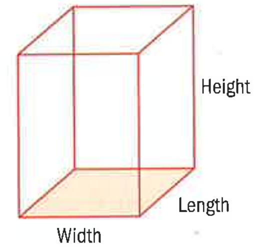
This formula is in the Formula booklet.

A cuboid is a prism with cross-section a rectangle.

Volume of a cuboid = area of cross-section × height

Area of cross-section = length × width

so Volume of a cuboid = length × width × height



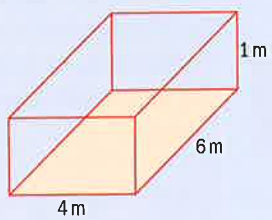
> → Volume of a cuboid is
> $V = l \times w \times h$,
> where l is the length, w is the width, h is the height.

This formula is in the Formula booklet.

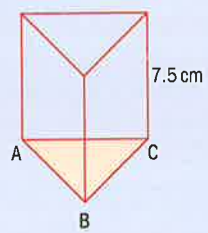
Example 10

Calculate the volumes of these prisms.

a

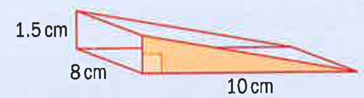


b



Area of ABC is $12\,cm^2$

c



Answers

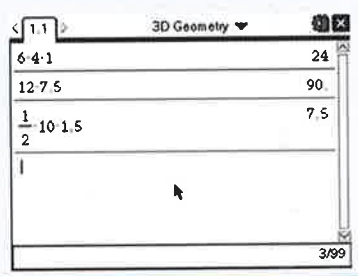
a Volume $= l \times w \times h$
$= 6 \times 4 \times 1$
$= 24\,m^3$

b Volume = area of cross-section × height
$= 12 \times 7.5 = 90\,cm^3$

c Area of cross-section $= \frac{1}{2}(b \times h)$
$= \frac{1}{2}(10 \times 1.5) = 7.5\,cm^2$
Volume = area of cross-section × height
$= 7.5 \times 8 = 60\,cm^3$

First calculate the area of the cross-section.

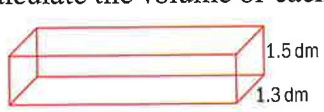
Area of triangle $= \frac{1}{2}(b \times h)$



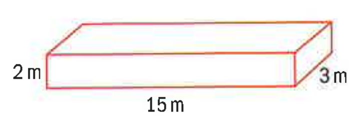
Exercise 10G

1 Calculate the volume of each prism.

a

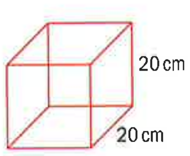


b

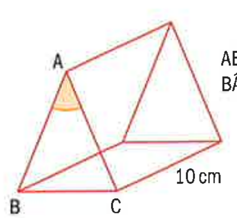


In part **a** the volume will be in dm^3.

c

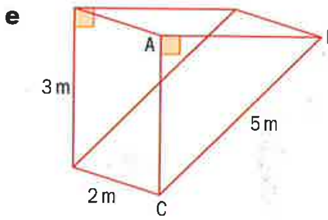


d

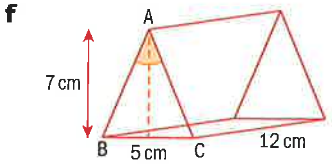


AB = AC = 8 cm
BÂC = 30°

e

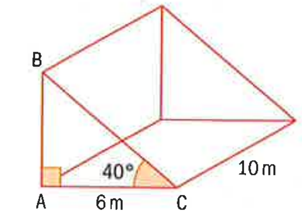


f

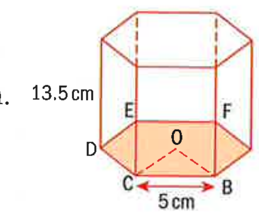


EXAM-STYLE QUESTIONS

2 The diagram shows a triangular prism.
Angle CAB = 90°.
a Calculate the length of AB.
b Calculate the area of triangle ABC.
c Calculate the volume of the prism.



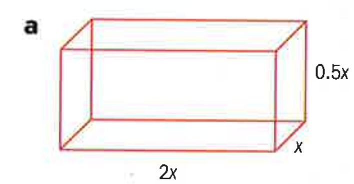
3 The diagram shows a prism with ABCDEF a regular hexagon. Each side of the hexagon is 5 cm and the height of the prism is 13.5 cm.
a What size is angle COB?
b Find the area of triangle COB.
c Find the area of the regular hexagon ABCDEF.
d Find the volume of the prism.



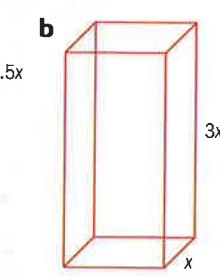
What type of triangle is OCB?

4 Find expressions for the volume, V, of each of these prisms.
Give answers in their simplest form.
(All the dimensions are in cm.)

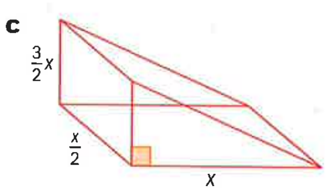
a



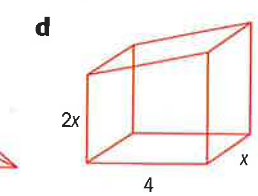
b



c

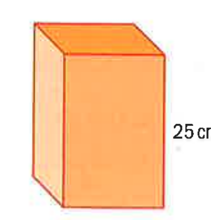


d



EXAM-STYLE QUESTION

5 A box with a square base has a volume of $11\,025\,cm^3$ and a height of 25 cm. Each side of the base is x cm.
a Write down an expression in terms of x for the volume of the box.
b Hence write down an equation in x.
c Find the value of x.



EXAM-STYLE QUESTION

6 An open box is cubical in shape. It has no lid. The volume of the box is 9261 cm^3.

a Find the length of one edge of the box.

b Find the total external surface area of the box.

Extension material on CD: *Worksheet 10 - Volume of a truncated cone*

Volume of a cylinder

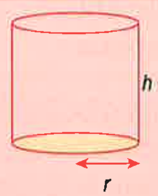
A cylinder is a prism with a circular cross-section.

Volume of cylinder = area of cross-section × height

→ Volume of a cylinder is

$V = \pi r^2 h,$

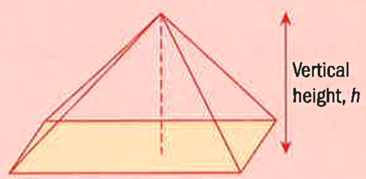
where r is the radius and h is the height



This formula is in the Formula booklet.

Volume of a pyramid

→ Volume of a pyramid $= \frac{1}{3}$(area of base × vertical height)

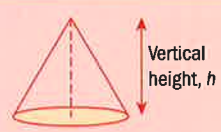


This formula is in the Formula booklet.

Volume of a cone

→ Volume of a cone $= \frac{1}{3}\pi r^2 h,$

where r is the radius and h is the vertical height

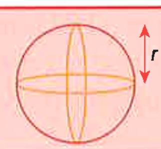


This formula is in the Formula booklet.

Volume of a sphere

→ Volume of a sphere $= \frac{4}{3}\pi r^3,$

where r is the radius

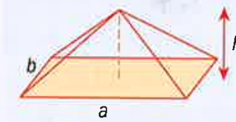


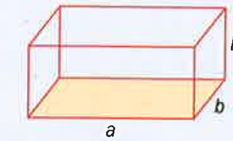
This formula is in the Formula booklet.

Investigation – relationships between volumes

Write an expression for the volume of each solid.

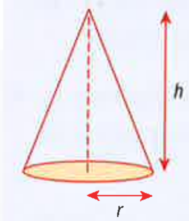
What do you notice?

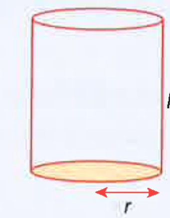




What can you say about the volume of a cuboid and the volume of a pyramid with the same base and height as that cuboid?

What is the relationship between the volumes of these two solids?





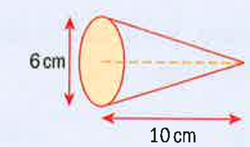
Take a cone and cylinder of the same height and radius. Fill the cone with rice. Pour it into the cylinder. How many times do you have to do this to fill the cylinder?

Example 11

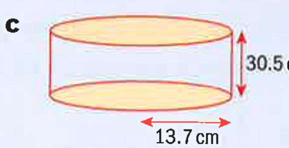
Calculate the volume of each solid.

a

b



c



Answers

a Volume of pyramid

$= \frac{1}{3}(3.7 \times 5 \times 2.6)$

$= 16.0\text{ m}^3$ (3 sf)

b Volume of cone

$= \frac{1}{3}\pi \times 3^2 \times 10 = 30\pi$

$= 94.2\text{ cm}^3$ (3 sf)

c Volume of cylinder

$= \pi \times 13.7^2 \times 30.5$

$= 18\,000\text{ cm}^3$ (3 sf)

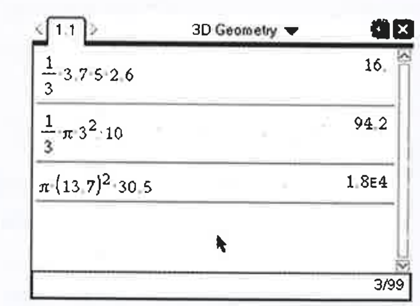
Volume of pyramid

$= \frac{1}{3}$*(area of base × vertical height)*

Area of base $= 3.7 \times 5$

Volume of cone $= \frac{1}{3}\pi r^2 h$

Volume of cylinder $= \pi r^2 h$

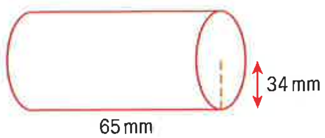


GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 10H

1 Calculate the volume of each solid.

a



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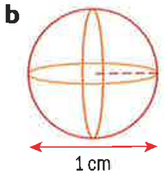

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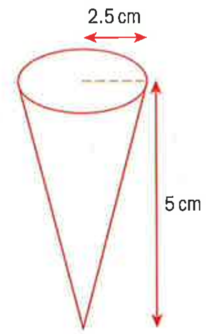

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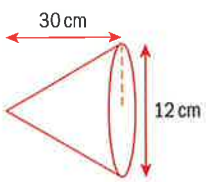

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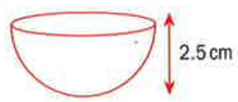

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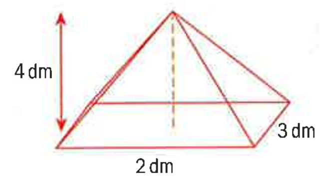


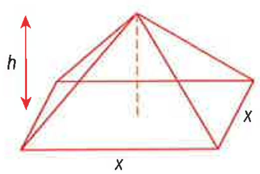
EXAM-STYLE QUESTION

2 A cylindrical water tank has height 3 m and base radius 1.2 m.
- **a** Calculate the volume of the tank in m^3.
- **b** Give your answer to part **a** in dm^3.
- **c** Hence find, in litres, the capacity of the tank.

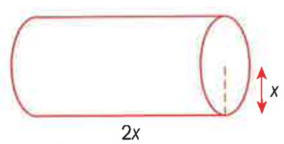
Capacity is the amount of liquid a container can hold when it is full.

3 Find an expression for the volume, V, of each solid. Give each answer in its simplest form.

a



b



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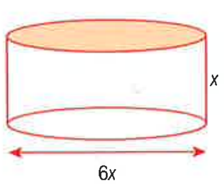

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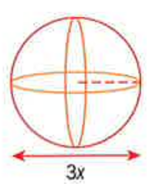
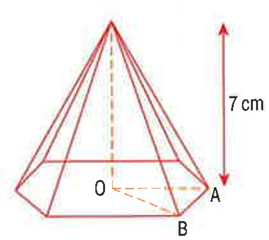


EXAM-STYLE QUESTION

4 The diagram shows a right pyramid with a regular hexagonal base. The volume of the pyramid is $84\,cm^3$ and the height is 7 cm. O is the center of the base.
- **a** Calculate the area of the base.
- **b** Calculate the area of triangle AOB.
- **c** What size is angle AOB?
- **d** Calculate the length of AB.



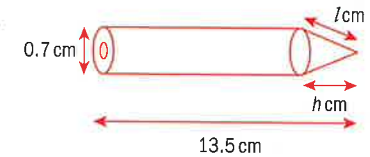
5 A spherical ball has a volume of $200\,cm^3$.
- **a** Find the radius of the ball.
- **b** Give your answer to part **a** correct to the nearest mm.

EXAM-STYLE QUESTIONS

6 A cylindrical container has base radius 15 cm and height 30 cm. It is full of sand.
- **a** Calculate the volume of sand in the container.

The sand is poured into a second container in the shape of a cuboid. The length of the cuboid is 60 cm, the width is 20 cm, and the height is 17 cm.
- **b** Is the second container big enough for all the sand? Justify your decision.

7 A cyclindrical pencil is 13.5 cm long with diameter 0.7 cm. It is sharpened to a cone as shown in the diagram.



The length of the cyclindrical part is now 12.3 cm.
The height of the cone is h cm and its slant height is l cm.
- **a** **i** Write down the value of h.
  - **ii** Find the value of l.
- **b** Hence find
  - **i** the total surface area of the pencil
  - **ii** the volume of the pencil.

  Give your answers to 3 sf.

Sketch the cone. Use Pythagoras to find l.

The pencils are packed in boxes. The boxes are cuboids of width 5.6 cm, height 1.4 cm and length 13.5 cm.
- **c** Show that the maximum number of pencils that will fit in the box is 16.
- **d** Find the space in a full box that is **not** occupied by the pencils.
- **e** Write your answer to part **d** as a percentage of the volume of the box. Give your answer correct to 2 sf.

Sketch the box.

Review exercise

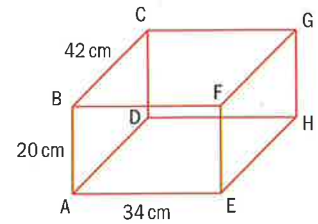
Paper 1 style questions

EXAM-STYLE QUESTIONS

1 The cuboid ABCDEFGH is shown in the diagram.
AB = 20 cm, BC = 42 cm and AE = 34 cm.

a Calculate the surface area of the cuboid.

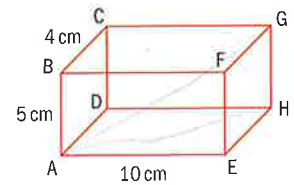
b Calculate the volume of the cuboid, giving your answer in dm^3.



2 The cuboid ABCDEFGH is shown in the diagram.
AB = 5 cm, BC = 4 cm and AE = 10 cm.

a Calculate the length of AH.

b Calculate the angle that AG makes with the face ADHE.

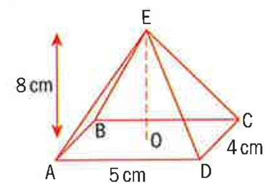


3 The diagram shows a rectangular-based right pyramid ABCDE. The height of the pyramid is 8 cm. The base is 5 cm long and 4 cm wide.
Calculate

a the length of AC

b the length of EC

c the angle AEC.

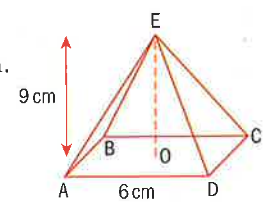


4 The diagram shows a square-based right pyramid ABCDE. The height of the pyramid is 9 cm. Each edge of the base is 6 cm.
Calculate

a the distance between the midpoint of DC and E

b the area of triangle DCE

c the surface area of the pyramid.



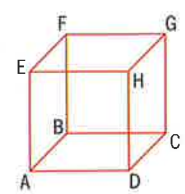
5 The diagram shows a hollow cube ABCDEFGH. Its volume is 512 cm^3.

a Write down the length of any edge of the cube.

b Find the distance AC.

Rosaura puts a pencil in the cube. The pencil is 13.5 cm long.

c Does the pencil fit in the cube? Justify your decision.

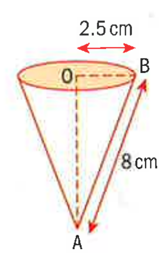


6 A cone has the dimensions shown in the diagram.
Point B is on the circumference of the base, point O is the center of the base and point A is the apex of the cone.

a Calculate the size of the angle that AB makes with the base of the cone.

b **i** Calculate the height of the cone.

ii Calculate the volume of the cone.

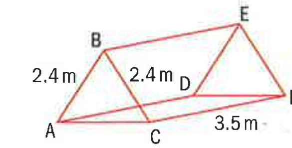


EXAM-STYLE QUESTION

7 The diagram represents a tent in the shape of a prism.
The front of the tent, ABC, is an isosceles triangle with AB = BC = 2.4 m and $A\hat{B}C = 110°$. The tent is 3.5 m long.

a Calculate the area of the front face of the tent ABC.

b Calculate the space inside the tent.



Paper 2 style questions

EXAM-STYLE QUESTIONS

1 An office tower is shown in the diagram. It consists of a cuboid with a square base and a square-based right pyramid.

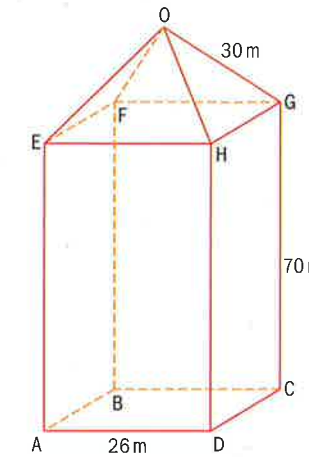
a Calculate the distance from O to M, the midpoint of HG.

b Calculate the height of the tower.

c Find the angle that OM makes with the plane EFGH.

A cleaning services company charges US$ 78 per m^2 to clean the outside of a building.

d Calculate the cost of cleaning the tower, giving your answer correct to the nearest US$.



2 A solid sculpture consist of a hemisphere of radius 3 cm and a right circular cone of slant height l as shown in the diagram.

a Show that the volume of the hemisphere is $18\pi cm^3$.

The volume of the hemisphere is two-thirds that of the cone.

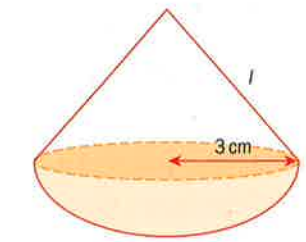
b Find the vertical height of the cone.

c Calculate the slant height of the cone.

d Calculate the angle between the slanting side of the cone and the flat face of the hemisphere.

The sculpture is made of a material that weighs 10.8 g per cm^3.

e Calculate the weight of the sculpture, giving your answer in kg.



3 ABCDE is a solid glass right pyramid.
The base of the pyramid is a square of side 5 cm and center O.
The vertical height is 7 cm.

a Calculate the volume of the pyramid.

The glass weighs 8.7 grams per cm^3.

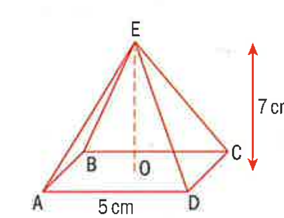
b Calculate the weight of the pyramid, giving your answer correct to the nearest g.

c Find the length of a sloping edge of the pyramid, giving your answer correct to 4 significant figures.

d Calculate the angle made between the edge ED and the base of the pyramid.

e Calculate the size of the angle AED.

f Calculate the total surface area of the pyramid.



EXAM-STYLE QUESTION

4 The diagram shows a square-based right pyramid ABCDV. The midpoint of DC is M and VM is inclined at 65° to the base. The sides of the base are 8 cm and O is the center of the base.

a Find the height of the pyramid, giving your answer correct to 3 significant figures.

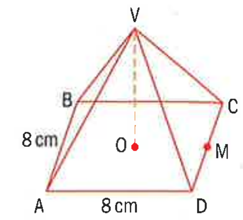
b Calculate

i the length of VM

ii the size of angle DVC.

c Find the total surface area of the pyramid.

d Find the volume of the pyramid, giving your answer correct to the nearest cm^3.



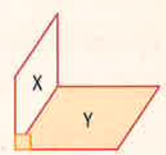
CHAPTER 10 SUMMARY

Geometry of three-dimensional solids

- In a **right prism** the end faces are the same shape and size and are parallel. All the other faces are rectangles that are **perpendicular** to the end faces.
- If you cut parallel to the end face of a right prism the **cross-section** will always be the same shape and size.
- The base of a **pyramid** is a polygon. The other faces are triangles that meet at a Point called the **apex**. In a **right pyramid** the apex is vertically above the center of the base.

Angles between two lines, or between a line and a plane

- When two faces of a solid, X and Y, are perpendicular, any line in face X is perpendicular to any line in face Y.

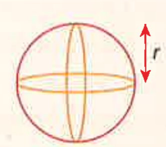


Surface areas of three-dimensional solids

- The **surface area** of a solid is the sum of the areas of all its faces. Surface area is measured in square units, e.g. cm^2, m^2.

- Area of curved surface of a cylinder = $2\pi rh$
  Total surface area of a cylinder = $2\pi rh + 2\pi r^2$

- Surface area of a sphere = $4\pi r^2$



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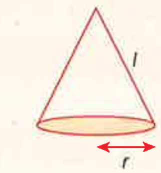
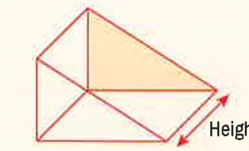
- Area of curved surface of a cone = πrl
- Total surface area of a cone = $\pi rl + \pi r^2$

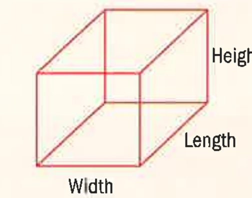


Volumes of three-dimensional solids

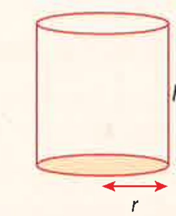
- The volume of a solid is the amount of space it occupies and is measured in cubic units, e.g. cm^3, m^3, etc.
- Volume of a prism is
  V = area of cross-section × height



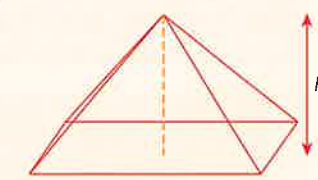
- Volume of a cuboid is
  $V = l \times w \times h$,
  where l is the length, w is the width, h is the height



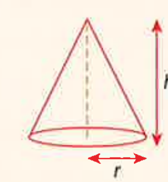
- Volume of a cylinder is
  $V = \pi r^2 h$,
  where r is the radius, h is the height



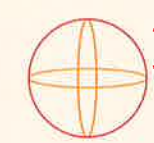
- Volume of a pyramid =
  $\frac{1}{3}$(area of base × vertical height)



- Volume of a cone = $\frac{1}{3}\pi r^2 h$,
  where r is the radius and h is the vertical height



- Volume of a sphere = $\frac{4}{3}\pi r^3$,
  where r is the radius



Mathematical proof

In mathematics, you can't just say a statement is true – you have to prove it.

A mathematical proof has to be rigorous. This means, it has to be true in all cases. In fact, one way of proving a statement isn't true is to find a counter example – just one example where it isn't true.

- Find a counter example to disprove the statement: **All prime numbers are odd.**

▲ Black Mondo Grass (*Ophiopogon planiscarpus nigrescens*) is native to Korea.

Searching for the truth

To produce a proof the mathematician starts with basic, self-evident assumptions called axioms. He or she then uses the rules of logic and deductive reasoning to reason toward proving a new theorem.

A new theorem then provides a base for further reasoning.

- How does mathematical proof differ from proof using 'good reasons' and 'sufficient evidence' in other areas of knowledge?

The Greek mathematician Euclid (c.300 B.C.E.) introduced the axiomatic method of proof. In the west, his book Elements was one of the standard geometry texts for students until the middle of the 20th century and forms the basis of what you learn in geometry today.

In the medieval period Islamic mathematicians developed further the ideas of arithmetic and algebra. These became the basis of more general proofs. In the 10th century C.E., the Iraqi mathematician Al-Hashimi provided general proofs for numbers and proved the existence of irrational numbers.

"A mathematician is a device for turning coffee into theorems."

Attributed variously to both the Hungarian mathematicians Paul Erdos (1913–86) and Alfréd Rényi (1921–70)

Proof problem

Complete a table of values like this for the equation $y = x^5 - 10x^4 + 35x^3 - 50x^2 + 5x$.

x	0	1	2	3	4
y					

- Predict the value of y when $x = 5$.
- Now work out the value when $x = 5$. Was your prediction correct?

A proof using mathematical induction shows that if one particular case is true, then so is the next one. It also shows that one particular base case is true.

- How did you use inductive reasoning to predict the value of y when $x = 5$?
- What were the problems with using inductive reasoning?

A mathematical proof

Prove the theorem:

The sum of any three consecutive even numbers is divisible by 6

Proof: Write the three consecutive even numbers as $2m$, $2m + 2$, $2m + 4$ where m is a whole number.

- What axioms have we used here?

Find their sum, S by adding them together:

$S = 2m + 2m + 2 + m + 4$

$= 6m + 6$

$= 6(m + 1)$

So S is a multiple of 6, and S is **always** divisible by 6.

- Use a similar method to show that the product of three consecutive even numbers is always divisible by 8.

Elegant and economical proof

Here are two solutions to the problem...

Prove that $(x + y + z)(x - y - z) = x^2 - (y + z)^2$

Solution 1

$(x + y + z)(x - y - z)$

$= x^2 - xy - xz + xy - y^2 - yz + xz - yz - z^2$

$= x^2 - y^2 - 2yz - z^2$

$= x^2 - (y^2 + 2yz + z^2)$

$= x^2 - (y + z)^2$

Solution 2

$(x + y + z)(x - y - z)$

$= (x + (y + z))(x - (y + z))$

$= x^2 - (y + z)^2$

- Which is the better solution?

Solution 1 and solution 2 both give the same answer, so neither is better than the other. However, solution 2 is more elegant and insightful.

"The mathematics are distinguished by a particular privilege, that is, in the course of ages, they may always advance and can never recede."

Edward Gibbon, *Decline and Fall of the Roman Empire*

A mathematician will seek to find a proof that is
* Economical – as short as possible.
* Elegant – with a surprise or moment of insight.