4 Mathematical models

CHAPTER OBJECTIVES:

6.1 Concept of a function, domain, range and graph; function notation; concept of a function as a mathematical model

6.2 Drawing accurate graphs and sketch graphs; transferring a graph from GDC to paper; reading, interpreting and making predictions using graphs

6.3 Linear models: linear functions and their graphs

6.4 Quadratic models: quadratic functions and their graphs (parabolas); properties of a parabola; symmetry, vertex; intercepts; equation of the axis of symmetry

6.5 Exponential models: exponential functions and their graphs; concept and equation of a horizontal asymptote

6.6 Use of a GDC to solve equations involving combinations of the functions above

Before you start

You should know how to:

1 Substitute values into a formula, e.g. Given that $x = -1$, find the value of $y = 3x^2 + 2x$.

$y = 3(-1)^2 + 2(-1) \Rightarrow y = 1$

2 Use your GDC to solve quadratic equations and simultaneous equations in two unknowns, e.g. Solve

a $3x^2 + 9x - 30 = 0 \Rightarrow x = 2, x = -5$

b $\begin{cases} x + y = 4 \\ -2x + y = 1 \end{cases} \Rightarrow x = 1, y = 3$

3 Find the gradient, m, of a line joining two points, e.g. A(3, 5) and B(1, 4).

$y - y_1 = m(x - x_1)$

$m = \dfrac{5-4}{3-1}$

$m = \dfrac{1}{2}$

Skills check

1 **a** Find the value of $y = 2.5x^2 + x - 1$ when $x = -3$.

b Find the value of $h = 3 \times 2^t - 1$ when $t = 0$.

c Find the value of $d = 2t^3 - 5t^{-1} + 2$ when $t = \dfrac{1}{2}$.

2 Use your GDC to solve:

a $x^2 + x - 3 = 0$

b $2t^2 - t = 2$

c $\begin{cases} x - 2y = 3 \\ 3x - 5y = -2 \end{cases}$

For help, see Chapter 12, Sections 1.1 and 1.2.

3 Find the gradient, m, of a line joining the two points:

a A(7, –2) and B(–1, 4)

b A(–3, –2) and B(1, 8)

The above photo shows the positions of a diver at various moments until he reaches the sea. Initially, the diver is at 40 m above sea level and it takes him 4.5 seconds to reach the sea. We can use mathematics to find a numerical relationship between the time in seconds, t, and the diver's height, h, in metres above sea level. The relationship linking the time, t, and the height, h, is a mathematical model. It can be described using a formula, a graph or a table of values.

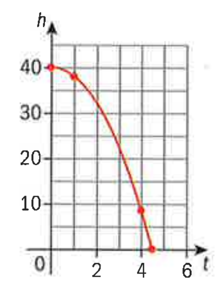
To construct a mathematical model we usually begin by making some assumptions. Here, we assume that the diver is initially at 40 m above sea level and it takes him 4.5 seconds to reach the sea. The formula linking the variables t and h is

$$h = -1.97\,(t^2 - 20.25) \quad \text{where } t \geq 0.$$

You can use this model to calculate the diver's height, h, above sea level at different times, t. Substitute the value of t into the formula to get the corresponding value for h. The table shows three pairs of values for t and h.

t (seconds)	h (metres)
0	40.0
1	38.0
4	8.37

The graph of $h = -1.97(t^2 - 20.25)$, $t \geq 0$, is shown.



The three pairs of values from the table are indicated with a •.

You can use the formula and/or the graph to answer questions such as:

At what height is the diver after 2 seconds?

How long does it take the diver to reach a height of 20 m above sea level?

In this chapter you will work with different types of mathematical models called **functions** to represent a range of practical situations. These functions help us to understand and predict the behavior of variables.

4.1 Functions

Mathematical models that link two variables are called functions.

→ A **function** is a relationship between two sets: a **first** set and a **second** set. Each element 'x' of the first set is related to **one and only one** element 'y' of the second set.

Example 1

Antonio and Lola are students at Green Village High School (GVHS). Miu is a student at Japan High School (JHS).
The set of students A = {Antonio, Lola, Miu}.
The set of schools B = {GVHS, JHS}.
Decide whether these relationships are functions. Justify your decisions.

a The relationship between the first set A and the second set B: 'x is a student at school y'.

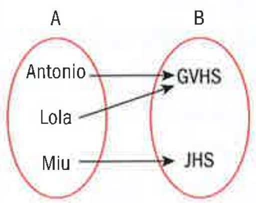
b The relationship between the first set B and the second set A: 'x is the school where y is a student'.

In Chapter 1 we saw sets where the elements were numbers. However, the elements of a set may be any kind of object.

Answers

a This relationship is a function because each element of the first set A is related to one and only one element of the second set B; that is, each student studies at only one school.

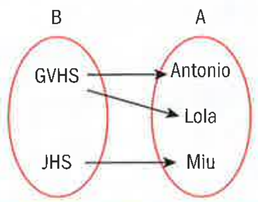
Draw a mapping diagram to show how the elements of set A, Antonio, Lola and Miu, are related to the elements of set B, GVHS, JHS.



A mapping diagram is a drawing used to show how the elements in the first set are matched to the elements in the second set.

b This relationship is not a function because one element of the first set B, GVHS, is related to more than one element in the second set A, Antonio and Lola.

Draw a mapping diagram to show how the elements of set B are related to the elements of set A.



Example 2

Let A = {1, −1, 0, 2, 4}, B = {1, 0, 4} and C = {1, 0, 4, 16}.
Decide whether these relationships are functions. Justify your decisions.

a The relationship between the first set A and the second set B: 'the square of x is y' or '$y = x^2$'.

b The relationship between the first set A and the second set C: 'the square of x is y' or '$y = x^2$'.

c The relationship between the first set C and the second set A: 'the square root of x is y' or '$y = \sqrt{x}$'.

In Example 2 as the elements of the sets are numbers, the relationships are **numerical**. In Mathematical Studies we work with numerical relationships that can be described using **equations**.

Answers

a It is not a function because one element of the first set A, 4, is not related to any element of the second set B.

Draw a table of values. The elements of set A are the values of x. Use these to work out the corresponding values of y given $y = x^2$. Check that the values of y match the elements of set B.

A	B
x	$y = x^2$
1	1
−1	1
0	0
2	4
4	

$4^2 = 16$; 16 is not an element of set B.

b It is a function because each element of the first set A is related to one and only one element of the second set C.

A	C
x	$y = x^2$
1	1
−1	1
0	0
2	4
4	16

c It is a function because each element of the first set C is related to one and only one element of the second set A.

C	A
x	$y = \sqrt{x}$
1	1
0	0
4	2
16	4

Think of everyday situations where you can define functions between two sets. For example, the relationship between a group of people and their names; the relationship between a tree and its branches; the relationship between the days and the mean temperature of each of these days, etc.

Exercise 4A

1 Mrs. Urquiza and Mr. Genzer both teach mathematics. Mick and Lucy are in Mrs. Urquiza's class. Lidia and Diana are in Mr. Genzer's class.
Let the set of students A = {Mick, Lucy, Lidia, Diana} and the set of teachers B = {Mrs. Urquiza, Mr. Genzer}.
Decide whether these relationships are functions. Justify your decisions.
   a The relationship between the first set A and the second set B: 'x is in y's mathematics class'.
   b The relationship between the first set B and the second set A: 'x is y's mathematics teacher'.

2 Let A = {3, 7, 50}, B = {12, 16, 49, 100} and C = {49, 100}.
Decide whether these relationships are functions. Justify your decisions.
   a The first set A, the second set B and the relationship 'x is a factor of y'.
   b The first set B, the second set A and the relationship 'x is a multiple of y'.
   c The first set C, the second set A and the relationship 'x is a multiple of y'.

3 Let A = {1, 2, 3, 4}, B = {2, 4, 6} and C = {1, 2, 4, 6}.
   a Decide whether these relationships are functions. Justify your decisions.
      i First set A, second set B and the relationship 'x is half of y'.
      ii First set A, second set C and the relationship 'x is half of y'.
      iii First set C, second set A and the relationship 'x is the double of y'.
      iv First set B, second set C and the relationship 'x is equal to y'.
      v First set C, second set A and the relationship 'x is equal to y'.
   b Draw a diagram to represent the relationships from part **a** that are functions.

4 Describe the following relationships between x and y using equations.
   a y is double x.
   b Half of x is y.
   c The cubic root of x is y.
   d Half of the cube of x is y.

The equation $y = x^2$ describes the relationship 'y is the square of x'.

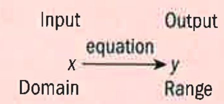
5 Decide whether these relationships are functions. Explain your decision in the cases in which they are not functions.
   a The first set is $\mathbb{R}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = 3x + 1$.
   b The first set is $\mathbb{R}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = x^2$.
   c The first set is $\mathbb{R}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = \sqrt{x}$.
   d The first set is A = $\{x \geq 0, x \in \mathbb{R}\}$, the second set is $\mathbb{R}$ and the relationship is defined by the equation $y = \sqrt{x}$.

$\mathbb{R}$ is the set of real numbers.

Domain and range of a function

A function is a relationship between two sets: a first set and a second set.

- The first set is called the **domain** of the function. The elements of the domain, often thought of as the 'x-values', are the **independent variables**.
- For each value of 'x' (input) there is one and only one output. This value is called the **image** of 'x'. The set of all the images (all the outputs) is called the **range** of the function. The elements of the range, often thought of as the 'y-values', are the **dependent variables**.



In Mathematical Studies the domain will always be the set of real numbers unless otherwise stated.

We write domain and range values as sets inside curly brackets:
Domain = {inputs}
Range = {images or outputs}

Example 3

Consider the function $y = x^2$.
a Find the image of **i** $x = 1$ **ii** $x = -2$.
b Write down the domain.
c Write down the range.

Answers

a **i** $y = 1$

i Substitute $x = 1$ into $y = x^2$
$y = (-1)^2 \Rightarrow y = 1$

ii $y = 4$

ii Substitute $x = -2$ into $y = x^2$
$y = (-2)^2 \Rightarrow y = 4$

b The domain is the set of real numbers, $\mathbb{R}$.

Squaring any real number produces another real number. So the domain is the set of all real numbers.

c The range is $y \geq 0$.

The square of any positive or negative number will be positive and the square of zero is zero. So the range is the set of all real numbers greater than or equal to zero.

The domain is assumed to be $\mathbb{R}$ unless there are any values which x cannot take.

Example 4

Consider the function $y = \frac{1}{x}, x \neq 0$.

a Find the image of: **i** $x = 2$ **ii** $x = -\frac{1}{2}$

b Write down the domain.

c **i** Decide whether $y = 0$ is an element of the range. Justify your decision.

ii Decide whether $y = -5$ is an element of the range. Justify your decision.

Answers

a **i** $y = \frac{1}{2}$

ii $y = \frac{1}{-\frac{1}{2}} = -2$

Substitute
i *$x = 2$ and*
ii *$x = -\frac{1}{2}$ into $y = \frac{1}{x}$.*

b The domain is the set of all real numbers except 0.

Since division by zero is not defined, the domain is the set of all real numbers except zero ($x \neq 0$).

c **i** $0 = \frac{1}{x}$

This equation has no solution. Therefore, $y = 0$ is not an element of the range.

*Substitute **i** $y = 0$ and*
ii *$y = -5$ into $y = \frac{1}{x}$.*

Is there an input value (x) that gives an output (y) of 0?

ii $-5 = \frac{1}{x}$

$x = -\frac{1}{5}$

So, $y = -5$ is an element of the range as it is the image of $x = -\frac{1}{5}$.

Is there an input value (x) that gives an output (y) of −5?

Gottfried Leibniz first used the mathematical term 'function' in 1673.

Exercise 4B

1 For each of the functions in **a–d**:

i Copy and complete the table. Put a × in any cells that cannot be completed.

ii Write down the domain.

iii Decide if $y = 0$ is in the range of the function. Justify your decision.

a $y = 2x$

x	$-\frac{1}{2}$	0	1	3.5	
y = 2x					12

b $y = x^2 + 1$

x	−3	0	2	$\frac{1}{4}$		
$y = x^2 + 1$					5	5

c $y = \frac{1}{x+1}, x \neq -1$

x	−2	−1	0	$\frac{1}{2}$	3	
$y = \frac{1}{x+1}$						$\frac{1}{6}$

d $y = \sqrt{x}, x \geq 0$

x	−3	0	$\frac{1}{4}$		9	
$y = \sqrt{x}$				1		10

2 Decide whether each statement is true or false. Justify each of your decisions.

a $y = 0$ is an element of the range of the function $y = \frac{2}{x}$.

b The equation $y = x^2$ cannot take the value −1.

c The equation $y = x^2 + 3$ cannot take the value 2.

d For the function $y = x^2 - 1$ there are two values of x when $y = 3$.

e For the function $y = \frac{x}{3} - 1$ the image of $x = -3$ is −2.

f For the function $y = 2(-x + 1)$ the image of $x = -1$ is $y = 0$.

Graph of a function

A graph can represent a function.

→ The graph of a function f is the set of points (x, y) on the Cartesian plane where y is the image of x through the function f.

We use different letters to name functions: *f*, *g*, *h*, etc.

Drawing graphs

- Draw a table of values to find some points on the graph.
- On 2 mm graph paper, draw and label the axes with suitable scales.
- Plot the points.
- Join the points with a straight line or a smooth curve.

Cartesian coordinates and the Cartesian plane are named after the Frenchman René Descartes (1596–1650).

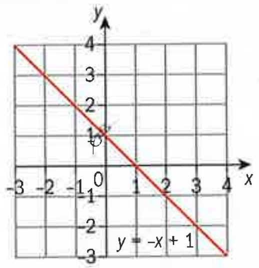
Example 5

a Draw the graph of the function $y = -x + 1$.
b Write down the coordinates of the point where the graph of this function intercepts the **i** x-axis **ii** y-axis.
c Decide whether the point A(200, −199) lies on the graph of this function.
d The point B(6, y) lies on the graph of this function. Find the value of y.

'Draw' means draw accurately on graph paper.

Answers

a



Draw a table of values. Use negative and positive values of x.
Use the values of x to work out the corresponding values of y.
When $x = -3$, $y = -(-3) + 1 = 4$.

x	−3	−1	0	1	3
y	4	2	1	0	−2

Use 2 mm graph paper.
Let 1 cm = 1 unit. Label the x- and y-axes.
Plot the coordinate points (−3, 4), (−1, 2), (0, 1), (1, 0) and (3, −2).
Join the points with a straight line.

b **i** x-intercept is (1, 0).
ii y-intercept is (0, 1).

i *To find the x-intercept, read off the point where the graph intersects the x-axis.*
ii *To find the y-intercept, read off the point where the graph intersects the y-axis.*

c $-199 = -200 + 1$
So A(200, 199) lies on the graph.

A(200, −199)
Substitute the x- and y-values into the equation of the line to see if they 'fit'.

A point P lies on the graph of a function if and only if the point satisfies the equation of the function.

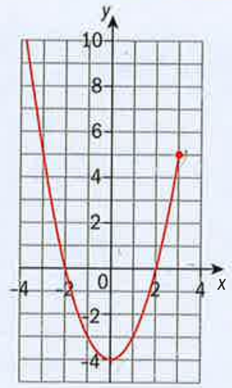
d B(6, y) lies on the graph, so $y = -6 + 1 = -5 \Rightarrow y = -5$

B(6, y)
Substitute x = 6 into the equation of the line to find the value of y at that point.

In the solution to the next example, the set notation $\{x \mid x \le 3\}$ is used. It is read as: the set of all x such that x is a real number less than or equal to 3.

Example 6

Here is the graph of a function f.
Use the graph to find
a the domain of f
b the range of f
c the points where the graph of f intersects the **i** x-axis and **ii** y-axis.

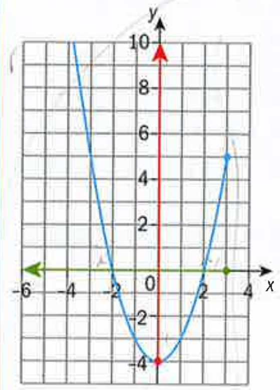


• indicates that an endpoint lies on the graph of a function. In Example 6 the point (3, 5) lies on the graph. To indicate that a point does not lie on the graph of a function use an empty dot, ○.

Answers

a Domain of $f = \{x \mid x \le 3\}$

To find the domain from a graph of a function, 'squash' or project the graph against the x-axis.



On the graph above, the domain is shown by the green line.

b Range of $f = \{y \mid y \ge -4\}$

To find the range from the graph of a function, 'squash' the graph against the y-axis.
On the graph in part a, the range is shown by the red line.

c **i** x-intercepts: (−2, 0) and (2, 0)
ii y-intercept: (0, −4)

i *On the x-axis the y-coordinate is zero.*
ii *On the y-axis the x-coordinate is zero.*

Is it possible for the graph of a function to cut the y-axis more than once?

Sketching a linear graph

- Draw and label the axes.
- Label the points where the graph crosses the x- or y-axis.

Example 7

Sketch a graph of the function $y = 3x - 1$.

Answer

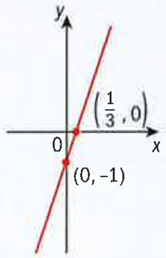
The x-axis intercept is at $\left(\frac{1}{3}, 0\right)$

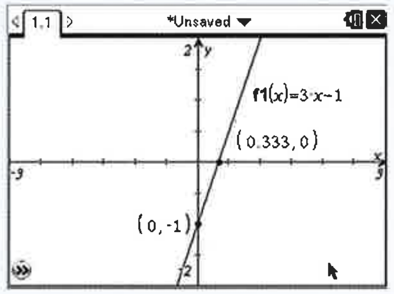
When $y = 0$, $x = \frac{1}{3}$

The y-axis intercept is at $(0, -1)$

When $x = 0$, $y = -1$

Draw the graph on your GDC.





Now sketch the graph:

1. *Draw and label the axes.*
2. *Copy the graph shown on the GDC.*
3. *Label the points where the graph crosses the axes.*

'Sketch' means give a general shape of the graph.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4C

EXAM-STYLE QUESTION

1 **a** Draw the graph of the function $y = 2x - 4$.

b Write down the coordinates of the point where the graph of this function meets

i the x-axis **ii** the y-axis.

c Decide whether the point A(250, 490) lies on the graph of this function. Justify your decision.

d The point B(-3, y) lies on the graph of this function. Find the value of y.

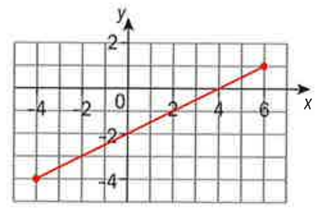
2 For each of the graphs of functions in **a**–**d** write down

i the domain **ii** the range

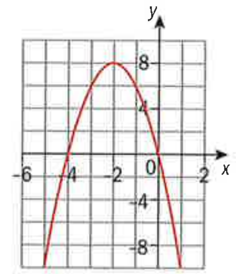
iii the points where the graph meets the x-axis (where possible)

iv the point where the graph meets the y-axis (where possible).

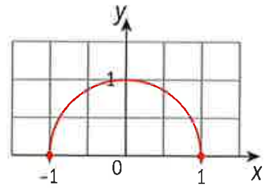
a



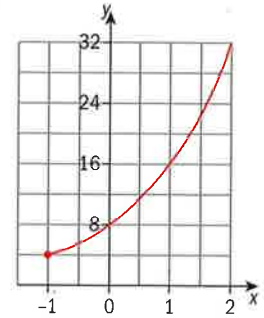
b



c



d



3 Decide whether these statements about the functions drawn in question 2 are true or false.

Function **a**

i Point $(1, -1)$ lies on this graph.

ii The image of $x = -2$ is 0. **iii** When $x = 6$, $y = 1$.

Function **b**

i There are two values of x for which $y = 8$.

ii There are two values of x for which $y = 4$.

iii There is a value of x for which $y = 9$.

Function **c**

i The line $x = 0.5$ intersects the graph of this function twice.

ii The line $y = 0.5$ intersects the graph of this function twice.

iii The image of $x = 0.2$ is the same as the image of $x = -0.8$.

Function **d**

i The line $y = 1$ intersects the graph of this function once.

ii When $x = 16$, $y = 1$.

iii As the values of x increase, so do their corresponding values of y.

4 Sketch a graph for each of these functions.

a $y = 2x + 3$ **b** $y = -x + 2$ **c** $y = 3x - 4$

Function notation

→ $y = f(x)$ means that the image of x through the function f is y. x is the independent variable and y is the dependent variable.

So, for example, if $f(x) = 2x - 5$

- $f(3)$ represents the image of $x = 3$.
  To find the value of $f(3)$ substitute $x = 3$: $f(3) = 2 \times 3 - 5 = 1$
- $f(-1)$ represents the image of $x = -1$.
  To find the value of $f(-1)$ substitute $x = -1$: $f(-1) = 2 \times -1 - 5 = -7$

We use different variables and different letters for functions; for example, $d = v(t)$, $m = C(n)$, etc.

$f(3) = 1$ can be read as: 'f at 3 equals 1' or 'f of 3 equals 1'.

Example 8

Consider the function $f(x) = -x^2 + 3x$.
a Find the image of $x = -2$. **b** Find $f(1)$.
c Show that the point (4, −4) lies on the graph of f.

Answers

a $f(-2) = -(-2)^2 + 3 \times -2 = -10$	*Substitute $x = -2$ into $f(x) = -x^2 + 3x$.*
b $f(1) = -1^2 + 3 \times 1 = 2$	*Substitute $x = 1$ into $f(x)$.*
c $f(4) = -4^2 + 3 \times 4 = -4$ So (4, −4) lies on the graph.	*If (4, −4) lies on the graph of f then $f(4) = -4$. Substitute $x = 4$.*

One of the first mathematicians to study the concept of function was French Philosopher Nicole Oresme (1323–82). He worked with independent and dependent variable quantities.

Exercise 4D

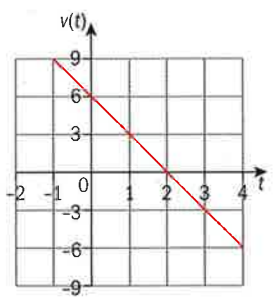
1 Consider the function $f(x) = x(x - 1)(x + 3)$.
  a Calculate $f(2)$. **b** Find the image of $x = \frac{1}{2}$.
  c Show that $f(-3) = 0$.
  d Decide whether the point (−1, −4) lies on the graph of f. Justify your decision.

2 Consider the function $d(t) = 5t - t^2$.
  a Write down the independent variable of this function.
  b Calculate $d(2.5)$.
  c Calculate the image of $t = 1$.
  d Show that $d(1)$ and $d(4)$ take the same value.

3 Consider the function $C(n) = 100 - 10n$.
  a Calculate $C(2)$.
  b The point $(3, b)$ lies on the graph of the function C. Find the value of b.
  c The point $(a, 0)$ lies on the graph of the function C. Find the value of a.

EXAM-STYLE QUESTION

4 Here is the graph of the function $v(t) = -3t + 6$.
  a Write down the value of **i** $v(1)$ **ii** $v(3)$.
  b The point $(m, 9)$ lies on the graph. Find the value of m.
  c Find the value of t for which $v(t) = 0$.
  d Find the set of values of t for which $v(t) < 0$.



EXAM-STYLE QUESTION

5 Consider the function $f(x) = 0.5(3 - x)$.
  a Draw the graph of f.
  b Find the point A where the graph of f meets the x-axis.
  c Find the point B where the graph of f meets the y-axis.
  d Solve the equation $f(x) = 2$.

6 Consider the function $h(x) = 3 \times 2^{-x}$.
  a Calculate **i** $h(0)$ **ii** $h(-1)$. **b** Find x if $h(x) = 24$.

$h(x) = 3 \times 2^{-x}$ is an exponential function. You will learn more about these in section 4.4.

Functions as mathematical models

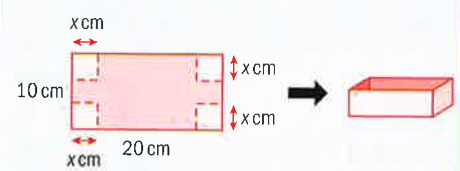
We can use functions to describe real-life situations.

Translate the situation into mathematical language and symbols. → Find the solution using mathematics. → Interpret the solution in the context of the problem.

Example 9

A rectangular piece of card measures 20 cm by 10 cm. Squares of length x cm are cut from each corner. The remaining card is then folded to make an open box. Write a function to model the volume of the box.

Answer



First draw a diagram to represent the information given in the question. Carefully label the dimensions of the open box:
length $(20 - 2x)$ cm
width $(10 - 2x)$ cm
height x cm

$V(x) = (20 - 2x)(10 - 2x)x$

The volume of the box, V, will depend on the value of x.
Volume of cuboid = length × width × height.

Look at Example 9.
1 What is the domain of the function $V(x)$? Can x take any value? Why? Try different values and draw a conclusion.
2 How could the function help you to find the maximum possible volume?

Exercise 4E

EXAM-STYLE QUESTION

1 A rectangular piece of card measures 30 cm by 15 cm. Squares of length x cm are cut from each corner. The remaining card is then folded to make an open box of length l cm and width w cm.
  a Write expressions, in terms of x, for
    i the length, l **ii** the width, w.
  b Find an expression for the volume of the box, V, in terms of x.
    i Explain in words the meaning of $V(3)$.
    ii Find the value of $V(3)$. **iii** Find the value of $V(3.4)$.
    iv Is $x = 8$ in the domain of the function $V(x)$? Justify your decision.

EXAM-STYLE QUESTIONS

2 The perimeter of a rectangle is 24 cm and its length is x cm.

a Find the width of the rectangle in terms of the length, x.

b Find an expression for the area of the rectangle, A, in terms of x.

c **i** Explain the meaning of $A(2)$.

ii Calculate $A(2)$.

d Is $x = 12$ in the domain of the function $A(x)$? Justify your decision.

You can use mathematical functions to represent things from your own life. For example, suppose the number of pizzas your family eats depends on the number of football games you watch. If you eat 3 pizzas during every football game, the function would be 'number of pizzas' (p) = 3 times 'number of football games' (g) or $p = 3g$. Can you think of another real-life function? It could perhaps be about the amount of money you spend or the number of minutes you spend talking on the phone.

3 The Simpsons rent a holiday house costing 300 USD for the security deposit plus 150 USD per day. Let n be the number of days they stay in the house and C the cost of renting the house.

a Write a formula for C in terms of n.

b How much does it cost to rent the house for 30 days?

The Simpsons have 2300 USD to spend on the rent of the house.

c **i** Write down an inequality using your answer to part **a** to express this condition.

ii Hence, decide whether they have enough money to rent the house for two weeks.

iii Write down the maximum number of days that they can rent the house.

4 An Australian company produces and sells books. The monthly **cost**, in AUD, for producing x books is modeled by $C(x) = 0.4x^2 + 1500$. The monthly **income**, in AUD, for selling x books is modeled by $I(x) = -0.6x^2 + 160x$.

a Show that the company's monthly profit can be calculated using the function $P(x) = -x^2 + 160x - 1500$.

Profit = Income – Cost

b What profit does the company make on producing and selling six books? Comment on your answer.

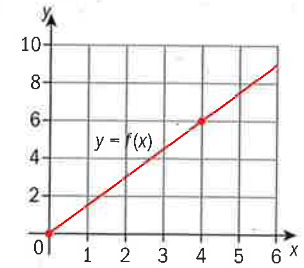
c **i** What profit does the company make on producing and selling 40 books?

ii Find the selling price of one book when 40 books are produced and sold. (Assume that all the books have the same price.)

d Use your GDC to find the number of books for which $P(x) = 0$.

4.2 Linear models

Linear models of the form $f(x) = mx$



The straight line shown here has a positive gradient and the function $y = f(x)$ is increasing.

$f(0) = 0$ and the line passes through the origin (0, 0).

The gradient of the line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Using two points on the line, (4, 6) and (0, 0), the gradient is

$m = \frac{6-0}{4-0} = \frac{3}{2} = 1.5$.

So $f(x) = 1.5x$.

This type of linear model is used in **conversion graphs**. The two variables which have a fixed relationship between them are in direct proportion, so their graphs are straight lines with a positive gradient passing through the origin.

Conversion graphs can be used to convert one currency to another, or one set of units to another: for example, kilometres to miles, or kilograms to pounds.

Example 10

1 mile is equivalent to 1.6 km.

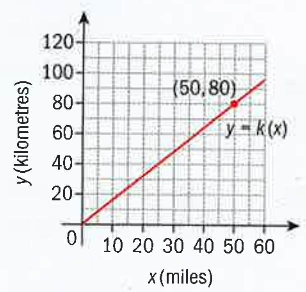
a Draw a conversion graph of miles to km.

b Find the gradient of the line.

c Hence, write down a model for $k(x)$, where $k(x)$ is the distance in km and x is the distance in miles.

Answers

a



Use 2 mm graph paper.
Put miles on the x-axis.
Put kilometres on the y-axis.
Find two points to draw a straight line: 0 miles = 0 km so (0, 0) is on the line.
50 miles is equivalent to 1.6 × 50 = 80 km so (50, 80) is on the line.
Plot the two points and join with a straight line.

b Gradient, $m = \frac{80-0}{50-0} = 1.6$

Use the two points from part **a** *to find the gradient,* $m = \frac{y_2 - y_1}{x_2 - x_1}$.

c The equation of the line is $y = 1.6x$.
Hence, $k(x) = 1.6x$, where $k(x)$ is the distance in km, and x is the distance in miles.

For a general linear function through the origin, $f(x) = mx$. Here, the function is $k(x) = mx$.

The equation $y = 1.6x$ can be rearranged to $x = \frac{y}{1.6}$ or $x = \frac{1}{1.6}y = 0.625y$. Use this to convert km to miles.

Exercise 4F

1 1 kg is equivalent to 2.2 pounds.

- **a** Convert 50 kg into pounds.
- **b** Draw a conversion graph of pounds to kilograms. Use x-values from 0 kg to 100 kg and y-values from 0 pounds to 250 pounds.
- **c** Find the gradient of the line. Hence, write down the model for $p(x)$, where $p(x)$ is the weight in pounds and x is the weight in kg.
- **d** Find $p(75)$ and $p(125)$.
- **e** Find the model for $k(x)$, where $k(x)$ is the weight in kg and x is the weight in pounds.
- **f** Calculate $k(75)$ and $k(100)$.

> Plot the point you found in part **a**.

> Write the formula as $y = \dots$ and rearrange to make x the subject.

2 The exchange rate for pounds sterling (GBP) to Singapore dollars (SGD) is £1 = S$2.05.

- **a** Find the number of Singapore dollars equivalent to 50 GBP.
- **b** Draw a conversion graph of GBP to SGD. Use x-values from £0 to £100 and y-values from S$0 to S$250.
- **c** Find the gradient of the line. Hence, write down the model for $s(x)$, where $s(x)$ is the amount of money in SGD and x is the amount of money in GBP.
- **d** Find $s(80)$ and $s(140)$.
- **e** Find the model for $p(x)$, where $p(x)$ is the amount of money in GBP and x is the amount of money in SGD.
- **f** Calculate $p(180)$.

3 The exchange rate for pounds sterling (GBP) to US dollars (USD) is £1 = $1.55.

- **a** Find the number of US dollars equivalent to 60 GBP.
- **b** Draw a conversion graph of GBP to USD. Use the x-axis for GBP with $0 \le x \le 80$, and the y-axis for USD with $0 \le y \le 140$.
- **c** Find the gradient of the line. Hence, write down the model for $u(x)$, where $u(x)$ is the amount of money in USD and x is the amount of money in GBP.
- **d** Find $u(300)$ and $u(184)$.
- **e** Find the model for $p(x)$, where $p(x)$ is the amount of money in GBP and x is the amount of money in USD.
- **f** Calculate $p(250)$ and $p(7750)$.

Linear models of the form $f(x) = mx + c$

When two variables are not in direct proportion, their graphs are straight lines that do not pass through the origin, that is, **linear functions**.

> → A **linear function** has the general form
> $$f(x) = mx + c$$
> where m (the gradient) and c are constants.

> You have seen the equation of a line in Chapter 3, Section 3.2.

Example 11

In a chemistry experiment, a liquid is heated and the temperature at different times recorded.
The table of results for one student is shown.

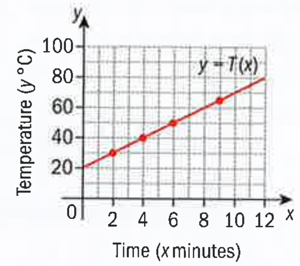
Time (x minutes)	2	4	6	9
Temperature (y °C)	30	40	50	65

- **a** Draw a graph for this data.
- **b** Find a model for $T(x)$, the temperature with respect to time, for these data.
- **c** Use the model to predict:
  - **i** the temperature of the liquid after 8 minutes
  - **ii** the time taken for the liquid to reach 57 °C.

> You can plot the graph on your GDC and find the model for $T(x)$. For help, see Chapter 12, Section 5.4.

Answers

a



Use 2 mm graph paper.
Put time on the x-axis.
Put temperature on the y-axis.
Plot the points from the table e.g. (2, 30) and join them with a straight line.

b Gradient, $m = \dfrac{65 - 40}{9 - 4}$

$= \dfrac{25}{5} = 5$

$T(x) = mx + c$
$T(x) = 5x + c$
$T(2) = 5 \times 2 + c = 30$
$10 + c = 30$
$c = 20$

Therefore, the model for the temperature is $T(x) = 5x + 20$.

The model will be in the form $T(x) = mx + c$. You need to find the constants m and c.
Use any two points from the table, e.g. (4, 40) and (9, 65), to find the gradient, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

To find the value of c use any point from the table, e.g. (2, 30), which means $T(2) = 30$.

c i At 8 minutes:
$T(8) = 5 \times 8 + 20 = 60$
So, the temperature of the liquid after 8 minutes is 60 °C.

*i A time of 8 minutes means $x = 8$. Substitute $x = 8$ into the function in part **b**.*

ii When $T(x) = 57$ °C:
$57 = 5x + 20$
$5x = 37$
$x = \dfrac{37}{5} = 7.4$
So, it takes 7.4 minutes for the liquid to reach 57 °C.

ii A temperature of 57 °C means $T(x) = 57$. Substitute $T(x) = 57$ and solve for x.

> For Example 11, the equation of the model was $T(x) = 5x + 20$.
> Compare the equation of the model with
> **a** the initial temperature
> **b** the average rise in temperature every minute.
> What conclusions can be drawn?

Exercise 4G

1 In a chemistry experiment, a liquid is heated and the temperature at different times is recorded. Here is a table of results.

Time (x minutes)	3	5	7	9
Temperature (y °C)	130	210	290	410

a Draw a graph for these data.
b What was the initial temperature of the liquid?
c Find the linear model, $T(x)$, for the temperature of the liquid with respect to time.

Use a scale up to 420 on the y-axis.

2 In a physics experiment, a spring is stretched by loading it with different weights, in grams. The results are given in the table.

Weight (x g)	40	50	75	90
Length of spring (y mm)	38	43	55.5	63

a Draw a graph for these data.
b Find the natural length of the spring.
c By how many mm does the spring stretch when the weight increases from 50 g to 90 g?
d Use the answer to part **c** to find the average extension of the spring in mm for each extra gram loaded.
e Find the equation of the linear model, $L(x)$, for the length of the spring with respect to load.

The natural length is the length of the spring with no loading.

3 The temperature of the water in a hot water tank is recorded at 15 minute intervals after the heater is switched on.

Time (x minutes)	15	30	45	60	75	90
Temperature (y °C)	20	30	40	50	60	70

a Plot a graph of these data on your GDC.
b Find the linear model, $T(x)$, for the temperature of the liquid with respect to time.
c Find the temperature of the water after 85 minutes.

Read off the value from your graph.

4 Different weights are suspended from a spring. The length of the spring with each weight attached is recorded in the table.

Weight (x g)	125	250	375	500
Length of spring (y cm)	30	40	50	60

a Plot a graph of these data on your GDC.
b Find the natural length of the spring.
c By how many cm does the spring stretch when the weight is increased from 125 g to 375 g?
d Find the weight that will stretch the spring to a length of 48 cm.
e Find the equation of the linear model, $L(x)$, for the length of the spring with respect to load.

Linear models involving simultaneous equations

Sometimes you cannot find the model from the given data. You may need to write equations to represent the situation and solve them simultaneously.

For a reminder on solving simultaneous equations, see Chapter 13, Section 2.4.

Example 12

A carpenter makes wooden tables and chairs.
He takes 10 hours to make a table and 4 hours to make a chair.
The wood costs $120 for a table and $40 for a chair.
Find a model for
a the time required to make the tables and chairs
b the cost of making the tables and chairs.

Answers

a Let t be the time required to make the tables and chairs.
The model for the time required is $t = 10x + 4y$.

Let x be the number of tables and y be the number of chairs.
Total number of hours for the tables: 10 hours per table, x tables $\Rightarrow 10 \times x$
Total number of hours for the chairs: 4 hours per chair, y chairs $\Rightarrow 4 \times y$

b Let c be the cost of making the tables and chairs.
The model for the cost is
$c = 120x + 40y$.

Total cost ($) for the tables: $120 per table, x tables $\Rightarrow 120 \times x$
Total cost ($) for the chairs: $40 per chair, y chairs $\Rightarrow 40 \times y$

The given values for a model are called **constraints**.

The simultaneous equations arise when you are given values that the model must satisfy.

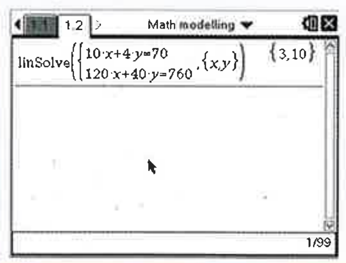
Example 13

The carpenter in Example 12 works 70 hours one week and spends $760 on wood.
How many tables and chairs can he make?

Answer

From Example 12:
$t = 10x + 4y$
$c = 120x + 40y$
$10x + 4y = 70$
$120x + 40y = 760$
Using a GDC: $x = 3$ and $y = 10$
The carpenter can make 3 tables and 7 chairs.

The model must work for the values: (time) $t = 70$ and (cost) $c = 760$.
Write a set of simultaneous equations.
Solve these either analytically or using a GDC.



For help with using a GDC to solve simultaneous equations see Chapter 12, Sections 1.1 and 3.4.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4H

1 To make a sponge cake you need 80 g of flour and 50 g of fat.
To make a fruit cake you need 60 g of flour and 90 g of fat.
Find a model for
- **a** the amount flour needed to make both cakes
- **b** the amount of fat needed to make both cakes.

Peter has 820 g of flour and 880 g of fat.
- **c** How many of each type of cake can he make?

2 It takes 8 hours to make a table and 3 hours to make a chair.
For a table the wood costs \$100. For a chair the wood costs \$30.
A carpenter has 51 hours and \$570.
How many tables and chairs can she make?

3 A van carries up to 3 people and 7 cases.
A car carries up to 5 people and 3 cases.
How many vans and cars do you need for 59 people and 70 cases?

4 A passenger plane carries 80 passengers and 10 tonnes of supplies.
A transport plane carries 50 passengers and 25 tonnes of supplies.
How many planes of each type do you need to carry 620 people and 190 tonnes of supplies?

5 A school mathematics department has 1440 euros to buy textbooks.
Maths for All volume 1 costs 70 euros. *Maths for All* volume 2 costs 40 euros.
The department wants twice as many copies of volume 1 as volume 2.
How many of each volume can they buy?

Extension material on CD: *Worksheet 4 - Equations*

4.3 Quadratic models

Quadratic functions and their graphs

→ A **quadratic function** has the form
$f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Why $a \neq 0$? What kind of function would you get if $a = 0$?

The domain of a quadratic function can be the entire set of real numbers ($\mathbb{R}$) or any subset of this.

Here are examples of some quadratic functions:

$f(x) = x^2 + 3x + 2$ $(a = 1, b = 3, c = 2)$

$f(x) = x - 3x^2$ $(a = -3, b = 1, c = 0)$

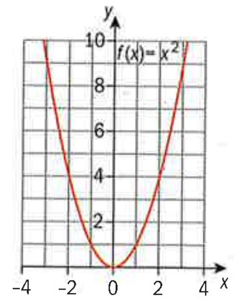
$f(x) = 3x^2 + 12$ $(a = 3, b = 0, c = 12)$

The simplest quadratic function is $f(x) = x^2$.

Here is a table of values for $f(x) = x^2$.

x	−3	−2	−1	0	1	2	3
f(x)	9	4	1	0	1	4	9

Plotting these values gives the graph shown here.



1. The graph is called a **parabola**.
2. The parabola has an **axis of symmetry** (the y-axis).
3. The parabola has a **minimum point** at (0, 0).
   The minimum point is called the **vertex** (or turning point) of the parabola.
4. The range of $f(x) = x^2$ is $y \geq 0$.

'Squash' the graph of $f(x) = x^2$ against the y-axis to confirm that the range is $y \geq 0$.

→ The graph of any quadratic function is a **parabola** – a ∪-shaped (or ∩-shaped) curve. It has an **axis of symmetry** and either a **minimum** or **maximum** point, called the **vertex** of the parabola.

The name 'parabola' was introduced by the Greek, Appollonius of Perga (c. 262–190 BCE) in his work on conic sections.

Investigation – the curve $y = ax^2$

1 Draw these curves on your GDC: $y = x^2$ and $y = -x^2$
How are these two curves related?

2 Now draw: $y = 2x^2$ $y = 3x^2$ $y = 0.5x^2$
$y = -2x^2$ $y = -3x^2$ $y = -0.5x^2$
Compare each of these six graphs to $y = x^2$.
Consider:
- **a** Is the curve still a parabola? Is the curve ∪-shaped or ∩-shaped?
- **b** Does it have a vertical line of symmetry?
- **c** What is its vertex? Is the vertex a minimum or maximum point?

3 What is the effect of changing the value of a?
Draw a few more graphs to test your conjecture. (Remember to use positive and negative values of a and also use fractions.)

For help with drawing graphs on your GDC see Chapter 12, Section 4.1.

Without drawing the graph, how do you know that it will be ∩-shaped?

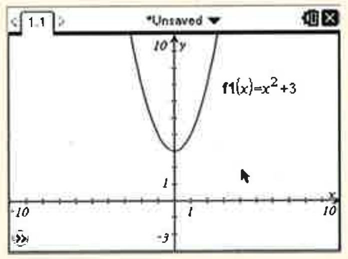
Investigation – the curve $y = x^2 + c$

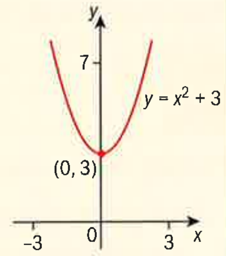
Draw these curves on your GDC: $y = x^2$ $y = x^2 + 2$ $y = x^2 - 4$
$y = x^2 + 3$ $y = x^2 - 2$

Compare each graph to the parabola $y = x^2$. (Use the list of considerations in the preceding investigation as a guide.)
What is the effect of changing the value of c?

Sketching a quadratic graph (1)

- Draw and label the axes.
- Mark any points where the graph intersects the axes (the *x*- and *y*-intercepts). Label these with their coordinates.
- Mark and label the coordinates of any maximum or minimum points.
- Show one or two values on each axis to give an idea of the scale.





Exercise 4I

Use your results from the two previous investigations to help you sketch these graphs.

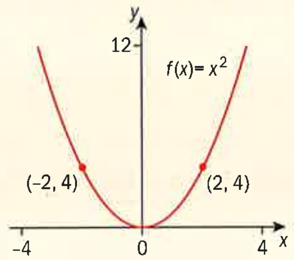
1 $y = 2x^2 + 1$

2 $y = -x^2 + 3$

3 $y = 3x^2 - 2$

4 $y = -2x^2 + 7$

Use this sketch graph of $f(x) = x^2$ to help you.



Investigation – the curves $y = (x + p)^2$ and $y = (x + p)^2 + q$

1 Use your GDC to draw these graphs:
$y = x^2$, $y = (x + 2)^2$, $y = (x + 3)^2$, $y = (x - 1)^2$, $y = (x - 0.5)^2$
Compare each graph to the graph of $y = x^2$.
What is the effect of changing the value of p?

2 Use your GDC to draw these graphs:
$y = (x + 2)^2 - 3$, $y = (x - 4)^2 + 2$, $y = (x - 1)^2 - 5$
What is the axis of symmetry of $y = (x + p)^2 + q$?
What are the coordinates of the vertex of $y = (x + p)^2 + q$?

Exercise 4J

For each graph, write down the coordinates of the vertex and the equation of the axis of symmetry.

1 $y = (x + 3)^2 - 2$

2 $y = (x + 5)^2 + 4$

3 $y = (x - 4)^2 - 1$

4 $y = (x - 5)^2 + 7$

5 $y = -(x + 3)^2 + 4$

The equation of the axis of symmetry must be given as '$x = \ldots$'.

Investigation – the curves $y = kx - x^2$ and $y = x^2 - kx$

Part A

1 Use your GDC to draw the graph of $y = 4x - x^2$.
What is the equation of its axis of symmetry?
What are the coordinates of the vertex?
What are the coordinates of the points at which the curve intersects the *x*-axis?

2 Draw these curves: $y = 2x - x^2$, $y = 6x - x^2$, $y = x - x^2$, $y = 5x - x^2$

3 What is the effect of varying the value of k?
What is the equation of the axis of symmetry of the curve $y = kx - x^2$?
What are the coordinates of the points at which the curve $y = kx - x^2$ intersects the *x*-axis?

Part B

Draw these curves: $y = x^2 - 2x$, $y = x^2 - 4x$, $y = x^2 - 6x$
Answer the same questions for these as you did for the curves in part A.

Investigation – curves of the form $y = (x - p)(x - q)$

1 Use your GDC to draw the graph of $y = (x - 1)(x - 3)$.
Where does it intersect the *x*-axis?
What is the equation of its axis of symmetry?
What are the coordinates of the vertex?

2 Answer the previous questions for the general curve $y = (x - p)(x - q)$.
(You may wish to draw more graphs of functions of this form.)

Exercise 4K

For each function, write down:

a the equation of the axis of symmetry
b the coordinates of the points at which the curve intersects the *x*-axis
c the coordinates of the vertex.

Do not draw the graphs.

1 $y = x(x - 4)$

2 $y = x(x + 6)$

3 $y = 8x - x^2$

4 $y = 3x - x^2$

5 $y = x^2 - 2x$

6 $y = x^2 - x$

7 $y = x^2 + 4x$

8 $y = x^2 + x$

9 $y = (x + 1)(x - 3)$

10 $y = (x - 5)(x + 3)$

11 $y = (x - 2)(x - 6)$

12 $y = (x + 2)(x - 4)$

Factorize and then use the same method as in questions 1 and 2.

Investigation – the general quadratic curve $y = ax^2 + bx + c$

Part A: *a* = 1

1 Use your GDC to draw the graph of $y = x^2 - 4x + 3$.
Where does it intersect the *x*-axis?
What is the equation of its axis of symmetry?
What are the coordinates of the vertex?

2 Answer the previous questions for the general curve $y = ax^2 + bx + c$.
(You may wish to draw more graphs of functions of this form.)

Part B: varying *a*

Use your GDC to draw the graph of $y = 2x^2 - 4x + 3$ as a starting point.
Consider graphs of this form and answer the questions from Part A.

Exercise 4L

For each function, write down:

a the equation of the axis of symmetry
b the coordinates of the points at which the curve intersects the *x*-axis
c the coordinates of the vertex.

1 $y = x^2 - 2x + 3$ **2** $y = x^2 + 4x - 5$
3 $y = x^2 + 6x + 4$ **4** $y = 3x^2 - 6x + 2$
5 $y = 2x^2 - 8x - 1$ **6** $y = 2x^2 + 6x - 7$
7 $y = 0.5x^2 - x + 2$ **8** $y = 0.5x^2 + 3x - 4$

The general form of a quadratic function is $f(x) = ax^2 + bx + c$.

→
- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- The curve intersects the *y*-axis at $(0, c)$.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$, $a \neq 0$.
- The *x*-coordinate of the vertex is $x = -\frac{b}{2a}$.

A ∪-shaped graph is 'concave up'.
A ∩-shaped graph is 'concave down'.

This formula is in the Formula booklet. You should have found it in the investigation above.

→ The factorized form of a quadratic function is

$f(x) = a(x - k)(x - l)$.

- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- The curve intersects the *x*-axis at $(k, 0)$ and $(l, 0)$.
- The equation of the axis of symmetry is $x = \frac{k+l}{2}$.
- The *x*-coordinate of the vertex is also $x = \frac{k+l}{2}$.

In a parabola, the axis of symmetry passes through the vertex.

→ **Finding the *x*-intercepts**
The function $f(x) = ax^2 + bx + c$ intersects the *x*-axis where $f(x) = 0$. The *x*-values of the points of intersection are the two solutions (or **roots**) of the equation $ax^2 + bx + c = 0$.
(The *y*-values at these points of intersection are zero.)

Example 14

Consider the function $f(x) = x^2 + 6x + 8$.

a Find
i the point where the graph intersects the y-axis
ii the equation of the axis of symmetry
iii the coordinates of the vertex
iv the coordinates of the point(s) of intersection with the *x*-axis.

b Use the information from part **a** to sketch this parabola.

Answers

a i The graph intersects the *y*-axis at (0, 8).

General form: $f(x) = ax^2 + bx + c$.
Here: $f(x) = x^2 + 6x + 8$
So: $a = 1$, $b = 6$, $c = 8$
The curve intersects the y-axis at (0, c).

ii The equation of the axis of symmetry is $x = -\frac{6}{2(1)} = -3$.

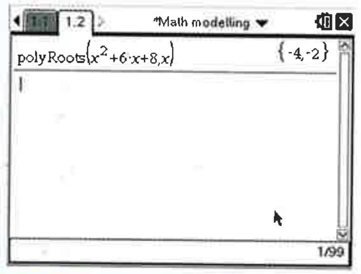
Use $x = -\frac{b}{2a}$, with $a = 1$ and $b = 6$.

iii The *x*-coordinate of the vertex is $x = -3$.
The *y*-coordinate of the vertex is:
$f(-3) = (-3)^2 + 6(-3) + 8$
$= -1$
So, the coordinates of the vertex are $(-3, -1)$.

*The x-coordinate of the vertex is $x = -\frac{b}{2a}$, which we found in part **b** so $x = -3$.*
To find the y-coordinate substitute $x = -3$ into the equation of the function.

iv $x^2 + 6x + 8 = 0$
$f(x) = 0$ when $x = -2$ or -4
The graph intersects the *x*-axis at $(-2, 0)$ and $(-4, 0)$.

The curve intersects the x-axis where $f(x) = 0$, so put $x^2 + 6x + 8 = 0$ and solve using a GDC.



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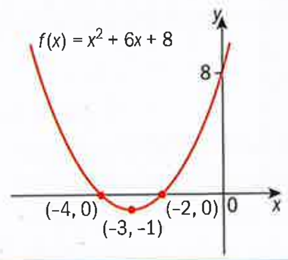
The Indian mathematician Sridhara is believed to have lived in the 9th and 10th centuries. He was one of the first mathematicians to propose a rule to solve a quadratic equation. Research why there is controversy about when he lived.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

b



$a > 0$ ($a = 1$) so the curve is ∪-shaped.
*The vertex is $(-3, -1)$ (part **a iii**)*
*The curve intersects the y-axis at $(0, 8)$ (part **a i**)*
*$x = -3$ is a line of symmetry (part **a ii**)*

For $f(x) = x^2 + 6x + 8$
- the vertex is $(-3, -1)$.
- the range of f is $y \geq -1$.

Exercise 4M

For each function $f(x)$ in 1–8:

a Find
- **i** the coordinates of the point of intersection with the y-axis
- **ii** the equation of the axis of symmetry
- **iii** the coordinates of the vertex
- **iv** the coordinates of the point(s) of intersection with the x-axis
- **v** the range of f.

b Sketch the graph of the function.

c Use your GDC to draw the graph to check your results.

1 $f(x) = x^2 + 2x - 3$ **2** $f(x) = x^2 + 8x + 7$
3 $f(x) = x^2 - 6x - 7$ **4** $f(x) = x^2 - 3x - 4$
5 $f(x) = x^2 - 3x - 10$ **6** $f(x) = 2x^2 + x - 3$
7 $f(x) = 2x^2 + 5x - 3$ **8** $f(x) = 3x^2 - x - 4$

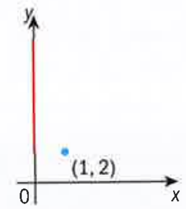
Sketching quadratic graphs

Example 15

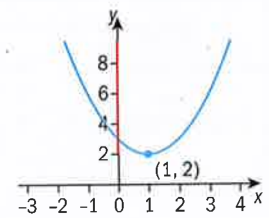
a Sketch the graph of a parabola with vertex $(1, 2)$ and range $y \geq 2$.
b Sketch the graph of a parabola with x-intercepts at $x = -2$ and $x = 3$ and y-intercept at $y = -1$.

Answers

a



Draw and label the axes.
Use a vertical line to show the range ($y \geq 2$) of the function on the y-axis (shown in red here).
Plot and label the vertex (1, 2).

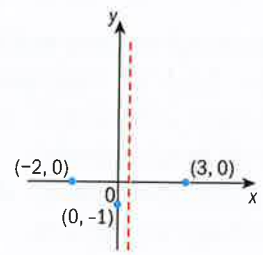


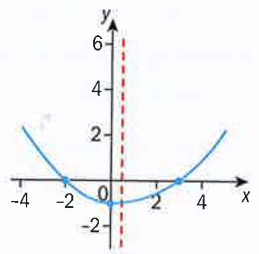
Draw a smooth curve through the point (1, 2). The curve is symmetrical about the vertical line through the vertex, that is, $x = 1$.

Is the parabola shown in Example 15 part **a** the only one that satisfies the information given? If not, how many are there?

▶ Continued on next page

b





Draw and label the axes.
Mark the x-intercepts at $(-2, 0)$ and $(3, 0)$. The axis of symmetry is halfway between the two x-intercepts, at $x = \frac{1}{2}$. Draw it in (as shown by the dashed red line here).
Mark the y-intercept at $(0, -1)$.
Draw a smooth curve through the marked points.
The curve is symmetrical about $x = \frac{1}{2}$, and the axis of symmetry passes through the vertex.

The x-intercepts are the points where the graph crosses the x-axis. The y-values at these points are zero. The values at the x-intercepts are called the 'zeros' of the function.
The y-intercepts are the points where the graph crosses the y-axis. The x-values at these points are zero.

Sketching a quadratic graph (2)
- If you are given the function, use your GDC to draw the graph and then copy the information on to a sketch.
- If you are not given the function, use the information you are given and what you know about quadratic curves, that is:
  - They are ∪-shaped or ∩-shaped.
  - They have an axis of symmetry that passes through the vertex.

If a quadratic function only takes negative values between $x = m$ and $x = n$, what can you tell about $x = m$ and $x = n$? What happens at the points where x takes those negative values? Is the parabola ∪-shaped or ∩-shaped?
What can you tell about a quadratic function that only takes positive values between $x = m$ and $x = n$?

Exercise 4N

Sketch the graph of:

1 A parabola with vertex $(1, -3)$ and x-intercepts at -1 and 3.

2 A parabola with vertex $(-1, 2)$ and range $y \leq 2$.

3 A parabola with an axis of symmetry at $x = 0$ and range $y \leq -1$.

4 A parabola with x-intercepts at $x = 3$ and $x = 0$ and range $y \leq 1$.

5 A parabola passing through the points $(0, -2)$ and $(4, -2)$ with a maximum value at $y = 2$.

6 A quadratic function f that takes negative values between $x = 2$ and $x = 5$, and $f(0) = 4$.

Intersection of two functions

→ Two functions $f(x)$ and $g(x)$ intersect at the point(s) where $f(x) = g(x)$.

To find the coordinates of the points of intersection

- use a GDC, or
- equate the two functions algebraically, rearrange to equal zero, and then solve on the GDC.

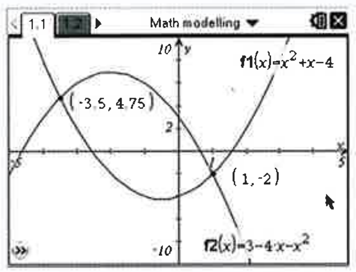
Example 16

Find the points of intersection of $f(x) = x^2 + x - 4$ and $g(x) = 3 - 4x - x^2$.

Answers

Method 1: Graphical

The points of intersection are $(-3.5, 4.75)$ and $(1, -2)$.



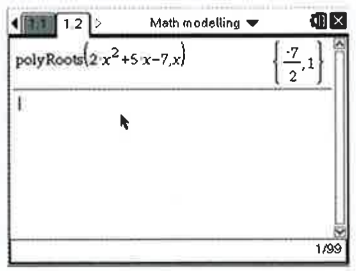
Method 2: Algebraic

$f(x) = g(x)$

$x^2 + x - 4 = 3 - 4x - x^2$ — *Equate f(x) and g(x).*

$2x^2 + 5x - 7 = 0$ — *Rearrange to equal zero.*

$x = 1, x = -\frac{7}{2}$ — *Solve using a GDC.*



$f(1) = (1)^2 + (1) - 4 = -2$

$f\left(-\frac{7}{2}\right) = \left(-\frac{7}{2}\right)^2 + \left(-\frac{7}{2}\right) - 4 = \frac{19}{4}$

Substitute the values of x into the function f (x) to find the y-coordinate of each point.

So, the points of intersection are $(1, -2)$ and $\left(-\frac{7}{2}, \frac{19}{4}\right)$.

Write as coordinate pairs.

For help with using a GDC to find the points of intersection of two curves see Chapter 12, Section 4.5.

For help with using a GDC to solve a quadratic equation see Chapter 12, Section 1.2.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4O

1 Here are two functions $f(x) = x^2 + 3x - 5$ and $g(x) = x - 2$ for the domain $-5 \le x \le 2, x \in \mathbb{R}$.

a Use a GDC to draw the graphs of these functions and find the coordinates of their points of intersection.

b Write down $f(x) = g(x)$ and solve it for x. Do you get the same answers as you did for part **a**?

The function $h(x) = 2x - 3$ has the same domain.

c Find the points of intersection of $f(x)$ and $h(x)$

i algebraically **ii** graphically.

2 Find the coordinates of the points of intersection of the graph of $f(x) = x^2 + 3x - 5$ for the domain $-5 \le x \le 2, x \in \mathbb{R}$, and the line $x + y + 5 = 0$.

3 Find the points of intersection of the graphs of:

a $f(x) = 5 + 3x - x^2$ and $g(x) = 1$

b $f(x) = 5 + 3x - x^2$ and $h(x) = 2x + 3$

4 **a** Use a GDC to draw the graphs of the functions $f(x) = 2x^2 - x - 3$ and $g(x) = x + 1$ for the domain $-3 \le x \le 3, x \in \mathbb{R}$.

b State the ranges of f and g on this domain.

c Find the x-coordinates of the points of intersection of the two functions.

d On the same axes, and for the same domain, draw the graph of the function $h(x) = 2x + 2$.

e Solve the equation $f(x) = h(x)$ both graphically and algebraically.

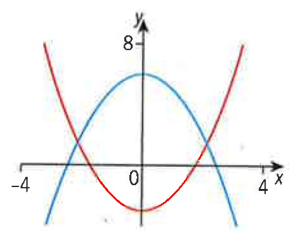
f Find the coordinates of the points of intersection of the graph of $y = f(x)$ and the line $x + y = 5$.

Find the points 'graphically' means draw the graphs on a GDC and read off the coordinates of the points of intersection.

First rearrange the linear equation to make y the subject.

EXAM-STYLE QUESTION

5 The diagram shows the graphs of the functions $f(x) = x^2 - 3$ and $g(x) = 6 - x^2$ for values of x between -4 and 4.



a Find the coordinates of the points of intersection.

b Write down the set of values of x for which $f(x) < g(x)$.

Finding the equation of a quadratic function from its graph

To find the equation of a graph of a quadratic function, use these facts:

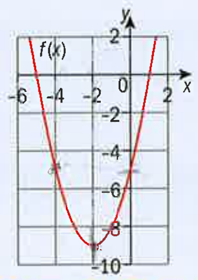
For the graph of $f(x) = ax^2 + bx + c$

- the point of intersection of the function with the y-axis is $(0, c)$
- the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

> You can use your GDC to find the equation of a quadratic function from its graph. For help see Chapter 12, Section 4.6.

Example 17

Find the equation of the quadratic function shown in the graph.



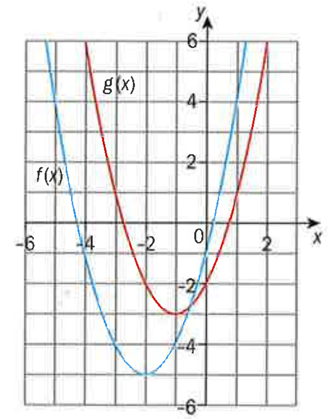
Answer

The general form of a quadratic function is given by $f(x) = ax^2 + bx + c$.	
The function intersects the y-axis at the point $(0, -5)$. So $c = -5$. $\Rightarrow f(x) = ax^2 + bx - 5$	*The function intersects the y-axis at the point (0, c). Read off the value of c from the graph.*
The equation of the axis of symmetry is $x = -2$. So: $-2 = -\frac{b}{2a}$; $-b = -4a$; $b = 4a$	*The equation of the axis of symmetry is given by $x = -\frac{b}{2a}$. Substitute the value of x.*
At the vertex, $x = -2$, $y = -9$. So: $f(-2) = a(-2)^2 + b(-2) - 5 = -9$; $4a - 2b - 5 = -9$; $4a - 2b = -4$	*Read off the coordinates of the vertex from the graph: (−2, −9). Substitute the x- and y-values into $f(x) = ax^2 + bx - 5$.*
$\left.\begin{array}{r} b = 4a \\ 4a - 2b = -4 \end{array}\right\} \Rightarrow$ $4a - 2(4a) = -4$; $4a - 8a = -4$; $-4a = -4 \Rightarrow a = 1$; $b = 4a \Rightarrow b = 4$	*Solve the simultaneous equations.*
Therefore, the equation of the quadratic function is: $f(x) = x^2 + 4x - 5$	*Substitute the values of a = 1, b = 4 and c = −5 into $f(x) = ax^2 + bx + c$.*

Exercise 4P

Find the equations of these quadratic functions.

1



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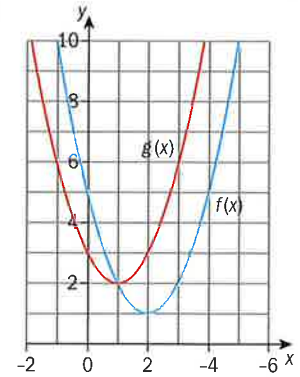

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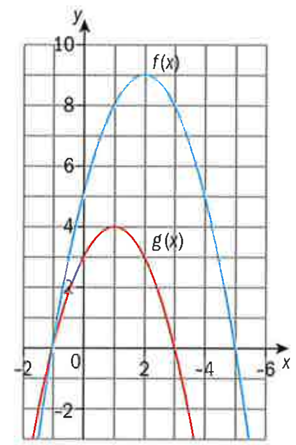

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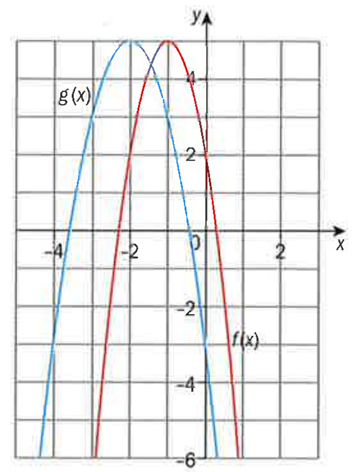
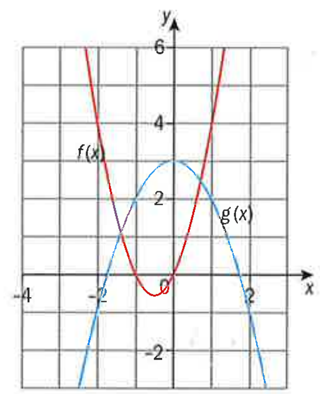


5

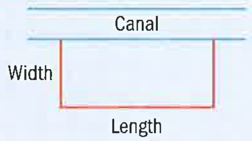


Quadratic models

Many real-life situations can be modeled using a quadratic function.

Example 18

A farmer wishes to fence off the maximum area possible to make a rectangular field.
She has 150 metres of fencing.
One side of the land borders a canal.
Find the maximum area of the field.



Answers

There are three variables:

- the length of the rectangle, l
- the width of the rectangle, w
- the area of the rectangle, A

Start by naming the variables in the problem.

The area of the rectangle $A = lw$.

Area = length × width.

As the total length of fencing is 150 m,

$l + 2w = 150$

$l = 150 - 2w$

Write down an equation for the perimeter of the field. Make l the subject.

Substitute the expression for l into the area formula.

So,

$A = lw$

$A = (150 - 2w)w$

$A = 150w - 2w^2$

Method 1: Using a GDC

The width, w, is 37.5 m.

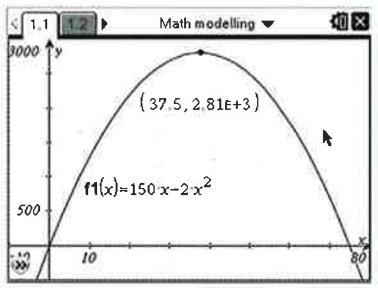
$l = 150 - 2w = 150 - 75 = 75$ m

Maximum area,

$A = lw = 75 \times 37.5$

$= 2812.5\ \text{m}^2$

Graph $A(x) = 150x - 2x^2$ on your GDC and read off the x-coordinate of the vertex: 37.5. This is the value of width, w, that gives the maximum value of A.



Method 2: Algebraic

$w = -\dfrac{150}{2(-2)} = 37.5$

$A = 150 \times 37.5 - 2 \times 37.5^2$

$= 2812.5\ \text{m}^2$

For the quadratic function $f(x) = ax^2 + bx + c$ the x-coordinate of the vertex is given by $x = -\dfrac{b}{2a}$.

The x-coordinate gives us the width, w. Here, the function is $150w - 2w^2$ so $a = -2$ and $b = 150$.

Ancient Babylonians and Egyptians studied quadratic equations like these thousands of years ago, to find solutions to problems involving areas of rectangles.

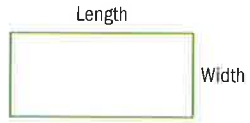
2.81E3 means $2.81 \times 10^3 = 2810$.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

You can use $A = lw$ or $A = 150w - 2w^2$ to work out the area.

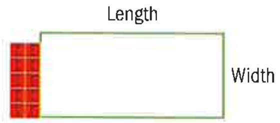
Exercise 4Q

1 a A farmer has 170 metres of fencing to fence off a rectangular area.



Find the length and width that give the maximum area.

b A farmer has 110 metres of fencing to fence off a rectangular area.
Part of one side is a wall of length 15 m.



Find the dimensions of the field that give the maximum area.

1. Identify and name the variables.
2. Use the constraint to find a model for the 'length' (this model will be linear).
3. Find a model for the area (this model will be quadratic).

2 A company's weekly profit, in riyals, is modeled by the function
$P(u) = -0.032u^2 + 46u - 3000$
where u is the number of units sold each week.
Find

a the maximum weekly profit

b the loss for a week's holiday period, where no units are sold

c the number of units sold each week at break-even point for the company.

At break-even point there is no profit and no loss, so $P(u) = 0$.

EXAM-STYLE QUESTION

3 A rocket follows a parabolic trajectory.
After t seconds, the vertical height of the rocket above the ground, in metres, is given by
$H(t) = 37t - t^2$.

a Find the height of the rocket above the ground after 10 seconds.

b Find the maximum height of the rocket above the ground.

c Find the length of time the rocket is in the air.

The trajectory is the path followed by an object.

4.4 Exponential models

Exponential functions and their graphs

→ In an **exponential function**, the independent variable is the **exponent** (or **power**).

Here are some examples of **exponential functions:**

$$f(x) = 2^x,\quad f(x) = 5(3)^x + 2,\quad g(x) = 5^{-x} - 3,\quad h(x) = \left(\frac{1}{3}\right)^x + 1$$

Investigation – exponential graphs

1 The number of water lilies in a pond doubles every week. In week one there were 4 water lilies in the pond.
Draw a table and write down the number of water lilies in the pond each week up to week 12.
Plot the points from the table on a graph of number of lilies against time.
Draw a smooth curve through all the points.

> Time is the dependent variable, so it goes on the horizontal axis.

> The graph is an example of an increasing exponential graph.

2 A radioactive substance has a half-life of two hours. This means that every two hours its radioactivity halves.
A Geiger counter reading of the radioactive substance is taken at time $t = 0$. The reading is 6000 counts per second.
Two hours later ($t = 2$) the reading is 3000 counts per second.
What will the readings be at $t = 4$, $t = 6$, $t = 8$ and $t = 10$?
Plot the points on a graph of counts per second against time and join them with a smooth curve.

> Could the number of water lilies in a pond keep doubling forever? Will the radioactivity of the substance ever reach zero?

> This graph is an example of a decreasing exponential graph.

> Does the shape of a ski slope form an exponential function? Investigate ski slopes on the internet to find out what the function is.

Graphs of exponential functions $f(x) = a^x$ where $a \in \mathbb{Q}^+$, $a \neq 1$

> $\mathbb{Q}^+$ is the set of positive rational numbers.

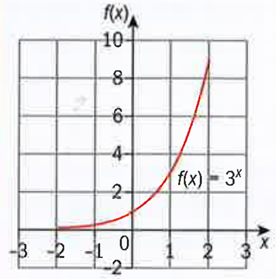
> Why can $a \neq 1$? What kind of function would you get if $a = 1$?

Example 19

Draw a graph of the function $f(x) = 3^x$ for $-2 \le x \le 2$

Answers

Method 1: By hand

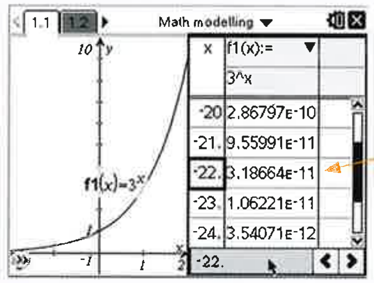


Draw a table of values.

x	−2	−1	0	1	2
f(x)	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Plot the points.
Draw a smooth curve through all the points.
This is an increasing exponential function.

Method 2: Using a GDC



For help with graphing exponential functions on your GDC see Chapter 12, Section 4.3.

> 3.1866E–11 = 0.000 000 000 031 866

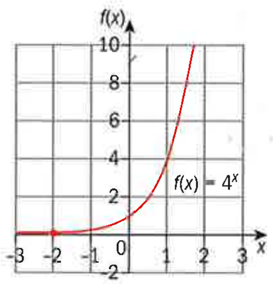
> **GDC help on CD:** *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

> You can check what happens when the values of x get very small or very large using the table of values on your GDC.

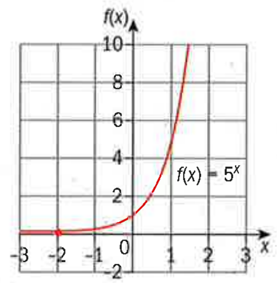
Look at the graph in Example 19. As the values of x get smaller, the curve gets closer and closer to the x-axis. The x-axis ($y = 0$) is a horizontal **asymptote** to the graph. At $x = 0$, $f(x) = 1$. As the values of x get very large, $f(x)$ gets larger even more quickly. We say that $f(x)$ tends to infinity. The function is an **increasing** exponential function.

> An asymptote is a line that the curve approaches but never touches.

Here are some more graphs of **increasing** exponential functions.



▲ $f(x) = 4^x$



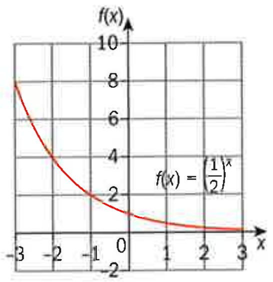
▲ $f(x) = 5^x$

All these graphs pass through the point (0, 1) and have $y = 0$ (the x-axis) as a horizontal asymptote.

Graphs of exponential functions $f(x) = a^x$ where $0 < a < 1$

What happens if a is a positive proper fraction?

Here is the graph of $y = \left(\frac{1}{2}\right)^x$.



A **proper fraction** is a fraction where the numerator is smaller than the denominator.

This graph also passes through the point (0, 1) and has $y = 0$ (the x-axis) as a horizontal asymptote. However, this is an example of a **decreasing** exponential function.

- For an increasing exponential function, the y-values increase as the x-values increase from left to right.
- For a decreasing exponential function, the y-values decrease as the x-values increase from left to right.

Exercise 4R

Draw the graphs of these functions using your GDC.
For each, write down the coordinates of the point where the curve intersects the y-axis and the equation of the horizontal asymptote.

1 $f(x) = 2^x$ **2** $f(x) = 6^x$ **3** $f(x) = 8^x$

4 $f(x) = \left(\frac{1}{3}\right)^x$ **5** $f(x) = \left(\frac{1}{5}\right)^x$

Investigation – graphs of $f(x) = ka^x$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

Use your GDC to draw the graphs of

1 $f(x) = 2(3)^x$ **2** $f(x) = 3\left(\frac{1}{2}\right)^x$ **3** $f(x) = -3(2)^x$

For each graph, write down

a the value of k in the equation $f(x) = ka^x$
b the point where the graph crosses the y-axis
c the equation of the horizontal asymptote.

What do you notice?

Investigation – graphs of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$

Use your GDC to draw the graphs of

1 $f(x) = 2^x + 3$ **2** $f(x) = 3\left(\frac{1}{2}\right)^x - 4$ **3** $f(x) = -2(3)^x + 5$
for $-3 \le x \le 3$.

For each graph, write down

a the values of k and c in the equation $f(x) = ka^x + c$
b the point where the graph crosses the y-axis
c the equation of the horizontal asymptote.
Work out $k + c$ for each graph. What do you notice?

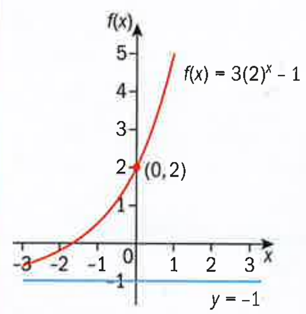
→ In general, for the graph of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$
- the line $y = c$ is a **horizontal asymptote**
- the curve passes through the point $(0, k + c)$.

Sketching an exponential graph
- Draw and label the axes.
- Label the point where the graph crosses the y-axis.
- Draw in the asymptotes.

Example 20

Sketch the graph of the function $f(x) = 3(2)^x - 1$

Answer



Comparing $f(x) = 3(2)^x - 1$ *to* $f(x) = ka^x + c$:
$k = 3$
$a = 2$
$c = -1$
$y = c$ is a horizontal asymptote $\Rightarrow$
$y = -1$
The curve passes through the point $(0, k + c) \Rightarrow (0, 3 - 1)$ *or* $(0, 2)$.

Exercise 4S

For each function, write down

a the coordinates of the point where the curve cuts the y-axis
b the equation of the horizontal asymptote.

Hence, sketch the graph of the function.

1 $f(x) = 2^x$ **2** $f(x) = 6^x$

3 $f(x) = \left(\frac{1}{3}\right)^x$ **4** $f(x) = \left(\frac{1}{5}\right)^x$

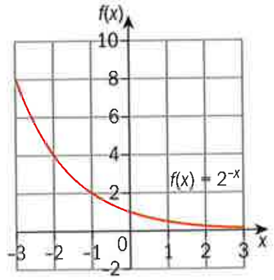
5 $f(x) = 3(2)^x + 4$ **6** $f(x) = -2(4)^x - 1$

7 $f(x) = -1(2)^x + 3$ **8** $f(x) = 4(3)^x - 2$

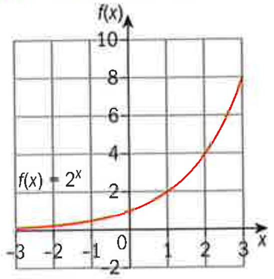
9 $f(x) = 0.5(2)^x + 3$ **10** $f(x) = 2(0.5)^x + 1$

11 $f(x) = 0.4^x + 1$ **12** $f(x) = 2(0.1)^x - 1$

Graphs of $f(x) = a^{-x} + c$ where $a \in \mathbb{Q}^+$ and $a \neq 1$

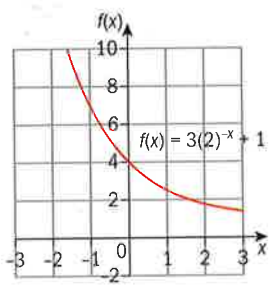


▲ Graph of $f(x) = 2^{-x}$.

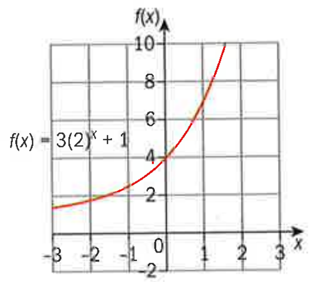


▲ Graph of $f(x) = 2^x$.

The graph of $f(x) = 2^{-x}$ is a reflection in the y-axis of the graph of $f(x) = 2^x$.



▲ Graph of $f(x) = 3(2)^{-x} + 1$.



▲ Graph of $f(x) = 3(2)^x + 1$.

$k = 3$ and $c = 1$.
Notice that $3 + 1 = 4$.

The curves pass through the point (0, 4) and the horizontal asymptote is $y = 1$.

→ In general, for the graph of $f(x) = ka^{-x} + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$
- the line $y = c$ is a horizontal asymptote
- the curve passes through the point $(0, k + c)$
- the graph is a reflection in the y-axis of $g(x) = ka^x + c$.

Exercise 4T

For each function, write down

a the coordinates of the point where the curve cuts the y-axis

b the equation of the horizontal asymptote.

Hence, sketch the graph of the function.

1 $f(x) = 4(2)^{-x} + 2$ **2** $f(x) = -4^{-x} + 1$

3 $f(x) = -2(2)^{-x} + 3$ **4** $f(x) = 3(2)^{-x} - 2$

5 $f(x) = 0.5(3)^{-x} + 2$ **6** $f(x) = 0.5^{-x} + 1$

7 $f(x) = 2(0.1)^{-x} - 1$ **8** $f(x) = 0.4^{-x} + 2$

9 $f(x) = 3(0.2)^{-x} + 4$ **10** $f(x) = 5(3)^{-x} - 2$

Applications of exponential functions

Many real-life situations involving growth and decay can be modeled by exponential functions.

Example 21

The length, l cm, of a pumpkin plant increases according to the equation

$$l = 4(1.2)^t$$

where t is the time in days.

a Copy and complete the table. Give your answers correct to 3 sf.

t	0	2	4	6	8	10	12	14	16
l									

b Draw a graph of l against t for $0 \le t \le 20$ and $0 \le l \le 100$.

c How long is the pumpkin plant when $t = 0$?

d How long will the pumpkin plant be after 3 weeks?

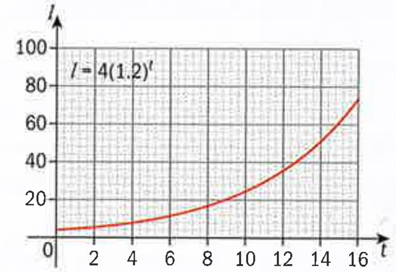
Answers

a

t	0	2	4	6	8	10	12	14	16
l	4	5.8	8.3	11.9	17.2	24.8	35.7	51.4	74

Substitute each value of t into the equation to find the corresponding value of l.

b



Draw and label the axes.
Put t on the horizontal axis.
Put l on the vertical axis.
Plot the points from the table and join with a smooth curve.

c When $t = 0$, $l = 4$ cm.

Read the value of l that corresponds to t = 0 from the table.

d 3 weeks = 21 days
So, $l = 4(1.2)^{21} = 184$ cm (to 3 sf).

For the equation, time is given in days, so convert from weeks.
Substitute t = 21 into the equation.

Example 22

Hubert invests 3000 euros in a bank at a rate of 5% per annum compounded yearly.
Let y be the amount he has in the bank after x years.

a Draw a graph to represent how much Hubert has in the bank after x years. Use a scale of 0–10 years on the x-axis and 2500–5000 euros on the y-axis.

b How much does he have after 4 years?

c How many years is it before Hubert has 4000 euros in the bank?

Answers

a The compound interest formula is:

$$y = 3000\left(1 + \frac{5}{100}\right)^x$$

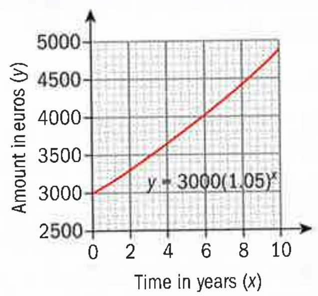
$$y = 3000(1.05)^x$$

where x = number of years.

This problem can be represented by a compound interest function.

Draw a table of values.

Time (x years)	Amount (y euros)
0	3000
2	3307.50
4	3646.52
6	4020.29
8	4432.37
10	4886.68



Draw and label the axes.

Plot the points and join them with a smooth curve.

b After 4 years Hubert has $3000(1.05)^4 = 3646.52$ euros.

Substitute $x = 4$ into the formula.

c Hubert has 4000 euros in the bank after 6 years.

You need to find the value of x for $y = 4000$ euros.
*From the table of values in part **a** you can see that after 6 years the amount is 4020.29.*
Check the amount after 5 years:
$y = 3000(1.05)^5 = 3828.84$
This is less than 4000 euros.

The **compound interest** formula is an exponential (growth) function.

You will learn more about compound interest in Chapter 7.

Exercise 4U

EXAM-STYLE QUESTIONS

1 Sketch the graphs of $f(x) = 2^x + 0.5$ and $g(x) = 2^{-x} + 0.5$ for $-3 \le x \le 3$.

a Write down the coordinates of the point of intersection of the two curves.

b Write down the equation of the horizontal asymptote to both graphs.

2 The value of a car decreases every year according to the function

$$V(t) = 26\,000x^t$$

where V is the value of the car in euros, t is the number of years after it was first bought and x is a constant.

a Write down the value of the car when it was first bought.

b After one year the value of the car is 22 100 euros. Find the value of x.

c Calculate the number of years that it will take for the car's value to fall to less than 6000 euros.

3 The equation $M(t) = 150(0.9)^t$ gives the amount, in grams, of a radioactive material kept in a laboratory for t years.

a Sketch the graph of the function $M(t)$ for $0 \le t \le 100$.

b Write down the equation of the horizontal asymptote to the graph of $M(t)$.

c Find the mass of the radioactive material after 20 years.

d Calculate the number of years that it will take for the radioactive material to have a mass of 75 grams.

4 The area, A m^2, covered by a certain weed is measured at 06:00 each day.
On the 1st June the area was 50 m^2.
Each day the area of the weeds grew by the formula

$$A(t) = 50(1.06)^t$$

where t is the number of days after 1st June.

a Sketch the graph of $A(t)$ for $-4 \le t \le 20$.

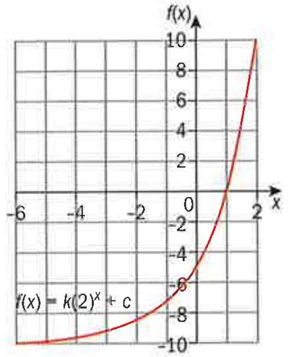
b Explain what the negative values of t represent.

c Calculate the area covered by the weeds at 06:00 on 15th June.

d Find the value of t when the area is 80 m^2.

EXAM-STYLE QUESTIONS

5 The graph shows the function $f(x) = k(2)^x + c$.
Find the values of c and k.



6 The temperature, T, of a cup of coffee is given by the function
$$T(t) = 18 + 60(2)^{-t}$$
where T is measured in °C and t is in minutes.

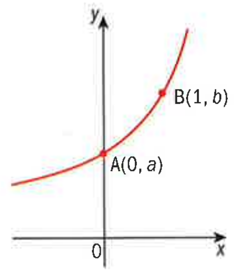
- **a** Sketch the graph of $T(t)$ for $0 \le t \le 10$.
- **b** Write down the temperature of the coffee when it is first served.
- **c** Find the temperature of the coffee 5 minutes after serving.
- **d** Calculate the number of minutes that it takes the coffee to reach a temperature of 40 °C.
- **e** Write down the temperature of the room where the coffee is served. Give a reason for your answer.

7 The value, in USD, of a piece of farm machinery depreciates according to the formula
$D(t) = 18\,000(0.9)^t$ where t is the time in years.

- **a** Write down the initial cost of the machine.
- **b** Find the value of the machine after 5 years.
- **c** Calculate the number of years that it takes for the value of the machine to fall below 9000 USD.

8 The graph of the function $f(x) = \frac{2^x}{a}$ passes through the points $(0, b)$ and $(2, 0.8)$. Calculate the values of a and b.

9 The diagram shows the graph of $y = 2^x + 3$. The curve passes through the points A(0, a) and B(1, b).



- **a** Find the value of a and the value of b.
- **b** Write down the equation of the asymptote to the curve.

10 A function is represented by the equation $f(x) = 2(3)^x + 1$.
Here is a table of values of $f(x)$ for $-2 \le x \le 2$.

x	−2	−1	0	1	2
$f(x)$	1.222	a	3	7	b

- **a** Calculate the value a and the value of b.
- **b** Draw the graph of $f(x)$ for $-2 \le x \le 2$.
- **c** The domain of $f(x)$ is the real numbers. What is the range?

4.5 Graphs of functions of the form $f(x) = ax^m + bx^n + \ldots,\ m, n \in \mathbb{Z}$

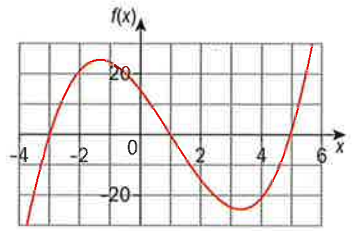
In Sections 4.2 and 4.3, you have seen examples of linear and quadratic functions. What happens when the power of x is an integer larger than 2 or smaller than 0?

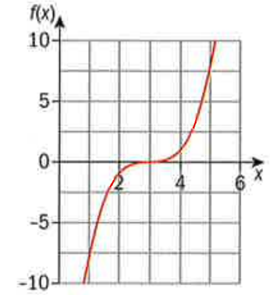
Cubic functions

When the largest power of x is 3 the function is called a **cubic** function.

→ A cubic function has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. The domain is $\mathbb{R}$, unless otherwise stated.

Here are two examples of graphs of cubic functions.





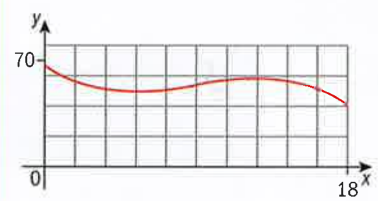
Example 23

The number of fish, F, in a pond from the period 1995 to 2010 is modeled using the formula
$$F(x) = -0.030x^3 + 0.86x^2 - 6.9x + 67$$
where x is the number of years after 1995.

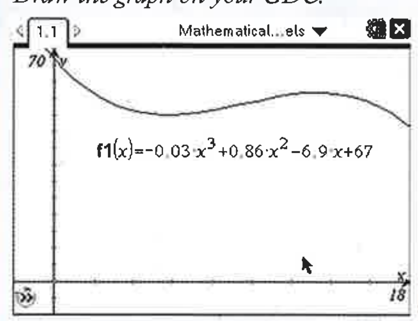
- **a** Use your GDC to sketch the function for $0 \le x \le 18$.
- **b** Find the number of fish in the pond after 6 years.
- **c** Find the number of fish in the pond after 13 years.

Answers

a



Draw the graph on your GDC.



Copy the details on to a sketch graph.

▶ Continued on next page

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b $F(6) = -0.030(6)^3 + 0.86(6)^2 - 6.9(6) + 67$
$= -6.48 + 30.96 - 41.4 + 67$
$= 50.08$
So, after 6 years, there are 50 fish in the pond.

Substitute $x = 6$ into the equation. Or, you can use your GDC table of values or the Trace function.

c $F(13)$
$= -0.030(13)^3 + 0.86(13)^2 - 6.9(13) + 67$
$= -65.91 + 145.34 - 89.7 + 67$
$= 56.73$
So, after 13 years, there are 56 fish in the pond.

Substitute $x = 13$ into the equation. Or, you can use your GDC table of values.

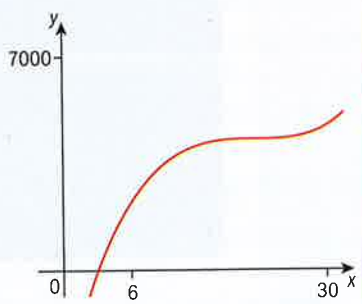
Example 24

A pandemic is modeled using the equation
$$y = (x - 20)^3 + 5000$$
where x is the number of weeks after the outbreak started and y is the number of cases reported.

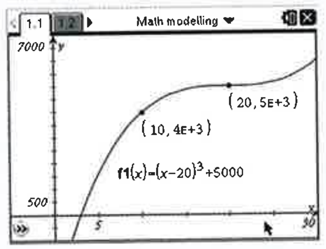
a Use your GDC to sketch the function for $0 \le x \le 30$.
b Find the number of cases after 10 weeks.
c Find the number of cases after 20 weeks.
d Is this a good model to represent the number of cases of a pandemic?

Answers

a



Draw the graph on your GDC.



Copy the details on to a sketch graph.

b $y = (10 - 20)^3 + 5000 = 4000$
So, after 10 weeks, there are 4000 cases.

Substitute $x = 10$ into the equation.

c $y = (20 - 20)^3 + 5000 = 5000$
So, after 20 weeks, there are 5000 cases.

Substitute $x = 20$ into the equation.

d No, because the number of cases starts to rise again after 20 weeks and will keep on rising.

Consider: Does the graph keep increasing? Would you expect the pandemic to increase forever?

A pandemic is an epidemic of an infectious disease that spreads over several continents.

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Can mathematical models accurately model the real world?

Investigation – quartic functions

When the largest power of x is 4 then the function is called a **quartic** function.

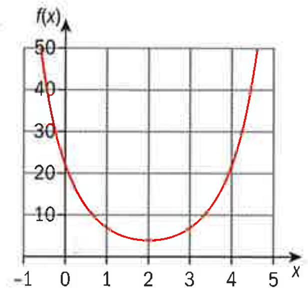
A quartic function has the form
$f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$. The domain is $\mathbb{R}$, unless otherwise stated.

Substitute different values of a, b, c, d and e into the equation
$f(x) = ax^4 + bx^3 + cx^2 + dx + e$.
Use your GDC to draw the functions.
What can you say about the shape of a quartic graph?

Exercise 4V

1 The times of high and low tides one day are modeled by the function
$$f(x) = -0.0015x^4 + 0.056x^3 - 0.60x^2 + 1.65x + 4$$
where x is the number of hours after midnight.
a Use your GDC to sketch the function for $0 \le x \le 20$.
b Find the time of the low tides.
c Find the times of the high tides.

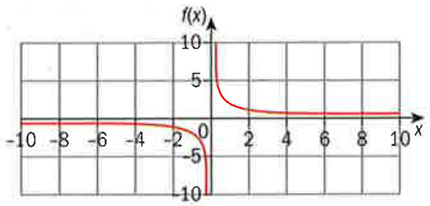
2 Here is the graph of the function $f(x) = (x - 2)^4 + 6$

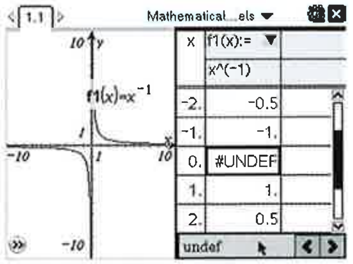


a Find the value of $f(x)$ when $x = 2$.
b Find the values of x when $y = 6$.
c Write down the range of this function.

Graphs of functions when the power of x is a negative integer

Here is the graph of $y = x^{-1}$, $x \neq 0$, for $-10 \leq x \leq 10$





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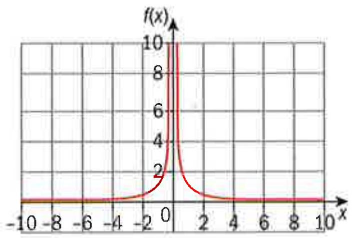
The graph has two branches that do not overlap or cross the y-axis.

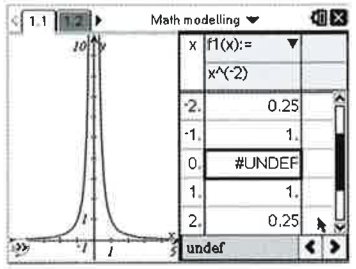
There is no value for y when $x = 0$. We call $x = 0$ a **vertical asymptote**.

When you look at the table of values on the GDC, you usually see UNDEF in the column for y whenever you have a vertical asymptote.

A vertical asymptote occurs when the value of y tends to infinity as x tends to zero. This means that when x approaches 0 from either the negative side or the positive side then y approaches either a very big negative number or a very big positive number.

Here is the graph of $y = x^{-2}$, $x \neq 0$, for $-10 \leq x \leq 10$





There is no value for y when $x = 0$, so $x = 0$ is a vertical asymptote. However, in this graph y tends to a very large positive number when x approaches 0 from either the negative side or the positive side.

Investigation – graphs of $y = ax^{-n}$

1 Use your GDC to draw the graphs of:

- $y = x^{-3}$ for $-10 \leq x \leq 10$
- $y = x^{-4}$ for $-10 \leq x \leq 10$

Compare them to the graphs of $y = x^{-1}$ and $y = x^{-2}$.
What do you notice?

2 Draw the graphs of:

- $y = 2x^{-3}$ for $-10 \leq x \leq 10$
- $y = 3x^{-4}$ for $-10 \leq x \leq 10$

Compare these graphs to the others.
What do you notice?

Example 25

A rectangle has an area of $1.5\,\text{m}^2$.
Let the length of the rectangle be y and the width be x.

a Show that $y = \frac{1.5}{x}$.

b Use your GDC to draw the graph of $y = \frac{1.5}{x}$ for $0 < x \leq 10$.

c What happens when x gets closer to 0?

d What happens when x gets closer to 10?

e Write down the equations of the vertical and horizontal asymptotes.

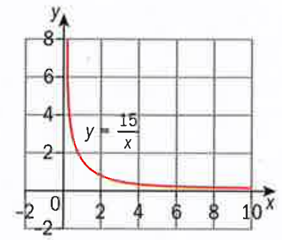
How many different rectangles could you draw with an area of $1.5\,\text{m}^2$?

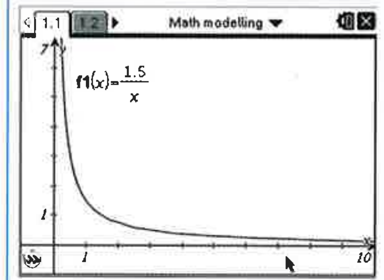
Answers

a $x \times y = 1.5 \Rightarrow y = \frac{1.5}{x}$

Area = length × width.
Rearrange the formula to make y the subject.

b





GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

c When x gets closer to 0 then y becomes a very large positive number.

d When x gets closer to 10 then y becomes a very small positive number.

e The vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

Which lines does the curve approach but never meet?

Exercise 4W

1 The temperature of water as it cools to room temperature is modeled by the function

$$f(x) = 21 + \frac{79}{x},\ x \neq 0,$$

where x is the time in minutes and $f(x)$ represents the temperature in °C.

a Use your GDC to sketch the graph of the function for $0 < x \leq 15$.

b Calculate the temperature of the water after 10 minutes.

c How many minutes does it take for the temperature to cool down to 50 °C?

d Write down the equation of the vertical asymptote.

e Write down the equation of the horizontal asymptote.

f Write down the room temperature.

2 Oil is heated on a stove. The temperature is modeled by the function

$$f(x) = 100 - \frac{100}{x},\ x \neq 0,$$

where x is the time in minutes from when the oil began to heat and $f(x)$ represents the temperature in °C.

a Use your GDC to sketch the graph of the function for $0 < x \leq 50$.

b Find the temperature of the oil after 10 minutes.

c Find the number of minutes that it takes the temperature to reach 30 °C.

d Write down the maximum temperature that the oil can reach.

3 **a** Use your GDC to sketch the graph of $f(x) = \frac{5}{x^2}$, $x \neq 0$.

b Write down the values of x when $y = 8$.

c Write down the equations of the vertical and horizontal asymptotes to the graph.

d Given that the domain of f is the real numbers, $x \neq 0$, write down the range of f.

4 **a** Use your GDC to sketch the graph of $f(x) = 3 + \frac{6}{x}$, $x \neq 0$, for $-10 \leq x \leq 10$.

b Find the value of $f(x)$ when $x = 8$.

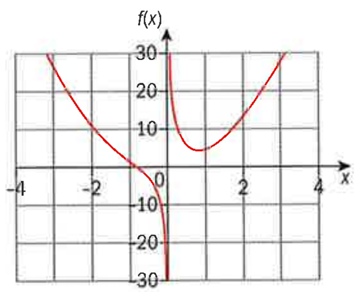
c Find the value of x when $y = 5$.

d Write down the equations of the vertical and horizontal asymptotes to the graph.

e Given that the domain of f is the real numbers, $x \neq 0$, write down the range of f.

Graphs of more complex functions

Here is the graph of $f(x) = 3x^2 + \frac{2}{x}$, $x \neq 0$, for $-4 \leq x \leq 4$.



The graph has two separate branches.
$x = 0$ is a vertical asymptote.
The domain is $-\infty \leq x < 0,\ 0 < x \leq +\infty$.

English mathematician John Wallis (1616–1703) introduced the symbol ∞ for infinity.

Example 26

A taxi company's fares depend on the distance, in kilometres, traveled.
The fares are calculated using the formula

$$f(x) = 2x + \frac{50}{x^2}$$

where x is the number of kilometres traveled ($x \neq 0$) and $f(x)$ is the fare in euros.

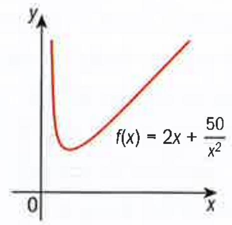
a Sketch the graph of the function for $0 < x \leq 20$.

b Find the cost for a journey of 10 kilometres.

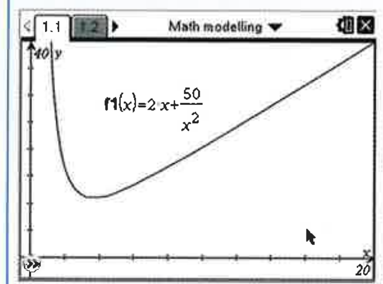
c Find the number of kilometres traveled that gives the cheapest fare.

Answers

a



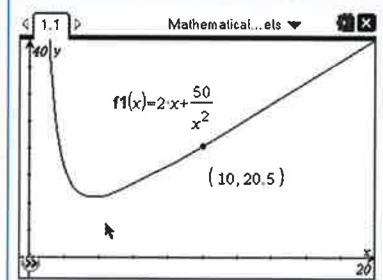
Draw the graph on your GDC.



Copy the details onto a sketch graph.

b The cost for a journey of 10 kilometres is 20.50 euros.

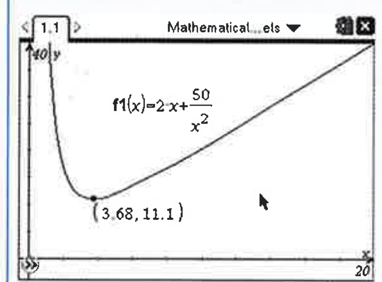
Use a GDC:



Use Trace (or the table) to find the value of f(x) when x = 10.

c The cheapest fare is achieved by a journey of 3.68 kilometres.

Use a GDC:



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

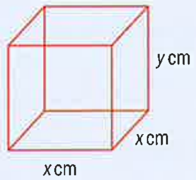
For help with finding the minimum value using a GDC see Chapter 12, Section 4.2, Example 20.

Example 27

A closed cuboid of height y cm has a square base of length x cm. The volume of the cuboid is 500 cm^3.

a Write down an expression for the volume of the cuboid.

b Hence, find an expression for the surface area, A, of the cuboid in terms of x. Simplify your answer as much as possible.



c Use your GDC to draw a graph of the area function for $0 < x \le 30$.

d Use your GDC to find the dimensions that give a minimum surface area.

Answers

a Volume $= x^2y$

Volume = length × width × height

b $A = 2x^2 + 4xy$

$= 2x^2 + 4x \times \dfrac{500}{x^2}$

$= 2x^2 + \dfrac{2000}{x}$

2 square faces each with area $x^2 \Rightarrow 2x^2$

4 rectangular faces each with area $xy \Rightarrow 4xy$

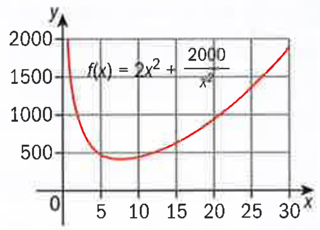
*From part **a**:*

Volume $= x^2y$

$500 = x^2y \Rightarrow y = \dfrac{500}{x^2}$

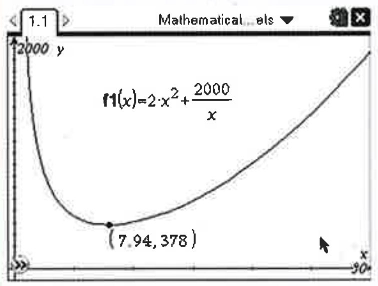
Substitute the expression for y into the formula for A.

c



The area function is:

$f(x) = 2x^2 + \dfrac{2000}{x}$



d The minimum surface area is obtained when $x = 7.937$ and $y = \dfrac{500}{7.937^2} = 7.937$.

Using a GDC the minimum value of base length $x = 7.937$.

Substitute the value of x into the rearranged volume formula to find y.

GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

For help with finding a minimum value using a GDC see Chapter 12, Section 4.2, Example 20.

Exercise 4X

1 One section of a rollercoaster ride can be modeled by the equation

$$f(x) = \frac{20}{x} + 2x^2,\ x \ne 0,$$

where x is the time in seconds from the start of the ride and $f(x)$ is the speed in m s^{-1}.

a Use your GDC to sketch the graph of the function for $0 < x \le 10$.

b Find the minimum value on the graph.

c Find the speed when $x = 6$.

d Find the times when the speed is 50 m s^{-1}.

2 An open box has the following dimensions:
length $= x$ cm, breadth $= 2x$ cm and height $= y$ cm.
The volume of the box is 300 cm^3.

a Write down an expression for the volume of the box.

b Find an expression for the surface area of the open box in terms of x only.

c Use your GDC to sketch the graph of the area function for $0 < x \le 20$.

d Find the dimensions that make the surface area a minimum.

3 A pyramid has a square base of side x metres. The perpendicular height of the pyramid is h metres. The volume of the pyramid is 1500 m^3.

a Find an expression for the volume of the pyramid using the information given.

b Show that the height of each of the triangular faces is $\sqrt{\left(h^2 + \left(\dfrac{x}{2}\right)^2\right)}$.

c Hence, find an equation for the total surface area of the pyramid.

d Write the equation in part **c** in terms of x only.

e Use your GDC to sketch the graph of this equation for $0 < x \le 30$.

f Find the dimensions that make this surface area a minimum.

4 A fish tank in the shape of a cuboid has length 320 cm. Its length is twice that of its width.
To enhance viewing, the area of the four vertical faces should be maximized.
Find the optimum 'viewing area' of a fish tank that is fixed to the wall so that the area of three faces only should be considered.

Example 28

Consider the function $f(x) = \frac{3x-12}{x}$, $x \neq 0$.

a Write down the domain of $f(x)$.

b Copy and complete the table of values for $f(x)$. Give your answers correct to two significant figures.

x	−24	−12	−4	−1	0	1	2	4	8	12	24
$f(x)$											

c Draw the graph of $f(x)$ for $-24 \leq x \leq 24$. Use a scale of 1 cm to represent 4 units on the horizontal axis and 1 cm to represent 2 units on the vertical axis.

d Write down the equation of the vertical asymptote to the graph of $f(x)$.

Answers

a The domain of f is the real numbers, $x \neq 0$.

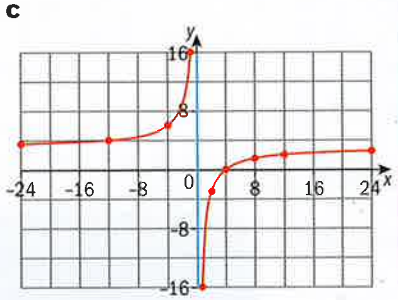
The only value excluded is $x = 0$ (as division by zero is not defined).

b

x	$f(x)$
−24	3.5
−12	4
−4	6
−1	15
0	
1	−9
2	−3
4	0
8	1.5
12	2
24	2.5

Substitute each value of x into f(x) to find the corresponding value of f(x). x = 0 has no image.

c



*Draw and label the axes. Plot the points from the table in part **b**. The graph has **two** branches. Points to the right of x = 0 are joined up with a smooth curve. Points to the left of x = 0 are joined up with another smooth curve.*

d $x = 0$

*Which vertical line does the curve approach but never meet? (Shown in blue on graph in part **c**.)*

As x gets very large in absolute value the graph of f(x) gets closer and closer to a horizontal line. What is the equation of this line?

Large in absolute value means very large positive numbers (1000, 10 000, etc.) or very large negative numbers (−1000, −10 000, etc.).

For more on absolute value, see Chapter 13, section 2.8.

Exercise 4Y

1 Consider the function $f(x) = 1 + \frac{2}{x}$, $x \neq 0$.

a Write down the domain of $f(x)$.

b Copy and complete the following table.

x	−10	−5	−4	−2	−1	−0.5	−0.2	0	0.2	0.5	1	2	4	5	10
$f(x)$															

c Draw the graph for $-10 \leq x \leq 10$. Use a scale of 1 unit to represent 1 cm on each of the axes.

d **i** Draw the vertical asymptote.

ii Write down the equation of the vertical asymptote.

e **i** Draw the horizontal asymptote.

ii Write down the equation of the horizontal asymptote.

2 Consider the function $f(x) = 8x^{-1} + 3$, $x \neq 0$.

a Write down the domain of $f(x)$.

b Copy and complete the following table.

x	−10	−8	−5	−4	−2	−1	0	1	2	4	5	8	10
$f(x)$													

c Draw the graph of $f(x)$ for $-10 \leq x \leq 10$. Use a scale of 1 cm to represent 2 units on both axes.

d **i** Draw the vertical asymptote.

ii Write down the equation of the vertical asymptote.

e **i** Draw the horizontal asymptote.

ii Write down the equation of the horizontal asymptote.

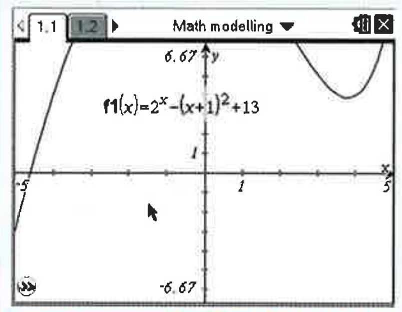
Sketching more complex graphs

Example 29

Sketch the graph of the function $f(x) = 2x - (x+1)^2 + 13$ for $-5 \leq x \leq 5$.

Answer

Using a GDC: Enter the function and adjust the window settings for x.



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

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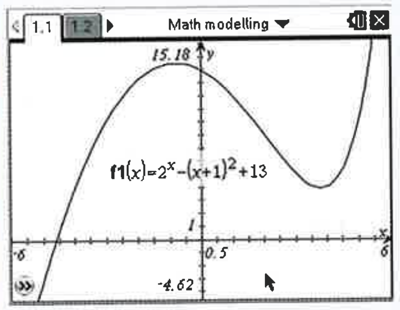
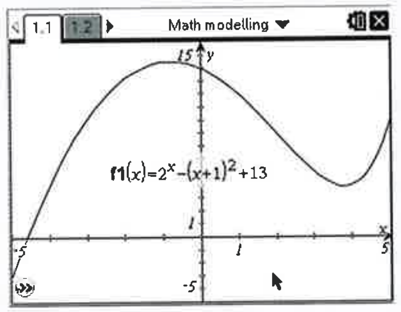
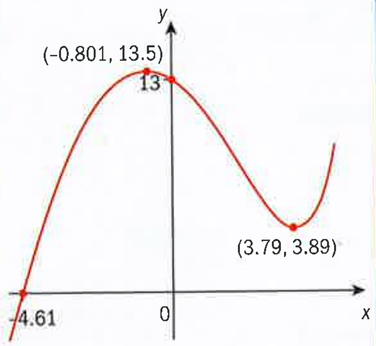
Use Zoom–Fit to adjust the y-axis to include points on the graph.



Choose some integer values to define the window.
For x
Minimum: −5, Maximum: 5
For y
Minimum: −5, Maximum: 15



Copy the details onto a sketch graph.



The range of the function in Example 29 is ℝ.
You can use a table on a GDC to give you an idea of the range of the function.

x	f1(x):= 2^x−(x+1)…
-5.	-2.96875
-4.	4.0625
-3.	9.125
-2.	12.25
-1.	13.5

x	f1(x):= 2^x−(x+1)…
-1.	13.5
0.	13.
1.	11.
2.	8.
3.	5.

Exercise 4Z

Use your GDC to help you sketch the graph of these functions.
Give the range for each function.

1 $f(x) = -0.5x + 1 + 3^x$

2 $f(x) = 2^x - x^2$

3 $f(x) = x(x-1)(x+3)$

4 $f(x) = x^4 - 3x^2 + 1$

5 $f(x) = 0.5^x - x^{-1}, x \neq 0$

4.6 Using a GDC to solve equations

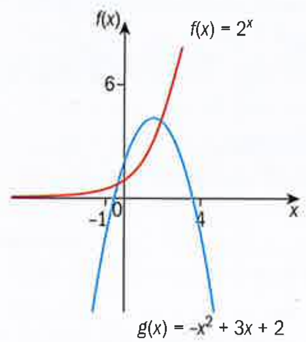
Example 30

a Use your GDC to sketch the graphs of $f(x) = 2^x$ and $g(x) = -x^2 + 3x + 2$.
b Hence, solve the equation $2^x + x^2 - 3x - 2 = 0$.

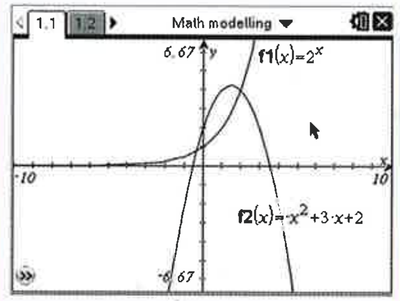
'Hence' means that you should try to use the previous part to answer this part of the question.

Answers

a



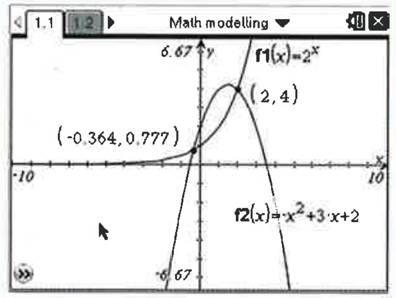
Put $Y_1 = 2^x$ and $Y_2 = -x^2 + 3x + 2$.



A standard window has been used here.

b The solutions are $x = -0.364$ or $x = 2$.

The equation $2^x + x^2 - 3x - 2 = 0$ is the same as $2^x = -x^2 + 3x + 2$. There are 2 points of intersection and we need to find them both.



GDC help on CD: *Alternative demonstrations for the TI-84 Plus and Casio FX-9860GII GDCs are on the CD.*

Exercise 4AA

1 **a** On the same graph, sketch the curves $y = x^2$ and $y = 4 - \frac{1}{x}$ for values of x from −8 to 8 and values of y from −2 to 8. Show scales on your axes.

b Find the coordinates of the points of intersection of these two curves.

EXAM-STYLE QUESTIONS

2 The functions f and g are defined by

$f(x) = 1 + \frac{4}{x}, x \in \mathbb{R}, x \neq 0$

$g(x) = 3x, x \in \mathbb{R}$

a Sketch the graph of f for $-8 \leq x \leq 8$.

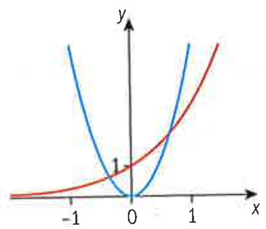
b Write down the equations of the horizontal and vertical asymptotes of the function f.

c Sketch the graph of g on the same axes.

d Hence, or otherwise, find the solutions of $1 + \frac{4}{x} - 3x = 0$.

e Write down the range of function f.

3 The diagram shows the graphs of the functions $y = 5x^2$ and $y = 3^x$ for values of x between −2 and 2.



a Find the coordinates of the points of intersection of the two curves.

b Write down the equation of the horizontal asymptote of the exponential function.

4 Two functions $f(x)$ and $g(x)$ are given by $f(x) = \frac{3}{x}, x \in \mathbb{R}, x \neq 0$ and $g(x) = x^3, x \in \mathbb{R}$.

a On the same diagram sketch the graphs of $f(x)$ and $g(x)$ using values of x between −4 and 4, and values of y between −4 and 4. You must label each curve.

b State how many solutions exist for the equation $\frac{3}{x} - x^3 = 0$.

c Find a solution of the equation given in part **b**.

5 Sketch the graphs of $y = 3x - 4$ and $y = x^3 - 3x^2 + 2x$. Find all the points of intersection of the graphs.

6 Sketch the graphs of $y = 2^x$ and $y = x^3 + x^2 - 6x$. Find the coordinates of all the points of intersection.

7 Sketch the graphs of $y = x + 2$ and $y = \frac{5}{x}, x \neq 0$.

a Find the solutions of the equation $\frac{5}{x} = x + 2$.

b Write down the equation of the horizontal asymptote to $y = \frac{5}{x}$.

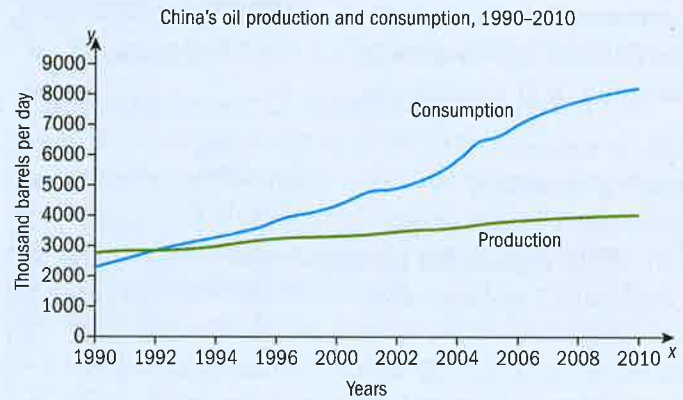
c Write down the equation of the vertical asymptote to $y = \frac{5}{x}$.

4.7 Graphs of real-life situations

Linear and non-linear graphs can be used to represent a range of real-life situations.

Example 31

The graph below shows China's oil production and consumption from 1990 to 2010.



Source: US Energy Information Administration, *International Energy Annual 2006*, 'Short term energy outlook (July 2009)'

a What are the two variables represented by this graph?

b What does the blue curve represent?

c What does the green curve represent?

d Explain the meaning of the point where both curves meet. What is the year at that point?

e Explain what happens before and after 1992.

f What is the tendency of the oil consumption in China?

Answers

a The variables are year and number of thousands of barrels per day.

b The blue curve represents oil consumption per day in China from 1990 to 2010.

c The green curve represents oil production per day in China from 1990 to 2010.

d At the point that the curves meet, oil production and consumption in China were equal. This occurred in 1992.

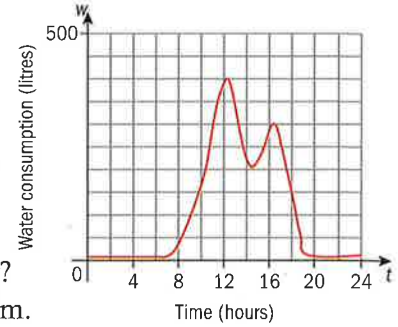
e Before 1992, oil consumption was less than oil production. After 1992, oil consumption was greater than oil production.

f Oil consumption in China tends to keep increasing.

Can you deduce any further information from this graph?

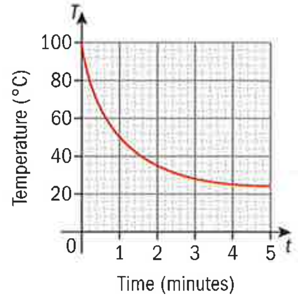
Exercise 4AB

1 The water consumption in Thirsty High School is represented in the graph.



a Write down are the two variables represented by this graph.
b During what period of time is Thirsty High School open?
c During what intervals of time is consumption increasing?
d During what intervals of time is consumption decreasing?
e Find the time at which the consumption is at a maximum.
f Find the time at which the consumption is at a minimum.

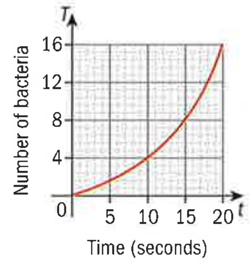
2 The graph represents the temperature, in degrees Celsius, of some coffee after Manuela has heated it.



a Write down the two variables represented by this graph.
b Write down the initial temperature of the liquid after heating.
c Write down the temperature of the liquid 2 minutes after heating.
d Find the time it takes for the temperature to reach 68 °C.
e Decide whether the liquid reaches 22 °C during the 5-minute period shown on the graph.
f Write down the room temperature.

EXAM-STYLE QUESTION

3 Under certain conditions the number of bacteria in a particular culture doubles every 5 seconds as shown by the graph.

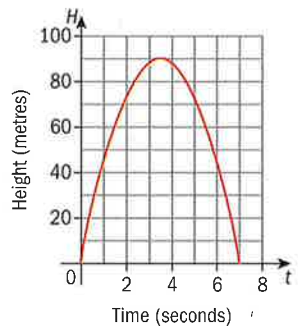


a Copy and complete the table below.

Time (t seconds)	0	5	10	15	20
Number of bacteria (N)	1				

b Write down the time it takes for the culture to have 6 bacteria.
c Calculate the number of bacteria in the culture after 1 minute if the conditions remain the same.

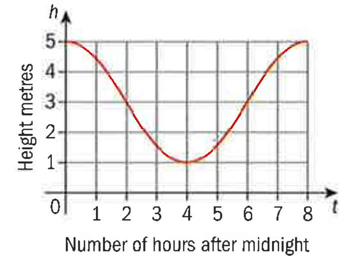
4 In a physics experiment a ball is projected vertically into the air from ground level.
The diagram represents the height of the ball at different times.



a Write down the height of the ball after one second.
b Find out how many seconds after being thrown the ball is at 60 metres.
c Write down the interval of time in which the ball is going up.
d Write down the interval of time in which the ball is coming down.
e Write down the maximum height reached by the ball and the time it takes the ball to reach that height.
f Explain what happens at $t = 7$.

EXAM-STYLE QUESTIONS

5 The graph shows the tide heights, h metres, at time t hours after midnight for Blue Coast Harbor.

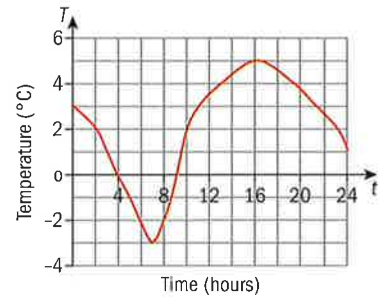


a Use the graph to find
i the height of the tide at 01:30
ii the height of the tide at 05:30
iii the times when the height of the tide is 3 metres.

The best time to catch fish in Blue Coast Harbor is when the tide is below 3 metres.

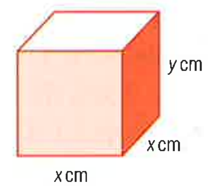
b Find this best time, giving your answer as an inequality in t.

6 The temperature (°C) during a 24-hour period in a certain city is represented in the graph.



a Determine how many times the temperature is exactly 0 °C during this 24-hour period.
b Write down the interval of time in which the temperature falls below 0 °C.
c Write down the time at which the temperature reaches its maximum value.
d Write down the maximum temperature registered during this 24-hour period.
e Write down the interval of time in which the temperature increases from 3 °C to 5 °C.
f Write down the times at which the temperature is 4 °C.
g Can you deduce from this graph whether the behavior of the temperature in the following day will be exactly the same as this day? Why?

7 The diagram represents a box with volume 16 cm³.
The base of the box is a square with sides x cm. The height of the box is y cm.



a Write an expression for the height, y, in terms of x.
b Copy and complete the table below for the function $y = f(x)$ from part **a**. Give your answers correct to two significant figures.

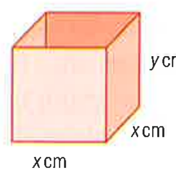
For part **a** use the formula: volume = length × width × height.

x	0.5	1	2	4	8	10
$y = f(x)$						

c Draw the graph of f for $0 < x \le 10$. Use a scale of 1 cm to represent 1 unit on the horizontal axis and 1 cm to represent 10 units on the vertical axis.
d What happens to the height of the box as the values of x tend to infinity?

EXAM-STYLE QUESTION

8 The diagram represents an open container with a capacity of 3 litres.
The base of the container is a square with sides x cm. The height of the container is y cm.



a Write down the volume of the box in cm³.

b Find an expression for the height, y, in terms of x.

c Find an expression for the surface area of the container, A, in terms of x.

d Copy and complete the table below. Give your answers correct to 2 significant figures.

x(cm)	5	10	15	20	25	30	35
A(x)(cm²)							

e Draw the graph of A for $0 < x \le 35$. Use a scale of 2 cm to represent 5 units on the horizontal axis and 1 cm to represent 400 units on the vertical axis.

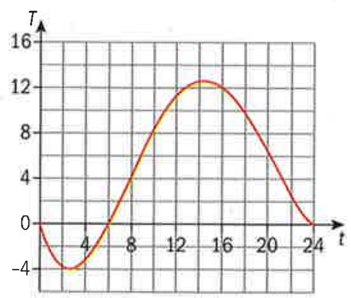
f Use your graph to decide if there is a value of x that makes the surface area of the container a minimum. If there is, write down this value of x.

Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

1 The graph represents the temperature in °C in a certain city last Tuesday.



a Write down the interval of time in which the temperature was below 0 °C.

b Write down the interval of time in which the temperature was above 11 °C.

c Write down the maximum temperature last Tuesday. Give your answer correct to the nearest unit.

2 The cost c, in Singapore dollars (SGD), of renting an apartment for n months is a linear model

$c = nr + s$

where s is the security deposit and r is the amount of rent per month.

Wan Ning rented the apartment for 6 months and paid a total of 35 000 SGD.
Tanushree rented the same apartment for 2 years and paid a total of 116 000 SGD.
Find the value of

a r, the rent per month **b** s, the security deposit.

EXAM-STYLE QUESTIONS

3 Given that $f(x) = x^2 + 5x$

a factorize $x^2 + 5x$

b sketch the graph of $y = f(x)$. Show on your sketch

i the coordinates of the points of intersection with the axes

ii the equation of the axis of symmetry

iii the coordinates of the vertex of the parabola.

4 A signal rocket is fired vertically from ground level by a gun. The height, in metres, of the rocket above the ground is a function of the time t, in seconds, and is defined by:

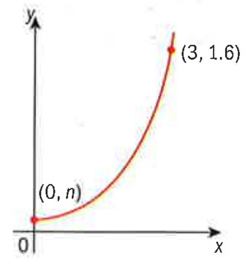
$h(t) = 30t - 5t^2,\ 0 \le t \le 6.$

a Find the height of the rocket above the ground after 4 seconds.

b Find the maximum height of the rocket above the ground.

c Use your GDC to find the length of time, in seconds, for which the rocket is at a height of 25 m or more above the ground.

5 The graph of the function $f(x) = \frac{2^x}{m}$ passes through the points (3, 1.6) and (0, n).

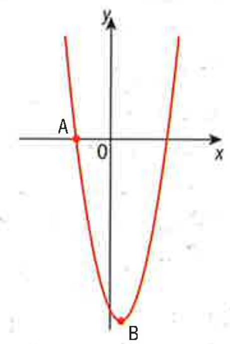


a Calculate the value of m.

b Calculate the value of n.

Find $f(2)$.

6 The diagram shows the graph of $y = x^2 - 2x - 15$.
The graph crosses the x-axis at the point A, and has a vertex at B.



a Factorize $x^2 - 2x - 15$.

b Find the coordinates of the point

i A **ii** B.

7 Consider the graphs of the following functions.

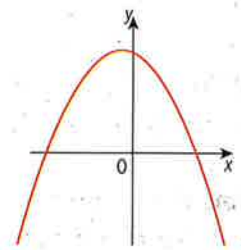
i $y = 8x + x^2$

ii $y = (x - 3)(x + 4)$

iii $y = x^2 - 2x + 5$

iv $y = 5 - 4x - 3x^2$

Which of these graphs



a has a y-intercept below the x-axis

b passes through the origin

c does not cross the x-axis

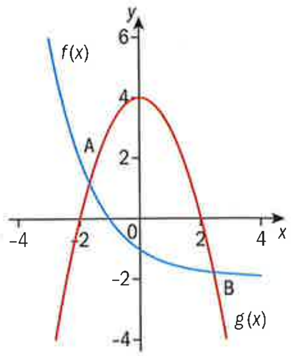
d could be represented by this diagram?

EXAM-STYLE QUESTIONS

8 The figure shows the graphs of the functions

$f(x) = (0.5)^x - 2$ and $g(x) = -x^2 + 4$

for values of x between −3 and 3. The two graphs meet at the points A and B.



a Find the coordinates of

i A **ii** B.

b Write down the set of values of x for which $f(x) < g(x)$.

c Write down the equation of the horizontal asymptote to the graph of $f(x)$.

9 Gabriel is designing a rectangular window with a perimeter of 4.40 m. The length of the window is x m.

a Find an expression for the width of the window in terms of x.

b Find an expression for the area of the window, A, in terms of x.

Gabriel wants to make the amount of light passing through this window a maximum.

c Find the value of x that meets this condition.

10 a On the same graph sketch the curves $y = 3x^2$ and $y = \frac{1}{x}$ for values of x from −4 to 4 and values of y from −4 to 4.

b Write down the equations of the vertical and horizontal asymptotes of $y = \frac{1}{x}$.

c Solve the equation $3x^2 - \frac{1}{x} = 0$.

Paper 2 style questions

EXAM-STYLE QUESTIONS

1 The number (n) of bacteria after t hours is given by the formula $n = 1500(1.32)^t$.

a Copy and complete the table below for values of n and t.

Time (t hours)	0	1	2	3	4
Number of bacteria (n)	1500		2613	3450	

b On graph paper, draw the graph of $n = 1500(1.32)^t$. Use a scale of 2 cm to represent 1 hour on the horizontal axis and 2 cm to represent 1000 bacteria on the vertical axis. Label the graph clearly.

c Find

i the number of bacteria after 2 hours 30 minutes. Give your answer to the nearest ten bacteria.

ii the time it will take to form approximately 5000 bacteria. Give your answer to the nearest 10 minutes.

2 The functions f and g are defined by

$f(x) = \frac{4}{x}, x \in \mathbb{R}, x \neq 0$

$g(x) = 2x, x \in \mathbb{R}$

a Sketch the graph of $f(x)$ for $-8 \leq x \leq 8$.

b Write down the equations of the horizontal and vertical asymptotes of the function f.

c Sketch the graph of g on the same axes.

d Find the solutions of $\frac{4}{x} = 2x$.

e Write down the range of function f.

3 A function is represented by the equation $f(x) = 2(1.5)^x + 3$. The table shows the values of $f(x)$ for $-3 \leq x \leq 2$.

x	−3	−2	−1	0	1	2
f(x)	3.59	3.89	*a*	5	6	*b*

a Calculate the values for a and b.

b On graph paper, draw the graph of $f(x)$ for $-3 \leq x \leq 2$, taking 1 cm to represent 1 unit on both axes.

The domain of the function $f(x)$ is the real numbers, $\mathbb{R}$.

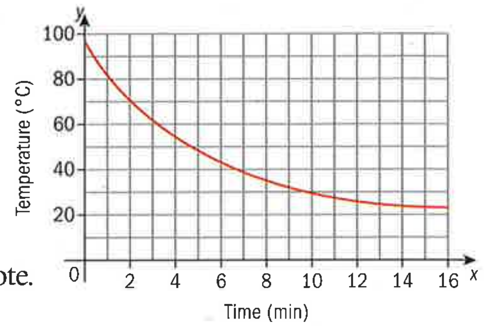
c Write down the range of $f(x)$.

d Find the approximate value for x when $f(x) = 10$.

e Write down the equation of the horizontal asymptote of $f(x) = 2(1.5)^x + 3$.

EXAM-STYLE QUESTIONS

4 The graph shows the temperature, in degrees Celsius, of Leonie's cup of hot chocolate t minutes after pouring it. The equation of the graph is $f(t) = 21 + 77(0.8)^t$ where $f(t)$ is the temperature and t is the time in minutes after pouring the hot chocolate out.



a Find the initial temperature of the hot chocolate.

b Write down the equation of the horizontal asymptote.

c Write down the room temperature.

d Find the temperature of the hot chocolate after 8 minutes.

5 Consider the functions

$f(x) = x^2 - x - 6$ and $g(x) = -2x + 1$

a On the same diagram draw the graphs of $f(x)$ and $g(x)$ for $-10 \le x \le 10$.

b Find the coordinates of the local minimum of the graph of $f(x)$.

c Write down the gradient of the line $g(x)$.

d Write down the coordinates of the point where the graph of $g(x)$ cuts the y-axis.

e Find the coordinates of the points of intersection of the graphs of $f(x)$ and $g(x)$.

f Hence, or otherwise, solve the equation $x^2 + x - 7 = 0$.

6 **a** Sketch the graph of $f(x) = x^2 - \frac{3}{x}$, for $-4 \le x \le 4$.

b Write down the equation of the vertical asymptote of $f(x)$.

c On the same diagram draw the graph of $g(x) = -3(2)^x + 9$, for $-4 \le x \le 4$.

d Write down the equation of the horizontal asymptote of $g(x)$.

e Find the coordinates of the points of intersection of $f(x)$ and $g(x)$.

EXAM-STYLE QUESTIONS

7 The profit (P) in euros made by selling homemade lemonade is modeled by the function

$$P = -\frac{x^2}{10} + 10x - 60$$

where x is the number of glasses of lemonade sold.

a Copy and complete the table.

x	0	10	20	30	40	50	60	70	80	90
P		30			180			150	100	

b On graph paper draw axes for x and $P(x)$, placing x on the horizontal axis and $P(x)$ on the vertical axis. Draw the graph of $P(x)$ against x by plotting the points.

c **Use your graph** to find

i the maximum possible profit

ii the number of glasses that need to be sold to make the maximum profit

iii the number of glasses that need to be sold to make a profit of 160 euros

iv the amount of money initially invested.

8 **a** Sketch the graph of the function $f(x) = x^2 - 7$, $x \in \mathbb{R}$, $-4 \le x \le 4$. Write down the coordinates of the points where the graph of $y = f(x)$ intersects the axes.

b On the same diagram sketch the graph of the function $g(x) = 7 - x^2$, $x \in \mathbb{R}$, $-4 \le x \le 4$.

c Solve the equation $f(x) = g(x)$ in the given domain.

d The graph of the function $h(x) = x + c$, $x \in \mathbb{R}$, $-4 \le x \le 4$, where c is a positive integer, intersects twice with both $f(x)$ and $g(x)$ in the given domain.
Find the possible values for c.

9 The functions f and g are defined by $f(x) = \frac{x^2}{2}$ and $g(x) = -\frac{x^2}{2} + 2x$, $x \in \mathbb{R}$.

a Calculate the coordinates of the points of intersection of the graphs $f(x)$ and $g(x)$.

b Find the equation of the axis of symmetry of the graph of $y = g(x)$.

c The straight line with equation $y = k$, $k \in \mathbb{R}$, is a tangent to the graph of g. Find the value of k.

d Sketch the graph of $f(x)$ and the graph of $g(x)$, using a rectangular Cartesian coordinate system with 1 cm as a unit. Show the coordinates of any points of intersection with the axes.

e Find the values of x for which $f(x) < g(x)$.

CHAPTER 4 SUMMARY

Functions

- A **function** is a relationship between two sets: a **first** set and a **second** set. Each element 'x' of the first set is related to **one and only one** element 'y' of the second set.
- The first set is called the **domain** of the function. The elements of the domain, the 'x-values', are the **independent variables**.
- For each value of 'x' (input) there is one and only one output. This value is called the **image** of 'x'. The set of all the images (all the outputs) is called the **range** of the function. The elements of the range, the 'y-values', are the **dependent variables**.
- The graph of a function f is the set of points (x, y) on the Cartesian plane where y is the image of x through the function f.
- $y = f(x)$ means that the image of x through the function f is y. x is the independent variable and y is the dependent variable.

Linear models

- A **linear function** has the general form $f(x) = mx + c$, where m (the gradient) and c are constants.
- When $f(x) = mx$ the graph passes through the origin, (0, 0).

Quadratic models

- A **quadratic function** has the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.
- The graph of any quadratic function is a **parabola** – a ∪-shaped (or ∩-shaped) curve. It has an **axis of symmetry** and either a **minimum** or **maximum** point, called the **vertex** of the parabola.
- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- The curve intersects the y-axis at $(0, c)$.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$, $a \neq 0$.
- The x-coordinate of the vertex is $x = -\frac{b}{2a}$.
- The factorized form of a quadratic function is $f(x) = a(x - k)(x - l)$.
- If $a > 0$ then the graph is ∪-shaped; if $a < 0$ then the graph is ∩-shaped.
- A ∪-shaped graph is 'concave up'. A ∩-shaped graph is 'concave down'.
- The curve intersects the x-axis at $(k, 0)$ and $(l, 0)$.
- The equation of the axis of symmetry is $x = \frac{k+l}{2}$.
- The x-coordinate of the vertex is also $x = \frac{k+l}{2}$.
- The function $f(x) = ax^2 + bx + c$ intersects the x-axis where $f(x) = 0$. The x-values of the points of intersection are the two solutions (or **roots**) of the equation $ax^2 + bx + c$. (The y-values at these points of intersection are zero.)
- Two functions $f(x)$ and $g(x)$ intersect at the point(s) where $f(x) = g(x)$.

Continued on next page

Exponential models

- In an **exponential function,** the independent variable is the **exponent** (or **power**).
- In general, for the graph of $f(x) = ka^x + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$
  - the line $y = c$ is a **horizontal asymptote**
  - the curve passes through the point $(0, k + c)$.
- In general, for the graph of $f(x) = ka^{-x} + c$ where $a \in \mathbb{Q}^+$ and $k \neq 0$ and $a \neq 1$
  - the line $y = c$ is a horizontal asymptote
  - the curve passes through the point $(0, k + c)$
  - the graph is a reflection in the y-axis of $g(x) = ka^x + c$.

Cubic functions

- A cubic function has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. The domain is $\mathbb{R}$, unless otherwise stated.

The language of mathematics

Mathematics is described as a language. It has vocabulary (mathematical symbols with precise meaning) and grammar (an order in which we combine these symbols together to make them meaningful).

- Mathematics is often considered a 'universal language'. Can a language ever be truly universal?

Precise and concise

Mathematical language is precise and explicit, with no ambiguity. It uses its own set of rules for manipulating its statements, so it is completely abstract.

$3 \leq x < 7$

$k = t^3 - 6t^2 + 12t + 2$

$D = \{(x, y) \mid x + y = 5\}$

$2 + 2 = 4$

$6 \times 9 = 54$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x + 18$

Mathematics can describe and represent ideas that are not easily expressed by conventional written or spoken words.

These two statements are equivalent:

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

(Euclid, Elements, II.4, c.300 BCE)

<=>

$(a + b)^2 = a^2 + b^2 + 2ab$

Mathematics says it far more simply!

- Draw and label a diagram to show that these statements are equivalent.

Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say.

Bertrand Russell, *The Scientific Outlook*, 1931

'Mathematics is the abstract key which turns the lock of the physical universe.'

John Polkinghorne, *One World: The Interaction of Science and Theology*, 2007

Abstract language

- What does '1' mean?

You can probably answer that with confidence. '1' is part of our language, we use it every day. Its meaning is clear to us. We can easily picture '1' banana.

But the language of mathematics has continued to expand to encompass more abstract concepts. Mathematicians call the square root of -1, '*i*'.

- What does this mean? Can you use *i* in everyday life?

What about pi (π)? Lots of people know this number.

It is the ratio $\frac{\text{circumference of a circle}}{\text{diameter of a circle}}$.

- What does this 'mean'? Can you picture 'π' bananas?
- Do π and *i* exist?

1 is an abstract concept of mathematics that has also become part of our everyday, English language too. *i* or π are also abstract concepts of mathematics, but have not become part of everyday language. Mathematicians need and use these numbers. They are not any more abstract than the number 1. They appear in a mathematical context and allow us to think mathematically and communicate these ideas, to perform manipulations, to express results and model real-life occurrences in a simple way.

Simple and beautiful equations that model the world

Here are some famous equations

Einstein's equation: $E = mc^2$

Newton's second law: $F = ma$

Boyle's law: $V = \frac{k}{p}$

Schrödinger's equation: $\hat{H}\psi = E\psi$

Newton's law of universal gravitation: $F = G\frac{m_1 m_2}{r^2}$

These are simple equations (although they were not simple to derive!). Isn't it startling that so much of what happens in the universe can be described using equations like these?

These equations have helped to put a man on the moon and bring him back, develop wireless internet and understand the workings of the human body.

- Do you think that mathematics and science will one day discover the ultimate 'theory of everything'? A theory that fully explains and links together all known physical phenomena? A theory that can predict the outcome of any experiment that could be carried out?
- What will mathematicians and scientists do then?