8 Sets and probability

CHAPTER OBJECTIVES:

3.5 Basic concepts of set theory: elements; subsets; intersection; union; complement; Venn diagram

3.6 Sample space; event A; complementary event A'; probability of an event; probability of a complementary event; expected value (a fair game)

3.7 Probability of combined events, mutually exclusive events, independent events; use of tree diagrams, Venn diagrams, sample space diagrams, and tables of outcomes; probability using 'with replacement' and 'without replacement'; conditional probability

Before you start

You should know how to

1 Use the terms integer, rational number, real number:

-2, 5, and 127 are integers

$\frac{1}{5}$ and $0.\dot{6} = \frac{2}{3}$ are rational numbers

$\sqrt{44}$ and 1.356 724 967... are real numbers

2 Use and interpret inequalities such as $3 \le x \le 7$, $3 < x < 7$ or $3 \le x < 7$, e.g. if x is an integer and $3 \le x \le 7$, the possible values of x are 3, 4, 5, 6, 7.

3 Identify factors and prime factors, e.g.

List the factors of 18:
1, 2, 3, 6, 9, 18.

List the prime factors of 18:
2, 3.

Skills check

1 Determine whether each of the following is an integer, a rational number or a real number. If it is rational, write it as a fraction.

a 5 **b** 1.875 **c** 0.333

d 0.303 003 000 3... **e** $\sqrt{0.5625}$

f $\sqrt[3]{2.744}$ **g** π^2

2 x is an integer. For each inequality, write down the possible values of x.

a $-2 \le x \le 3$ **b** $-3 < x \le 3$

c $-2 \le x < 4$ **d** $-3 < x < 4$

3 a List the factors of

i 12 **ii** 8 **iii** 17 **iv** 25 **v** 24

b List the prime factors of

i 12 **ii** 8 **iii** 17 **iv** 25 **v** 24

c One of the numbers in part **b** is prime. Which?

d How many factors does zero have? Is zero an integer? Rational? Real? Prime?

Here is one of the two 'integrated resorts' that have been built in Singapore. They are also known as casinos – big businesses that contribute over USD 1 000 000 000 to the country's tax revenues. Imagine the income for the companies when the **tax** they pay on it is one billion dollars!

Their business is gambling, and gambling is all about understanding the probability of winning and losing – and ensuring (as far as possible) that the casino always wins **overall**. The casino managers need to understand the laws of probability and be able to manipulate these in their favor, so that the casino makes a profit.

But if the casino **always** wins then the gamblers **always** lose, and this does not seem to be 'fair'.

In this chapter, you will investigate 'fair' games and how this idea of fairness relates to the probability of winning and losing. To do this you need to understand the fundamentals of probability theory. You will see that, although an intuitive approach is often helpful, sometimes intuition fails and you need the theory to fully understand the probability of an event.

The roots of probability lie in set theory, which can help you to visualize the problem, so this chapter begins with set theory and then goes on to apply it to probability theory.

Investigation – a contradiction?

A teacher asks her class how many of them study Chemistry. She finds that there are 15. She then asks how many study Biology and finds that there are 13.
Later, she remembers that there are 26 students in the class.
But 15 + 13 = 28.
Has she miscounted?

How is it possible that the two totals are different?

What is the apparent contradiction in this problem?
How can you resolve it?

Once you have resolved the contradiction, try to answer these questions:

1 How many people study both Chemistry and Biology?
2 How many people study Chemistry but do not study Biology?
3 How many study neither subject?

Investigation – intuition

We all have a feeling about whether something is fair or not.

For example, in a football match the referee tosses a coin to decide which team starts with possession of the ball. One team captain calls heads and if the coin lands heads up, that team has possession. This, we feel, is fair. But why?

Write down *why* you think this is a fair way of deciding.

1 Are these scenarios 'fair'?

a To determine initial possession in a match between Team A and Team B, the captain of Team A tosses a coin, then the captain of Team B tosses the same coin. The team whose captain first tosses 'heads' gets possession.

b To determine initial possession in a match between Team A and Team B, the captain of Team A chooses a number from 1 to 6. An ordinary dice is rolled, and if that number turns up, Team A gets possession. Otherwise, Team B gets possession.

c To determine initial possession in a match between Team A and Team B, the captains of both teams roll an ordinary dice once. The team whose captain rolled the higher score gets possession.

Is there a **guaranteed** higher score? What happens if there isn't?

2 The idea of what is fair changes when money is involved.
Are these situations involving two players, Paul and Jade, fair?
What makes a situation fair or not?

a Paul and Jade each place a bet of $1. Then an unbiased coin is tossed. If the coin lands 'heads', Paul wins the $2; if the coin lands 'tails', Jade wins the $2.

An **unbiased** coin has an **equal probability** of landing on either 'heads' or 'tails'.

Continued on next page

b Paul and Jade each place a bet of $1. Then each takes turns to toss an unbiased coin, Paul going first. The first player to toss 'heads' wins the $2.

c Paul and Jade take turns to toss an unbiased coin, Paul going first. Each places a bet of $1 immediately before their turn. The first player to toss 'heads' wins the accumulated sum of money.

d Paul and Jade each place a bet of $1. Then an unbiased dice is rolled. If the dice shows a six, Paul wins the $2; if the dice does not show a six, Jade wins the $2.

e Paul and Jade each place a bet; Paul bets $1 and Jade bets $5. Then an unbiased cubical dice is rolled. If the dice shows a six, Paul wins the $6; if the dice does not show a six, Jade wins the $6.

Casinos analyse all the gambling games played and ensure that they are **not** fair. Probability theory is the key to this. By the end of this chapter, you will be able to judge the merit of the statement: 'gambling is a tax on the mathematically ignorant'.

8.1 Basic set theory

→ A **set** is simply a collection of objects. The objects are called the **elements** of the set.

We usually consider sets that have numbers as their elements; however, a set can be a collection of **any** type of object.

Some sets are so commonly used that they have their own symbols:

$\mathbb{Z}$ the set of integers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
$\mathbb{Z}^+$ the set of positive integers $\{1, 2, 3, ...\}$
$\mathbb{N}$ the set of natural numbers $\{0, 1, 2, 3, ...\}$
$\mathbb{Q}$ the set of rational numbers (fractions)
$\mathbb{R}$ the set of real numbers

Is '5' a rational number? Is '–5'? Is zero a rational number?

We usually use capital letters for sets, and lower case letters for their elements.

There are several ways to describe which objects belong to a set:

$A = \{1, 2, 3, 4, 5\}$
$B = \{4, 5, 6, 7\}$
$C = $ {Names of the students absent from your school today}

You can explicitly list the elements of a set.

You can describe the properties of the set's elements.

You can use **set builder notation**:

$D = \{x \mid 0 \le x \le 5\}$ The set of all numbers between 0 and 5, inclusive
$E = \{(x, y) \mid x + y = 5\}$ The set of pairs of numbers that add up to 5
$F = \{p \mid p$ is a prime number and a multiple of 10$\}$ The set of prime numbers which are a multiple of 10

Set builder notation describes the properties of the elements of a set, using mathematical notation. The '|' symbol means 'such that', for example, the definition of set *D* is read as 'the set of all *x* such that *x* is greater than or equal to 0 and less than or equal to 5'.

$G = \{x \mid x$ is a square number less than $50\}$	The set of square numbers less than 50.
$H = \{x \mid x \leq 200\}$	The set of all numbers less than or equal to 200.

As we shall see, the expressions used for sets D to H are not precise enough – they do not specify what sort of numbers the set elements should be. For example, if x in the definition for set D is an integer then D has six elements. If x is real, how many elements are there in D?

→ The number of elements in a set A is denoted as $n(A)$.

Set G has seven elements (assuming that x is an integer).

We write $n(G) = 7$, which is read as 'the number of elements in G is seven'.

Similarly, $n(A) = 5$ and $n(B) = 4$.
Set F has no elements, so $n(F) = 0$ and F is called the **empty set**. The empty set is written as $\varnothing$ (or sometimes as $\{\ \}$).

Note that the set $\{0\}$ is **not** the empty set since it has **one** element – the number zero.

Sets A, B and G are examples of **finite sets**; they each contain a finite number of elements.

However, $n(\mathbb{Z}^+) = \infty$, so $\mathbb{Z}^+$ is an example of an **infinite set**.

Now consider set $D = \{x \mid 0 \leq x \leq 5\}$. This is read as '$x$ is any number that lies in between 0 and 5, inclusive'.

In this case it is impossible to list the elements of D, since, x has not been properly defined – it has not been stated whether x is an integer, a positive integer, a real number or a rational number.

1 If x is an **integer**, then $D = \{0, 1, 2, 3, 4, 5\}$ and $n(D) = 6$.
In set builder notation, D is properly defined as
$D = \{x \mid 0 \leq x \leq 5, x$ is an integer$\}$.

2 Suppose x is a **positive integer**. Then $D = \{1, 2, 3, 4, 5\}$ and $n(D) = 5$.
In set builder notation, D is properly defined as
$D = \{x \mid 0 \leq x \leq 5, x$ is a positive integer$\}$.

3 Suppose x is a **rational number**. Then D cannot be listed; it is an infinite set, $n(D) = \infty$.

Write down the elements of G to check how many there are.

Why is set F empty? Because, by definition, a prime number *cannot* be a multiple of 10.

Is $\{\varnothing\}$ the empty set?

Zero is an integer, but zero is not **positive**.

We can use mathematical notation to replace statements such as 'x is a positive integer' or, more precisely, 'x is an element of the set of positive integers'.

$\in$ means 'is an element of'
$\notin$ means 'is not an element of'

So $x \in \mathbb{Z}^+$ means 'x is a positive integer'.

$1 \in A, \quad 49 \in G, \quad 8 \notin B, \quad (3, 5) \notin E, \quad \pi \notin G,$

using the sets on pages 331–332.

Example 1

Decide whether each set is well defined. Give reasons for your answer.

a $E = \{(x, y) \mid x + y = 5\}$
b $F = \{p \mid p$ is a prime number and a multiple of $10\}$
c $H = \{x \mid x \leq 200\}$

Answers

a E is **not** well defined, since we don't know which sets x and y belong to.

E becomes well defined if it is specified, for example, that $x \in \mathbb{Z}^+$, $y \in \mathbb{Z}^+$, so that $E = \{(x, y) \mid x + y = 5, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\}$. Then $n(E) = 4$, since $E = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

b $F = \{p \mid p$ is a prime number and a multiple of $10\}$ **is** well defined since all multiples of 10 are integers and all prime numbers are positive integers.

$n(F) = 0$, however, since there is no prime multiple of 10.

c H is **not** well defined, since the set to which x belongs has not been specified.

$H = \{x \mid x \leq 200, x \in \mathbb{N}\}$ is now well defined, and $n(H) = 201$.

Example 2

Write the set $\{5, 6, 7, 8, 9\}$ using set builder notation.

Answer

There are many different correct answers, including:

$\{x \mid 5 \leq x \leq 9, x \in \mathbb{Z}\}$	or	$\{x \mid 5 \leq x \leq 9, x \in \mathbb{Z}^+\}$
$\{x \mid 5 \leq x < 10, x \in \mathbb{Z}\}$	or	$\{x \mid 5 \leq x < 10, x \in \mathbb{Z}^+\}$
$\{x \mid 4 < x \leq 9, x \in \mathbb{Z}\}$	or	$\{x \mid 4 < x \leq 9, x \in \mathbb{Z}^+\}$
$\{x \mid 4 < x < 10, x \in \mathbb{Z}\}$	or	$\{x \mid 4 < x < 10, x \in \mathbb{Z}^+\}$

Exercise 8A

1 For each of the sets given below,

a list its elements, if this is possible

b state the number of elements in the set.

$M = \{x \mid 2 \le x < 5, x \in \mathbb{Z}\}$
$N = \{x \mid 0 < x \le 5, x \in \mathbb{Z}\}$
$P = \{x \mid -2 \le x < 6, x \in \mathbb{Z}^+\}$
$S = \{(x, y) \mid x + y = 5, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\}$
$T = \{(x, y) \mid x + y = 5, x \in \mathbb{N}, y \in \mathbb{N}\}$
$V = \{p \mid p$ is a prime number and a multiple of $4\}$
$W = \{x \mid x$ is a factor of $20\}$
$X = \{x \mid x < 200, x \in \mathbb{R}\}$

2 Here are three sets:
$A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{3, 5, 7, 9, 11\}$.
List the elements of the sets given by:

a $\{x \mid x > 3, x \in A\}$ **b** $\{x \mid x \le 6, x \in B\}$
c $\{x \mid 5 < x < 12, x \in C\}$ **d** $\{x \mid x = 2y + 1, y \in B\}$
e $\{(x, y) \mid x = y, x, y \in B\}$ **f** $\{(x, y) \mid x = 2y, x \in B, y \in C\}$

3 Write these sets using set builder notation.

a $\{2, 4, 6, 8, \ldots\}$ **b** $\{2, 3, 5, 7, 11, 13, \ldots\}$
c $\{-2, -1, 0, 1, 2\}$ **d** $\{2, 3, 4, 5, 6, 7, 8\}$
e $\{-2, 0, 2, 4, 6, 8\}$ **f** $\{3, 6, 9, 12, 15, 18\}$

8.2 Venn diagrams

Universal set

It is important to know what sort of elements are contained in a set.

In other words, in order to properly define a set, we need to define the **universal set**, those elements that are under consideration.

→ The **universal set** (denoted U), must be stated to make a set well defined.

The universal set is shown in diagrammatic form as a rectangle:

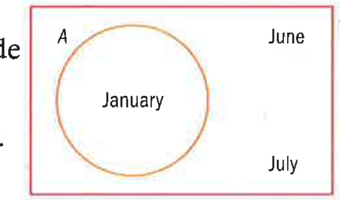
This type of set diagram is called a **Venn diagram**. Any set under consideration is shown as a circle inside the universal set.

The diagram is named after the English mathematician John Venn, who first used it.

Suppose that, as part of a problem, we were considering the months of the year that (in English) begin with the letter 'J'. Then the universal set, U, would be {January, June, July}.

Set A is defined as the set of all months which end in '…uary'.

Representing this on a Venn diagram, set A is a **subset** of U and is drawn inside the rectangle. This is written as $A \subset U$.



Since January $\in A$, it is written inside A. Since June, July $\notin A$, but June, July $\in U$, they are written inside the rectangle (U) but outside the circle (A).

From the definition of set A, 'February' could be one of its elements. But February is **not in the universal set**, so it **cannot** be an element of A.

For set $D = \{x \mid 0 \le x \le 5\}$ on page 331, its elements can only be defined properly when we define U. The three cases we considered were $U = \mathbb{Z}$, $U = \mathbb{Z}^+$, and $U = \mathbb{Q}$, respectively.

$\mathbb{Q}$, the set of rational numbers, is properly defined as $\left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \ne 0\right\}$

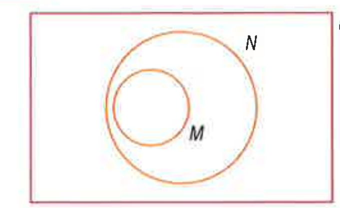
Subsets

→ If every element in a given set, M, is also an element of another set, N, then M is a **subset** of N, denoted $M \subseteq N$

→ A **proper subset** of a given set is one that is **not identical** to the original set.
If M is a proper subset of N (denoted $M \subset N$) then
1 **every** element of M also lies in N and
2 there are some elements in N that do not lie in M.

If M is a proper subset of N then we write $M \subset N$.
If M could be equal to N then we write $M \subseteq N$.
Clearly, M and N are both subsets of the universal set U.

The Venn diagram on the right shows $M \subset N \subset U$.

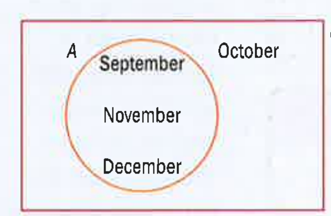


Example 3

Let U = {months of the year that end (in English) with '…ber'}
Let A = {months of the year that begin with a consonant}
Let B = {months of the year that have exactly 30 days}
Draw a Venn diagram to show
a sets U and A **b** sets U and B **c** sets U, A and B.

Answers

a



First, write down the sets
U = {September, October, November, December}
A = {September, November, December}
B = {September, November}
Note that since set A is not identical to U, we write $A \subset U$.

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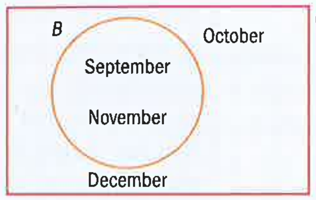
b



$B \subset U$

c

$B \subset A$

> **Every** element of B is also an element of A, so B is a subset of A, $B \subseteq A$.

> There is an element of A (December) that is not an element of B, so A and B are not identical: B is a **proper** subset of A, $B \subset A$.

Exercise 8B

Consider these sets:

$M = \{x \mid 2 \le x < 5, x \in \mathbb{Z}\}$
$N = \{x \mid 0 < x \le 5, x \in \mathbb{Z}\}$
$P = \{x \mid -2 \le x < 6, x \in \mathbb{Z}^+\}$
$S = \{(x, y) \mid x + y = 5, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\}$
$T = \{(x, y) \mid x + y = 5, x \in \mathbb{Z}, y \in \mathbb{Z}\}$
$V = \{p \mid p$ is a prime number and a multiple of 4$\}$
$W = \{x \mid x$ is a factor of 20$\}$
$X = \{x \mid x < 200, x \in \mathbb{R}\}$

State whether each statement is true or false:

1 $N \subseteq M$ **2** $S \subseteq T$ **3** $P \subseteq M$ **4** $W \subseteq X$
5 $N \subseteq P$ **6** $P \subseteq N$ **7** $\varnothing \subseteq W$ **8** $W \subseteq W$

In Exercise 8B you should have found that the last two examples were true:

For question 7, $\varnothing \subseteq W$ since every element of $\varnothing$ is in W. The fact that there are no elements in $\varnothing$ makes this certain!

Alternatively, there is no element in $\varnothing$ which is **not** in W, therefore W must contain $\varnothing$. Hence, $\varnothing$ is a subset of W.

This argument is valid for sets M, N, P, S too. In fact, it is valid for all sets.

> → The empty set $\varnothing$ is a subset of every set.

For question 8, since every element of W is in W, $W \subseteq W$. And the same argument is valid for all sets.

> → Every set is a subset of itself.

> When considering subsets you don't usually need to include either the empty set or the original set itself. The empty set and the set itself are not proper subsets of any set.

Exercise 8C

1 Consider these sets:

$M = \{x \mid 2 \le x < 5, x \in \mathbb{Z}\}$
$N = \{x \mid 0 < x \le 5, x \in \mathbb{Z}\}$
$P = \{x \mid -2 \le x < 6, x \in \mathbb{Z}^+\}$
$S = \{(x, y) \mid x + y = 5, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+\}$
$T = \{(x, y) \mid x + y = 5, x \in \mathbb{Z}, y \in \mathbb{Z}\}$
$V = \{p \mid p$ is a prime number and a multiple of 4$\}$
$W = \{x \mid x$ is a factor of 20$\}$
$X = \{x \mid x < 200, x \in \mathbb{R}\}$

State whether each statment is true or false:

a $N \subset M$ **b** $S \subset T$ **c** $P \subset M$ **d** $W \subset X$
e $M \subset P$ **f** $P \subset N$ **g** $\varnothing \subset T$ **h** $V \subset W$

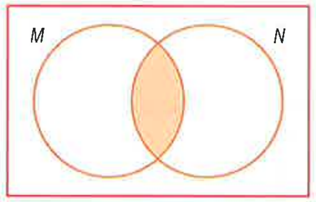
2 **a** List all the subsets of
i $\{a\}$ **ii** $\{a, b\}$ **iii** $\{a, b, c\}$ **iv** $\{a, b, c, d\}$
b How many subsets does a set with n members have?
c How many subsets does $\{a, b, c, d, e, f\}$ have?
d A set has 128 subsets. How many elements are there in this?

3 **a** List all the proper subsets of
i $\{a\}$ **ii** $\{a, b\}$ **iii** $\{a, b, c\}$ **iv** $\{a, b, c, d\}$
b How many proper subsets does a set with n members have?
c How many proper subsets has $\{a, b, c, d, e, f\}$?
d A set has 254 subsets. How many elements are there in this?

Intersection

> → The **intersection** of set M and set N (denoted $M \cap N$) is the set of all elements that are in **both** M **and** N.

$M \cap N$ is the shaded region on the Venn diagram:



Example 4

Given the sets:
$A = \{1, 2, 3, 4, 5\}$
$B = \{x \mid 0 < x \leq 5, x \in \mathbb{Z}\}$
$C = \{p \mid p$ is a prime number and a multiple of 10$\}$
$D = \{4, 5, 6, 7\}$
$E = \{x \mid x$ is a square number less than 50$\}$
write down the sets
a $A \cap D$ **b** $A \cap B$ **c** $D \cap E$ **d** $C \cap D$

Answers

First, list the elements of each set:
$A = \{1, 2, 3, 4, 5\}$
$B = \{1, 2, 3, 4, 5\}$
$C = \varnothing$
$D = \{4, 5, 6, 7\}$
$E = \{1, 4, 9, 16, 25, 36, 49\}$

Compare the sets
$A = \{1, 2, 3, 4, 5\}$ *and*
$D = \{4, 5, 6, 7\}$.

a $A \cap D = \{4, 5\}$.

Sets A and B are identical.

b $A \cap B = \{1, 2, 3, 4, 5\}$.

$D = \{4, 5, 6, 7\}$ *and*
$E = \{1, 4, 9, 16, 25, 36, 49\}$.

c The element 4 lies in both sets, hence $D \cap E = \{4\}$.

C does not contain any elements; hence there is no element that lies in both sets.

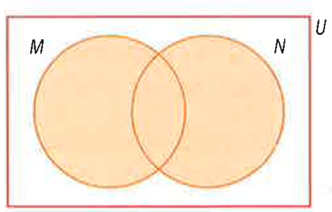
d $C \cap D = \varnothing$.

Is it always true that for any set X:
$\varnothing \cap X = \varnothing$ and
$X \cap X = X$?

Union

→ The **union** of set M and set N (denoted $M \cup N$) is the set of all elements that are in **either** M **or** N or **both**.

$M \cup N$ is the shaded region on the Venn diagram:



$M \cup N$ includes those elements that are in **both** M and N. This is important!

Example 5

Given the sets:
$A = \{1, 2, 3, 4, 5\}$
$B = \{1, 2, 3, 4, 5\}$
$C = \varnothing$
$D = \{4, 5, 6, 7\}$
$E = \{1, 4, 9, 16, 25, 36, 49\}$
Write down the sets
a $A \cup D$ **b** $A \cup B$ **c** $C \cup D$

Answers

$A = \{1, 2, 3, 4, 5\}$ and $D = \{4, 5, 6, 7\}$. To write down $A \cup D$ list ***every*** *element of each set, but* ***only once****.*

a $A \cup D = \{1, 2, 3, 4, 5, 6, 7\}$

A and B are identical.

b $A \cup B = \{1, 2, 3, 4, 5\}$

$C = \varnothing$ and $D = \{4, 5, 6, 7\}$. $C \cup D = D$, since there are no extra elements to list from C.

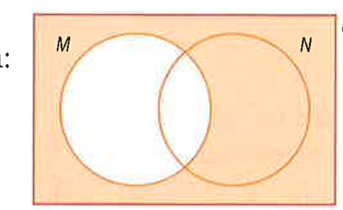
c $C \cup D = \{4, 5, 6, 7\}$

Is it always true that for any set X:
$\varnothing \cup X = X$ and
$X \cup X = X$?

Complement

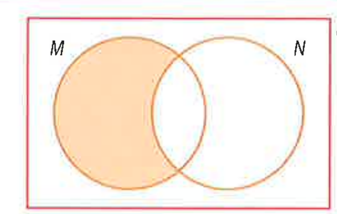
→ The **complement** of set M, denoted as M', is the set of all the elements in the universal set that **do not** lie in M.

M' is the shaded part of this Venn diagram:



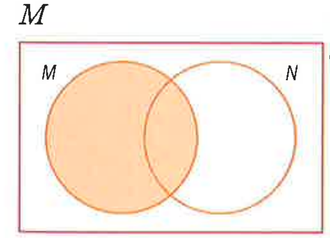
→ The complement of the universal set, U', *is the empty set,* $\varnothing$.

We can use Venn diagrams to represent different combinations of set complement, intersection and union. For example, $M \cap N'$ is shown here:

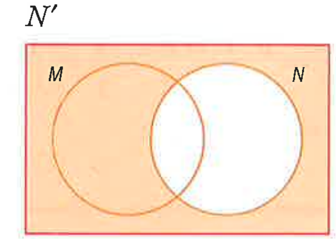


To see this in more detail, look at at the separate diagrams of M and N':

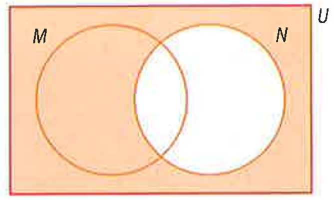
M



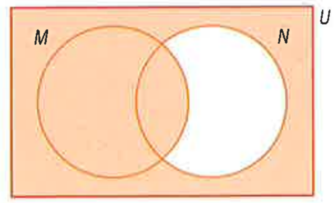
N'



Combining these for the intersection $M \cap N'$ gives shading only in the area common to both diagrams.

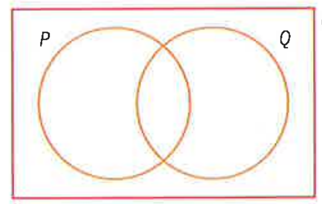


This diagram shows the set $M \cup N'$. Since it is the region that satisfies **either** M **or** N', it includes the shading from both diagrams.

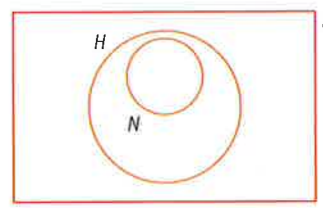


Exercise 8D

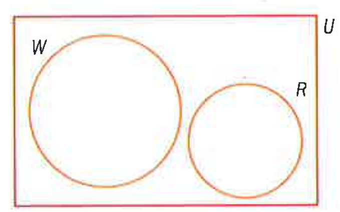
1 Copy the Venn diagram for sets P and Q.
Shade the region that represents
a $P \cup Q'$ **b** $P \cap Q'$ **c** $P' \cup Q'$
d $P' \cap Q'$ **e** $(P \cup Q)'$ **f** $(P \cap Q)'$



2 Copy the Venn diagram for sets H and N.
Shade the region that represents
a H' **b** $H \cap N'$ **c** N'
d $H' \cup N'$ **e** $H' \cap N'$ **f** $H \cup N'$



3 Copy the Venn diagram for sets W and R.
Shade the region that represents
a W' **b** $W' \cap R'$ **c** $W' \cap R$
d $W' \cup R'$ **e** $(W \cup R)'$ **f** $(W' \cap R)'$



4 U is defined as the set of all integers. Consider the following sets:
$A = \{1, 2, 3, 4, 5\}$
$B = \{x \mid 0 \le x < 5, x \in \mathbb{Z}\}$
$C = \{p \mid p$ is an even prime number$\}$
$D = \{4, 5, 6, 7\}$
$E = \{x \mid x$ is a square number less than $50\}$

Write down the sets:
a $A \cap B$ **b** $B \cap E$ **c** $C \cap D$ **d** $C \cap E$ **e** $B \cap D$
f $A \cup B$ **g** $B \cup A$ **h** $C \cup D$ **i** $C \cup A$ **j** $B \cup D$
Decide whether each statment is true or false.
k $A \subset B$ **l** $B \subset A$ **m** $C \subset A$ **n** $C \subset D$ **o** $(C \cap D) \subset E$

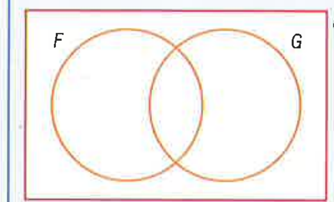
Venn diagrams can show individual set elements as well.

Example 6

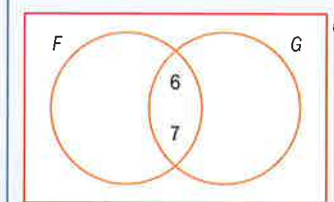
$U = \{4, 5, 6, 7, 8, 9, 10\}$, $F = \{4, 5, 6, 7\}$ and $G = \{6, 7, 8, 9\}$.
a Draw a Venn diagram for F, G and U.
b Write down these sets:
i F' **ii** $F \cap G'$ **iii** $(F \cap G)'$

Answers

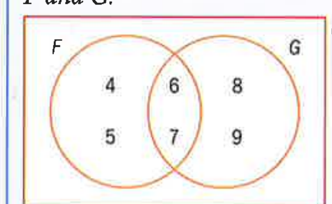
Sketch the empty Venn diagram.



$F \cap G = \{6, 7\}$. Add 6 and 7 to the diagram.

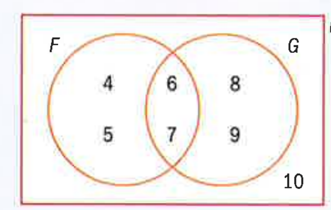


Add the remaining elements of F and G.



Add the remaining elements of U.

a



Use the diagram to write down the elements of these sets.

b **i** $F' = \{8, 9, 10\}$
ii $F \cap G' = \{4, 5\}$
iii $(F \cap G)' = \{4, 5, 8, 9, 10\}$

Note that $F \cap G' \neq (F \cap G)'$. You must be very precise when using brackets.

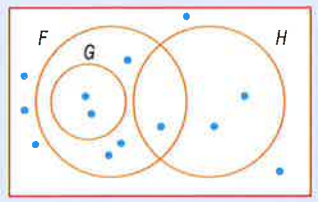
You can use Venn diagrams to work out the **number of elements** in each set without writing them all down.

Example 7

In this Venn diagram, each dot represents an element.

Write down:

a $n(G)$
b $n(F)$
c $n(G \cap F)$
d $n(H')$
e $n(F \cap H)$
f $n(G \cap H)$



Is each statement true or false?

g $n(F \cup H) = n(F) + n(H)$
h $n(G \cup H) = n(G) + n(H)$

Answers

a $n(G) = 2$
b $n(F) = 6$
c $n(G \cap F) = 2$
d $n(H') = 10$
e $n(F \cap H) = 1$
f $n(G \cap H) = 0$
g The statement is false.
h The statement is true.

Count the dots in each set.

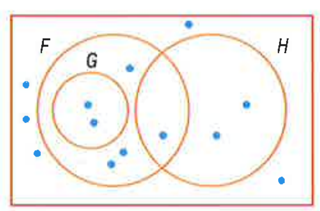
$n(F \cup H) = 8$, $n(F) = 6$, $n(H) = 3$
$n(G \cup H) = 5$, $n(G) = 2$, $n(H) = 3$.

The statements in **e** and **f** help you decide whether statements **g** and **h** are true or false.

Exercise 8E

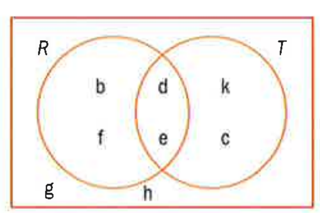
1 Is each statement true or false?

a $F \subset G$ **b** $n(F \cup G) = 6$
c $n(G') = 8$ **d** $n(F \cup H) = 6$
e $H \cup F = G'$ **f** $F' \subset H$
g $n(G' \cap H) = 5$ **h** $n(F' \cap G) = 5$



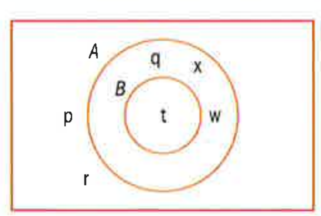
2 List the elements of

a U
b R
c R'
d T
e T'



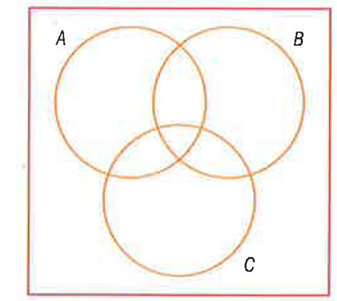
3 List the elements of

a A
b A'
c $A \cup B'$
d $A \cap B'$
e $A' \cup B'$



8.3 Extending to three sets

This Venn diagram shows a general three-set problem.



Use the same notation for three sets. But take great care using brackets to describe the sets.

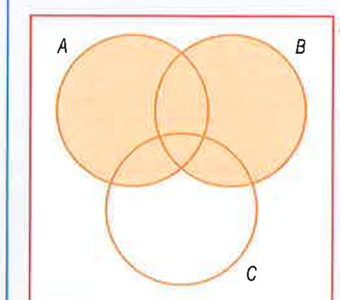
Example 8

Shade the region on a Venn diagram that shows the sets:

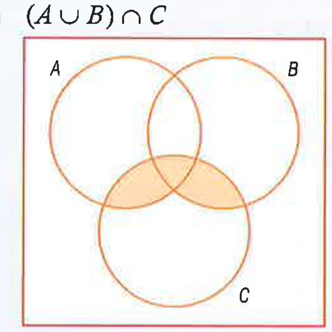
a $(A \cup B) \cap C$ **b** $A \cup (B \cap C)$

Answers

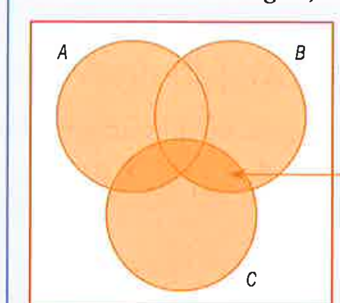
First shade the region in the brackets $(A \cup B)$:



a $(A \cup B) \cap C$



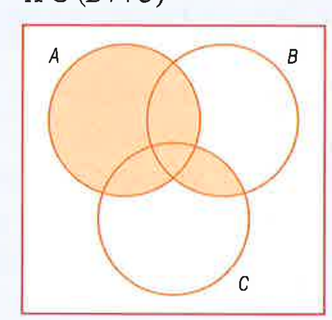
Then shade the other region, C:

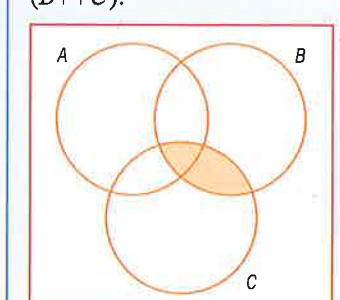


The dark region is the intersection

First shade the region in the brackets $(B \cap C)$:

b $A \cup (B \cap C)$



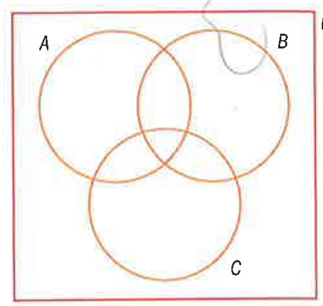


Then shade the other region A.

The union is *all* the shaded areas

Note that the statement $A \cup B \cap C$ has no mathematical meaning. The brackets are required to remove the ambiguity.

Exercise 8F

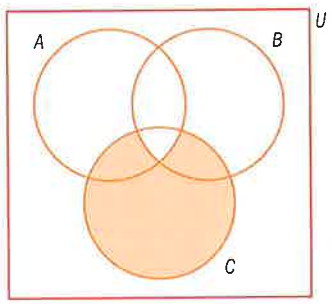


1 Shade the region on a three-set Venn diagram that shows each set:

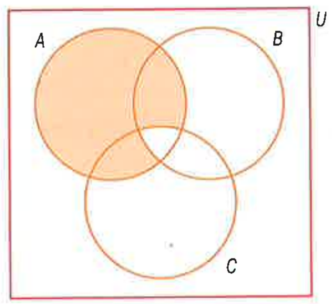
	i	ii
a	$(A \cup B) \cup C$	$A \cup (B \cup C)$
b	$(A \cap B) \cap C$	$A \cap (B \cap C)$
c	$(A \cup C) \cap B$	$A \cup (C \cap B)$
d	$C \cap (A \cup B)$	$B \cup (C \cap A)$
e	$(A \cup B) \cup C'$	$A \cup (B \cup C')$
f	$(A \cap B') \cap C$	$A \cap (B' \cap C)$
g	$(A \cup C) \cap B'$	$A \cup (C \cap B')$

2 Use set notation to name the shaded region in each Venn diagram.

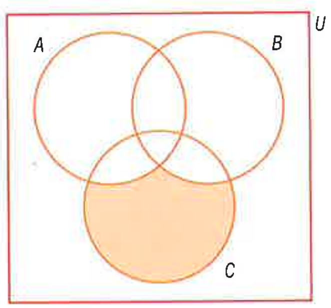
a



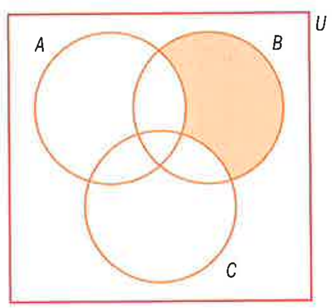
b



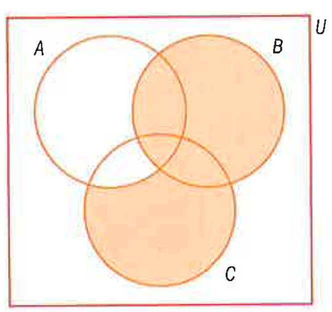
c



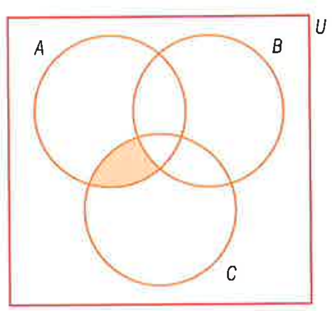
d



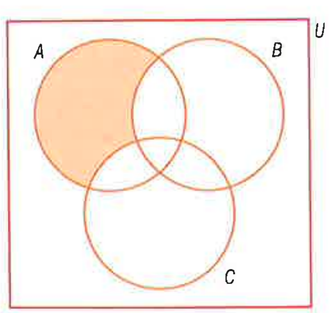
e



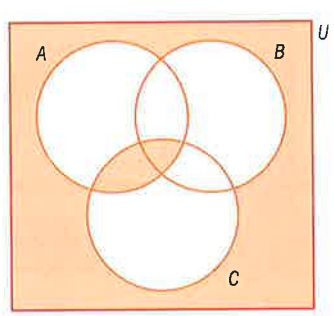
f



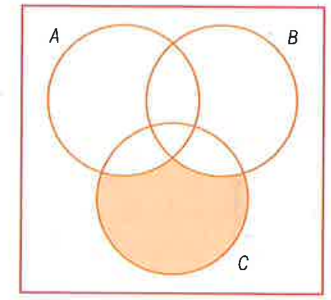
g



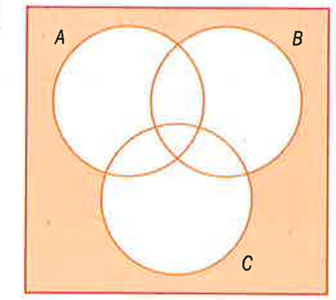
h



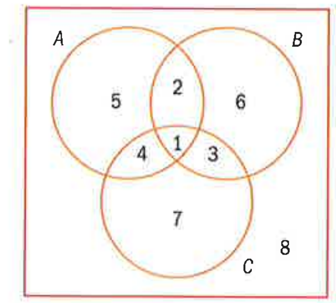
i



j



3 In this Venn diagram, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
List the elements of:



a $A \cap B \cap C$
b $A' \cap B \cap C$
c $A \cap B' \cap C$
d $A \cap B \cap C'$
e $A' \cap B' \cap C$
f $A' \cap B \cap C'$
g $A \cap B' \cap C'$
h $A' \cap B' \cap C'$

4 For the Venn diagram in question 3, list the elements of:

a	$A \cap (B \cup C)$	**b**	$A' \cap (B \cup C)$
c	$(A \cup B') \cap C$	**d**	$(A \cup B) \cap C'$
e	$(A' \cup B') \cap C$	**f**	$(A' \cup B) \cap C'$
g	$B \cap (A' \cup C')$	**h**	$B' \cap (A' \cup C)$

8.4 Problem-solving using Venn diagrams

Here is the problem from the first investigation in this chapter:

Investigation – a contradiction?

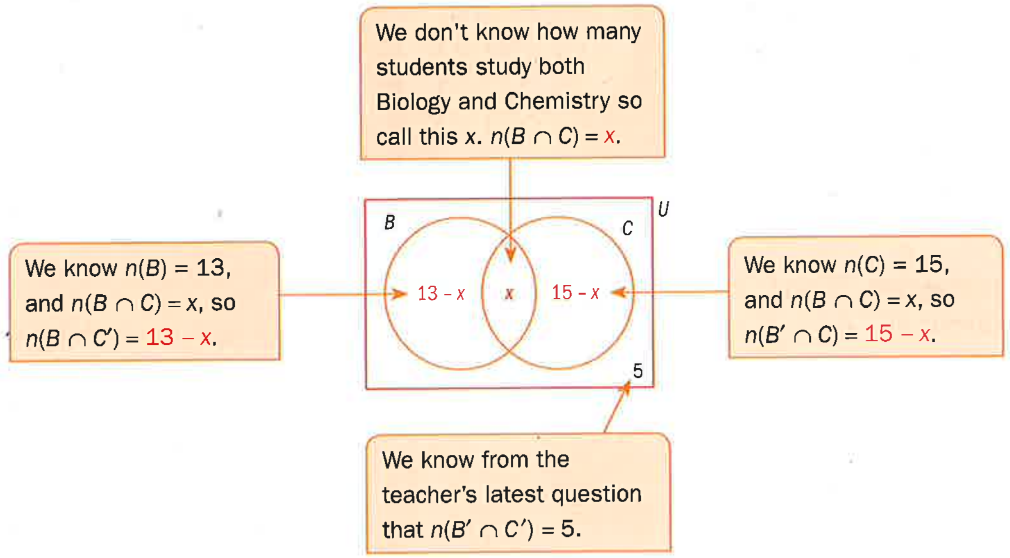
A teacher asks her class how many of them study Chemistry.
She finds that there are 15. She then asks how many study Biology and finds that there are 13.
Later, she remembers that there are 26 students in the class.
But 15 + 13 = 28. Has she miscounted?

We can represent this problem on a Venn diagram.

Let B be the set of students studying Biology, and C be the set of students studying Chemistry. Then $n(B) = 13$, $n(C) = 15$ and $n(U) = 26$.

The teacher asks another question and finds out that 5 of the students study neither Biology nor Chemistry, so $n(B' \cap C') = 5$.

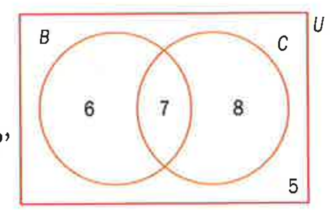
We can put what we know, and what we don't know, on a Venn diagram:



We also know that $n(U) = 26$. From the Venn diagram we can write

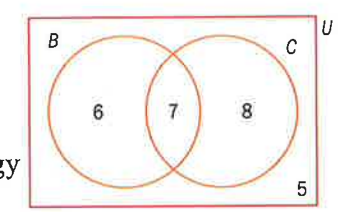
$$(13 - x) + x + (15 - x) + 5 = 26$$
$$33 - x = 26$$
$$x = 7$$

So now we can substitute for x on the Venn diagram, and answer questions like 'How many students study Chemistry but not Biology?'



Exercise 8G

Use the Venn diagram to answer these questions:



1 How many students study Biology **only**? (That is, 'Biology, but not Chemistry'.)

2 How many students study **exactly** one science? (That is, 'Biology or Chemistry, but not **both**'.)

3 How many students study **at least** one science? (That is, 'Biology or Chemistry, or **both**'.)

If you study two sciences, then you necessarily must study one!

4 How many students study one science? (That is, 'Biology or Chemistry, or **both**'.)

5 How many students do not study Biology?

6 How many students do not study Chemistry?

7 How many Chemists study Biology?

8 How many Biologists do not study Chemistry?

9 How many science students do not study both Biology and Chemistry?

Example 9

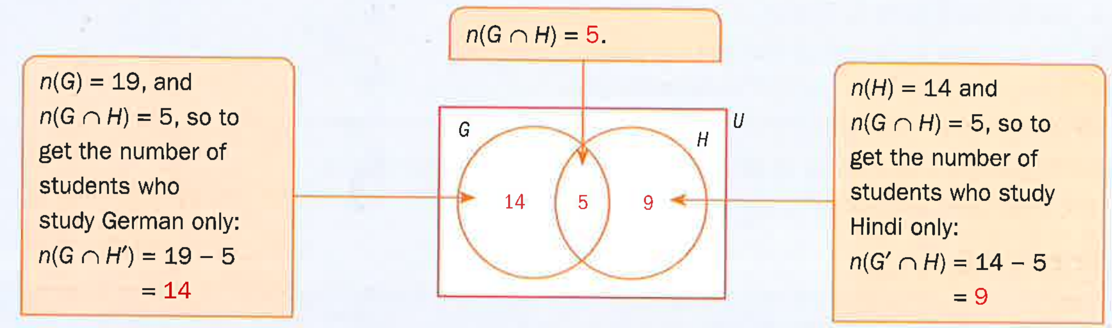
In a class of 29 students, 19 study German, 14 study Hindi and 5 study both languages. Work out the number of students that study neither language.

Answer

Let G be the set of students who study German, and H the set of those studying Hindi. From the information in the question

$n(U) = 29$ $\quad n(G) = 19$ $\quad n(H) = 14$ $\quad n(G \cap H) = 5$

Draw a Venn diagram:



The number of students who study neither language is $n(G' \cap H')$.
$n(G' \cap H') = 29 - 14 - 5 - 9 = 1$
1 student studies neither language.

$n(U) = 29$

Exercise 8H

1 There are 25 students in a class. 17 study French, 12 study Malay, and 10 study both languages.
Show this information on a Venn diagram.
Find the number of students who:
- **a** study French only
- **b** study Malay or French or both
- **c** study neither subject
- **d** do not study both subjects.

2 In a class 20 people take Geography, 17 take History, 10 take both subjects, and 1 person takes neither subject.
Draw a Venn diagram to show this information.
Find the number of students who:
- **a** are in the class
- **b** do not study History
- **c** study Geography but not History
- **d** study Geography or History but not both.

3 Of the 32 students in a class, 18 play the violin, 16 play the piano, and 7 play neither. Find the number of students who:
- **a** play the violin but not the piano
- **b** do not play the violin
- **c** play the piano but not the violin
- **d** play the piano or the violin, but not both.

4 There are 30 students in a mathematics class. 20 of the students have studied probability, 14 have studied set theory, and two people have studied neither.
Find the number of students who:
- **a** have studied both topics
- **b** have studied exactly one of these subjects
- **c** have studied set theory, but not probability.

5 There are 25 girls in a PE group. 13 have taken aerobics before and 17 have taken gymnastics. One girl has done neither before.
Find the number of girls who:
- **a** have taken both activities
- **b** have taken gymnastics but not aerobics
- **c** have taken at least one of these activities.

You can use the same ideas to draw Venn diagrams with more sets; see the following example.

Example 10

145 people answered a survey to find out which flavor of fruit juice, orange, apple or grape, they preferred.
The replies showed:

15 liked none of the three	35 liked orange and apple
55 liked grape	20 liked orange and grape
80 liked apple	30 liked apple and grape
75 liked orange	

Find the number of people who liked all three types of juice.

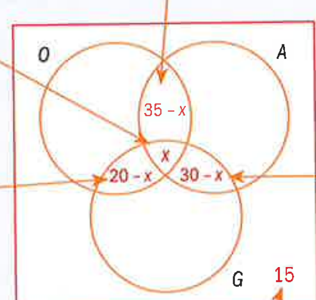
Answer

number of people who liked all three juices is $n(O \cap A \cap G) = x$

35 people liked both orange and apple, so $n(O \cap A \cap G') = 35 - x$

20 people liked both orange and grape, so $n(O \cap A' \cap G) = 20 - x$

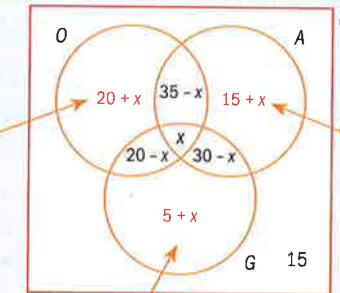
30 people liked apple and grape, so $n(O' \cap A \cap G) = 30 - x$



15 people liked none of the juices, so $n(O' \cap A' \cap G') = 15$

▶ Continued on next page

75 people liked orange, so $n(O \cap A' \cap G') = 75 - ((35 - x) + x + (20 - x)) = 20 + x$



80 people liked apple, so $n(O' \cap A \cap G') = 80 - ((35 - x) + x + 30 - x) = 15 + x$

55 people liked grape, so $n(O' \cap A' \cap G) = 55 - ((20 - x) + x + (30 - x)) = 5 + x$

To find x, use $n(U) = 145$

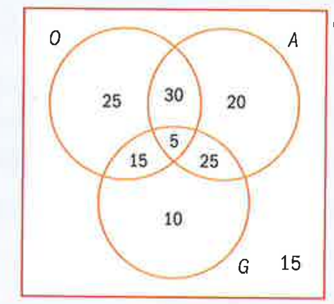
$145 = n(O) + (15 + x) + (30 - x) + (5 + x) + 15$

We know that $n(O) = 75$, so

$145 = 75 + (15 + x) + (30 - x) + (5 + x) + 15$

$145 = 140 + x$

$x = 5$



There are a number of ways of combining the various parts of the Venn diagram to make U. They will all give the same answer.

In the IB examination you won't be asked to draw a Venn diagram with more than three sets.

Exercise 8I

Use the information from Example 10 to answer these questions.

1 Find the number in the survey above who
- **a** liked exactly two of the three flavors of juice
- **b** did not like orange juice
- **c** liked one flavor of juice only
- **d** did not like either orange or apple juice
- **e** did not like orange juice and did not like apple juice
- **f** liked at least two of the three flavors of juice
- **g** liked fewer than two of the three flavors of juice.

Find the number of orange juice drinkers who
- **h** liked apple juice
- **i** did not like grape juice
- **j** liked no other flavors of juice
- **k** liked exactly one other flavor of juice.

2 In a group of 105 students, 70 students passed Mathematics, 60 students passed History and 45 students passed Geography; 30 students passed Mathematics and History, 35 students passed History and Geography, 25 passed Mathematics and Geography, and 15 passed all three subjects.
Draw a Venn diagram to illustrate this information.
Find the number of students who

- **a** passed at least one subject
- **b** passed exactly two subjects
- **c** passed Geography and failed Mathematics
- **d** passed all three subjects given that they passed two
- **e** failed Mathematics given that they passed History.

3 In a youth camp, each participant must take part in at least one of the following activities: chess, backgammon or dominoes. Of the total of 55 in the camp, 25 participants participated in chess, 24 in backgammon, and 30 in dominoes; 15 in both chess and backgammon, 10 in both backgammon and dominoes, 5 in both chess and dominoes, and 2 in all three events.

Draw a Venn diagram to show this information.

How many of the participants are not taking part in at least one activity?

Find the number of participants who

- **a** take part in one activity only
- **b** take part in exactly two activities
- **c** do not take part in at least two activities
- **d** take part in chess, given that they take part in dominoes
- **e** take part in backgammon, given that they do not take part in dominoes.

4 Fatty's Delight sells chicken, duck, and barbecued pork rice. Of the 160 customers one day, 57 had chicken rice, 60 had duck rice and 48 had barbecued pork rice. 30 customers ordered chicken and duck rice, 25 ordered duck and barbecued pork rice, 35 ordered chicken and barbecued pork rice, and 20 ordered all three types.
Draw a Venn diagram to show these data.

Find the number of customers who

- **a** ordered more than one type of rice
- **b** did not order a rice dish from Fatty's Delight
- **c** did not order chicken rice
- **d** ordered duck rice and one other rice dish.

5 In a community center in Buona Vista there are 170 youths. Of these, 65 take up climbing, 65 bouldering and 50 swimming; 15 take up climbing and bouldering, 10 bouldering and swimming, and 5 swimming and climbing. 17 youths take up other activities.

Let x be the number of youths who take up all three activities.
Show the above information in a Venn diagram.
Show clearly the number in each separate region in terms of x.
Form an equation satisfied by x, and hence find its value.

Find the number of youths who

- **a** take up one activity only
- **b** take up at least two activities
- **c** take part in fewer than two activities
- **d** take up bouldering given that they have already taken up climbing
- **e** take up one other activity given that they have already taken up swimming.

6 65 elderly men failed a medical test because of defects in at least one of these organs: the heart, lungs or kidneys. 29 had heart disease, 28 lung disease and 31 kidney disease. 8 of them had both lung and heart diseases, 11 had lung and kidney diseases, while 12 had kidney and heart diseases.

Draw a Venn diagram to show this information. You will need to introduce a variable.

Find the number of men who

- **a** suffer from all three diseases
- **b** suffer from at least two diseases
- **c** suffer from lung disease and exactly one other disease
- **d** suffer from heart disease and lung disease but not kidney disease
- **e** suffer from lung disease only.

7 Each of the 116 students in the Fourth Year of a school studies at least one of the subjects History, English and Art.
Of the 50 students who study Art,
 15 also study History and English,
 12 study neither History nor English, and
 17 study English but not History.
Of the 66 students who do not study Art,
 39 study both History and English,
 x study History only, and
 $2x$ study English only.

Draw a Venn diagram showing the number of students in each subset.
Hence find

a the value of x **b** the total number of students studying English.

8.5 Basic probability theory

Probability is the branch of mathematics that analyses random experiments. A **random experiment** is one in which we cannot predict the precise outcome. Examples of random experiments are 'tossing a coin' or 'rolling a dice' or 'predicting the gold, silver, and bronze medalists in a 100 m sprint'.

It is impossible to predict the outcome of a random experiment **precisely** but it is possible to

a list the set of all possible outcomes of the experiment

b decide how likely a particular outcome may be.

There are some assumptions being made:

1 the coin is unbiased

2 the dice is unbiased

3 all sprinters are evenly matched

When tossing a coin, there are two possible outcomes: **heads** (H) and **tails** (T).

Also, the likelihood of getting a head is the same as getting a tail, so the probability of getting a head is one chance out of two. The probability of getting a tail is the same.

In other words, the set of equally likely outcomes is $\{H, T\}$ and $P(H) = P(T) = \frac{1}{2}$.

When rolling a dice, the set of equally likely possible outcomes has six elements and is $\{1, 2, 3, 4, 5, 6\}$.

As all six outcomes are equally likely, $P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$.

Let event A be 'rolling an even number'.

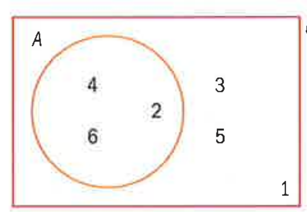
To find $P(A)$, consider the set of equally likely outcomes $\{1, 2, 3, 4, 5, 6\}$.

There are six equally likely outcomes and three of these are even numbers, so $P(A) = \frac{3}{6}$.

Let B be the event 'rolling a prime number'.

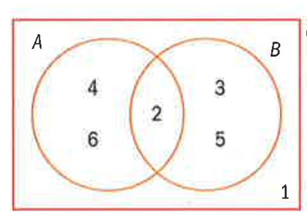
To find $P(B)$, look again at the set of outcomes. There are three prime numbers: 2, 3, and 5 so, $P(B) = \frac{3}{6}$.

We can show the equally likely possible outcomes of rolling a dice on a Venn diagram using $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{\text{even numbers}\}$.



$$P(A) = \frac{n(A)}{n(U)} = \frac{3}{6}$$

Set B can be added to the Venn diagram to represent the event B.



$$P(B) = \frac{n(B)}{n(U)} = \frac{3}{6}$$

→ If all of the equally likely possible outcomes of a random experiment can be listed as U, the universal set, and an event A is defined and represented by a set A, then:

$$P(A) = \frac{n(A)}{n(U)}$$

There are three consequences of this law:

1 $P(U) = \frac{n(U)}{n(U)} = 1$ (the probability of a **certain** event is 1)

2 $P(\varnothing) = \frac{n(\varnothing)}{n(U)} = 0$ (the probability of an **impossible** event is 0)

3 $0 \le P(A) \le 1$ (the probability of any event **always** lies between 0 and 1)

Example 11

Find the probability that these events occur for the random experiment 'rolling a fair dice'.

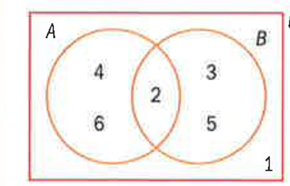
a Rolling an odd number

b Rolling an even prime number

c Rolling an odd prime number

d Rolling a number that is either prime or even

Unless stated otherwise, we will always be talking about a cubical dice with faces numbered 1 to 6.



Answers

Use the Venn diagram drawn earlier, where A is the event 'rolling an even number' and B is the event 'rolling a prime number'.

a $P(A') = \frac{n(A')}{n(U)} = \frac{3}{6}$

A is the event 'rolling an even number', so the probability of rolling an odd number is P(A'). From the Venn diagram, A' = {1, 3, 5}.

b $P(A \cap B) = \frac{n(A \cap B)}{n(U)} = \frac{1}{6}$

A is the event 'rolling an even number', and B is the event 'rolling a prime number', so the probability of rolling an even prime number is P(A ∩ B).

c $P(A' \cap B) = \frac{n(A' \cap B)}{n(U)} = \frac{2}{6}$

The probability of rolling an odd prime number is P(A' ∩ B).

d $P(A \cup B) = \frac{n(A \cup B)}{n(U)} = \frac{5}{6}$

The probability of rolling a number that is either prime or even is P(A ∪ B).

This example illustrates the basics of probability theory: list the equally likely possible outcomes of a random experiment and count. Drawing a Venn diagram may clarify the situation.

Two further laws of probability:

- For complementary events, $P(A') = 1 - P(A)$
- For combined events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Use the Venn diagram to illustrate these laws.

Exercise 8J

1 A random experiment is: roll an unbiased six-faced dice.
Let A be the event 'roll a square number' and let B be the event 'roll a factor of 6'.
- **a** List the elements of set A.
- **b** List the elements of set B.
- **c** Show sets A and B on a Venn diagram.
- **d** Write down $P(A)$.
- **e** Write down $P(B)$.
- **f** Find the probability that the number rolled is not a square number.
- **g** Find the probability that the number rolled is both a square number and a factor of 6.
- **h** Find the probability that the number rolled is either a square number or a factor of 6 or both.
- **i** Verify that both $P(A') = 1 - P(A)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

2 The numbers 3, 4, 5, 6, 7, 8, 9, 10 are written on identical pieces of card and placed in a bag. A random experiment is: a card is selected at random from the bag.
Let A be the event 'a prime number is chosen' and let B be the event 'an even number is chosen'.
- **a** List the elements of set A.
- **b** List the elements of set B.
- **c** Show sets A and B on a Venn diagram.
- **d** Write down $P(A)$.
- **e** Write down $P(B)$.
- **f** Find the probability that the number rolled is composite (not a prime).
- **g** Find the probability that the number rolled is odd.
- **h** Find the probability that the number rolled is both even and prime.
- **i** Find the probability that the number rolled is either even or prime or both.
- **j** Verify that both $P(A') = 1 - P(A)$ and $P(B') = 1 - P(B)$.
- **k** Verify that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- **l** Find the probability that the number rolled is both odd and composite.
- **m** Find the probability that the number rolled is either odd or composite or both.
- **n** Verify that $P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$

3 The numbers 2, 3, 4, 5, 6, 7, 8, 9 are written on identical pieces of card and placed in a bag. A random experiment is: a card is selected at random from the bag.
Let A be the event 'an odd number is chosen' and let B be the event 'a square number is chosen'.
- **a** List the elements of set A.
- **b** List the elements of set B.
- **c** Show sets A and B on a Venn diagram.
- **d** Write down $P(A)$.
- **e** Write down $P(B)$.
- **f** Find the probability that an odd square number is chosen.
- **g** Find the probability that either an odd number or a square number is chosen.
- **h** Verify that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The first book written on probability was *The Book of Chance and Games* by Italian philosopher and mathematician Jerome Cardan (1501–75). It explained techniquies on how to cheat and catch others at cheating.

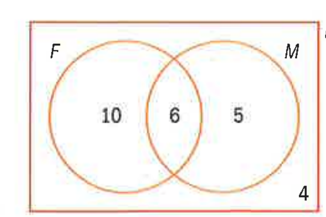
4 A random experiment is: toss two unbiased coins.
- **a** List the set of four equally likely possible outcomes.
- **b** Find P(two heads show), P(one head shows), P(no heads show).

5 A random experiment is: toss three unbiased coins.
- **a** List the set of eight equally likely possible outcomes.
- **b** Find P(no heads), P(one head), P(two heads), P(three heads).

6 A random experiment is: toss four unbiased coins.
- **a** Find P(no heads).
- **b** Find P(four heads).
- **c** Find P(one head).
- **d** Find P(three heads).
- **e** Use the answers **a** to **d** to deduce P(two heads).
- **f** List the equally likely possible outcomes.

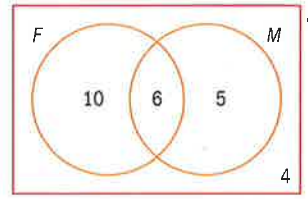
8.6 Conditional probability

In a class of 25 students, 16 students study French, 11 students study Malay and 4 students study neither language. This information can be shown in a Venn diagram.

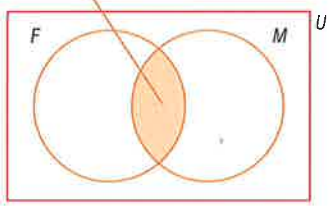


Suppose a student is chosen at random from the class. We can use the techniques we have looked at already to find the probability that
- **a** the student studies French and Malay
- **b** the student studies exactly one language
- **c** the student does not study two languages
- **d** the student does not study French.

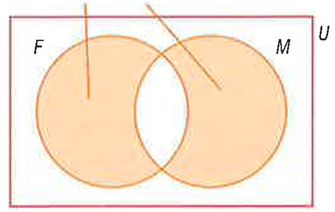
Using the Venn diagram on the right:



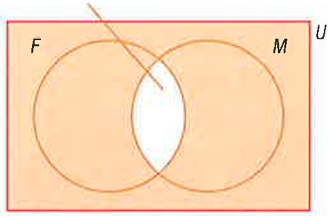
a $\frac{6}{25}$



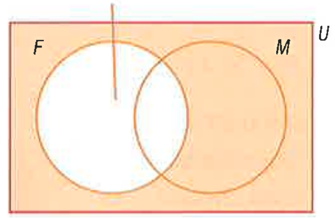
b $\frac{10}{25}+\frac{5}{25}=\frac{15}{25}$



c $1-\frac{6}{25}=\frac{19}{25}$



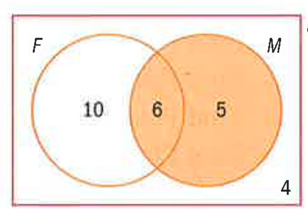
d $1-\frac{16}{25}=\frac{9}{25}$



What is the probability that a student chosen at random studies French, **given that** the student studies Malay?

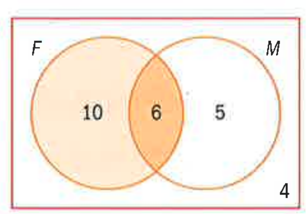
This requires a different approach because there is an extra condition: the student studies Malay.

The probability that a student studies French given that the student studies Malay is an example of a **conditional probability**. It is written P(F | M).



Given that *M* has definitely occurred, then we are restricted to set M (the shaded area), rather than choosing from the universal set (the rectangle).

If we now want to determine the probability that F has also occurred, then we consider that part of F which also lies within M – the intersection of F and M (darkest shading).



The conditional probability, the probability that a student studies French given that the student studies Malay, is

$$P(F \mid M)=\frac{n(F\cap M)}{n(M)}=\frac{6}{11}$$

→ The conditional probability that A occurs given that B has occurred is written as P(A | B) and is defined as:

$$P(A \mid B)=\frac{P(A\cap B)}{P(B)}$$

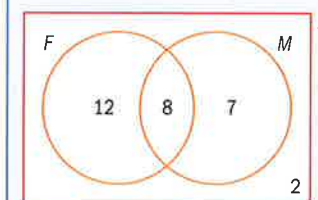
Example 12

In a class of 29 students, 20 students study French, 15 students study Malay, and 8 students study both languages. A student is chosen at random from the class.

Find the probability that the student

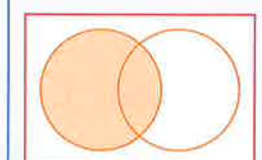
a studies French
b studies neither language
c studies at least one language
d studies both languages
e studies Malay given that they study French
f studies French given that they study Malay
g studies both languages given that they study at least one of the languages.

Answers

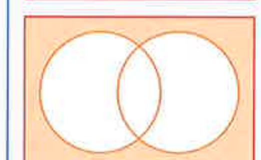


First draw a Venn diagram to show the information.

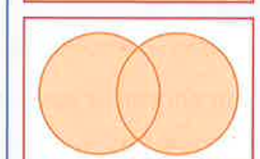
a P(studies French) = $\frac{20}{29}$



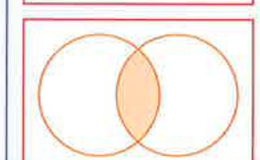
b P(studies neither language) = $\frac{2}{29}$



c P(studies at least one language) = $\frac{27}{29}$

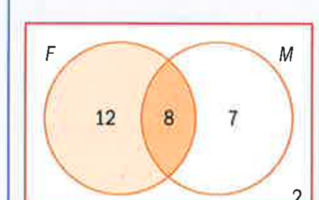


d P(studies both languages) = $\frac{8}{29}$



e P(studies Malay given that they study French)

$= P(M \mid F)=\frac{n(M\cap F)}{n(F)}=\frac{8}{20}$

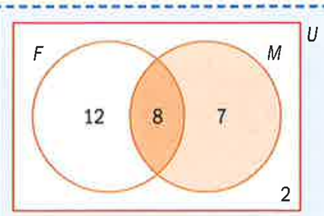


Probabilities **e** to **g** are conditional, and require more care

▶ Continued on next page

f P(studies French given that they study Malay)

$= P(F \mid M) = \frac{n(F \cap M)}{n(M)} = \frac{8}{15}$



Looking at the Venn diagram you can see that $(F \cap M) \cap (F \cup M) = (F \cap M)$

g P(studies both languages given that they study at least one language)

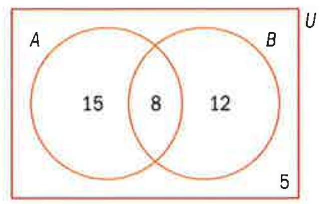
$= P(F \cap M \mid F \cup M)$

$= \frac{n([F \cap M] \cap [F \cup M])}{n(F \cup M)} = \frac{8}{27}$

Exercise 8K

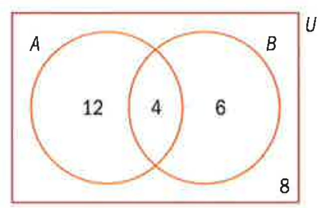
The numbers in each set are shown on the Venn diagrams.

1 Find the probability that a person chosen at random:



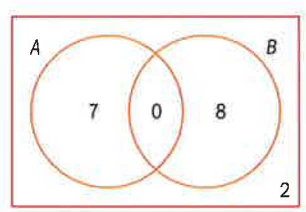
a is in A
b is not in either A or B
c is not in A and not in B
d is in A, given that they are not in B
e is in B, given that they are in A
f is in both A and B, given that they are in A.

2 Find the probability that a person chosen at random:



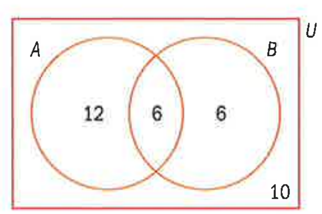
a is not in A
b is neither in A nor in B
c is not in both A and B given that they are in B
d is not in A given that they are not in B
e is in B given that they are in A
f is in both A and B, given that they are not in A.

3 Find the probability that a person chosen at random:



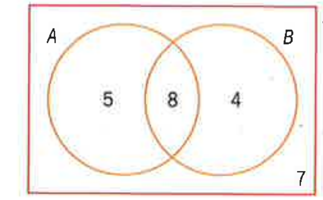
a is in B but not in A
b is not in A or B
c is in B and not in A
d is in A given that they are not in B
e is in B given that they are in A
f is not in both A and B, given that they are in A.

4 Find the probability that a person chosen at random:



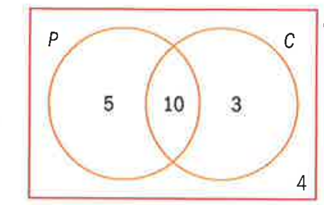
a is in A but not in both A and B
b is not in A and not in both
c is not in both A and B
d is in A given that they are not in B
e is in B given that they are in A
f is not in A given that they are not in B.

5 The Venn diagram shows the number of students who take Art and/or Biology in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



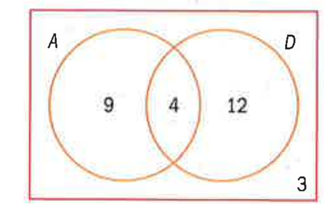
a takes Art
b takes Biology but not Art
c takes both Art and Biology
d takes at least one of the two subjects
e takes neither subject
f takes Biology
g takes exactly one of the two subjects.

6 The Venn diagram shows the number of students who take Physics and/or Chemistry in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



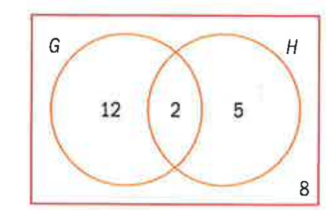
a takes Physics but not Chemistry
b takes at least one of the two subjects
c takes Chemistry given that the student takes Physics
d is a Chemist given that the student takes exactly one of the two subjects.

7 The Venn diagram shows the number of students who take Art and/or Drama in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



a takes Drama but not Art
b takes Drama given that they take Art
c takes both subjects given that they take Drama
d takes neither subject
e takes Drama given that they take exactly one of the two subjects.

8 The Venn diagram shows the number of students who take Geography and/or History in a class.
Use the Venn diagram to find the probability that a student chosen at random from the class:



a takes Geography but not History
b takes Geography given that they do not take History
c takes History given that they take at least one of the two subjects
d takes Geography given they take History
e takes Geography given that they take exactly one of the two subjects.

8.7 Two special cases: mutually exclusive and independent events

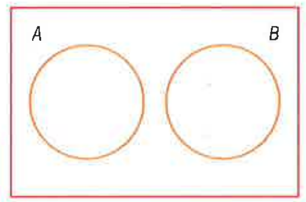
Two events, A and B, are **mutually exclusive** if whenever A occurs it is impossible for B to occur and, similarly, whenever B occurs it is impossible for A to occur.

Events A and A' are the most obvious example of mutually exclusive events – either one or the other must occur, but A and A' cannot occur at the same time.

For example, in tossing a coin, the events 'a head is tossed' and 'a tail is tossed' are mutually exclusive.

Here is the Venn diagram for mutually exclusive events A and B.

As the two sets do not overlap, $A \cap B = \varnothing$.



→ Events A and B are mutually exclusive if and only if $P(A \cap B) = 0$.

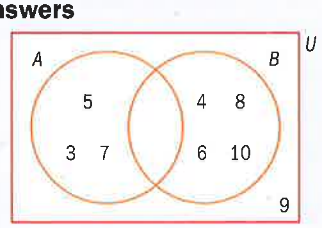
Example 13

The numbers 3, 4, 5, 6, 7, 8, 9, 10 are each written on an identical piece of card and placed in a bag. A random experiment is: a card is selected at random from the bag.
Let A be the event 'a prime number is chosen' and B the event 'an even number is chosen'.

a Draw a Venn diagram that describes the random experiment.
b Determine whether the events A and B are mutually exclusive.

Answers

a



Draw a Venn diagram to show the sets A and B.

$A \cap B = \varnothing$, so $P(A \cap B) = 0$.

The intersection $A \cap B$ is empty.

b A and B are mutually exclusive.

In 1933, the Russian Mathematician Andrey Nikolaevich Kolmogorov (1903–1987) defined probability by these axioms:

- The probability of all occurrences is 1
- Probability has a value which is greater than or equal to zero
- When occurrences cannot coincide their probabilities can be added

The mathematical properties of probability can be deduced from these axioms. Kolmogorov used his probability work to study the motion of the planets and the turbulent flow of air from a jet engine.

What is an axiom? Find out more about Euclid's axioms for geometry, written 2000 years ago.

Exercise 8L

In each experiment, determine whether the events A and B are mutually exclusive.

1 Roll an unbiased six-faced dice.
Let A be the event 'roll a square number' and let B be the event 'roll a factor of six'.

2 Roll an unbiased six-faced dice.
Let A be the event 'roll a four' and let B be the event 'roll a six'.

3 Roll an unbiased six-faced dice.
Let A be the event 'roll a prime number' and let B be the event 'roll an even number'.

4 Roll an unbiased six-faced dice.
Let A be the event 'roll a square number' and let B be the event 'roll a prime number'.

5 Each of the numbers 3, 4, 5, 6, 7, 8, 9, 10 are written on identical pieces of card and placed in a bag. A card is selected at random from the bag.
Let A be the event 'a square number is chosen' and let B be the event 'an odd number is chosen'.

6 Each of the numbers 5, 6, 7, 8, 9, 10 are written on identical pieces of card and placed in a bag. A card is selected at random from the bag.
Let A be the event 'a square number is chosen' and let B be the event 'an even number is chosen'.

7 Each of the numbers 2, 3, 4, 5, 6, 7, 8, 9 are written on identical pieces of card and placed in a bag. A card is selected at random from the bag.
Let A be the event 'an even number is chosen' and let B be the event 'a multiple of three is chosen'.

8 Two unbiased coins are tossed.
Let A be the event 'two heads show' and let B be the event 'one head shows'.

If two events, A and B, are mutually exclusive, the effect of the first event, A, on the second, B, could not be greater – if A occurs, then it is impossible that B can occur (and vice versa). The occurrence of one event completely prevents the occurrence of the other.

The other extreme is when the occurrence of the one event does not affect in any way the occurrence of the other. Then the two events are **mathematically independent** of each other.

Another way to put this is that the probability that A occurs, $P(A)$, remains the same given that B has occurred. Writing this as an equation, A and B are independent whenever $P(A) = P(A \mid B)$.

The definition of $P(A \mid B)$ is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Thus whenever A and B are independent:

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

Rearranging, $P(A \cap B) = P(A) \times P(B)$

→ A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$.

For example, if a one-euro coin is tossed and then a one-dollar coin is tossed, the fact that the euro coin landed 'heads' does not affect in any way whether the dollar coin lands 'heads' or 'tails'. The two events are independent of each other.

If you are asked to determine whether two events are independent, this is the test you must use.

Example 14

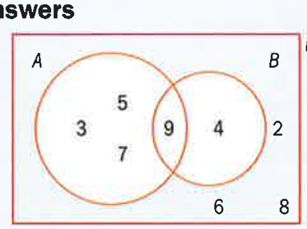
The numbers 2, 3, 4, 5, 6, 7, 8, 9 are each written on identical pieces of card and placed in a bag.
A card is selected at random from the bag.
Let A be the event 'an odd number is chosen' and let B be the event 'a square number is chosen'.

a Draw a Venn diagram to represent the experiment.

b Determine whether A and B are independent events.

Answers

a



The event $A \cap B$ is 'an odd number is chosen ***and*** *a square number is chosen' or 'an odd square number is chosen'.*

b $P(A) \times P(B) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

From the Venn diagram,
$P(A) = \frac{4}{8} = \frac{1}{2}$ $\quad P(B) = \frac{2}{8} = \frac{1}{4}$

$P(A \cap B) = \frac{1}{8}$

$A \cap B = \{9\}$, hence $P(A \cap B) = \frac{1}{8}$

So A and B are independent events.

Now, consider the definition for (mathematical) independence: $P(A \cap B) = P(A) \times P(B)$.

This work links to the chi-squared test for independence that you studied in Chapter 5. Recall that to calculate the expected frequencies, the row total is multiplied by the column total and then divided by the overall total of frequencies. This is a direct consequence of the definition of mathematical independence.

Exercise 8M

For each experiment determine whether the events A and B are independent.

1 The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are each written on identical cards and placed in a bag.
A card is selected at random from the bag.

Let A be the event 'an odd number is chosen' and let B be the event 'a square number is chosen'.

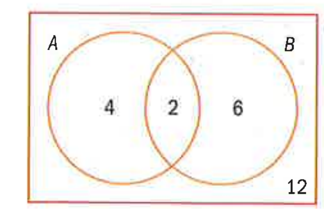
2 The numbers 1, 2, 3, 4, 5, 6 are each written on identical cards and placed in a bag.
A card is selected at random from the bag.

Let A be the event 'an even number is chosen' and let B be the event 'a square number is chosen'.

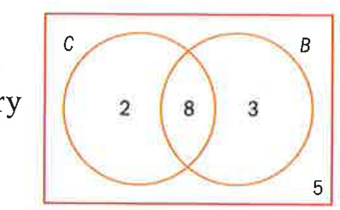
3 The numbers 2, 3, 4, 5, 6, 7, 8, 9, 10 are each written on identical cards and placed in a bag.
A card is selected at random from the bag.

Let A be the event 'a prime number is chosen' and let B be the event 'a multiple of three is chosen'.

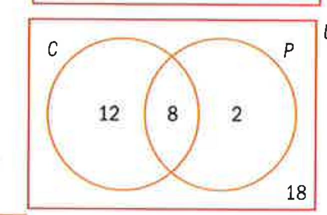
4 The Venn diagram shows the number of students who take Art and/or Biology in a class.
Use the Venn diagram to determine whether taking Art and taking Biology are independent events.



5 The Venn diagram shows the number of students who take Chemistry and/or Biology in a class.
Use the Venn diagram to determine whether taking Chemistry and taking Biology are independent events.



6 The Venn diagram shows the number of students who take Chemistry and/or Physics in a class.
Use the Venn diagram to determine whether taking Chemistry and taking Physics are independent events.

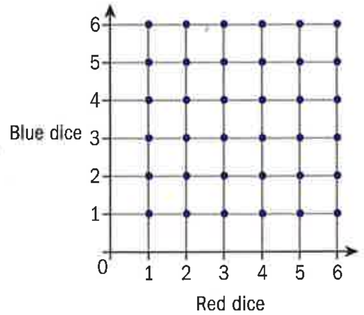


8.8 Sample space diagrams

A sample space diagram is a graphical way of showing the possible equally likely outcomes of an experiment rather than listing them.

One red dice and one blue dice are rolled together. Both dice are fair.

You can show all the possible outcomes on a grid.



There are 36 possible outcomes, $n(U) = 36$. You can use the sample space diagram to calculate probabilities.

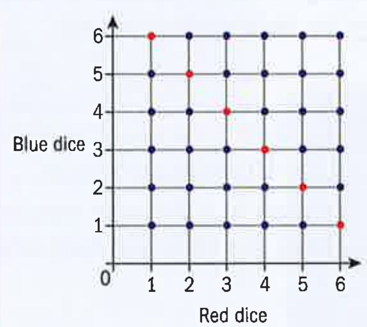
Example 15

A red and a blue dice are rolled together. Calculate the probability that:

a The total score is 7.
b The same number comes up on both dice.
c The difference between the scores is 1.
d The score on the red dice is less than the score on the blue dice.
e The total score is a prime number.

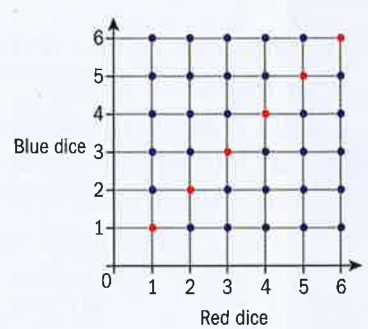
Answers

a P(the total score is 7) = $\frac{6}{36}$



Red dots show the scores that add to make 7.

b P(the same number comes up on both dice) = $\frac{6}{36}$



Red dots show all the identical number pairs.

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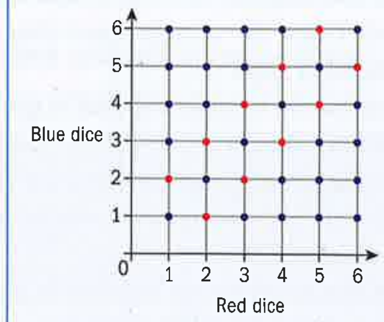
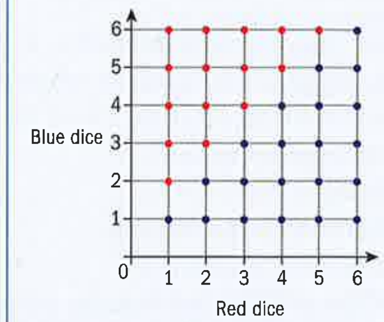
c P(the difference between the scores is 1) = $\frac{10}{36}$

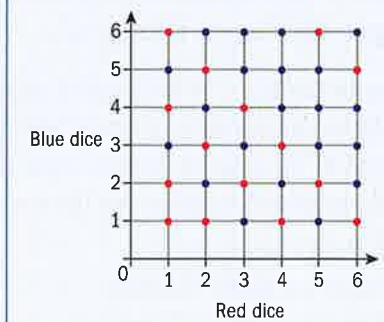


Red dots show the outcomes to include.

d P(the score on the red dice is less than the score on the blue dice) = $\frac{15}{36}$



e P(the total score is a prime number) = $\frac{15}{36}$



Exercise 8N

1 Draw the sample space diagram for this experiment: two tetrahedral dice, one blue and the other red, each numbered 1 to 4, are rolled. Find the probability that

a the number on the red dice is greater than the number on the blue dice
b the difference between the numbers on the dice is one
c the red dice shows an odd number and the blue dice shows an even number
d the sum of the numbers on the dice is prime.

2 A tetrahedral dice (numbered 1 to 4) and an ordinary six-sided dice are rolled. Draw the sample space diagram for this experiment. Find the probability that:

a the number on the tetrahedral dice is greater than the number on the ordinary dice
b the difference between the numbers on the dice is more than one
c the ordinary dice shows an odd number and the tetrahedral dice shows an even number
d the sum of the numbers on the dice is prime
e the two dice show the same number.

3 A box contains three cards numbered 1, 2, 3.
A second box contains four cards numbered 2, 3, 4, 5. A card is chosen at random from each box. Draw the sample space diagram for the experiment.
Find the probability that:

a the cards have the same number
b the largest number drawn is 3
c the sum of the two numbers on the cards is less than 7
d the product of the numbers on the cards is at least 8
e at least one even number is chosen.

4 Six cards, numbered 0, 1, 2, 3, 4, and 5, are placed in a bag. One card is drawn at random, its number noted, and then the card is replaced in the bag. A second card is then chosen. Draw the sample space diagram for the experiment.

Find the probability that:

a the cards have the same number
b the largest number drawn is prime
c the sum of the two numbers on the cards is less than 7
d the product of the numbers on the cards is at least 8
e at least one even number is chosen.

5 Six cards, numbered 0, 1, 2, 3, 4, and 5, are placed in a bag. One card is drawn at random and is **not** replaced. A second card is then chosen. Draw the sample space diagram for the experiment.

Find the probability that:

a the cards bear the same number
b the larger number drawn is prime
c the sum of the two numbers on the cards is less than 7
d the product of the numbers on the cards is at least 8
e at least one even number is chosen.

> Be careful: this is **not** the same sample space as for question 4.

8.9 Tree diagrams

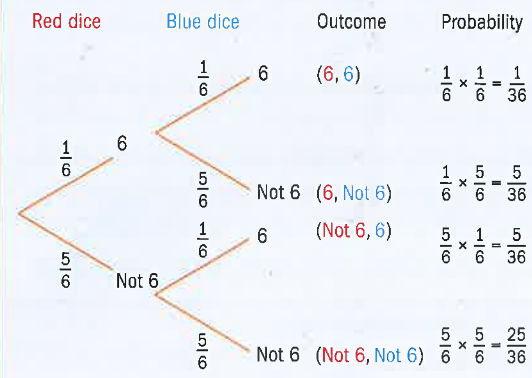
Tree diagrams are another way of representing and calculating probabilities.

Example 16

Two fair dice are rolled, one red and one blue.
Using a tree diagram, find the probability that:

a a double six is rolled **b** no sixes are rolled
c exactly one six is rolled **d** at least one six is rolled.

Answers



First, break the experiment into two simple experiments:
One: *roll the red dice and note if the score is a six or not*
Two: *then roll the blue dice and note if its score is a six or not*
Draw a tree diagram to show the outcomes.
Add probabilities to the branches.
For the red dice,
$P(6)=\frac{1}{6}$, $P(Not\ 6)=\frac{5}{6}$
For the blue dice,
$P(6)=\frac{1}{6}$, $P(Not\ 6)=\frac{5}{6}$

a P(double six) = P(6, 6)
$= \frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$

Rolling a six on each dice are independent events, so multiply the probabilities.

b P(no sixes) = P(Not 6, Not 6)
$= \frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$

Not rolling a six on each dice are independent events.

c P(exactly one six)
P(6, Not 6) = $\frac{5}{36}$
P(Not 6, 6) = $\frac{5}{36}$
P(exactly one six) = $\frac{5}{36}+\frac{5}{36}=\frac{10}{36}$

There are two ways this can happen:
(6, Not 6) **or** *(Not 6, 6)*

> Instead of P(6, Not 6) you could write P(6, 6′).

Add the probabilities.

d P(at least one six)
$= \frac{5}{36}+\frac{1}{36}+\frac{5}{36}=\frac{11}{36}$

P(at least one six) =
P(6, Not 6) + P(6, 6) + P(Not 6, 6)

> Notice that P(at least one six) = 1 – P (no sixes)

You can use tree diagrams for conditional probabilities too.

Example 17

For the experiment in Example 16, find the probability that, given that at least one six was rolled, then the red dice showed a six.

Answer

P(six on red dice | at least one six rolled)

$= \dfrac{\text{P(six on red dice and at least one six rolled)}}{\text{P(at least one six rolled)}}$

Use the definition of conditional probability.

$= \dfrac{\text{P}(6,\ 6)+\text{P}(6,\ \text{Not }6)}{\text{P}(6,\ 6)+\text{P}(6,\ \text{Not }6)+\text{P}(\text{Not }6,\ 6)}$

$= \dfrac{\left(\frac{1}{36}+\frac{5}{36}\right)}{\left(\frac{1}{36}+\frac{5}{36}+\frac{5}{36}\right)} = \dfrac{6}{11} = 0.545$

Read the probabilities from the final column of the tree diagram in Example 16.
Use your GDC to evaluate this – change to a fraction or give the answer correct to 3 significant figures.

Exercise 8O

1 A bag contains 6 red and 5 blue counters. One is chosen at random. Its color is noted and it is put back into the bag. Then a second counter is chosen.
   a Find the probability that exactly one red counter is chosen.
   b Find the probability that at least one blue counter is chosen.
   c Find the probability that one of each color is chosen.
   d If one of each color was chosen, what is the probability that the blue was chosen on the second pick?
   e If at least one blue was chosen, what is the probability that the blue was chosen on the first pick?

These are conditional probabilities

2 A 5-sided dice is numbered 1, 2, 3, 4, 5. It is rolled twice.
   a Find the probability that exactly one prime number is rolled.
   b Find the probability that at least one prime number is rolled.
   c Given that at least one prime number was rolled, find the probability that two primes are rolled.
   d Given that at least one prime was rolled, find the probability that a prime was rolled on the first attempt.

3 To get to work I must go through two sets of traffic lights – first at Sixth Avenue and then at Dover Road. I get delayed at Sixth Avenue with probability $\frac{7}{10}$ and at Dover Road with probability $\frac{3}{5}$.
   Draw a tree diagram to show the possible delays on my journey to work.
   a Find the probability that I get delayed only once.
   b Find the probability that I do not get delayed at all.
   c Given that I get delayed exactly once, what is the probability that it was at Sixth Avenue?
   d Given that I get delayed, what is the probability that it was at Sixth Avenue?

4 On a journey to school a teacher has to pass through two sets of traffic lights (A and B). The probabilities that he will be stopped at these are $\frac{2}{7}$ and $\frac{1}{3}$ respectively. The corresponding delays are one minute and three minutes. Without these delays his journey takes 30 minutes.
   Draw a tree diagram to illustrate the possible delays.
   a Find the probability that the journey takes no more than 30 minutes.
   b Find the probability that the teacher has only one delay.
   c Given that the teacher gets delayed, what is the probability that it happened at A?
   d On a particular morning, the teacher has only 32 minutes to reach school on time. Find the probability that he will be late.

5 The probability that it will rain today is 0.2. If it rains today, the probability that it will rain tomorrow is 0.15. If it is fine today then the probability that it will be fine tomorrow is 0.9.
   a Find the probability that at least one day will be fine.
   b Given that at least one day is fine, what was the probability that it was today?
   c Given that at least one day is fine, what is the probability that both are fine?

Extension material on CD: *Worksheet 8 - A game*

'Without replacement' problems

A classic probability problem involves picking a ball from a bag, noting its color and *not* replacing it, then picking another ball.

This means that the probability of choosing the next ball from the bag will be different from the probability of choosing the first.

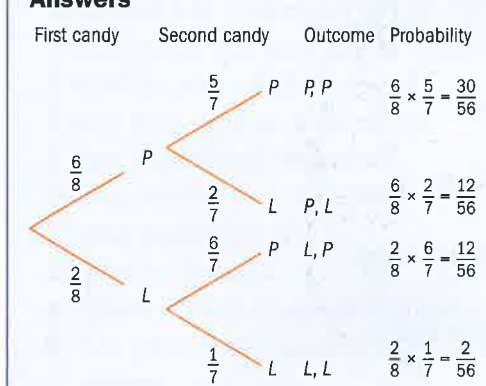
You can use a tree diagram for this type of problem.

Example 18

There are 6 peppermints (P) and 2 liquorice (L) candies in a bag. A candy is picked **and not replaced in the bag**. Then a second candy is picked.
a Find the probability that one of each type is chosen.
b **Given that** one of each type was chosen, find the probability that the first one chosen was a peppermint.

This 'without replacement' problem uses candies instead of balls.

Answers



Draw a tree diagram. Break the experiment into
1 *pick the first candy*
2 *pick the second candy*
On the second pick, only 7 candies remain.
If a peppermint is chosen first time, only 5 peppermints remain.

Continued on next page

a P(one of each type) = P(P, L) + P(L, P)

$$= \frac{12}{56} + \frac{12}{56} = \frac{24}{56} = \frac{3}{7}$$

The outcomes corresponding to 'one of each type is chosen' are P, L and L, P.

b $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Let A be the event that the first candy chosen is peppermint, and B be the event that one of each type is picked. Then we need P(A | B).

$P(A \cap B) = P(P, L) = \frac{12}{56} = \frac{3}{14}$ and $P(B) = \frac{3}{7}$

*P(B) is the probability from part **a**.*

So, $P(A \mid B) = \frac{\frac{3}{14}}{\frac{3}{7}} = \frac{1}{2}$

Exercise 8P

1 A bag contains 6 red and 5 blue counters. One is chosen at random. Its color is noted, **and it is not put back in the bag**. Then a second counter is chosen.

- **a** Find the probability that exactly one red counter is chosen.
- **b** Find the probability that at least one blue counter is chosen.
- **c** Find the probability that one of each color is chosen.
- **d** If one of each color was chosen, what is the probability that the blue was chosen on the second pick?
- **e** If at least one blue was chosen, what is the probability that the blue was chosen on the first pick?

2 A bag contains 5 faulty and 7 working pens. A boy and then a girl each need to take a pen.

- **a** What is the probability that both take faulty pens?
- **b** Find the probability that at least one takes a faulty pen.
- **c** Given that exactly one faulty pen was taken, what is the probability that the girl took it?

3 To get to the school I can take one of two routes, via Kent Ridge or via Sunny Vale. I take the Kent Ridge route on average 3 times in a 5 day week. If I take this route the probability that I am delayed is 0.25. If I take the Sunny Vale route the probability that I am delayed is 0.5.
Draw a tree diagram that shows my journey to school.

- **a** Find the probability that I get delayed.
- **b** Find the probability that I go by Sunny Vale and I do not get delayed.
- **c** Given that I am delayed, what is the probability that I went via Kent Ridge?
- **d** Given that I am not delayed, what is the probability that I went via Sunny Vale?

4 The probability that it will snow today is 0.9. If it does snow today then the probability it will snow tomorrow is 0.7. However, if it does not snow today then the probability that it will snow tomorrow is 0.6.
Draw a tree diagram which shows the possible weather conditions for the two days.

- **a** Find the probability of two snowy days.
- **b** Find the probability of exactly one snowy day.
- **c** Given that there is exactly one snowy day, what is the probability that it is today?
- **d** Given that there is at least one snowy day, what is the probability that it is today?

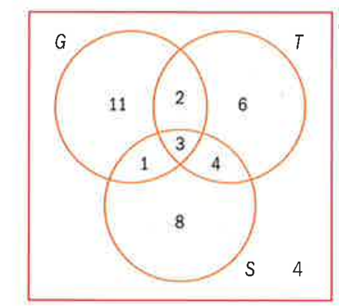
5 There are eight identical discs in a bag, five of which are black and the other three are red. The random experiment is: choose a disc at random from the bag, do not return the disc to the bag, then choose a **second** disc from the bag. Find the probability that the second disc chosen is red.

Review exercise

Paper 1 style questions

EXAM-STYLE QUESTIONS

1 The activities offered by a school are golf (G), tennis (T), and swimming (S). The Venn diagram shows the numbers of people involved in each activity.



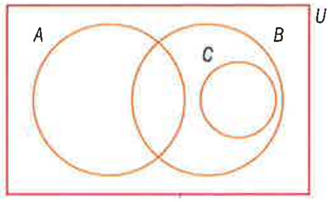
- **a** Write down the number of people who
  - **i** play tennis only
  - **ii** play both tennis and golf
  - **iii** play at least two sports
  - **iv** do not play tennis.
- **b** Copy the diagram and shade the part of the Venn diagram that represents the set $G' \cap S$.

2 A group of 40 children are surveyed to find out which of the three sports volleyball (A), basketball (B) or cricket (C) they play. The results are as follows:
7 children do not play any of these sports
2 children play all three sports
5 play volleyball and basketball
3 play cricket and basketball
10 play cricket and volleyball
15 play basketball
20 play volleyball.

- **a** Draw a Venn diagram to illustrate the relationship between the three sports played.
- **b** On your Venn diagram indicate the number of children that belong to each **disjoint** region.
- **c** Find the number of children that play cricket only.

EXAM-STYLE QUESTIONS

3 The following Venn diagram shows the sets U, A, B and C.



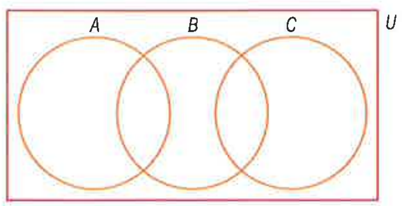
State whether the following statements are true or false for the information illustrated in the Venn diagram.

a $A \cup C = \varnothing$ **b** $C \subset (C \cup B)$

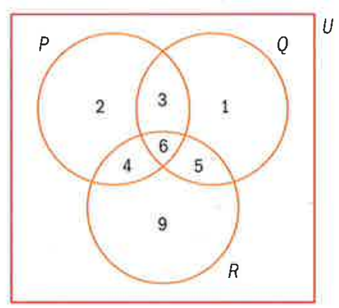
c $C \cap (A \cup B) = \varnothing$ **d** $C \subset A'$

e $C \cap B = C$ **f** $(A \cup B)' = A' \cap B'$

4 **a** Copy this diagram and shade $A \cup (B \cap C')$.



b In the Venn diagram on the right, the number of elements in each region is given. Find $n((P \cup Q) \cap R)$.



c U is the set of positive integers, $\mathbb{Z}^+$.
E is the set of odd numbers.
M is the set of multiples of 5.

i List the first four elements of the set M.

ii List the first three elements of the set $E' \cap M$.

5 $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers.

a Write down an element of $\mathbb{Z}$. **b** Write down an element of $\mathbb{R} \cap \mathbb{Z}'$.

c Write down an element of $\mathbb{Q}$. **d** Write down an element of $\mathbb{Q} \cup \mathbb{Z}'$.

e Write down an element of $\mathbb{Q}'$. **f** Write down an element of $\mathbb{Q}' \cap \mathbb{Z}'$.

6 The table below shows the number of left- and right-handed tennis players in a sample of 60 males and females.

	Left-handed	Right-handed	Total
Male	8	32	40
Female	4	16	20
Total	12	48	60

If a tennis player was selected at random from the group, find the probability that the player is

a female and left-handed **b** male or right-handed

c right-handed, given that the player selected is female.

EXAM-STYLE QUESTIONS

7 A bag contains 3 red, 4 yellow and 8 green sweets.
You Jin chooses one sweet out of the bag at random and eats it.
She then takes out a second sweet.

a Write down the probability that the first sweet chosen was red.

b Given that the first sweet was not red, find the probability that the second sweet is red.

c Find the probability that both the first and second sweets chosen were yellow.

8 Ernest rolls two cubical dice. One of the dice has three red faces and three black faces. The other dice has the faces numbered from 1 to 6. By means of a sample space diagram, or otherwise, find

a the number of different possible combinations he can roll

b the probability that he will roll a black and an even number

c the probability that he will roll a number more than 4.

9 The table below shows the number of words in the extended essays of an IB class.

Number of words	$3100 \le w < 3400$	$3400 \le w < 3700$	$3700 \le w < 4000$	$4000 \le w < 4300$
Frequency	7	20	18	5

a Write down the modal group.

b Write down the probability that a student chosen at random writes an extended essay with a number of words in the range: $4000 \le w < 4300$.

The maximum word count for an extended essay is 4000 words.
Find the probability that a student chosen at random:

c does not write an extended essay that is on or over the word count

d writes an extended essay with the number of words in the range $3400 \le w < 3700$ given that it is not on or over the word count.

Paper 2 style questions

EXAM-STYLE QUESTIONS

1 Let $U = \{x \mid 8 \le x < 13, x \in \mathbb{N}\}$.
P, Q and R are the subsets of U such that
$P = \{\text{multiples of four}\}$
$Q = \{\text{factors of 24}\}$
$R = \{\text{square numbers}\}$

a List the elements of U.

b **i** Draw a Venn diagram to show the relationship between sets P, Q and R.

ii Write the elements of U in the appropriate places on the Venn diagram.

c List the elements of:

i $P \cap R$ **ii** $P' \cap Q \cap R$

d Describe in words the set $P \cup Q$.

e Shade the region on your Venn diagram that represents $(P \cup R) \cap Q'$.

EXAM-STYLE QUESTIONS

2 In a club with 70 members, everyone attends either on Tuesday for Drama (D) or on Thursday for Sports (S), or on both days for Drama and Sports.
One week it is found that 48 members attend for Drama, 44 members attend for Sports, and x members attend for both Drama and Sports.

a **i** Draw and **label fully** a Venn diagram to illustrate this information.
ii Find the number of members who attend for both Drama and Sports.
iii Describe, in words, the set represented by $(D \cap S)'$.
iv What is the probability that a member selected at random attends for Drama only or Sports only?

The club has 40 female members, 10 of whom attend for both Drama and Sports.

b What is the probability that a member of the club selected at random
i is female and attends for Drama only or Sports only?
ii is male and attends for both Drama and Sports?

3 On a particular day 50 children are asked to make a note of what they drank that day.
They are given three choices: water (P), fruit juice (Q) or coffee (R).
2 children drank only water.
4 children drank only coffee.
12 children drank only fruit juice.
3 children drank all three.
4 children drank water and coffee only.
5 children drank coffee and fruit juice only.
15 children drank water and fruit juice only.

a Represent the above information on a Venn diagram.
b How many children drank none of the above?
c A child is chosen at random. Find the probability that the child drank
i fruit juice
ii water or fruit juice but not coffee
iii no fruit juice, given that the child did drink water.
d Two children are chosen at random. Find the probability that both children drank all three choices.

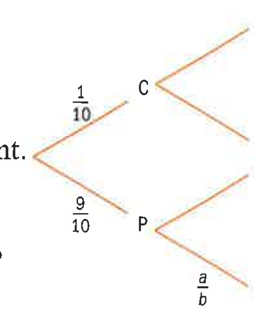
EXAM-STYLE QUESTIONS

4 The sets P, Q, and R are subsets of U. They are defined as follows:
U = {positive integers less than 13}
P = {prime numbers}
Q = {factors of 18}
R = {multiples of 3}

a List the elements (if any) of
i P **ii** Q **iii** R **iv** $P \cap Q \cap R$
b **i** Draw a Venn diagram showing the relationship between the sets U, P, Q and R.
ii Write the elements of sets U, P, Q and R in the appropriate places on the Venn diagram.
c From the Venn diagram, list the elements of
i $P \cup (Q \cap R)$ **ii** $(P \cup R)'$ **iii** $(P \cup Q)' \cap R'$
d Find the probability that a number chosen at random from the universal set U will be
i a prime number
ii a prime number, but **not** a factor of 18
iii a factor of 18 or a multiple of 3, but **not** a prime number
iv a prime number, given that it is a factor of 18.

5 There are two biscuit tins on a shelf. The **red** tin contains four chocolate biscuits and six plain biscuits. The **blue** tin contains one chocolate biscuit and nine plain biscuits.
A child reaches into the **red** tin and randomly selects a biscuit. The child returns that biscuit to the tin, shakes the tin, and then selects another biscuit.

a Draw a tree diagram that shows the possible outcomes. Place the appropriate probability on each branch of the tree diagram.
b Find the probability that
i both biscuits chosen are chocolate
ii one of the biscuits is plain and the other biscuit is chocolate.
c A second child chooses a biscuit from the **blue** tin. The child eats the biscuit and chooses another one from the **blue** tin. The tree diagram on the right represents the possible outcomes for this event.



i Write down the value of a and of b.
ii Find the probability that both biscuits are chocolate.
iii What is the probability that *at least* one of the biscuits is plain?
d Suppose that, before the two children arrived, their brother randomly selected one of the biscuit tins and took out one biscuit. Calculate the probability that this biscuit was chocolate.

EXAM-STYLE QUESTION

6 The data in the table below refer to a sample of 60 randomly chosen plants.

Growth rate	Classification by environment			
	desert	temperate	waterlogged	total
high	4	7	13	24
low	9	11	16	36
total	13	18	29	60

a **i** Find the probability of a plant being in a desert environment.
ii Find the probability of a plant having a low growth rate and being in a waterlogged environment.
iii Find the probability of a plant not being in a temperate environment.

b A plant is chosen at random from the above group.
Find the probability that the chosen plant has
i a high growth rate or is in a waterlogged environment, but not both
ii a low growth rate, given that it is in a desert environment.

c The 60 plants in the above group were then classified according to leaf type. It was found that 15 of the plants had type A leaves, 36 had type B leaves and 9 had type C leaves.
Two plants were randomly selected from this group. Find the probability that
i both plants had type B leaves
ii neither of the plants had type A leaves.

CHAPTER 8 SUMMARY

Basic set theory

- A **set** is simply a collection of objects. The objects are called the **elements** of the set.
- The number of elements in a set A is denoted as $n(A)$.

Venn diagrams

- The **universal set** (denoted U), must be stated to make a set well defined.
- If every element in a given set, M, is also an element of another set, N, then M is a **subset** of N, denoted $M \subseteq N$
- A **proper subset** of a given set is one that is **not identical** to the original set.
  If M is a proper subset of N (denoted $M \subset N$) then
  1 **every** element of M also lies in N and
  2 there are some elements in N that do not lie in M.
- The empty set $\varnothing$ is a subset of every set.
- Every set is a subset of itself.
- The **intersection** of set M and set N (denoted $M \cap N$) is the set of all elements that are in **both** M **and** N.

Continued on next page

- The **union** of set M and set N (denoted $M \cup N$) is the set of all elements that are in **either** M **or** N or **both**.
- The **complement** of set M, denoted as M', is the set of all the elements in the universal set that **do not** lie in M.
- The complement of the universal set, U', is the empty set, $\varnothing$.

Basic probability theory

- If all of the equally likely possible outcomes of a random experiment can be listed as U, the universal set, and an event A is defined and represented by a set A, then:

$$P(A) = \frac{n(A)}{n(U)}$$

There are three consequences of this law:

1 $P(U) = \frac{n(U)}{n(U)} = 1$ (the probability of a **certain** event is 1)

2 $P(\varnothing) = \frac{n(\varnothing)}{n(U)} = 0$ (the probability of an **impossible** event is 0)

3 $0 \le P(A) \le 1$ (the probability of any event **always** lies between 0 and 1)

- For complementary events, $P(A') = 1 - P(A)$
- For combined events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional probability

- The conditional probability that A occurs given that B has occurred is written as $P(A \mid B)$ and is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Two special cases: mutually exclusive and independent events

- Events A and B are mutually exclusive if and only if $P(A \cap B) = 0$.
- A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$.

It's not fair!

A certain degree of uncertainty

In mathematics we can be certain that we have the right answer – certain about what we know. Probability deals with situations that are uncertain.

- How can a theory involving uncertainty be quantified?
- Mathematics is an exact science.
  So how can probability considered to be mathematics?

Gambling dice

Probability theory began to develop in France in the 17th century, as mathematicians Blaise Pascal, Antoine Gombaud (also known as the Chevalier de Méré), and Pierre de Fermat discussed how to bet in a dice game.

The Chevalier de Méré asked: Which is more likely – rolling one 'six' in four throws of one dice, or rolling a 'double six' in 24 throws with two dice?

- Which option seems intuitively correct?
- Can you always trust your intuition?

At the time it was thought that the better option was to bet on the double six because so many more throws are allowed. The mathematicians analyzed the probabilities and proved rolling a six in four rolls of one dice is more likely.

- Can you prove this?

A moral question

While French mathematicians were developing probability theory, the English view was that 'as gambling is immoral, probability must not be studied'.

- How valid is this view?
- Discuss the statement 'Mathematics transcends morality; it cannot be immoral.'
- Why do people gamble? Why do people do things that they know will be self-destructive?

Fair games

In a game, X and Y toss a coin. If the coin lands heads then X wins. If the coin lands tails then Y wins.

- Is this a fair game?
- What do we mean by a fair game?

- Are these games fair?

1. Two people, X and Y, toss a coin. If the coin lands heads then X pays Y \$5. If the coin lands tails then Y pays X \$1.
2. X and Y roll a dice. If the die shows 'one' then X pays Y \$1. If the dice does not show 'one' then Y pays X \$1.
3. X and Y roll a dice. If the dice shows 'one' then X pays Y \$5. If the dice does not show 'one' then Y pays X \$1.

All the coins and dice in this section are 'unbiased'.

"Gambling is a tax on the mathematically ignorant."

- Do you agree or disagree?

Mathematical fairness

The mathematical definition of a fair game is a game in which the expected gain for each player is zero.

In a casino, **no game is fair**, from a mathematical standpoint. The casino has to make enough money to pay for the building, the electricity, its employees and taxes, as well as making a profit.

Who wins in roulette?

In roulette you can bet on single numbers, groups of numbers, rows, columns, whether the number that turns up is odd or even, red or black, or 'passé' or 'manqué'.

Passé means that the number is in the range 19 to 36 inclusive; manqué means that the number is in the range 1 to 18 inclusive.

Player X places a bet of \$1 on manqué; if the ball lands in the range 1 to 18 inclusive, the casino pays X \$1 (and X retains the original \$1), if the ball lands outside this range, X loses the \$1 to the casino. Is the game fair?

Intuitively you might think that the ball is equally likely to land on 'passé' or 'manqué', so the game is fair. But look at the picture of the roulette wheel: there are two outcomes colored green, labelled 0 and 00.

So there are 36 + 2 = 38 possible equally likely outcomes.

X would expect to lose $\$\frac{2}{38}$ per play (paying out \$1 on 20 out of 38 plays and gaining \$1 on 18 out of 38 plays)

The game is not fair. The $\$\frac{2}{38}$ is called the '**house edge**' and is the casino's profit margin.

As a percentage this is $\frac{2}{38} \times 100 = 5.26\%$.

The house edge ensures that the casino can be run as any other business – to make money.

◀ A roulette table.