11 Modelling periodic phenomena: trigonometric functions

Many natural, and human-made, phenomena repeat themselves regularly over time. Tides rise and fall, the sun rises and sets, the moon changes its appearance, a clock's pendulum swings, crystals vibrate when an electrical current is applied. All of these things happen in a predictable fashion.

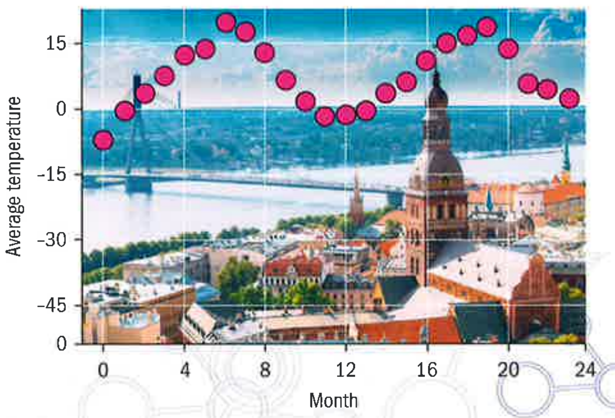
Concepts

- Modelling
- Representation

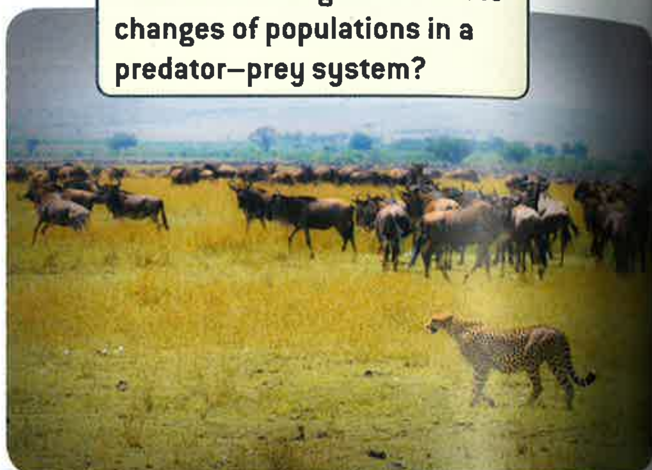
Microconcepts

- Periodic functions
- Sinusoidal functions
- Features of sinusoidal functions: amplitude, period and the principal axis

How can you model features of weather systems that change with time and geographical location?



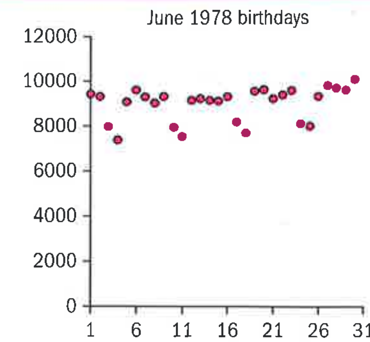
How can a biologist model the changes of populations in a predator–prey system?



Given the times of high and low tides on a given day, how could you decide when there would be enough water in a harbour to enter with a boat?

If there is a full moon tonight, how can we predict when the next new moon will occur?

The graph on the previous page shows the average monthly temperature for Riga, Latvia, compiled in 2014 and 2015. The graph on the right shows the number of babies born on each day of June 1978 in the USA.



- What patterns can you see in these graphs?
- To what extent is your perception of these patterns dependent on the viewing window that has been chosen for you?
- How often do any repeating patterns repeat?
- What reasons could there be for any repeating patterns and the shape of the functions?
- If you developed models for these data sets, what would you expect to happen outside the domain of your functions?

Developing inquiry skills

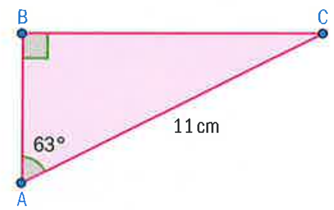
Which other climate phenomena could have a repeating pattern? What questions could you ask for your country? What data would you need?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Calculate the length of the shortest side in triangle ABC.



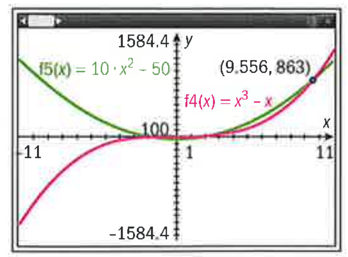
Since $\hat{C} = 27°$, AB will be the shortest side.

$AB = 11\sin 27° = 4.99$ cm to 3sf

2 Use technology to solve the equation $x^3 - x = 10x^2 - 200$ where $x > 5$.

a Graph each side of the equation as a function.

b Find the coordinates of the point(s) required.



c Write the solution correct to three significant figures: $x = 9.56$.

Skills check

Click here for help with this skills check

1 Calculate all the angles and sides in the triangle PQR where $\hat{R} = 90°$, PQ = 13 m and QR = 5 m.

2 Karolina stands 73 m from the base of an apartment block and measures the angle of elevation to her apartment as 43°. What is the height of Karolina's apartment?

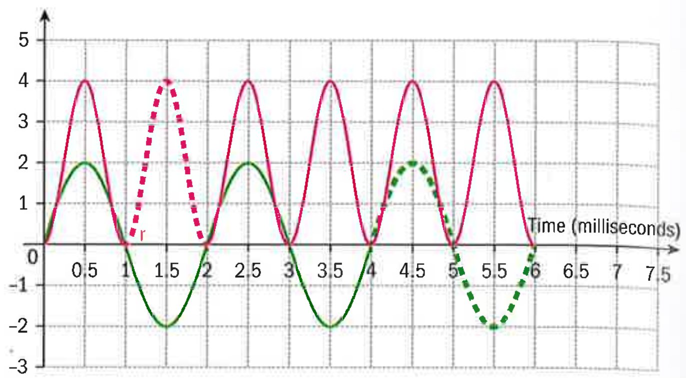
3 Use technology to solve the equations:

a $2^x - 17 = -10x - x^2$ where $x \le 0$

b $19 + \dfrac{3}{t-11} = t^4$ where $t > 0$.

11.1 An introduction to periodic functions

Periodic functions help you to model a set of data in the special case where there is a repeating pattern of **identical** y-values. For example, the periodic functions shown to the right come from the study of electricity. The values of the pink function quantify power (watts) and the values of the green quantify current (amps). The independent variable is time in milliseconds.



Notice that the values of the functions repeat regularly. The entire graph is a series of congruent shapes, which form a continuous curve. The **period** of the function is the length of the **shortest** interval for which the function values when repeated give the **entire** graph. Hence the period of the power function is 1 millisecond and of the current 2 milliseconds, as shown in the diagram by the dashed lines. This means that the relationship between time and power repeats itself each millisecond, and the relationship between time and current repeats itself every 2 milliseconds.

TOK

Is there always a trade-off between accuracy and simplicity?

In the following investigation you will explore a physical structure that gives rise to a periodic function.

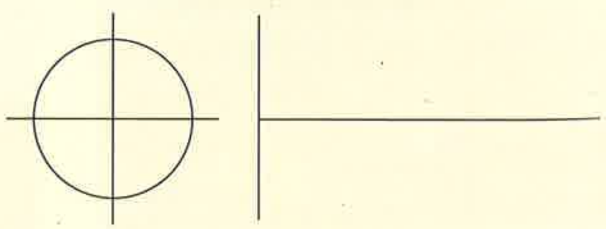
For example, a new Ferris wheel attraction is planned for Alphapark to carry passengers in pods. Safety laws require that the operators of the wheel must know the exact position of each pod: the extent to which it is above or below the x-axis and the extent to which it is to the right or left of the y-axis.

You will model this context physically, and then with mathematics.

Investigation 1

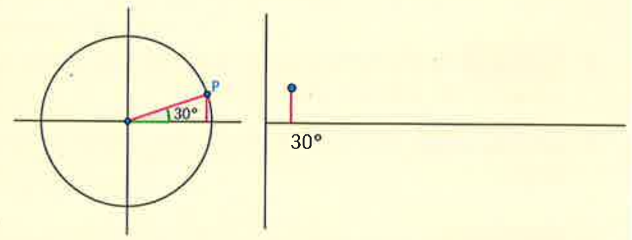
Part A: Construction of a spaghetti curve

You will need: spaghetti (or drinking straws), protractor, string, ball of colourful wool, scissors, glue and poster paper. Work in groups of two or three.



1. At one end of the paper, construct a circle on a set of axes with a radius of one spaghetti length.

2. Use a protractor to measure and mark every 30° around the circle.
3. To the right of the circle draw a graph with an x-axis that is about 6.5 spaghetti lengths long and a y-axis 2 spaghetti lengths tall.
4. Place the string around the circle with the end at 0°. Mark the 30° marks onto the string. Then put the string on the x-axis of the right-hand set of axes and transfer the marks labelling every 30°.
5. Place a piece of spaghetti from the origin to the 30° mark on the circle. Take another piece of spaghetti and measure the vertical distance from the pod (P) to the x-axis. Cut the piece of spaghetti where it crosses the x-axis and glue this piece at 30° on the x-axis of the graph as shown. (Diagram not to scale.)

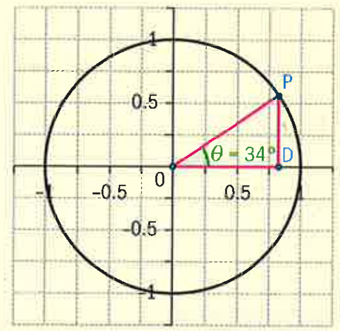


6. Now repeat for 30°, 60° and so on, until you have gone completely around the circle to 360°. If the pod is above the x-axis of the circle, then place the spaghetti above the x-axis of the graph; otherwise place it below the x-axis.
7. Take the wool and glue it to the graph on your poster from zero degrees, along the top of the spaghetti pieces to form a smooth curve.

Part B: Reflection on the properties of your spaghetti curve

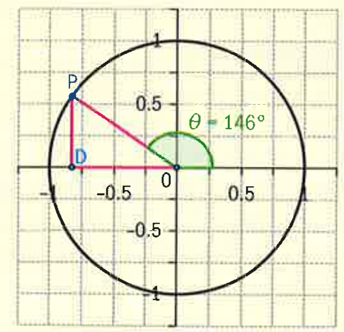
1. **Factual** Your spaghetti curve is a periodic function. Write down its period.
2. **Factual** What do the y-values of your spaghetti curve represent in the context of the Ferris wheel?
3. **Factual** What are the maximum and minimum values of your curve and at what values of x do they occur? At what values of x is the value of your curve zero?
4. Graph the function $f(x) = \sin x$ on your GDC. Compare and contrast your spaghetti curve with the curve shown on your GDC.

Represent the angle shown in the **right-angled triangle** formed by the pod, the origin and the intersection with the x-axis as θ and let the radius of the circle be 1.



5. **Factual** Find the height of the pod above the x-axis in terms of θ.
6. **Factual** For what values of θ is your answer to **5** valid?

Now imagine the pod moving over the y-axis so that θ is greater than 90° but less than 180°.



7. **Factual** Compare and contrast the position of the pod when $\theta = 146°$ and when $\theta = 34°$.

What does this suggest about $\sin 34°$ and $\sin 146°$?

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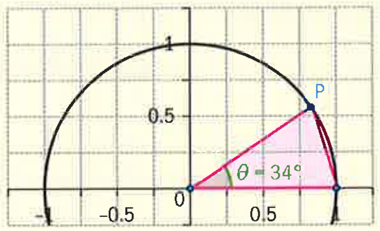
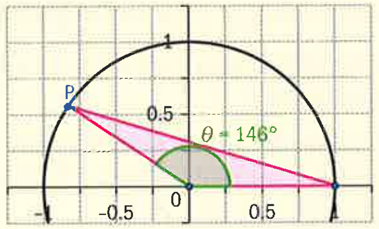
Functions

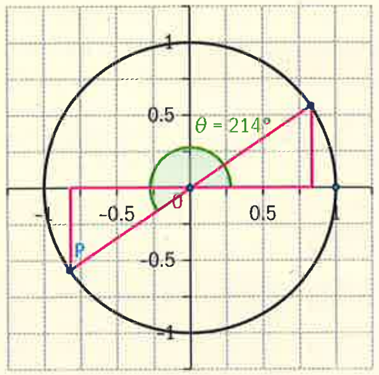




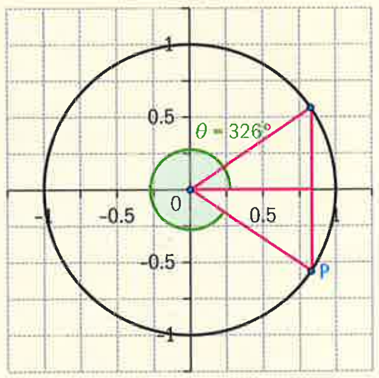
8 **Factual** Apply the formula for the area of a triangle to the two triangles shown here. What does this prove for $\sin 34°$ and $\sin 146°$?

9 **Conceptual** How can you define the sine of an obtuse angle in terms of the sine of an acute angle?

10 **Factual** When the pod is at position P, use geometry and trigonometry to find how far it is below the x-axis.



11 **Factual** When the pod is at position P, use geometry and trigonometry to find how far it is below the x-axis.



12 **Conceptual** How can you interpret the negative values of $\sin \theta$ when θ is greater than 180° but less than 360° in the context of the Ferris wheel?

13 **Conceptual** How can you extend the concept of the trigonometric ratio $\sin \theta$ in a right-angled triangle to give all values of the periodic function $f(x) = \sin x$?

You have discovered the properties of the periodic function $f(x) = \sin x$. There are five other trigonometric ratios that are also periodic functions. In the next investigation you will explore one of these functions.

Investigation 2

Part A: Construction of a new spaghetti curve

You will make a new periodic graph. Follow the same steps as in Investigation 1 part A, except this time measure the **horizontal** distance from the pod (P) to the y-axis. If the pod is to the right of the y-axis, assign a positive sign, if to the left of the y-axis, assign a negative sign.

Part B: Reflection on the properties of your new spaghetti curve

1 **Factual** Compare and contrast your function with that of $f(x) = \sin x$.

2 **Conceptual** Why should your new periodic function be periodic?

Now use your GDC to graph $g(x) = \cos x$ and $h(x) = \tan x$. Compare and contrast the graphs of these functions and your new spaghetti curve.

3 **Factual** Which of $g(x) = \cos x$ and $h(x) = \tan x$ model your new spaghetti curve?

4 **Factual** What are the maximum and minimum values of your curve and at what values of x do they occur? At what values of x is the value zero?

5 **Conceptual** Why is your new spaghetti curve modelled so well by one of these functions but not the other?

6 **Conceptual** How can you define the cosine of an obtuse angle in terms of the cosine of an acute angle?

Definition: $f(x) = \sin x$ and $g(x) = \cos x$ are examples of **sinusoidal functions**.

You can use technology to solve simple trigonometric equations in a given domain and hence solve problems.

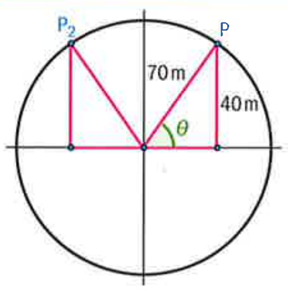
Example 1

Given that the radius of the Ferris wheel planned for Alphapark is 70 m and that θ represents the angle measured anticlockwise from the x-axis, find all the angles for which the pod P is:

a a distance of 40 m vertically above the x-axis

b a distance of 20 m horizontally from the y-axis.

a



$\sin \theta = \dfrac{40}{70}$ can be graphed on the domain $0° \le \theta \le 360°$:

Sketching a labelled diagram is a good problem-solving strategy.

The diagram shows that there are two possible positions of the pod that are 40 m above the x-axis.

The problem involves the vertical distance, hence sin is the right trigonometrical function to apply.

This fact is consistent with the given quantities in the right-angled triangle sketched: the hypotenuse and the side opposite to the angle.

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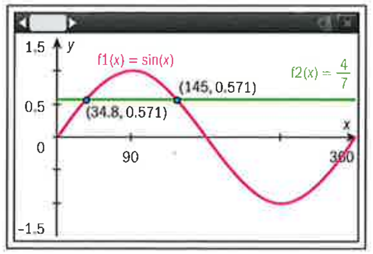
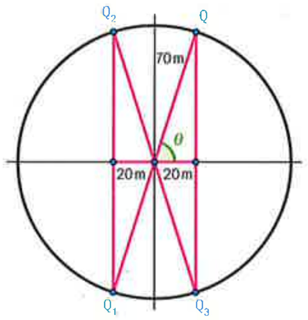
Functions

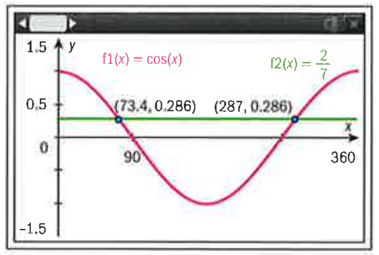


There are two values θ for which the pod is 40 m above the x-axis: 34.8° and 145° correct to three significant figures.

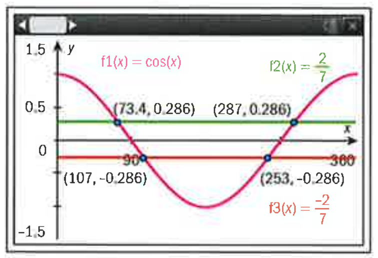
b



$\cos\theta = \frac{20}{70}$ can be graphed on the domain $0° \le \theta \le 360°$:



There are two values of θ for which the pod is 20 m to the right of the y-axis: and 73.4° and 287° correct to three significant figures.



Altogether, there are four values of θ for which the pod is 20 m from the y-axis: 73.4°, 107°, 253° and 287° correct to three significant figures.

Make sure the viewing window shows the domain and the range of the function so that you can identify all solutions.

It is essential that you ensure your GDC is in degree mode at all times for both calculations **and** for graphing.

Find the intersections of the sine curve with the line $y = 4/7$ in order to find the solutions.

Reflect that this solution is consistent with the diagram sketched at the start. The GDC rounded the angles to 3 sf, however, if the angles were given to 1 dp, the values would be 34.8° and 145.2°, which add to 180° as required by the diagram.

A sketch shows there are four solutions to be found.

The problem involves the horizontal distance, hence cos is the right trigonometrical function to apply.

This fact is consistent with the given quantities in the right-angled triangle sketched: the hypotenuse and the side adjacent to the angle.

Reflecting on the context of the problem helps you understand that more solutions can be found from the graph. When the pod is to the left of the y-axis, the value of the cosine function is negative.

Only now is the solution consistent with the context of the problem.

Exercise 11A

1 Given that the radius of the Ferris wheel planned for Alphapark is 70 m and that θ represents the angle measured anti-clockwise from the y-axis, find all the angles for which the pod P is:

  a a distance of 30 m from the x-axis

  b a distance of 20 m to the right of the y-axis.

2 **a** Solve the following equations.

  i $\sin\theta_1 = \cos\theta_1$ for $0° \le \theta_1 \le 360°$

  ii $\sin\theta_2 = 0.8$ for $0° \le \theta_2 \le 360°$

  iii $\cos\theta_3 = -0.1$ for $0° \le \theta_3 \le 360°$

  b Hence find the coordinates of the pod on the Alphapark Ferris wheel for each value of θ_1, θ_2 and θ_3 found in part **a**.

3 Jakub models the average monthly temperature T degrees Celsius in Warsaw with the function $T = -11\cos(30t) + 7.5$ on the domain $0 \le t \le 11$. t is time measured in months, with $T(0)$ representing the average temperature in January.

  a Predict the average temperature in May.

  b Predict when the average temperature would be zero.

4 Zuzanna models the depth D metres of sea water in a harbour h hours after midnight with the function $D = 1.8\sin(30t) + 12.3$.

  a Predict the depth of water in the harbour at 5.30am.

  b Predict when the depth of water in the harbour will be 10.9 m.

Functions

Developing inquiry skills

Return to the opening problem. Do the functions appear to be periodic? If so, state their period.

Identify the type of function you could use to model this data.

11.2 An infinity of sinusoidal functions

In the previous section, we dealt with one Ferris wheel, and used a radius of 70 m to represent it. How would the graph change if the Ferris wheel were twice as tall?

When you alter the parameters k, r and c of the function $f(x) = ke^{rx} + c$, an infinity of different exponential models are created. Similarly, when you alter the parameters a, b and d of the sinusoidal functions $f(x) = a\sin(bx) + d$ and $g(x) = a\cos(bx) + d$ you create an infinite number of sinusoidal models.

In this section you will explore how to model this and other more general sinusoidal phenomena.

TOK

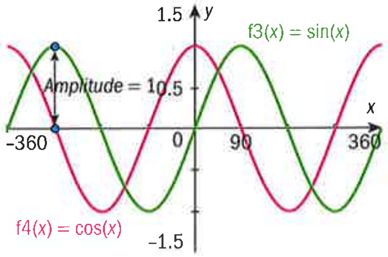
Solving an equation has given you an answer in mathematics, but how can an equation have an infinite number of solutions?

Definition: In a sinusoidal function with range $\min \le y \le \max$, the **principal axis** is the line $y = \frac{\min + \max}{2}$, the horizontal line located exactly in the middle of the range of the function, and the **amplitude** is the vertical distance from this line to any maximum or minimum points of the function.

For example,

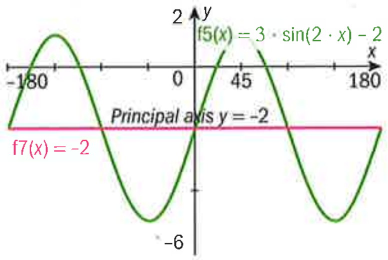
$f(x) = \sin x$ and $g(x) = \cos x$ both have:

- range $-1 \le y \le 1$
- domain $x \in \mathbb{R}$
- period 360°
- principal axis $y = 0$
- amplitude 1.



The function $h(x) = 3\sin(2x) - 2$ has:

- range $-5 \le y \le 1$
- domain $x \in \mathbb{R}$
- period 180°
- principal axis $y = \frac{-5+1}{2} = -2$
- amplitude 3.

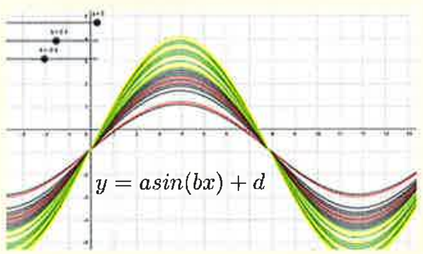


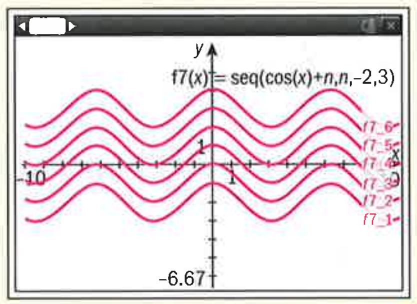
Investigation 3

1 Use technology to graph each set of functions and fill in the table:

	$y=2\sin(x)$	$y=2\sin(3x)$	$y=2\sin(3x)+1$
Amplitude			
Period			
Principal axis			
	$y=0.8\cos(x)$	$y=0.8\cos\left(\frac{x}{2}\right)$	$y=0.8\cos\left(\frac{x}{2}\right)-3$
Amplitude			
Period			
Principal axis			

Explore: Use dynamic geometry software or your GDC to create images like these and widen your exploration to include positive and negative values of a, b and d.





TOK

Why are there 360 degrees in a complete turn?

Write down more functions of the form $f(x) = a\sin(bx) + d$ and $g(x) = a\cos(bx) + d$. Try to predict the amplitude, period and principal axis before you graph with technology.

2 **Factual** Find the amplitude, period and principal axis of $y = 3.7\sin(2x) - 2$. Predict without technology and confirm with technology.

3 **Factual** Find the amplitude, period and principal axis of $y = -1.6\cos(0.25x) + 5$. Predict without technology and confirm with technology.

4 **Factual** Identify the correct statements about the sinusoidal functions $f(x) = a\sin(bx) + d$ and $g(x) = a\cos(bx) + d$ with range $\min \le y \le \max$.

a **i** The amplitude is $|a|$, **ii** The amplitude is a.

b **i** $a = \frac{\max - \min}{2}$ **ii** $|a| = \frac{\max - \min}{2}$

c The period is equal to: **i** b **ii** $\frac{360°}{|b|}$ **iii** $\frac{360°}{b}$

d The principal axis has equation: **i** $y = |d|$ **ii** $y = d$

e **i** $d = \frac{\max + \min}{2}$ **ii** $|d| = \frac{\max + \min}{2}$

5 **Factual** What effect does changing the sign of a have on the orientation of the graph?

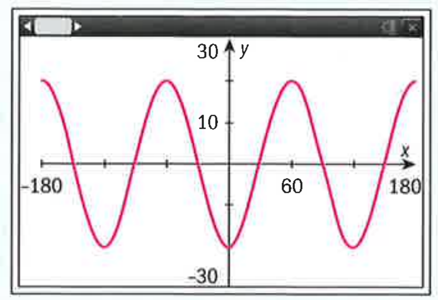
6 **Conceptual** How does changing the parameters of a sinusoidal function change its graph?

Example 2

The graph of a function of the form $y = -20\cos(3x)$ is shown on Clara's GDC.

a Write down the amplitude of the function and justify your answer.

b Write down the period of the function and justify your answer.

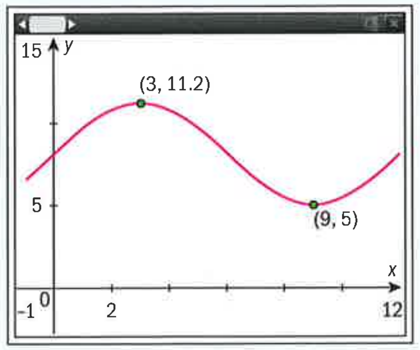


a Since the vertical distance from the principal axis to any turning point is 20 the amplitude is 20.	The graph is a reflection of the standard cosine graph in the x-axis; the amplitude is the same as that of $y = 20\cos(3x)$.
b It can be seen that the period is 120° from the graph. This is confirmed by the formula for the period $= \frac{360°}{3} = 120°$.	120° is the length of the shortest interval in which the values of the function when repeated give the entire graph.

Example 3

Simon explores tidal patterns by measuring the depth of water at the end of a pier. He finds that the data can be modelled by the function $f(x) = a\sin(bx) + d$ where x is the number of hours after 12 midnight. Simon's laptop crashes, but he finds a paper with some of his data:

Time	Coordinates	Description
3.00am	(3, 11.2)	Maximum
9.00am	(9, 5)	Minimum



a Determine the values of a, b and d in Simon's model.

b Interpret the values of a, b and d in context.

a The principal axis is $y = \frac{11.2+5}{2} = 8.1$. Hence d is 8.1. The amplitude is $\frac{11.2-5}{2} = 3.1$. Hence a is 3.1. One half of a cycle is $9 - 3 = 6$ hours long. Hence the period is 12 hours.	Apply the formulae for the principal axis and the amplitude.
$b = \frac{360}{12} = 30$	Apply the formula for the period.
Hence Simon's model is $f(x) = 3.1\sin(30x) + 8.1$.	This can be verified with technology—it gives the exact graph shown above.
b $d = 8.1$ is the average depth of the water. $a = 3.1$ is the maximum amount by which the depth differs from 8.1. $b = 30$ means that the function repeats 30 times in a domain of 360 hours. It is more useful to state that the tidal pattern repeats every 12 hours.	Reflect back to the original context.

TOK

"Beauty is the first test; there is no permanent place in the world for ugly mathematics,"—Godfrey Harold Hardy (1877–1947), a Professor of Geometry at the University of Oxford has been called the most important British pure mathematician of the first half of the 20th century.

What do we mean by elegance in mathematical proof?

Exercise 11B

1 For each function, predict the amplitude, the period, the equation of the principal axis and the range.

a $f(x) = 3\sin(4x) + 1$

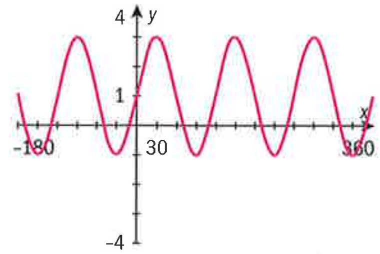
b $f(x) = 0.5\sin(0.5x) - 3$

c $f(x) = 7.1\cos(3x) + 1$

d $f(x) = -5\cos\left(\frac{x}{2}\right) + 8.1$

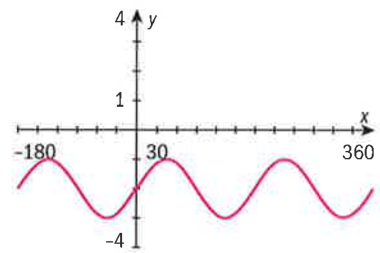
2 For the following graphs, determine the numerical values of the unspecified constants:

a



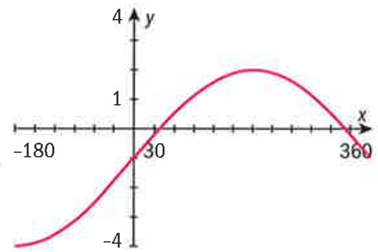
$y = a\sin(3x) + 1$

b



$y = a\sin(bx) - 2$

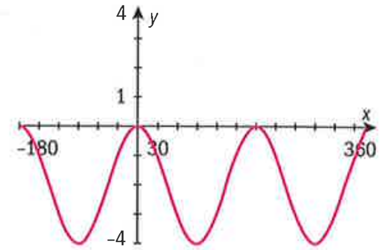
c



$y = a\sin(bx) + d$

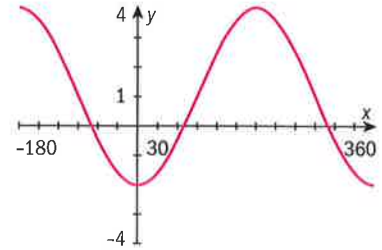
3 For the following graphs, determine the numerical values of the unspecified constants:

a



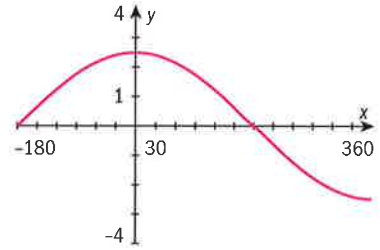
$y = 2\cos(bx) - 2$

b



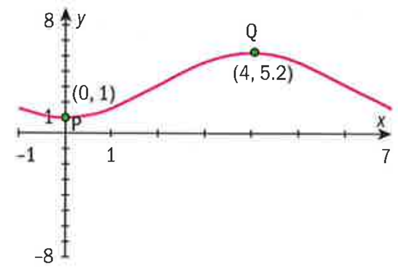
$y = a\cos(x) + d$

c



$y = a\cos(bx) + d$

4 **a** A function in the form $y = t + s\cos(rx)$ is shown. The point Q is a maximum and P is a minimum. Use this information to find the values of t, s and r.



Functions

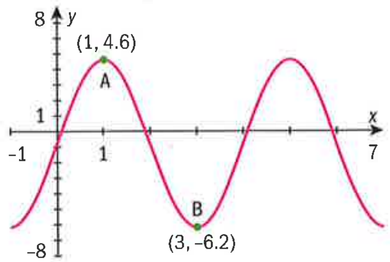
b This function models the temperature in a refrigerator unit. Given that y is measured in degrees Celsius and x is time measured in hours, determine which of t, s and r are related to the following quantities:

  i the average temperature in the unit

  ii the difference between the minimum temperature in the unit and the average temperature

  iii the time taken for the unit to complete an entire cycle of operation.

5 **a** A function in the form $y = a\sin(bx) + d$ is shown. The point A is a maximum and B is a minimum. Use this information to find the values of a, b and d.



b This function models the height above or below sea level of a part of a renewable energy generator. Given that y is measured in metres and x is time measured in minutes, interpret each of the values of a, b and d.

6 Pam explores ocean level patterns influenced by currents by measuring how far the water level varies above and below a point on an oil rig. She finds that the data can be modelled by the function $f(x) = r\cos(sx)$ where x is the number of hours after 12 midnight. Due to safety concerns, Pam can only observe the first two minimum points and she defines the principal axis as $y = 0$.

Time	Coordinates	Description
5.00am	(2.5, −10)	Minimum
11am	(7.5, −10)	Minimum

a Sketch a graph of Pam's model for the domain $0 \le x < 15$.

b Determine the values of r and s in Pam's model.

7 The parameters of the function $g(x) = a\cos(bx) + d$ are determined by throwing dice. A fair tetrahedral die numbered 1, 2, 3, 4 determines the value of a and of b. A fair three-sided die numbered 1, 2, 3 determines the value of d.

Find the probability that the resulting function has a maximum value greater than 5.

Developing inquiry skills

If one of the functions in the opening problem can be modelled by a sinusoidal function, does it have the form $y = a\sin(bx) + d$ or $y = a\cos(bx) + d$?

11.3 A world of sinusoidal models

TOK

Sine curves model musical notes and the ratios of octaves. Does this mean that music is mathematical?

In this section you will apply your knowledge and understanding of periodic functions to explore whether a certain context can be modelled by a sinusoidal function. As an example, consider the following:

1 **Situation:** Maria is planning an ecosystems trip to Verkhoyansk, a Russian town near the Arctic Circle, in order to study the permafrost. She knows that Verkhoyansk has a very cold climate and that the temperatures vary greatly.

2 **Problem:** To aid her planning, Maria wants to predict:

  a At what times in the year does the temperature change fastest? This has implications for the clothing she needs to pack.

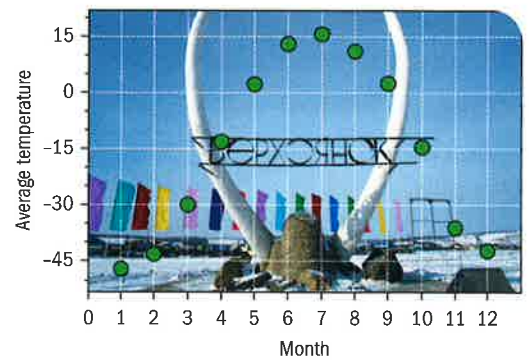
  b At what dates will the temperature be between −20°C and −5°C? This has implications for the design of her experiments.

3 **Data:** Maria gathered this data from the internet:

Month	Average temperature, °C
1	−47
2	−43
3	−30
4	−13
5	2.4
6	12.9
7	15.5
8	11.1
9	2.5
10	−14.5
11	−35.9
12	−42

4 **Develop a model**

Maria graphed her data using technology and noticed the data resembles a sinusoidal function with a period of 12 months. She then applied the knowledge you learned in the previous section to calculate the parameters:



$$|a| = \frac{\max - \min}{2} = \frac{15.5 - -47}{2} = 31.25$$

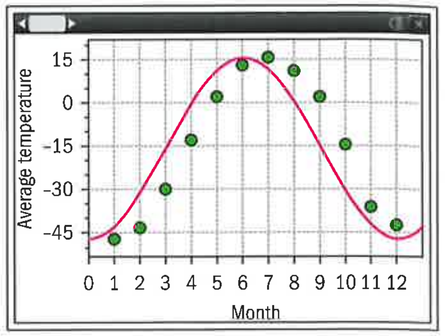
$$b = \frac{360^\circ}{12} = 30$$

$$d = \frac{\max + \min}{2} = \frac{15.5 + -47}{2} = -15.75$$

Functions

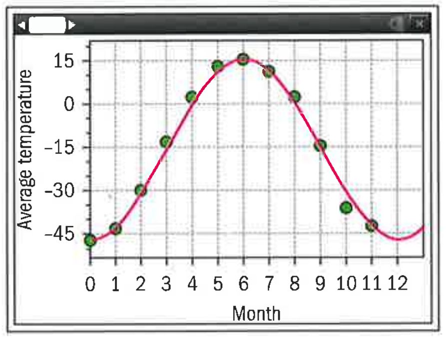
5 Test the model using technology

Maria notices that the data follows the orientation of a cosine graph reflected in the x-axis so she proposes that $y = -31.25\cos(30x) - 15.75$ would be an appropriate model and she tests it with technology.



Maria reflects on the fact that the model seems to be the correct shape but not in the correct position. She then remembers that the maximum value of $\cos x$ occurs at $x = 0$; however, her data set starts at $x = 1$.

So Maria represents January with $x = 0$, February with $x = 1$ and so on in order to make the function easier to apply.



Finally, Maria is satisfied with her model.

6 Apply the model

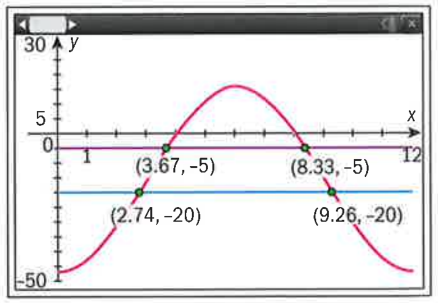
a From the graph, Maria sees that between the x-coordinates of 2 and 4 the gradient of the graph increases by approximately 15°C per month and between 8 and 10 there is a decrease of approximately 15°C per month. So, during the periods March–May and September–November she can expect considerable changes in temperature and needs to pack accordingly.

b Allowing for the x-axis starting at 0, Maria can estimate that between 3.74 and 4.67 months the temperatures will be between −20°C and −5°C.

Since

$0.74 \times 31 \approx 23$ and $0.67 \times 30 \approx 20$

this corresponds to the dates 23 March and 20 April. Similarly, 9.33 and 10.26 correspond to 10 September and 8 October.



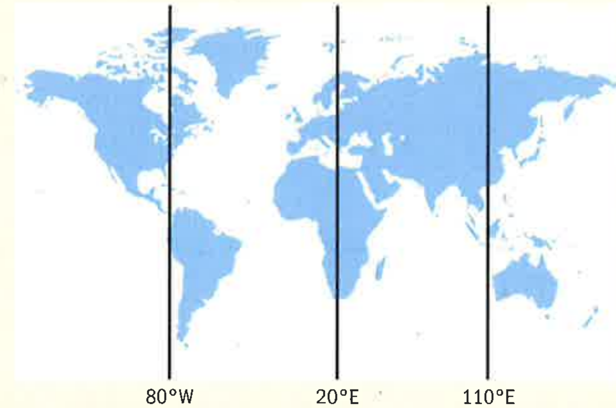
Reflection

The data used represents average monthly temperature. Therefore, the predictions cannot be considered any more precise than an average is, even if the graph of the function fits the data well. As a consequence, the dates found are only guidelines. The advantage of this model lies in simplicity not precision. Maria could more reasonably focus her planning on the end of March and April and the start of September and October than on precise dates.

Investigation 4

Aim: Collaborate to create a world map annotated with temperature graphs across different latitudes and longitudes in order to look for patterns, practise the modelling process and consider temperature data in a global perspective.

Three longitudinal lines for 80°W, 20°E and 110°E have been added to this map of the world in order to guide the sample of temperature data from large vertical landmasses. Pick one or more lines.



In a group, choose cities close to your line with a wide range of latitudes. For example, Ushuaia is the most southerly city in the world and Quito is very close to the equator. Choose one city each and follow the steps **3**, **4** and **5** of Maria's permafrost example.

Then **apply the model** to predict **a** the temperature in your chosen city on 31 October this year; **b** the temperature in your chosen city on your 40th birthday.

1 Factual Which of **a** and **b** is a more valid answer and why?

Collaborate: Collect all your graphs together in order to look for patterns in temperature data models in relation to latitude. Use technology to create a presentation of a sequence of temperature graphs to present to your peers.

2 Factual In which longitudes does the sinusoidal model fit best? Are there any cities in which the sinusoidal model is less useful? What factors influence the data apart from latitude?

3 Factual How does latitude affect the amplitude, period and principal axis of the models?

4 Factual When using technology to present a sequence of functions, is it always best to use the window chosen by default by the device? Explain your answer.

5 Conceptual When using technology, how does the choice of viewing window affect your ability to communicate patterns effectively?

6 Which natural phenomena can you think of that could be modelled by sinusoidal functions?

Functions

TOK

Trigonometry was developed by successive civilizations and cultures.

To what extent is mathematics a product of human social interaction?

Developing your toolkit

Now do the Modelling and investigation activity on page 520.

Exercise 11C

1 a Plot this data set.

x	10	11	12	13	14	15	16	17	18	19	20
y	13.4	11.2	8.3	5.2	4.1	5.2	8.5	11.8	12.7	11.5	8.3

b Find a model for the data using a function of the form $y = a\sin(bx) + d$.

2 a This data set represents water depth measurements at the mouth of a harbour. T is measured in hours from midnight and d in metres. Plot the data set.

T	0	1	2	3	4	5	6	7	8	9	10	11	12
d	10	9.68	8.55	7	5.45	4.32	4	4.32	5.45	7	8.55	9.68	10.1

b Show that a sinusoidal function models this data.

c Sam's boat needs the water to be at least 6 m deep in order for it to safely enter the harbour. Predict the range of hours between which he can sail in and out of the harbour.

3 a This data set represents measurements of part of a sound wave. Time, t, is measured in seconds and D in decibels. Use technology to plot the data set.

t	0	1.5×10^{-4}	3×10^{-4}	4.5×10^{-4}	6×10^{-4}	7.5×10^{-4}	9×10^{-4}	0.00105	0.0012	0.00135
D	0	0.4029	0.7375	0.9471	0.9961	0.8763	0.6079	0.2365	−0.175	−0.5569

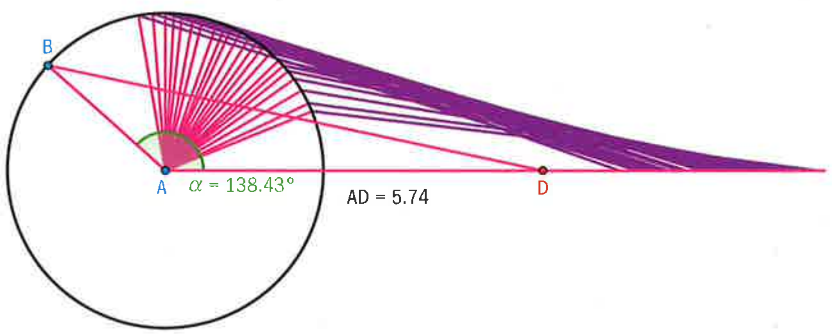
b Find a model for the data using a function of the form $y = a\sin(bx) + d$.

c Hence predict the decibels of the sound after 0.00011 seconds.

d Hence predict the decibels of the sound after 0.002 seconds.

e Comment on which of your answers to **c** and **d** are more reliable, giving a reason for your answer.

4 Pam is researching old steam engines. The following diagram shows a representation of a mechanism she is exploring.



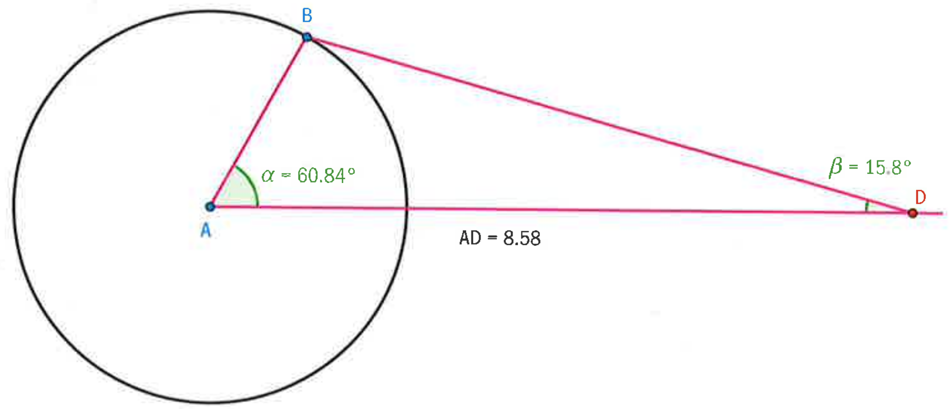
AB is fixed at 2.4 m and BD is fixed at 7.7 m. D can move only horizontally back and forth as B moves anticlockwise around the circle. AD is the distance from the point D to the centre of the circle. α is the angle $D\hat{A}B$.

The following data is collected.

α	1.06	12.2	18.2	26.6	45.9	72.1	100	149	174	196	224	252	277	295
AD	10.1	10.0	9.94	9.77	9.17	8.10	6.91	5.54	5.31	5.37	5.79	6.61	7.62	8.38

a Suggest and investigate a sinusoidal model for the data.

Pam realizes that the context can be represented in another way:



b Calculate AD in terms of the angles α and β and the lengths of AB and BD, using the cosine rule.

c Hence or otherwise comment critically on the fit of your model.

5 The times of sunrise and of sunset in Reykjavik, Iceland, on the 1st of each month are represented in this table using hours and minutes.

Month (t)	Rise	Set
0	11.20	15.42
1	10.09	17.13
2	8.37	18.43
3	6.47	20.16
4	5.01	21.49
5	3.24	23.27
6	3.05	23.57
7	4.34	22.34
8	6.09	20.47
9	7.35	19.00
10	9.09	17.13
11	10.44	15.49

a Calculate the times in decimal notation. Explain why this is necessary.

b Find sinusoidal models for sunrise $r(t)$ and for sunset $s(t)$.

c Draw the graph of $d(t) = s(t) - r(t)$, hence estimate:

i the number of days in the year with at least 18 hours of daylight

ii the fraction of a year with at least 12 hours of daylight but no more than 18.

6 Karim collects sunrise data for Cairo on the 15th of each month from www.sunrisesunset.com and represents them in this table as decimals.

Month (t)	Rise
0.5	6.87
1.5	6.58
2.5	6.08
3.5	5.48
4.5	5.03
5.5	4.88
6.5	5.07
7.5	5.37
8.5	5.65
9.5	5.95
10.5	6.33
11.5	6.72

a Find a sinusoidal model for the sunrise $r(t)$.

b Karim wishes to take time-lapse photographs at the Pyramid of Cheops on the Giza Plateau on the outskirts of Cairo. He wants to arrive there at least one hour before sunrise to set up his equipment and to capture a range of images as the sun rises. However, he does not want to arrive any earlier than 5.00am. Predict from your model between which dates he can arrive.

7 Zaida analyses the motion of a circular fan in order to calculate the speed at which it rotates. She places a coloured dot on one of the fan blades as shown and tracks its position with video tracking software, placing the origin at the centre of the fan. Zaida's laptop runs out of memory; however, she manages to collect the data shown:

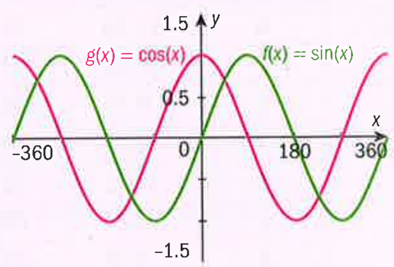
Time (t) in secs	y-coordinate of green dot (cm)
0	0
0.01	15
0.02	25.98076211
0.03	30
0.04	25.98076211
0.05	15
0.06	3.675×10^{-15}

a Find a model of the y-coordinate of the green dot as a function of time.

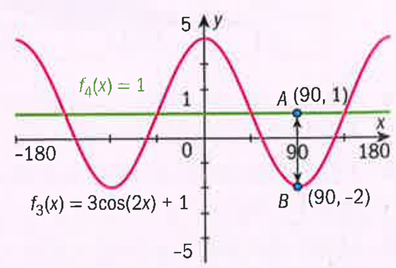
b Hence calculate the speed of the fan.

Chapter summary

- Periodic functions help you to model a set of data in the special case where there is a repeating pattern of **identical** y-values.
- The **period** of the function is the length of the **shortest** interval for which the function values when repeated give the entire graph.
- Two examples of periodic functions are $f(x) = \sin x$ and $g(x) = \cos x$. They both have range $-1 \le y \le 1$ and are examples of **sinusoidal functions.** Both have period 360°.



- In a sinusoidal function with range $\min \le y \le \max$, the **principal axis** is the line $y = \dfrac{\min + \max}{2}$ and the **amplitude** is the vertical distance from this line to any turning point of the function, as shown in this example.
- The graph of $f_3(x) = 3\cos(2x) + 1$ shown here has maximum value 4 and minimum value −2. Its principal axis $y = 1$ is shown by $f_4(x) = 1$. The amplitude is represented by the segment AB which has length 3 units. The period of $f_3(x)$ is 180°.



- The parameters a, b and d in the functions $f(x) = a\sin(bx) + d$ and $g(x) = a\cos(bx) + d$ enable you to **predict** the amplitude, period and principal axis respectively.
- To apply a sinusoidal model of the form $f(x) = a\sin(bx) + d$ or $g(x) = a\cos(bx) + d$ to a data set, the amplitude, period and principal axis must be **estimated** from the data.

Developing inquiry skills

Return to the opening problem. You were given a graph showing the average monthly temperature for Riga compiled over a period of two years and a graph showing the number of babies born on each day of June 1978 in the USA.

Now that you have learned about sinusoidal models, you can reflect on the fact that a sinusoidal model fits only one of these sets of data well.

Which one?

Why is that?

What kind of phenomena can be modelled by periodic functions?

What kind of phenomena can be modelled by sinusoidal models?

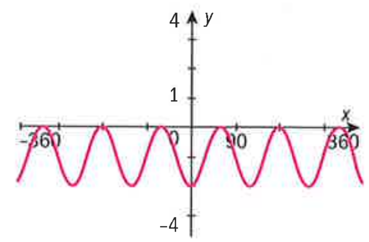
Chapter review

Click here for a mixed review exercise

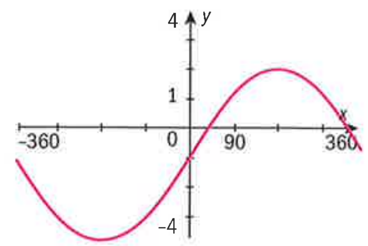
1 Solve the equation $-0.4\cos t + 1 = 1.3$ where $301° \le t \le 700°$.

2 For the following graphs, find the equation of the function.

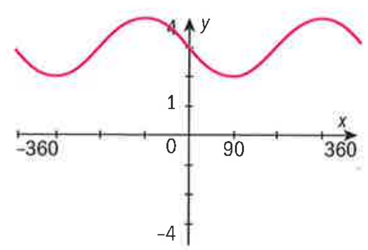
a



b



c

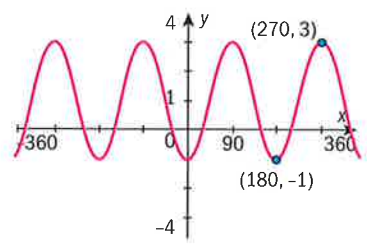


3 Given $f(x) = 3x$ and $g(x) = -2\cos x$, consider the functions

$a(x) = f(x) \times g(x)$, $b(x) = f(x) + g(x)$, $c(x) = f(g(x))$ and $d(x) = g(f(x))$.

Find the period, amplitude and equation of the principal axis for any of these functions that are periodic.

4 The graph of a sinusoidal function $f(x)$ on the domain [−360°, 360°] is shown below.



There is a maximum at (270, 3) and a minimum at (180, −1).

a Find the equation of the function.

b The equation $f(x) = \beta$ has exactly five solutions. Find the value of β.

5 A water wave passes a sensor in a physics laboratory. As the wave passes, the depth of the water in metres (d) after time t seconds is modelled by the function $d = 18.3 - 0.47\cos(20.9t)$.

a What are the maximum and minimum values of d?

b Find the first time after 19 seconds at which the depth is 18 m.

Functions

6 Teodora examines old photos of the Neue Elbbrücke bridge in Hamburg. She finds out that the three sections shown are each 102 m long and she estimates that the vertical distance from the surface of the road to the top of the bridge is 16 m and that the road is 6 m above the river. She uses this information to model the bridge with functions as shown below:

a Find a sinusoidal model of the bridge and draw a diagram like Teodora's.

b Hence or otherwise find an estimate for the coordinates of the points shown.

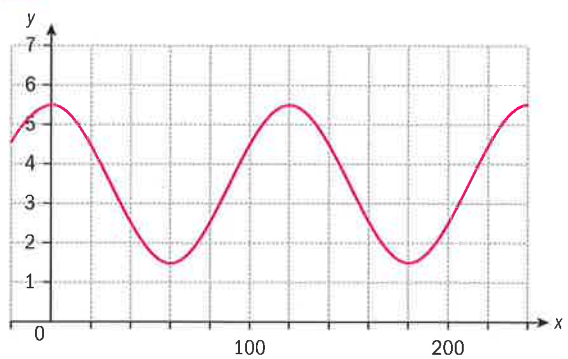
Exam-style questions

7 P1: Consider the curve given by the equation $y = 4.1 + 2.3\cos\left(\frac{x}{2}\right)$.

a Find the minimum and maximum values for y. (2 marks)

b Find the first two positive values of x for which $2.5 = 4.1 + 2.3\cos\left(\frac{x}{2}\right)$. (3 marks)

8 P1: The following shows a portion of the graph of $y = p + q\cos(rx)$.



Determine the values of the constants p, q and r. (6 marks)

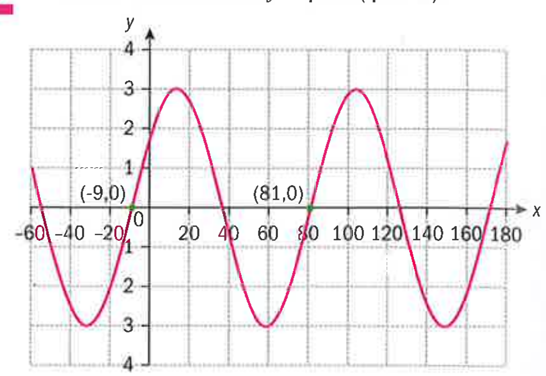
9 P1: A sinusoidal function is given by the equation $y = 20\sin(4x)$ for $0° \leq x \leq 180°$.

a Sketch the graph of $y = f(x)$. (3 marks)

b State the equation of the principal axis of the curve. (1 mark)

c Using technology, solve the inequality $f(x) < -5$. (4 marks)

10 P1: The following diagram shows a sinusoidal curve of the form $y = p\sin(qx + r)$.



Determine the values of the constants p, q and r. (8 marks)

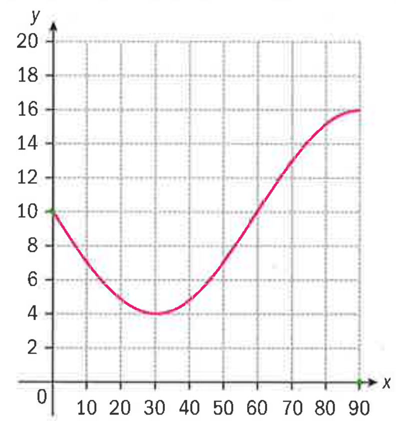
11 P1: A sinusoidal wave is given by the equation $y = 5.7 + 0.3\cos(12.5x)$.

a Write down the amplitude of the wave. (1 mark)

b Find the minimum value of y and first the value of x for when this occurs. (3 marks)

c Find the period of the wave. (2 marks)

12 P1: Below is a portion of the graph of $y = p - q\sin(rx)$ $(p > 0, q > 0, r > 0)$.



a On the same axes, sketch the graph of $y = p + \frac{q}{2}\sin(rx)$ (2 marks)

b Determine the values of the constants p, q and r (7 marks)

13 P1: A sinusoidal curve $y = a + b\sin(cx)$ $(a > 0, b > 0, c > 0)$ has two concurrent maximum points at coordinates $(60°, 5)$ and $(300°, 5)$.

Given that it also has an amplitude of 2 units, determine the values of a and b, and determine the least possible value of c. (8 marks)

14 P2: Over the course of a single December day in Limassol, Cyprus, the highest temperature was found to be 22°C. The lowest temperature was 12°C, which occurred at 03.00am

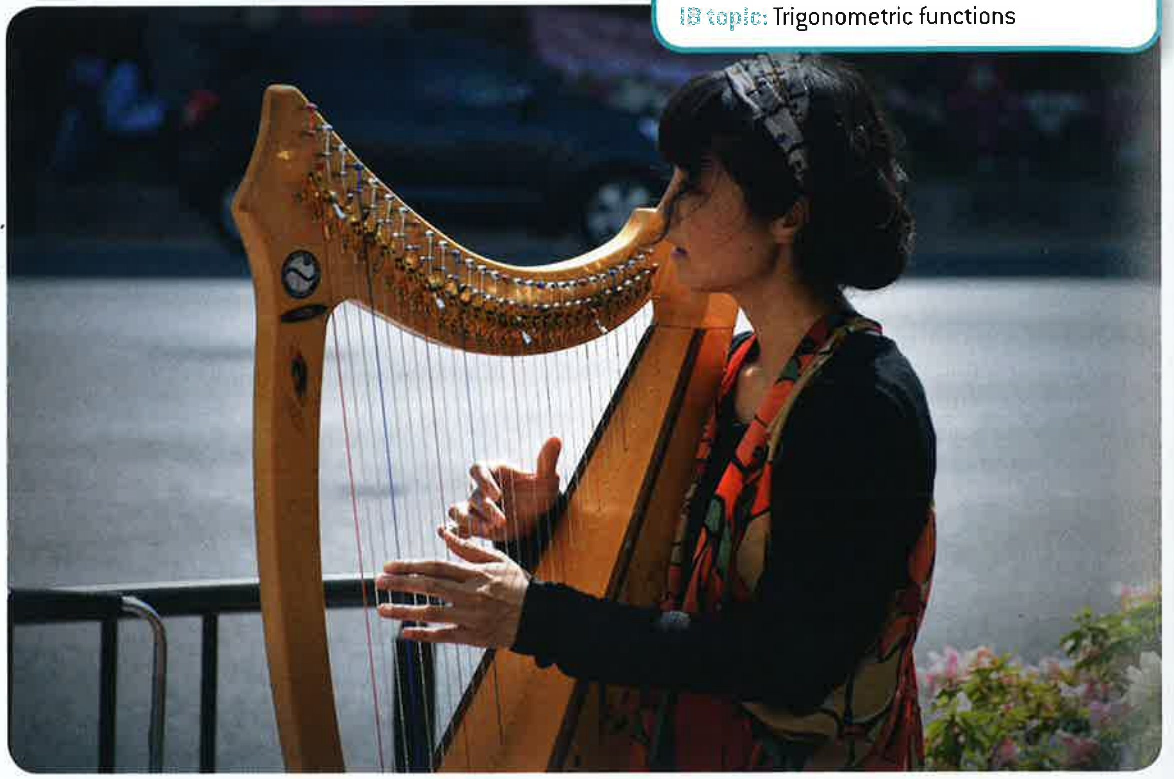
If t is the number of hours since midnight, the temperature T may be modelled by the equation $T = A\sin(B(t - C)) + D$.

a Find the values of A, B, C and D. (9 marks)

b Hence, by using technology, find for how many hours the daily temperature was above 20°C. (4 marks)

The sound of mathematics!

Approaches to learning: Research, Critical thinking, Using technology
Exploration criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)
IB topic: Trigonometric functions



Brainstorm

In small groups, brainstorm some ideas that link music and mathematics.

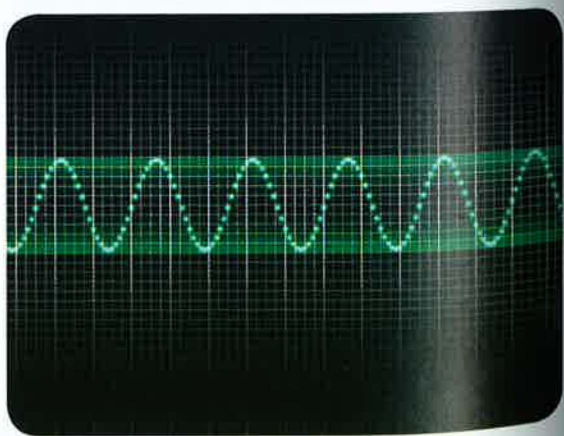
Construct a **mindmap** from your discussion with the topic "MUSIC" in the centre.

Share your mindmaps with the whole class and discuss.

This task concentrates on the relationship between **sound waves** and **trigonometric functions.**

Research

Research how sound waves and trigonometric functions are linked.



Think about:

How do vibrations cause sound waves?

How do sound waves travel?

What curve can be used to model a sound wave?

The fundamental properties of a basic sound wave are its **frequency** and its **amplitude.**

What is the frequency of a sound?

What units is it measured in?

What is the amplitude of a sound?

Use what you studied in this chapter to answer these questions:

If you have a sine wave with the basic form $y = a\sin(bt)$, where t is measured in seconds, how do you determine its period, frequency and its amplitude?

What do the values of a and b represent? What does y represent in this function?

With this information, determine the equivalent sine wave for a sound of 440 Hz.

Did you know?

This is in fact the sine wave equation for the note A.

Technology

There are a large number of useful programs that you can use to consider sound waves.

Using these programs it is possible to record or generate a sound and view a graphical representation of the sound wave with respect to time.

If you have a music department in your school and/or access to such a program, you could try this.

Design an investigation

Using the available technology and the information provided here, what could you investigate and explore further?

What experiment could you design regarding sounds?

Disucss your ideas with your group.

What exactly would be the aim of each investigation, exploration or experiment that your group thought of?

Select one of the ideas in your group and plan further.

What will you need to think about as you conduct this exploration?

How will you ensure that your results are reliable?

How will you know that you have completed the exploration and answered the aim?

Extension

Trigonometric functions also occur in many other areas that aid your understanding of the physical world.

Some examples are:

- temperature modelling
- tidal measurements
- the motion of springs and pendulums
- the electromagnetic spectrum.

They can be thought of in similar terms of waves with different frequencies, periods, amplitudes and phase shifts too.

Think about the task that you have completed here and consider how you could collect other data that can be modelled using a trigonometric function. You could use one of the examples here or research your own idea.